PACMAN RENORMALIZATION AND SELF-SIMILARITY OF THE MANDELBROT SET NEAR SIEGEL PARAMETERS

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Abstract. In the 1980s Branner and Douady discovered a surgery relating various limbs of the Mandelbrot set. We put this surgery in the framework of "Pacman Renormalization Theory" that combines features of quadratic-like and Siegel renormalizations. We show that Siegel renormalization periodic points (constructed by McMullen in the 1990s) can be promoted to pacman renormalization periodic points. Then we prove that these periodic points are hyperbolic with one-dimensional unstable manifold. As a consequence, we obtain the scaling laws for the centers of satellite components of the Mandelbrot set near the corresponding Siegel parameters.

Contents

1. Introduction

1.1. **Statements of the results.** Though the Mandelbrot set $\mathcal{M}$ is highly non-homogeneous, it possesses some remarkable self-similarity features. Most notable is the presence of baby Mandelbrot sets inside $\mathcal{M}$ which are almost indistinguishable from $\mathcal{M}$ itself. The explanation of this phenomenon is provided by the Renormalization Theory for quadratic-like maps, which has been a central theme in Holomorphic Dynamics since the mid-1980s (see [DH2, S, McM1, L1] and references therein).

By exploring the pictures, one can also observe that the Mandelbrot set has self-similarity features near its main cardioid. For instance, as Figure 2 indicates, near the (anti-)golden mean point, the $(\mathfrak{p}_n/\mathfrak{p}_{n+2})$-limbs of $\mathcal{M}$ scale down at rate λ^{-2n},

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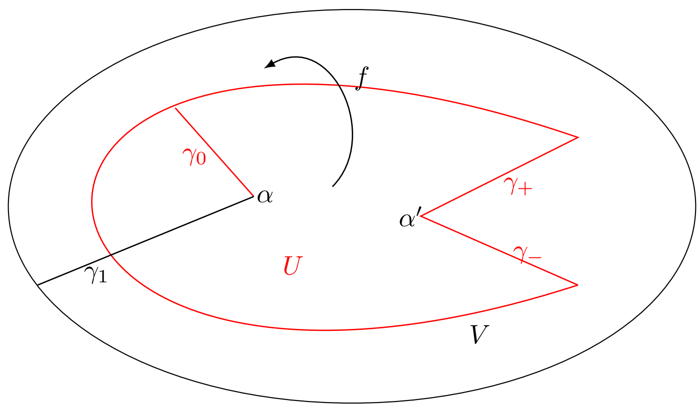

Figure 1. A (full) pacman is a $2:1$ map $f : U \to V$ such that the critical arc γ_1 has 3 preimages: γ_0, γ_+ and γ_-.

where $\lambda = (1+\sqrt{5})/2$ and $\mathfrak{p}_n$ are the Fibonacci numbers. The goal of this paper is to develop a renormalization theory responsible for this phenomenon.

Our renormalization operator acts on the space of "pacmen", which are holomorphic maps $f : (U,\alpha) \to (V,\alpha)$ between two nested domains, see Figure 1, such that $f : U \setminus \gamma_0 \to V \smallsetminus \gamma_1$ is a double branched covering, where γ_1 is an arc connecting α to ∂V. The *pacman renormalization* $\mathcal{R}f$ of f (see Figure 4) is defined by removing the sector S_1 bounded by γ_1 and its image γ_2, identifying its boundary components by the dynamics, and taking the second iterate of f on the sector $S_0 \subset f^{-1}(S_1)$ between γ_0 and γ_1, while keeping the original map on $U \smallsetminus f^{-1}(S_1)$. (See §2 for precise definitions.) Note that it acts on the rotation numbers (see Appendix A and, in particular, (A.2)) as

$$\theta \longrightarrow \frac{\theta}{1-\theta} \quad \text{if } 0 \le \theta \le \frac{1}{2}; \qquad \theta \longrightarrow \frac{2\theta - 1}{\theta} \quad \text{if } \frac{1}{2} \le \theta \le 1. \tag{1.1}$$

A pacman is called *Siegel* with rotation number θ if α is a Siegel fixed point with rotation number θ whose closed Siegel disk is a quasidisk compactly contained in U (subject of extra technical assumption, see Definition 3.1).

Theorem 1.1. *For any rotation number θ periodic under* (1.1) *(e.g. with periodic continued fraction expansion), the pacman renormalization operator $\mathcal{R}$ has a unique periodic point $f_\star$ which is a Siegel pacman with rotation number θ. This periodic point is hyperbolic with one-dimensional unstable manifold. Moreover, the stable manifold of $f_\star$ consists of all Siegel pacmen.*

Let $c(\theta)$, $\theta \in \mathbb{R}/\mathbb{Z}$, be the parameterization of the main cardioid $\mathcal{C}$ by the rotation number θ. At any parabolic point $c(\mathfrak{p}/\mathfrak{q})$, there is a satellite hyperbolic component

$\mathbf{\Delta}_{\mathfrak{p}/\mathfrak{q}}$ of $\mathcal{M}$ attached to $c(\mathfrak{p}/\mathfrak{q})$. Let $a_{\mathfrak{p}/\mathfrak{q}}$ be the *center* of this component, i.e., the unique superattracting parameter inside $\mathbf{\Delta}_{\mathfrak{p}/\mathfrak{q}}$.

In this paper, notation $\alpha_n \sim \beta_n$ will mean that $\alpha_n/\beta_n \to \text{const} \neq 0$.

Theorem 1.2. *Let θ be a rotation number periodic under* (1.1)*, and let $\mathfrak{p}_n/\mathfrak{q}_n$ be its continued fraction approximands. Then*

$$|c(\theta) - a_{\mathfrak{p}_n/\mathfrak{q}_n}| \sim \frac{1}{\mathfrak{q}_n^2}.$$

The above results can be generalize to the case of rotation numbers of *bounded* type. We conjecture that they extend to arbitrary combinatorics, which would provide us with a good geometric control of the *molecule* of the Mandelbrot set (see Appendix C).

1.2. **Outline of the proof.** We let:
– $\mathbf{e}(z) = e^{2\pi i z}$;
– $p_\theta : z \mapsto \mathbf{e}(\theta) z + z^2$;
– $\mathcal{P}_\theta$ be the set of pacmen with rotation number $\theta \in \mathbb{R}/\mathbb{Z}$;
– Θ_{per} be the set of *combinatorially periodic* rotation numbers (i.e., rotation numbers periodic under (1.1) – this includes numbers with periodic continued fraction expansion, or equivalently, periodic quadratic irrationals);
– Θ_{bnd} be the set of *combinatorially bounded* rotation numbers (i.e., rotation numbers with continued fraction expansion where all its coefficients are bounded).

Any holomorphic map $f : (U_f, \alpha) \to (\mathbb{C}, \alpha)$ whose fixed point α is neutral with rotation number $\theta \in \Theta_{\text{per}}$ is locally linearizable near α. Its maximal completely invariant linearization domain Z_f is called the *Siegel disk* of f. If $\overline{Z}_f$ is a quasidisk compactly contained in U_f whose boundary contains exactly one critical point, then f is called a (unicritical) *Siegel map*. For any $\theta \in \Theta_{\text{per}}$, the quadratic polynomial p_θ and any Siegel pacman (§3) give examples of Siegel maps.

There are two versions of the Siegel Renormalization theory: *holomorphic commuting pairs* renormalization and the *cylinder renormalization*. The former was developed by McMullen [McM2] who proved, for any rotation number $\theta \in \Theta_{\text{per}}$, the existence of a renormalization periodic point $f_\star$ and the exponential convergence of the renormalizations $\mathcal{R}_{\text{cp}}^n(p_\theta)$ to the orbit of $f_\star$. McMullen has also studied the maximum domain of analyticity for $f_\star$.

The cylinder renormalization $\mathcal{R}_{\text{cyl}}$ was introduced by Yampolsky who showed that $f_\star$ can be transformed into a periodic point for $\mathcal{R}_{\text{cyl}}$ with a *finite codimension* stable manifold $\mathcal{W}^s(f_\star)$ and *at least one-dimensional* unstable manifold $\mathcal{W}^u(f_\star)$ [Ya]. However, the conjecture that $f_\star$ is hyperbolic with $\dim \mathcal{W}^u(f_\star) = 1$ remained unsettled.

Let us now select our favorite $\theta \in \Theta_{\text{per}}$; it is fixed under some iterate of (1.1). Then the corresponding iterate of the Siegel renormalization fixes $f_\star$, so below we will refer to the $f_\star$ as "renormalization fixed points".

We start our paper (§2) by discussing an interplay between a "pacman" and a "prepacman". The latter (see Figure 5) is a piecewise holomorphic map with two branches $f_\pm : U_\pm \to S$, one of which is univalent while the other has "degree 1.5", with a single critical point. Such an object can be obtained from a pacman by cutting along the critical arc γ_1. For a technical reason, we "truncate" both pacmen and prepacmen by removing a small disk around the co-α point, see Figure 3.

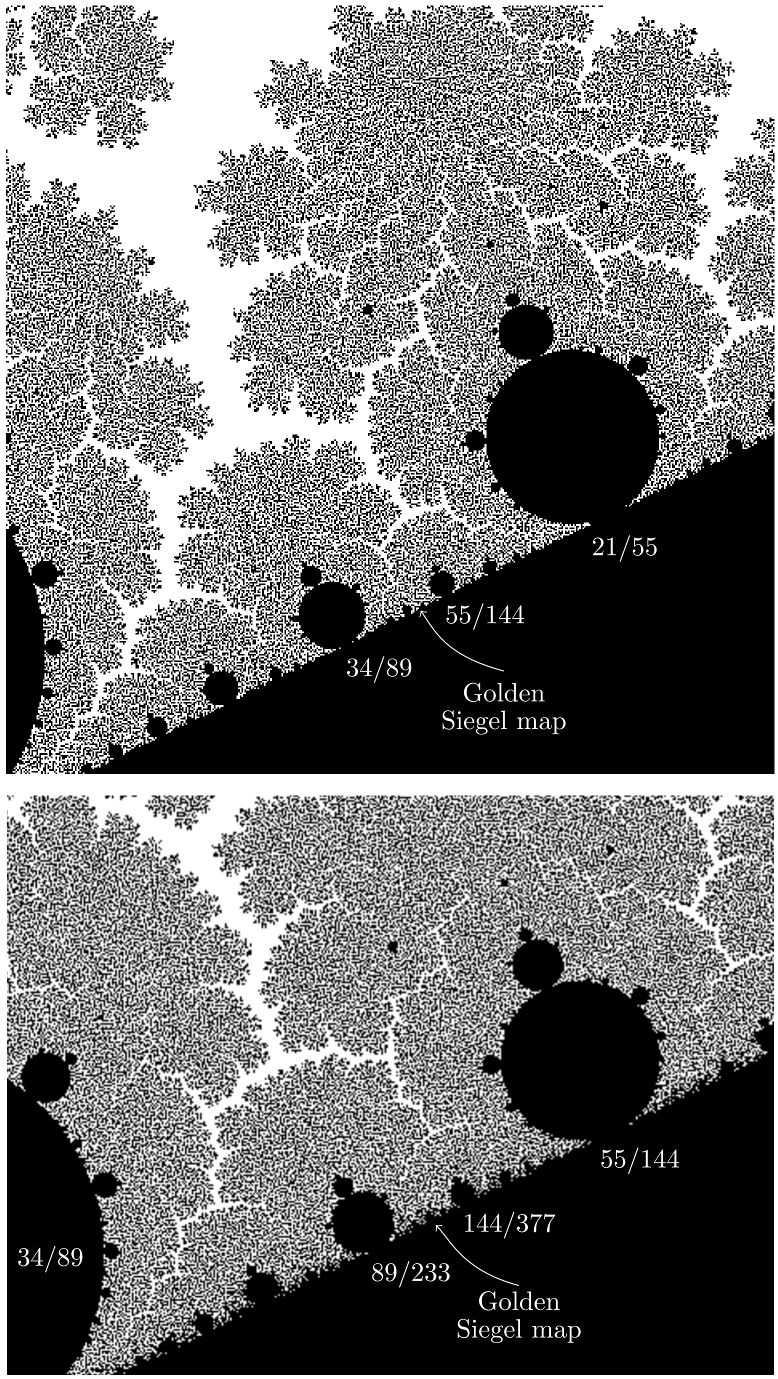

FIGURE 2. Limbs $8/21, 21/55, 55/144, 144/377, \ldots$ scale geometrically fast on the right side of the (anti-)golden Siegel parameter, while limbs $5/13, 13/34, 34/89, 89/233, \ldots$ scale geometrically fast on the left side. The bottom picture is a zoom of the top picture.

Then we define, in three steps, the pacman renormalization. First we define a "pre-renormalization" (Definition 2.3) as a prepacman obtained as the first return map to an appropriate sector S. Then, by gluing the boundary arcs of S, we obtain an "abstract" pacman. Finally, we embed this pacman back to the complex plane.

There are some choices involved in this definition. We proceed to show that near any renormalizable pacman f, the choices can be made so that we obtain a holomorphic operator $\mathcal{R}$ in a Banach ball (Theorem 2.7).

In Section 3 we analyze the structure of Siegel pacmen f. The key result is that any Siegel map can be renormalized to a Siegel pacman (Corollary 3.7), where the rotation number changes as an iterate of (1.1), see Lemma 3.17.

In case when $f = f_\star$ is the Siegel renormalization fixed point, this provides us with the pacman renormalization fixed point (§3.7). Moreover, the pacman renormalization $\mathcal{R}$ becomes a compact holomorphic operator in a Banach neighborhood of $f_\star$, with at least one-dimensional unstable manifolds $\mathcal{W}^u(f_\star)$, see Theorem 3.16.

Along the lines, we introduce and discuss the associated geometric objects (§3.1): the pacman "Julia sets" $\mathfrak{K}(f)$ and $\mathfrak{J}(f)$, "bubble chains", and "external rays". We also use them to show, via the Pullback Argument, that any two combinatorially equivalent Siegel pacmen are hybrid equivalent (Theorem 3.11), i.e. there is a qc conjugacy between them which is conformal on the Siegel disk.

For a Siegel pacman $f_\star$, any renormalization prepacman can be "spread around" (see Figure 6) to provide us with a dynamical tiling of a neighborhood of the Siegel disk, see §4.3 and Figure 13. Moreover, this tiling is robust under perturbations of $f_\star$, even when the rotation number gets changed, see Theorem 4.6. In this case, the domain filled with the tiles can be used as the central "bubble" for the perturbed map f, replacing for many purposes the original Siegel disk $Z_\star$ of $f_\star$. In particular, it allows us to control long-term f^n-pullbacks of small disks D centered at $\partial Z_\star$ (making sure that these pullbacks are not "bitten" by the pacman mouth). This is the crucial technical result of this paper (Key Lemma 4.8).

When $f_\star$ is the renormalization fixed point and the perturbed map f belongs to its unstable manifold $\mathcal{W}^u(f_\star)$ then we can apply this construction to the anti-renormalizations $\mathcal{R}^{-n} f$. This allows us to show that the maximal holomorphic extension of the associated prepacman is a σ-proper map $\mathbf{F} = (\mathbf{f}_\pm : \mathbf{X}_\pm \to \mathbb{C})$, where $\mathbf{X}_\pm$ are plane domains (Theorem 5.1).

Applying this result to a parabolic map $f \in W^u(f_\star)$, we conclude that its attracting Leau-Fatou flower contains the critical point, so the critical point is non-escaping under the dynamics (Corollary 6.4).

After this preparation, we are ready for proving Theorem 1.1, see §7. Assuming for the sake of contradiction that $\dim \mathcal{W}^u(f_\star) > 1$, we can find a holomorphic curve $\Gamma_\star \subset \mathcal{W}^u(f_\star)$ through $f_\star$ consisting of Siegel pacmen with the same rotation number. Approximating this curve with parabolic curves $\Gamma_n \subset \mathcal{W}^u(f_\star)$, we conclude that the critical point is non-escaping for $f \in \Gamma_\star$. This allows us to apply Yampolsky's holomorphic motions argument [Ya] to show that $\dim \mathcal{W}^u(f_\star) = 1$.

Finally, using the Small Orbits argument of [L1], we prove that $f_\star$ is hyperbolic under the pacman renormalization, completing the proof.

Along the lines we prove the stability of Siegel maps (see Corollary 7.9): if a small perturbation of a Siegel map f fixes the multiplier of the α-fixed point, then the new map g is again a Siegel map. Moreover, the Siegel quasidisk $\overline{Z}_g$ is in a small neighborhood of $\overline{Z}_f$.

To derive Theorem 1.2 from Theorem 1.1, we need to show that the centers of the hyperbolic components in question are represented on the unstable manifold $\mathcal{W}^u(f_\star)$. We first show that the roots of these components are represented on $\mathcal{W}^u(f_\star)$ which requires good control of the corresponding pacmen Julia sets (see §6.5), and robustness of the renormalization with respect to a particular choice of cutting arcs, see Appendix B. Then we use quasiconformal deformation techniques to reach the desired centers from the parabolic points, see §8.

Through out the paper we use Appendix B containing a topological preparation justifying robustness of the anti-renormalizations with respect to the choice of cutting arcs.

In Appendix C we formulate the Molecule conjecture on existence of pacman hyperbolic operator with the one-dimension unstable foliation whose horseshoe is parametrized by parameters from the boundary of the main molecule. A closely related conjecture is the upper semi-continuity of the mother hedgehog.

1.3. **More historical comments.** Renormalization of Siegel maps appeared first in the work by physicists (see [Wi, MN, MP]) as a mechanism for self-similarity of the golden mean Siegel disk near the critical point. A few years later, Douady and Ghys discovered a surgery that reduces previously unaccessible geometric problems for Siegel disks[1] of bounded type to much better understood problems for critical circle maps. This led, in particular, to the local connectivity result for Siegel Julia sets of bounded type (Petersen [Pe]) and also became a key to the mathematical study of the Siegel renormalization. In particular, McMullen-Yampolsky theory mentioned above is based upon this machinery.

On the other hand, in the mid 2000's, Inou and Shishikura proved the existence and hyperbolicity of Siegel renormalization fixed points *of sufficiently high combinatorial type* using a completely different approach, based upon the parabolic perturbation theory [IS].

The Siegel renormalization theory achieved further prominence when it was used for constructing examples of Julia sets of positive area (see Buff-Cheritat [BC] and Avila-Lyubich [AL2]).

A different line of research emerged in the 1980s in the work of Branner and Douady who discovered a *surgery* that embeds the 1/2-limb of the Mandelbrot set into the 1/3-limb [BD]. This surgery is the prototype for the pacman renormalization that we are developing in this paper.

Note also that according to the Yoccoz inequality, the $\mathfrak{p}/\mathfrak{q}$-limb of the Mandelbrot set has size $O(1/\mathfrak{q})$. It is believed, though, that $1/\mathfrak{q}^2$ is the right scale. The pacman renormalization can eventually provide an insight into this problem.

Remark 1.3. *Genadi Levin has informed us about his unpablished work where it is proven, by different methods, that*

$$|a_{\mathfrak{p}/\mathfrak{q}} - c(\mathfrak{p}/\mathfrak{q})| = O(1/\mathfrak{q}^2), \tag{1.2}$$

where $a_{\mathfrak{p}/\mathfrak{q}}$ *is the center of the* $\mathfrak{p}/\mathfrak{q}$*-satellite hyperbolic component and* $c(\mathfrak{p}/\mathfrak{q})$ *is its root. He has also informed us that* (1.2) *has been independently established by Mitsuhiro Shishikura.*

[1]The original surgery applies to Siegel polynomials only. Its extension to general Siegel maps leads to *quasicritical* circle maps, see [AL2].

1.4. **Notation.** We often write a partial map as $f\colon W \dashrightarrow W$; this means that $\operatorname{Dom} f \cup \operatorname{Im} f \subset W$.

A *simple arc* is an embedding a closed interval. We often say that a simple arc $\ell\colon [0,1] \to \mathbb{C}$ *connects* $\ell(0)$ and $\ell(1)$. A *simple closed curve* or a *Jordan curve* is an embedding of the unit circle. A *simple curve* is either a simple closed curve or a simple arc.

A *closed topological disk* is a subset of a plane homeomorphic to the closed unite disk. In particular, the boundary of a closed topological disk is a Jordan curve. A *quasidisk* is a closed topological disk qc-homeomorphic to the closed unit disk.

Given a subset U of the plane, we denote by $\operatorname{int} U$ the interior of U.

Let U be a closed topological disk. For simplicity we say that a homeomorphism $f\colon U \to \mathbb{C}$ is *conformal* if $f \mid \operatorname{int} U$ is conformal. Note that if U is a quasidisk, then such an f admits a qc extension through ∂U.

A *closed sector*, or *topological triangle* S is a closed topological disk with two distinguished simple arcs γ_- , γ_+ in ∂S meeting at the *vertex* v *of* S satisfying $\{v\} = \gamma_- \cap \gamma_+$. Suppose further that γ_- , $\operatorname{int} S, \gamma_+$ have clockwise orientation at v. Then γ_- is called the *left boundary of* S while γ_+ is called the *right boundary* of S. A closed *topological rectangle* is a closed topological disk with four marked sides.

Let $f : (W, \alpha) \to (\mathbb{C}, \alpha)$ be a holomorphic map with a distinguished α-fixed point. We will usually denote by λ the multiplier at the α-fixed point. If $\lambda = \mathbf{e}(\phi)$ with $\phi \in \mathbb{R}$, then ϕ is called the *rotation number of* f. If, moreover, $\phi = \mathfrak{p}/\mathfrak{q} \in \mathbb{Q}$, then $\mathfrak{p}/\mathfrak{q}$ is also the *combinatorial rotation number*: there are exactly $\mathfrak{q}$ local attracting petals at α and f maps the i-th petal to $i + \mathfrak{p}$ counting counterclockwise.

Consider a continuous map $f\colon U \to \mathbb{C}$ and let $S \subset \mathbb{C}$ be a connected set. An *f-lift* is a connected component of $f^{-1}(S)$. Let

$$x_0, x_1, \dots x_n, \quad x_{i+1} = f(x_i)$$

be an f orbit with $x_n \in S$. The connected component of $f^{-n}(S)$ containing x_0 is called the *pullback of* S *along the orbit* $x_0, \dots, x_n$.

To keep notations simple, we will often suppress indices. For example, we denote a pacman by $f\colon U_f \to V$, however a pacman indexed by i is denoted as $f_i\colon U_i \to V$ instead of $f_i\colon U_{f_i} \to V$.

Consider two partial maps $f\colon X \dashrightarrow X$ and $g\colon Y \dashrightarrow Y$. A homeomorphism $h\colon X \to Y$ is *equivariant* if

$$h \circ f(x) = g \circ h(x) \tag{1.3}$$

for all x with $x \in \operatorname{Dom} f$ and $h(x) \in \operatorname{Dom} g$. If (1.3) holds for all $x \in T$, then we say that h is *equivariant on* T.

We will usually denote an analytic renormalization operator as "$\mathcal{R}$", i.e. $\mathcal{R}f$ is a renormalization of f obtained by an analytic change of variables. A renormalization postcompose with a straightening will be denoted by "$\boldsymbol{R}$"; for example, $\boldsymbol{R}_s\colon \mathcal{M}_s \to \mathcal{M}$ is the Douady-Hubbard straightening map from a small copy $\mathcal{M}_s$ of $\mathcal{M}$ to the Mandelbrot set. The action of the renormalization operator on the rotation numbers will be denoted by "R".

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Figures 2 and 31 are made with W. Jung's program *Mandel*.

2. Pacman renormalization operator

Definition 2.1 (Full pacman)**.** Consider a closed topological disk $\overline{V}$ with a simple arc γ_1 connecting a boundary point of V to a point α in the interior. We will call γ_1 the *critical arc* of the pacman.

A *full pacman* is a map

$$f : \overline{U} \to \overline{V}$$

such that (see Figure 1)

- $f(\alpha) = \alpha$;
- $\overline{U}$ is a closed topological disks with $\overline{U} \subset V$;
- the critical arc γ_1 has exactly 3 lifts $\gamma_0 \subset U$ and $\gamma_-, \gamma_+ \subset \partial U$ such that γ_0 start at the fixed point α while γ_-, γ_+ start at the pre-fixed point α'; we assume that γ_1 does not intersect $\gamma_0, \gamma_-, \gamma_+$ away from α;
- $f : U \to V$ is analytic and $f : U \setminus \gamma_0 \to V \setminus \gamma_1$ is a two-to-one branched covering;
- f admits a locally conformal extension through $\partial U \setminus \{\alpha'\}$.

Since $f : U \setminus \gamma_0 \to V \setminus \gamma_1$ is a two-to-one branched cover, f has a unique critical point, called $c_0(f)$, in $U \setminus \gamma_0$. We denote by $c_1(f)$ the image of c_0.

We will mostly consider truncated pacmen or simply pacmen defined as follows. Consider first a full pacman $f : U \to V$ and let O be a small closed topological disk around $\alpha \in \operatorname{int} O \not\ni c_1(f)$ and assume that γ_1 cross-intersects ∂O at single point. Then $f^{-1}(O)$ consists of two connected components, call them $O_0 \ni \alpha$ and $O_0' \ni \alpha'$. We obtain a truncated pacman

$$f : (U \setminus O_0', O_0) \to (V, O). \tag{2.1}$$

A *pacman* is an analytic map as in (2.1) admitting a locally conformal extension through ∂U such that f can be topologically extended to a full pacman, see Figure 3. In particular, every point in $V \setminus O$ has two preimages while every point in O has a single preimage.

2.1. Dynamical objects.

Let us fix a pacman $f : U \to V$. Note that objects below are sensitive to small deformations of ∂U.

The *non-escaping* set of a pacman is

$$\mathfrak{K}_f := \bigcap_{n \geq 0} f^{-n}(\overline{U}).$$

The *escaping set* is $V \setminus \mathfrak{K}_f$.

We recognize the following two subsets of the boundary of U: the *external boundary* $\partial^{\text{ext}} U := f^{-1}(\partial V)$ and *the forbidden part of the boundary* $\partial^{\text{frb}} U := \overline{\partial U \setminus \partial^{\text{ext}} U}$.

Suppose $\ell_0 : [0, 1] \to \overline{V}$ is an arc connecting a point in $\mathfrak{K}_f$ to ∂V. We define inductively images $\ell_m : [0, 1] \to V$ for $m \leq M \in \{1, 2, \dots, \infty\}$ as follows. Suppose $t_m \leq 1$ be the maximal such that the image of $[0, t_m]$ under ℓ_m is within $\overline{U}$. If $\ell_m(t_m) \in \partial^{\text{ext}} U$, then we say ℓ_{m+1} is *defined* and we set $\ell_{m+1}(t) := f(\ell_m(t/t_m))$ for $t \leq 1$. Abusing notation, we write

$$\ell_m = f(\ell_{m-1}).$$

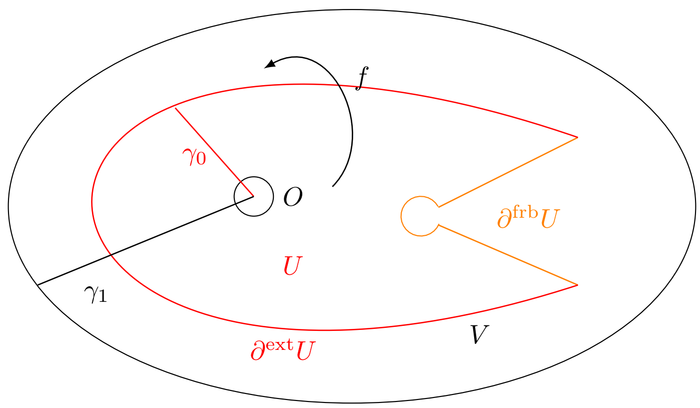


FIGURE 3. A pacman is a truncated version of a full pacman, see Figure 1; it is an almost $2:1$ map $f:(U,O_0)\to(V,O)$ with $f(\partial U)\subset\partial V\cup\gamma_1\cup\partial O$.

We define *external rays* of a pacman in the following way. Let us embed a rectangle $\mathfrak{R}$ in $\overline{V}\setminus U$ so that bottom horizontal side B is equal to $\partial^{\text{ext}}U$ and the top horizontal side T is a subset of ∂V. The images of the vertical lines within $\mathfrak{R}$ form a lamination of $\overline{V}\setminus U$. We pull back this lamination to all iterated preimages $f^{-n}(\mathfrak{R})$. Leaves of this lamination that start at ∂V are called *external ray segments* of f; infinite external ray segments are called *external rays* of f. Note that if γ is an external ray, then $f(\gamma)$, as defined in the previous paragraph, is also an external ray.

We have two maps from B to T: one is the natural identification π along the vertical lines, the other is the map $f\colon B\dashrightarrow T$ which is defined only on $f^{-1}(T)$. Composition thereof, $\phi=\pi^{-1}\circ f\colon B\dashrightarrow B$ is a partially defined two-to-one map. We consider the set $\mathcal{A}\subset B$ of all the points with whole forward orbits are well defined. Then $\mathcal{A}$ is completely invariant and there is a unique orientation preserving map $\theta\colon\mathcal{A}\to\mathbb{S}^1$ which semi-conjugates $\phi\colon\mathcal{A}\to\mathcal{A}$ to the doubling map of the circle. We say that $\theta(a)$ is the *angle* of the external ray segment passing through the point a.

An external ray segment passing through a point $a\in\mathcal{A}$ is infinite (i.e. it is an external ray) if and only if it hits neither an iterated precritical point nor an iterated lift of $\partial^{\text{frb}}U$. The latter possibility is a major technical issue we have to deal with.

2.2. **Prime pacman renormalization.** Let us first give an example of a prime renormalization of full pacmen where we cut out the sector bounded by γ_1 and γ_2, see Figure 4. This renormalization is motivated by the surgery procedure that

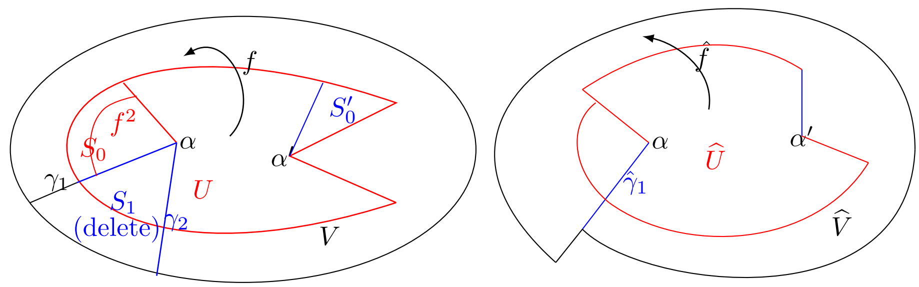


FIGURE 4. Prime renormalization of a pacman: delete the sector S_1, forget in U the sector S_0' attached to α', and iterate f twice on S_0. By gluing γ_1 and γ_2 along $f : \gamma_1 \to \gamma_2$ we obtain a new pacman $\hat{f} : \widehat{U} \to \widehat{V}$.

Branner and Douady [BD] used to construct a map between the Rabbit $\mathcal{L}_{1/3}$ and the Basilica $\mathcal{L}_{1/2}$ limbs of the Mandelbrot set, see Appendix C.1. Pacman renormalization will be defined in §2.3.

Recall that a sector S is a closed topological disk with two distinguished arcs in ∂S meeting at single point, called the vertex of S. Suppose $f : U \to V$ is a full pacman and

(A) γ_0, γ_1, and $\gamma_2 := f(\gamma_1)$ are mutually disjoint except for the fixed point α.

Denote by S_1 the closed sector of V bounded by $\gamma_1 \cup \gamma_2$ and not containing γ_0. Let us further assume that

(B) S_1 does not contain the critical value; and
(C) $\gamma_- \cup \gamma_+ \subset V \setminus S_1$.

Let $\widehat{V}$ be the Riemann surface with boundary obtained from $\overline{V} \setminus \operatorname{int} S_1$ by gluing $\gamma_1' := f^{-1}(\gamma_2) \cap \gamma_1$ and γ_2 along f. This means that there is a quotient map

$$\psi : \overline{V} \setminus \operatorname{int} S_1 \to \widehat{V}$$

such that ψ is conformal in $V \setminus S_1$ while $\psi(z) = \psi(f(z)) \in \widehat{V}$ for all $z \in \gamma_1'$. Let us select an embedding $\widehat{V} \hookrightarrow \mathbb{C}$.

The sector S_1 has two f-lifts; let S_0 be the lift of S_1 attached to α and let S_0' be the lift of S_1 attached to α'. Condition (B) implies that $\gamma_- \cup \gamma_+ \subset V \setminus S_0$. Define

$$\bar{f}(z) := \begin{cases} f(z), & \text{if } z \in U \setminus (S_1 \cup S_0 \cup S_0') \\ f^2(z) & \text{if } z \in S_0 \cap f^{-1}(U). \end{cases}.$$

Then the map $\bar{f}$ descends via ψ into a full pacman $\hat{f} : \widehat{U} \to \widehat{V}$ with the critical ray $\hat{\gamma}_1$.

2.3. **Pacman renormalization.** Let us start with defining an analogue of commuting pairs for pacmen.

A map $\psi\colon S \to \overline{V}$ from a closed sector (S, β_-, β_+) onto a closed topological disk $\overline{V} \subset \mathbb{C}$ is called a *gluing* if ψ is conformal in the interior of S, $\psi(\beta_-) = \psi(\beta_+)$, and ψ can be conformally extended to a neighborhood of any point in $\beta_- \cup \beta_+$ except the vertex of S.

Definition 2.2 (Prepacmen, Figure 5)**.** Consider a sector S with boundary rays β_-, β_+ and with an interior ray β_0 that divides S into two subsectors T_-, T_+. Let $f_-\colon U_- \to S$, $f_+\colon U_+ \to S$ be a pair of holomorphic maps, defined on $U_- \subset T_-, U_+ \subset T_+$. We say that $F = (S, f_-, f_+)$ is a *prepacman* if there exists a gluing ψ of S which projects (f_-, f_+) onto a pacman $f\colon U \to V$ where β_-, β_+ are mapped to the critical arc γ_1 and β_0 is mapped to γ_0.

The map ψ is called a *renormalization change of variables.*

The definition implies that f_- and f_+ commute in a neighborhood of β_0. Note that every pacman $f\colon U \to V$ has a prepacman obtained by cutting V along the critical arc γ_1.

Dynamical objects (such as the non-escaping set) of a prepacman F are preimages of the corresponding dynamical objects of f under ψ.

Definition 2.3 (Pacman renormalization, Figure 6)**.** We say that a pacman $f\colon U \to V$ is *renormalizable* if there exists a prepacman

$$G = (g_- = f^{\mathfrak{a}}\colon U_- \to S, \quad g_+ = f^{\mathbf{b}}\colon U_+ \to S)$$

defined on a sector $S \subset V$ with vertex at α such that g_-, g_+ are iterates of f realizing the first return map to S and such that the f-orbits of U_-, U_+ before they return to S cover a neighborhood of α compactly contained in U. We call G the *pre-renormalization* of f and the pacman $g\colon \widehat{U} \to \widehat{V}$ is the *renormalization* of f.

The numbers $\mathfrak{a}, \mathbf{b}$ are the *renormalization return times.*

The renormalization of f is called *prime* if $\mathfrak{a} + \mathbf{b} = 3$.

Similarly, a *pacman renormalization* is defined for any map $f\colon U \to V$ with a distinguished fixed point which will be called α. For example, we will show in Corollary 3.7 that any Siegel map is pacman renormalizable.

Combinatorially, a general pacman renormalization is an iteration of the prime renormalization – see details in Appendix A, in particular Lemma A.2.

We define $\mathbb{\Delta} = \mathbb{\Delta}_G$ to be the union of points in the f-orbits of $\overline{U}_-, \overline{U}_+$ before they return to S. Naturally, $\mathbb{\Delta}$ is a triangulated neighborhood of α, see Figure 6. We call $\mathbb{\Delta}$ a *renormalization triangulation* and we will often say that $\mathbb{\Delta}$ is obtained by *spreading around* U_-, U_+.

Definition 2.4 (Conjugacy respecting prepacmen)**.** Let f and g be any two maps with distinguished α-fixed points and let R and Q be two prepacmen in the dynamical plane of f and g defining some pacman renormalizations. Let h be a local conjugacy between f and g restricted to neighborhoods of their α-fixed points. Then h *respects* R *and* Q if h maps the triangulation $\mathbb{\Delta}_R$ to $\mathbb{\Delta}_Q$ so that the image of $(S_R, U_{R,\pm})$ is $(S_Q, U_{Q,\pm})$.

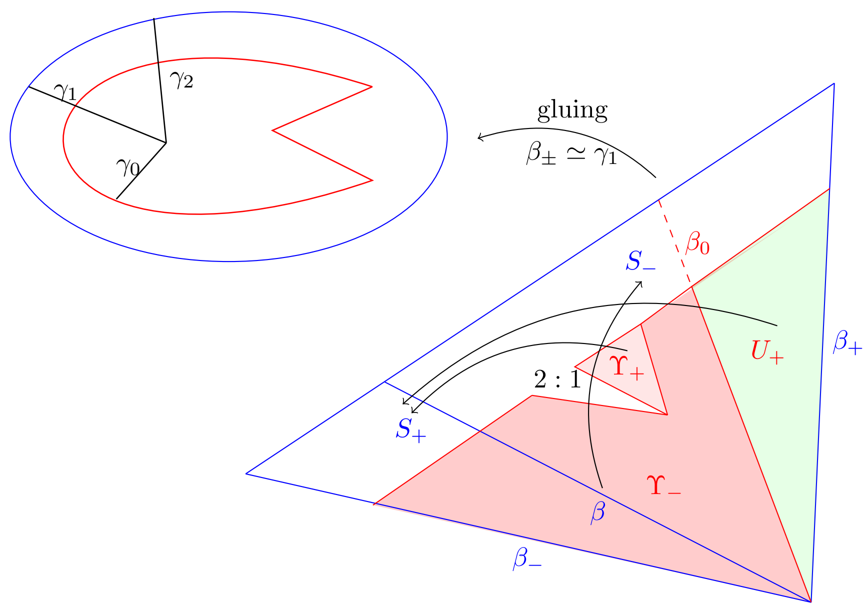


FIGURE 5. A (full) prepacman $(f_-\colon U_- \to S,\ \ f_+\colon U_+ \to S)$. We have $U_- = \Upsilon_- \cup \Upsilon_+$ and f_- maps Υ_- two-to-one to S_- and Υ_+ to S_+. The map f_+ maps U_+ univalently onto S_+. After gluing dynamically β_- and β_+ we obtain a full pacman: the arc β_- and β_+ project to γ_1, the arc β_0 projects to γ_0, and the arc β projects to γ_2.

2.4. **Banach neighborhoods.** Consider a pacman $f : U_f \to V$ with a non-empty truncation disk O. We assume that there is a topological disk $\widetilde{U} \Supset U_f$ with a piecewise smooth boundary such that f extends analytically to $\widetilde{U}$ and continuously to its closure. Choose a small $\varepsilon > 0$ and define $N_{\widetilde{U}}(f, \varepsilon)$ to be the set of analytic maps $g : \widetilde{U} \to \mathbb{C}$ with continuous extensions to $\partial\widetilde{U}$ such that

$$\sup_{z\in\widetilde{U}} |f(z) - g(z)| < \varepsilon.$$

Then $N_{\widetilde{U}}(f, \varepsilon)$ is a Banach ball.

We say a curve γ lands at α at a *well-defined angle* if there exists a tangent line to γ at α.

Lemma 2.5. *Suppose γ_0, γ_1 land at α at distinct well-defined angles. If $\varepsilon > 0$ is sufficiently small, then for every $g \in N_{\widetilde{U}}(f, \varepsilon)$ there is a domain $U_g \subset \widetilde{U}$ such that $g : U_g \to V$ is a pacman with the same critical arc γ_1 and truncation disk O.*

Proof. For $g \in N_{\widetilde{U}}(f, \varepsilon)$ with small ε, set $\gamma_0(g)$ to be the lift of γ_1 landing at α. Since $\gamma_0(f), \gamma_1$ land at distinct well-defined angles, so are $\gamma_0(g), \gamma_1$ if ε is small; i.e. $\gamma_0(g), \gamma_1$ are disjoint.

Set $g_\delta = f + \delta(g - f)$, where $\delta \in [0, 1]$. Define $\psi_\delta(z) = g_\delta^{-1} \circ f(z)$ on ∂U_f where the inverse branch is chosen so that $\psi_0(z) = z$ and $\psi_\delta(z)$ is continuous with respect to δ. We claim that ψ_δ is well defined and that $\psi_\delta(\partial U_f)$ is a simple closed curve for all $\delta \in [0,1]$. Indeed, let $A \Subset \widetilde{U}$ be a closed annular neighborhood of ∂U_f that contains no critical points of f. For ε small enough, the derivative of any $g \in N_{\widetilde{U}}(f,\varepsilon)$ is uniformly bounded and non-vanishing on a lightly shrank A; in particular g has no critical points in A.

It follows that $\psi_\delta \mid A$ has uniformly bounded derivative and (choosing yet smaller ε, if necessary) is close to the identity map, hence $\psi_\delta(\partial U_f) \subset A$ is well-defined for all δ. Since f has no critical values in A, it is locally injective, which implies that $\psi_\delta(x) \neq \psi_\delta(y)$ when x is sufficiently close to y. We conclude that ψ_δ is injective on ∂U_f. Therefore $\psi_1(\partial U_f)$ is a simple closed curve; let U_g be the disk enclosed by $\psi_1(\partial U_f)$. It is straightforward to check that $g : U_g \to V$ is a pacman with critical arc γ_1 and truncation disk O. □

Consider a pacman $f\colon U_f \to V$. Applying the λ-lemma, we can endow all $g\colon U_g \to V$ from a small neighborhood of f with a foliated rectangle $\mathfrak{R}_g$ as in §2.1 such that $\mathfrak{R}_g$ moves holomorphically and the holomorphic motion of $\mathfrak{R}_g$ is equivariant. As a consequence, an external ray R with a given angle depends holomorphically on g unless R hits an iterated lift of $\partial^{\mathrm{frb}} U_g$ or an iterated precritical point.

Lemma 2.6 (Stability of periodic rays)**.** *Suppose a periodic ray R lands at a repelling periodic point x in the dynamical plane of f. Then the ray R lands at x for all g in a small neighborhood of f. Moreover, the closure $\overline{R}(g)$ is in a small neighborhood of $\overline{R}(f)$.*

Proof. Since x is repelling periodic, it is stable by the implicit function theorem. By continuity, $R(g)$ is stable away from $x(g)$. Stability of $R(g)$ in a small neighborhood of $x(g)$ follows from stability of linear coordinates at x. □

2.5. **Pacman analytic operator.** Suppose that $\hat{f} : \widehat{U} \to \widehat{V}$ is a renormalization of $f : U_f \to V$ via a quotient map $\psi_f : S_f \to \widehat{V}$ that extends analytically through $\partial S_f \setminus \{\alpha\}$ (this actually follows from the definition of renormalization) where $S_f \subset V$ is the domain of a prepacman $\widehat{F}$ such that curves $\beta_0, \beta_+, \beta_-$ all land at α at pairwise distinct well-defined angles. We claim that there exists an analytic renormalization operator defined on a neighborhood of f.

We note that $\beta_\pm = f^{k_\pm}(\beta_0)$ for some integers k_+, k_-. For a map g that is sufficiently close to f, the fact that the three curves land at different angles implies that $\beta_0, g^{k_+}(\beta_0), g^{k_-}(\beta_0)$ are disjoint. Define $\tau_g\colon \beta_0 \cup \beta_- \cup \beta_+ \to \mathbb{C}$ by $\tau_g = \mathrm{id}$ on β_0 and $\tau_g = g^{k_\pm} \circ f^{-k_\pm}$ on $\beta_\pm$. Then τ_g is an equivariant holomorphic motion of $\beta_0 \cup \beta_- \cup \beta_+$ over a neighborhood of f. By the λ-lemma [BR], [ST] τ_g extends to a holomorphic motion of S_f over a possibly smaller neighborhood of f. Denote by μ_g the Beltrami differential of τ_g. Define a Beltrami differential ν_g on $\mathbb{C}$ as $\nu_g = (\psi_f)_* \mu_g$ on $\widehat{V}$ and $\nu_g = 0$ outside of $\widehat{V}$ and let ϕ_g be the solution of the Beltrami equation

$$\frac{\partial \phi_g}{\partial \bar{z}} = \nu_g \frac{\partial \phi_g}{\partial z}$$

that fixes α, ∞, and the critical value. We see that $\psi_g := \phi_g \circ \psi_f \circ \tau_g^{-1}$ is conformal on $S_g := \tau_g(S)$. It follows, that ψ_g depends analytically on g (see Remark on page 345 of [L1]).

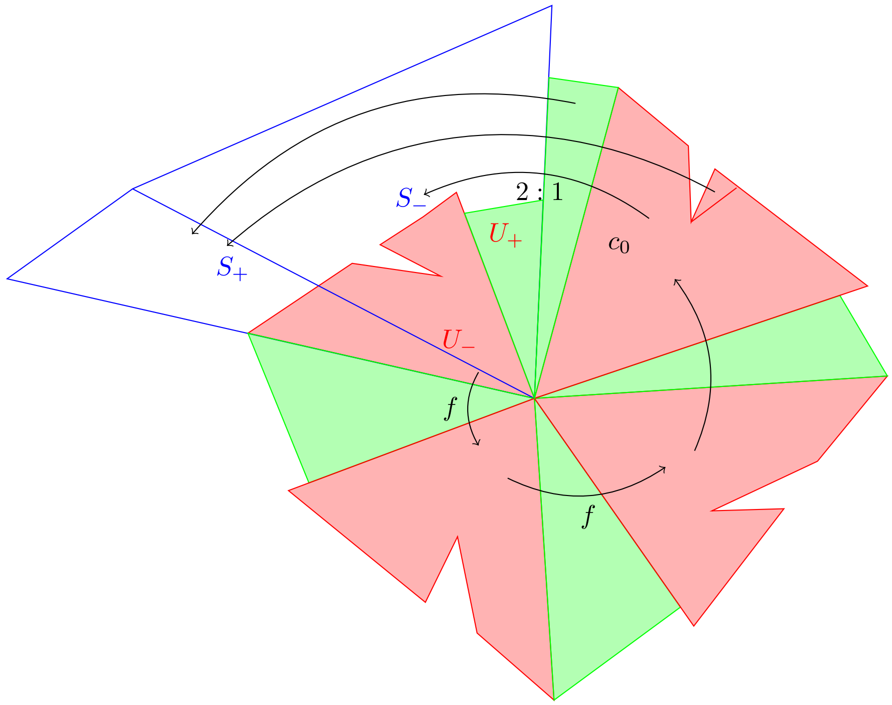


FIGURE 6. Pacman renormalization of f: the first return map of points in $U_- \cup U_+$ back to $S = S_- \cup S_+$ is a prepacman. Spreading around $U_\pm$: the orbits of U_- and U_+ before returning back to S triangulate a neighborhood $\triangle$ of α; we obtain $f\colon \triangle \to \triangle$, and we require that $\triangle \cup S$ is compactly contained in Dom f.

We claim now that $\widehat{G} = (S_g, g^{k_-}, g^{k_+})$ is a prepacman. Indeed, by definition of τ_g, we have $g^{k_\pm}(\tau_g(\beta_0)) = \beta_\pm$ and ψ_g glues $\widehat{G}$ to a map $\hat{g}$ which is close to $\hat{f}$. By Lemma 2.5, $\hat{g}$ restricts to a pacman with the same range as $\hat{f}$. We, thus, have proven the following:

Theorem 2.7 (Analytic renormalization operator)**.** *Suppose that $\hat{f} : \widehat{U} \to \widehat{V}$ is a renormalization of $f : U_f \to V$ via a quotient map $\psi_f : S_f \to \widehat{V}$. Assume that the curves $\beta_0, \beta_-, \beta_+$ (see Definition 2.2) land at α at pairwise distinct well-defined angles. Then for every sufficiently small neighborhood $N_{\widetilde{U}}(f, \varepsilon)$, there exists a compact analytic pacman renormalization operator $\mathcal{R}\colon g \mapsto \hat{g}$ defined on $N_{\widetilde{U}}(f, \varepsilon)$ such that $\mathcal{R}(f) = \hat{f}$. Moreover, the gluing map ψ_g, used in this renormalization, also depends analytically on g.* □

Proof. We have already shown that $\hat{g}$ depends analytically on $g \in N_{\widetilde{U}}(f, \varepsilon)$. Choose U_2 with $U_2 \Subset U_2 \Subset \widetilde{U}$ so that the operator $\mathcal{R}$ is the composition of the restriction operator $N_{\widetilde{U}}(f, \varepsilon) \to N_{U_2}(f, \varepsilon)$ and the pacman renormalization operator defined on $N_{U_2}(f, \varepsilon)$. Since $N_{\widetilde{U}}(f, \varepsilon) \to N_{U_2}(f, \varepsilon)$ is compact, we obtain that $\mathcal{R}$ is compact. □

3. Siegel pacmen

We say a holomorphic map $f\colon U \to V$ is *Siegel* if it has a fixed point α, a Siegel quasidisk $\overline{Z}_f \ni \alpha$ compactly contained in U, and a unique critical point $c_0 \in U$ that is on the boundary of Z_f. Note that in [AL2] a Siegel map is assumed to satisfy additional technical requirements; these requirements are satisfied by restricting f to an appropriate small neighborhood of $\overline{Z}_f$.

Let us foliated Siegel disk Z_f of f by equipotentials parametrized by their heights ranging from 0 (the height of α) to 1 (the height of ∂Z_f).

Definition 3.1. A pacman $f : U \to V$ is *Siegel* if

- f is a Siegel map with Siegel disk Z_f centered at α;
- the critical arc γ_1 is the concatenation of an external ray R_1 followed by an inner ray I_1 of Z_f such that the unique point in the intersection $\gamma_1 \cap \partial Z_f$ is not precritical; and
- writing $f\colon (U \setminus O_0', O_0) \to (V, O)$ as in (2.1), the disk O is a subset of Z_f bounded by its equipotential.

The *rotation number* of a Siegel pacman (or a Siegel map) is $\theta \in \mathbb{R}/\mathbb{Z}$ so that $\mathbf{e}(\theta)$ is the multiplier at α. It follows that the rotation number of Siegel map is in Θ_{bnd}. The level of *truncation* of f is the height of ∂O.

Since γ_1 is a concatenation of an external ray R_1 and an internal ray I_1, so is γ_0: it is a concatenation of an external ray R_0 and an internal ray I_0 with $f(R_0 \cup I_0) = R_1 \cup I_1$. Two Siegel pacmen $f\colon U_f \to V_f$ and $g\colon U_g \to V_g$ are combinatorially equivalent if they have the same rotation number and if $R_0(f_1)$ and $R_0(f_2)$ have the same external angles, see (2.1). Starting from §3.6 we will normalize γ_0 so that it passes through the critical value.

A *hybrid conjugacy* between Siegel maps is a qc-conjugacy that is conformal on the Siegel disks. A hybrid conjugacy between Siegel pacmen is defined in a similar fashion. We will show in Theorem 3.11 that combinatorially equivalent pacmen are hybrid equivalent.

We will often refer to the connected component Z_f' of $f^{-1}(Z_f) \setminus Z_f$ attached to c_0 as *co-Siegel disk*.

3.1. **Local connectivity and bubble chains.**

Consider a quadratic polynomial $p_\theta\colon z \mapsto \mathbf{e}(\theta)z + z^2$.

Theorem 3.2. *If $\theta \in \Theta_{\text{bnd}}$, then the closed Siegel disk $\overline{Z}$ of p_θ is a quasidisk containing the critical point of p_θ.*

Conversely, suppose a holomorphic map $f\colon U \to V$ with a single critical point has a fixed Siegel quasidisk $\overline{Z}_f \Subset U \cap V$ containing the critical point of f. Then f has a rotation number of bounded type.

The first part of Theorem 3.2 follows essentially from the Douady-Ghys surgery, see [D1]. By [AL2] there is a quasicritical map associated with f. Moreover, the linearizing map for $f \mid \overline{Z}_f$ must be quasi-symmetric, which implies that f has a rotation number of bounded type by the real *a priori* bounds.

Let us now fix a polynomial $p = p_\theta$ with $\theta \in \Theta_{\text{bnd}}$. A *bubble* of p is either

- $\overline{Z}_p$, or
- $\overline{Z}_p' = \overline{p^{-1}(Z_p) \setminus Z_p}$, or
- an iterated p-lift of $\overline{Z}_p'$.

The *generation* of a bubble Z_k is the smallest $n \geq 0$ such that $p^n(Z_k) \subset \overline{Z}_p$. In particular, $\overline{Z}_p$ has generation 0 and $\overline{Z}'_p$ has generation 1. If the generation of Z_k is at least 2, then $p\colon Z_k \to p(Z_k)$ admits a conformal extension through ∂Z_k (because $p(Z_k) \not\ni c_1$).

We say that a bubble Z_n is *attached* to a bubble Z_{n-1} if $Z_n \cap Z_{n-1} \neq \emptyset$ and the generation of Z_n is strictly grater than the generation of Z_{n-1}.

A *limb* of a bubble Z_k is the closure of a connected component of $\mathfrak{K}_p \setminus Z_k$ not containing the α-fixed point. A limb of $\overline{Z}_p$ is called *primary.*

Theorem 3.3 ([Pe])**.** *The filled-in Julia set $\mathfrak{K}_p$ is locally connected. Moreover, for every $\varepsilon > 0$ there is an $n \geq 0$ such that every connected component of $\mathfrak{K}_p$ minus all bubbles with generation at most n is less than ε.*

In particular the diameter of bubbles in $\mathfrak{K}_p$ tends to 0: for every $\varepsilon > 0$ there are at most finitely many bubbles with diameter grater than ε. Similarly, the diameter of limbs of any bubble tends to 0.

An (infinite) *bubble chain of* $\mathfrak{K}_p$ is an infinite sequence of bubbles $B = (Z_1, Z_2, \dots)$ such that Z_1 is attached to $\overline{Z}_p$ and Z_{n+1} is attached to Z_n.

As a consequence of Theorem 3.3 every bubble chain $B = (Z_1, Z_2, \dots)$ *lands*: there is a unique $x \in \mathfrak{K}_p$ such that for every neighborhood U of x there is an $m \geq 0$ such that $\bigcup_{i\geq m} Z_i$ is within U. Conversely, if $x \in \mathfrak{K}_p$ does not belong to any bubble, then there is a bubble chain $B = (Z_1, Z_2, \dots)$ landing at x. If x is periodic, then so is B: there is an $m > 1$ and $q \geq 1$ such that p^q maps $(Z_m, Z_{m+1}, \dots)$ to $(Z_1, Z_2, \dots)$.

Let $f\colon U \to V$ be a Siegel pacman. *Limbs, bubbles, and bubble chains for* f are define in the same way as for quadratic polynomials with Siegel quasidisks. In particular, a bubble of f is either $\overline{Z}_f$, or $\overline{Z}'_f = \overline{f^{-1}(Z_f) \setminus Z_f}$, or an f^{n-1}-lift of $\overline{Z}'_f$, where n is the *generation* of the bubble. Since $\overline{Z}_f$ is the only bubble intersecting $\{c_1\} \cup \gamma_1$, all bubbles of positive generation are conformal lifts of $\overline{Z}'_f$. We define the *Julia set* of f as

$$\mathfrak{J}_f := \overline{\bigcup_{n\geq 0} f^{-n}(\partial Z_f)}. \tag{3.1}$$

We will show in Theorem 3.12 that Theorem 3.3 holds for standard Siegel pacmen and that $\mathfrak{J}_f$ is the closure of repelling periodic points.

Limbs, bubbles, and bubble chains of a prepacman F are preimages of the corresponding dynamical objects of f.

3.2. **Siegel prepacmen.** A prepacman Q of a Siegel pacman q is also called *Siegel*; the *rotation number* and *level of truncation* of Q are those of q. Recall that Q consists of two commuting maps $q_-\colon U_- \to S_Q$, $q_+\colon U_+ \to S_Q$ such that U_- and U_+ are separated by β_0. Given a Siegel map f we say that f has a *prepacman Q around $x \in \partial Z_f$* if q_- , q_+ are iterates of f, the vertex of S_Q is at $\alpha(f)$, and $\beta_0(Q)$ intersects ∂Z_f at x.

Lemma 3.4. *Suppose that p is a Siegel quadratic polynomial with rotation number $\theta \in \Theta_{\rm bnd}$. Consider a point $x \in \partial Z_p$ such that x is neither the critical point of p nor its iterated preimage. Then for every $r \in (0,1)$ and every $\varepsilon > 0$, the map p has a Siegel prepacman*

$$Q = (q_-\colon U_- \to S_Q, \quad q_+\colon U_+ \to S_Q) \tag{3.2}$$

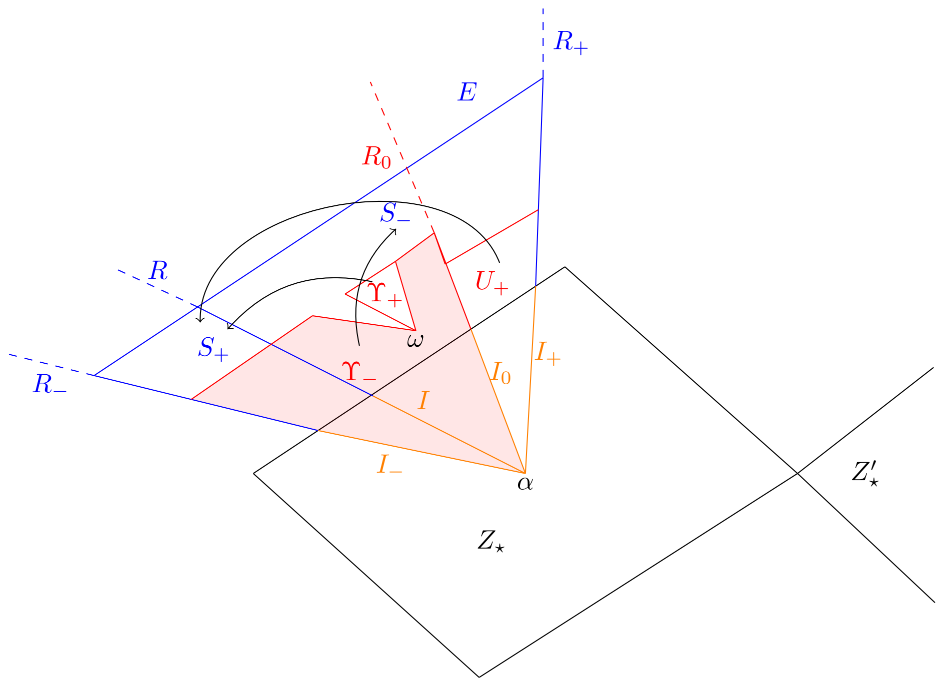


FIGURE 7. A full Siegel prepacman, compare with Figure 5. In the dynamical plane of a quadratic polynomial p, the sector $S_Q = S_- \cup S_+$ is bounded by $R_- \cup I_- \cup I_+ \cup R_+$ and truncated by an equipotential at small height. Pulling back S_- , S_+ along appropriate branches of $p^{\mathfrak{a}}, p^{\mathbf{b}}$ we obtain $U_- = \Upsilon_- \cup \Upsilon_+$ and U_+ so that $(p^{\mathfrak{a}} \mid U_- \ , \ p^{\mathbf{b}} \mid U_+)$ is a full prepacman(3.3). Truncating $(p^{\mathfrak{a}} \mid U_- \ , p^{\mathbf{b}} \mid U_+)$ at ω and at the vertex where R_+ meets E (see Figure 9) we obtain a required prepacman (3.2).

around x such that

- *the rotation number of Q is a renormalization of θ – iteration of* (A.2)*;*
- *for every $z \in U_- \cup U_+$ the orbit $z, p(z), \dots, p^k(z) = q_\pm(z)$ is in the ε-neighborhood of $\overline{Z}_p$; and*
- *r is the level of truncation of Q;*
- *every external ray segment (see §2.1) of Q is within an external ray of p.*

Before proceeding with the proof let us define:

Definition 3.5 (Sector renormalization of $p \mid \overline{Z}_p$ around $x \in \partial Z_p$)**.** Using notations from Appendix A, let $h\colon \overline{Z}_p \to \overline{\mathbb{D}^1}$ be the unique conformal conjugacy between $p \mid \overline{Z}_g$ and $\mathbb{L}_\theta \mid \overline{\mathbb{D}^1}$ normalized such that $h(x) = 1$. Consider a sector pre-renormalization $(\mathbb{L}^{\mathfrak{a}} \mid \mathbb{X}_- \ , \ \ \mathbb{L}^{\mathbf{b}} \mid \mathbb{X}_+)$ as in §A.2. Denote by δ the angle of $\mathbb{X} = \mathbb{X}_- \cup \mathbb{X}_+$ at 0. Pulling back $(\mathbb{L}^{\mathfrak{a}} \mid \mathbb{X}_- \ , \ \ \mathbb{L}^{\mathbf{b}} \mid \mathbb{X}_+)$ by h we obtain the commuting pair $(p^{\mathfrak{a}} \mid X_- \ , \ \ p^{\mathbf{b}} \mid X_+)$ with

- $X_- := h^{-1}(\mathbb{X}_-)$, $X_+ := h^{-1}(\mathbb{X}_+)$ and $X := h^{-1}(\mathbb{X}) = X_- \cup X_+$ are closed sectors of $\overline{Z}_f$,
- the internal ray $I_0 := X_- \cap X_+$ lands at x.

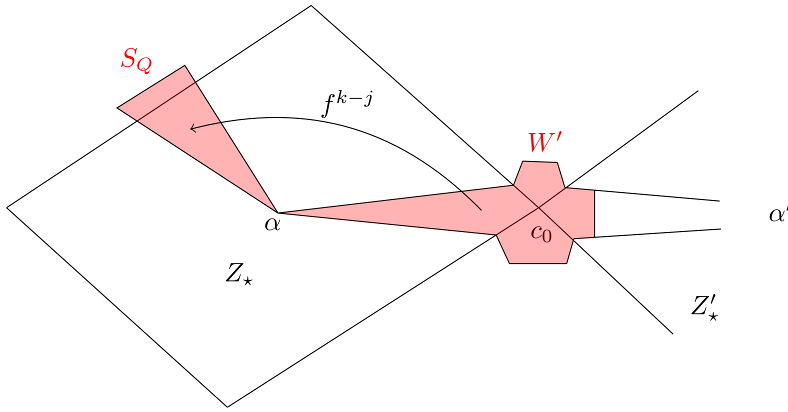


FIGURE 8. Since W' is truncated by an equipotential of Z'_p at small height, the point $p^j(z) \in W' \setminus \overline{Z}_p$ is in a small neighborhood of c_0.

The gluing map $z \to z^{1/\delta}$ from descents to

$$\psi_x := h^{-1} \circ [z \to z^{1/\delta}] \circ h$$

with $\psi_x(X) = \overline{Z}_f$.

Proof of Lemma 3.4. Consider the sector renormalization $(p^{\mathfrak{a}} \mid X_- \ , \ \ p^{\mathbf{b}} \mid X_+)$ from Definition 3.5 and assume that the angle δ of $\mathbb{X}$ is small. We will now extend $(p^{\mathfrak{a}} \mid X_- \ , \ \ p^{\mathbf{b}} \mid X_+)$ beyond $\overline{Z}_p$ to obtain a prepacman (3.2), see Figure 7. Set

$$I_- := p^{\mathbf{b}}(I_0), \quad I_+ := p^{\mathfrak{a}}(I_0), \quad I := f^{\mathfrak{a}+\mathbf{b}}(I_0).$$

Then the sector X_- is bounded by I_-, I_0 and X_+ is bounded by I_0, I_+.

Since x is not precritical, there are unique external rays $R_- \ , R_+ \ , R$ extending $I_- \ , I_+ \ , I$ beyond $\overline{Z}_p$. Let S_Q be the closed sector bounded by $R_- \cup I_- \cup I_+ \cup R_+$ and truncated by an external equipotential E at a small height $\sigma > 0$. The curve $R \cup I$ divides S into two closed sectors S_+ and S_- such that S_+ is between $R_- \cup I_-$ and $R \cup I$ while S_- is between $R \cup I$ and $R_+ \cup I_+$. We note that $p^{\mathfrak{a}}(X_-) \subset S_-$ and $p^{\mathbf{b}}(X_+) \subset S_+$.

Let us next specify $U_- \supset X_- \ , \ \ U_+ \supset X_+$ such that

$$Q = (q_- \ , q_+) = (p^{\mathfrak{a}} \mid U_- \ , \ \ p^{\mathbf{b}} \mid U_+) \tag{3.3}$$

is a full prepacman. Since the p-orbits of $X_- \ , X_+$ cover $\overline{Z}_\star$ before they return back to X, we see that $\partial X \cap \partial Z_p$ has a unique precritical point, call it c'_0, that travels though the critical point of p before it returns to X. Below we assume that $c'_0 \in X_-$; the case $c'_0 \in X_+$ is analogous. Then S_+ has a conformal pullback U_+ along $p^{\mathbf{b}} \colon X_+ \to S_+$. We have $U_+ \subset S_Q$ because rays and equipotentials bounding S_Q enclose U_+.

The sector S_- has a degree two pullback Υ_- along $p^{\mathfrak{a}} \colon X_- \to S_-$. Under $p^{\mathfrak{a}} \colon \Upsilon_- \to S_-$ the fixed point α has two preimages, one of them is α, we denote the other preimage by ω. Let Υ_+ be the conformal pullback of S_+ along the orbit $p^{\mathfrak{a}} \colon \{\omega\} \to \{\alpha\}$. We define $U_- := \Upsilon_- \cup \Upsilon_+ \subset S_Q$ and we observe that Q in (3.3) is a full prepacman.

By Theorem 3.2, primary limbs of $\mathfrak{K}_p$ intersecting a small neighborhood of x have small diameters. By choosing δ and σ sufficiently small we can guarantee that $S_Q \setminus \overline{Z}_p$ is in a small neighborhood of x.

Let us now truncate Q at level r and let us show that the orbit

$$z, p(z), \dots, p^k(z) = q_\pm(z), \qquad k \in \{\mathfrak{a}, \mathbf{b}\}$$

of any $z \in U_\pm$ is in a small neighborhood of $\overline{Z}_p$. The truncation of Q at level r removes points in $U_- = \Upsilon_- \cup \Upsilon_+$ with $p^{\mathfrak{a}}$- images in the subdisk of Z_p bounded by the equipotential at height $t \coloneqq r^\delta$. Since δ is small, we obtain that t is close to 1.

Since $\mathfrak{K}_p$ is locally connected (Theorem 3.2), all the external rays of p land. For $z \in U_\pm \setminus \mathfrak{K}_f$, define $\rho(z) \in \mathfrak{K}_p$ to be the landing point of the external ray passing through z. Since S_Q is truncated by an equipotential at a small height, the orbit of z stays close to that of $\rho(z)$. This reduces the claim to the case $z \in \mathfrak{K}_p \cap U_\pm$.

By Theorem 3.2, there is an $\ell \geq 0$ such that all of the *big* (with diameter at least ε) primary limbs of p are attached to one of $c_0, c_{-1}, \dots, c_{-\ell}$, where c_0 is the critical point of p and c_{-i} is the unique preimage of c_0 under $p^i \mid \overline{Z}_p$. Since δ is assumed to be small, the orbit of c'_0 travels through all $c_{-\ell}, \dots, c_0$ before it returns to S_Q.

Let us denote by L the primary limb of p containing z (the case $z \in \overline{Z}_p$ is trivial). If L is not attached to c'_0, then by the above discussion all $L, p(L), \dots, p^k(L) = q_\pm(L)$ are small and the claim follows.

Suppose that L is attached to c'_0. Denote by L_{-i} the connected component of $\mathfrak{K}_p \setminus \overline{Z}_p$ attached to c_{-i}. Since c'_0 travels through a critical point, we have $L = L_{-j}$ for some $j < k$.

Let W be the pullback of S_Q along

$$p^{k-j} \colon c_0 = p^j(c'_0) \to p^k(c'_0)$$

and let W' be W truncated by the equipotential of Z'_p at height t, see Figure 8. Since $t = r^\delta$ is close to 1, we obtain that $W' \cap L'_0$ is in a small neighborhood of c_0 because the angle of W at α' (the non-fixed preimage of α) is small – it is equal to δ. Therefore, $p^j(z)$ is close to c_0, and by continuity all $p^{j-1}(z), p^{j-2}(z), \dots, p^{j-\ell}(z)$ are close to $c_0, c_{-1}, \dots, c_{-\ell}$. Recall that $p^{j-i}(z) \in L_{-i}$. Since $L_0, L_{-1}, \dots, L_{-\ell}$ are the only big limbs, we obtain that the orbit $z, p(z), \dots, p^k(z)$ is in a small neighborhood of $\overline{Z}_p$.

It remains to specify external rays for Q. As it shown on the Figure 9 we slightly truncate S_Q at the vertex where R_+ meets the equipotential E and we slightly truncate $U_\pm$ such the truncations are respected dynamically and such that the preimage of the $\partial S_Q \setminus (R_- \cup R_+)$ under Q consists of exactly two curves that project to $\partial^{\text{ext}} U_q$, where $q \colon U_q \to V_q$ is the pacman of Q. We now can embed in $S_Q \setminus (U_- \cup U_+)$ two rectangles $\mathfrak{R}_-$ and $\mathfrak{R}_+$ that define external rays of Q as in §2.1. □

3.3. **Pacman renormalization of Siegel maps.** An immediate consequence of [AL2, Theorem 3.19, Proposition 4.3] is

Theorem 3.6. *Any two Siegel maps with the same rotation number are hybrid conjugate on neighborhoods of their closed Siegel disks.*

As a corollary Theorem 3.6 and Lemma 3.4 we obtain.

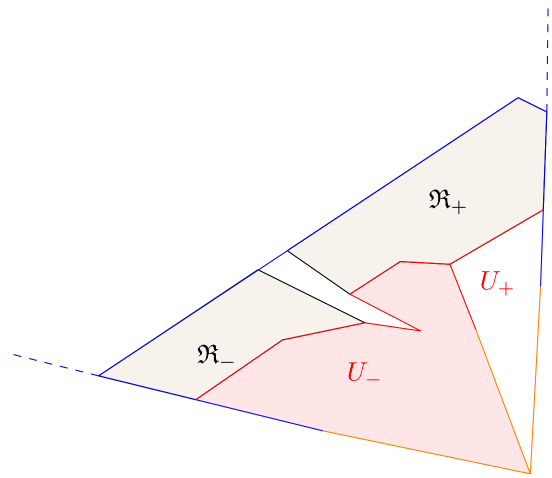


FIGURE 9. Truncating the prepacman from Figure 7 and embedding rectangles $\mathfrak{R}_\pm$ we endow the prepacman with external rays.

Corollary 3.7. *Every Siegel map $f\colon U \to V$ is pacman renormalizable.*

Moreover the following holds. Let f be a Siegel map and let p be the unique quadratic polynomial with the same rotation number as f. Let h be a hybrid conjugacy from a neighborhood of $\overline{Z}_f$ to a neighborhood of $\overline{Z}_p$ respectively. Then there are prepacmen R and Q in the dynamical planes of f and p respectively such that h respects R and Q in the sense of Definition 2.4.

Proof. Choose a small $\varepsilon > 0$ such that the ε-neighborhood of Z_f is in the domain of h. Then h pullbacks a prepacman Q from Lemma 3.4 to a prepacman R in the dynamical plane of f. This shows that f is pacman renormalizable. □

Lemma 3.8. *Suppose that a Siegel pacman f is a renormalization of a quadratic polynomial. Then the non-escaping set $\mathfrak{K}_f$ is locally connected.*

Moreover, for every $\varepsilon > 0$ there is an $n \geq 0$ such that every connected component of $\mathfrak{K}_f$ minus all the bubbles with generation at most n is less than ε. All the external rays of f land and the landing point in $\mathfrak{J}_f$. Conversely, every point in $\mathfrak{J}_f$ is the landing point of an external ray. The Julia set $\mathfrak{J}_f$ is the closure of repelling periodic points.

Proof. Follows from Theorem 3.3. Suppose that f is obtained from a quadratic polynomial p. Then every bubble Z_α of f is obtained from a bubble $\widetilde{Z}_\alpha$ of p by removing an open sector. All of the limbs of $\widetilde{Z}_\alpha$ attached to the removed sector are also removed. It follows from Theorem 3.3 that for $\varepsilon > 0$ there is an $n \geq 0$ such that every connected component of $\mathfrak{K}_f$ minus all of the bubbles with generation at most n is less than ε. Since bubbles of f are locally connected, so is $\mathfrak{K}_f$. The landing property of external rays is straightforward. □

3.4. **Rational rays of Siegel pacmen.** By a *rational point* we mean either a periodic or preperiodic point. Similarly, a periodic or preperiodic ray is *rational.*

Let us fix pacmen f, p and prepacmen R, Q as in Corollary 3.7. Let $\mathfrak{K}_R$ be the non-escaping set of R. By definition, $\mathfrak{K}_R \subset \mathfrak{K}_f$; spreading around $\mathfrak{K}_R$ we define the

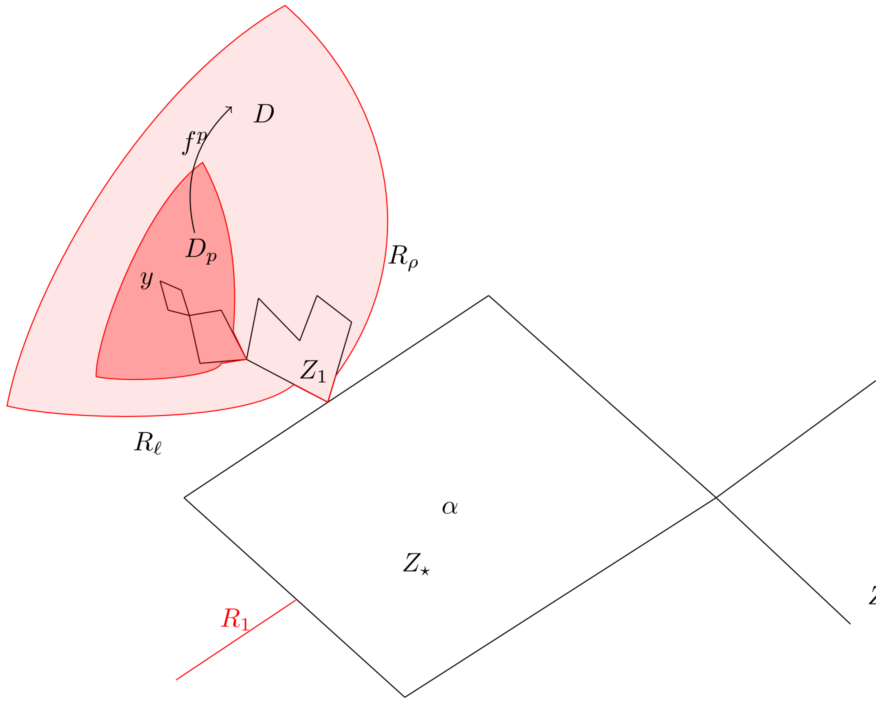


FIGURE 10. Illustration to the proof of Lemma 3.9. The ray R_1 has preimages R_ℓ and R_ρ that land at Z_1 such that $R_\ell \cup R_\rho$ together with together with appropriate arcs in $\partial V \cup Z_1$ bound a disk D containing y. The disk D has a univalent lift $D_p \Subset D$. By Schwarz lemma, $f^p \colon D_p \to D$ is expanding, which implies that there is an external ray landing at y.

local non-escaping set of f*:*

$$\mathfrak{K}_f^{\mathrm{loc}} := \bigcup_{n\geq 0} f^n(\mathfrak{K}_R). \tag{3.4}$$

This is the set of point that do not escape $\mathbb{A}_R$ under $f\colon \mathbb{A}_R \to \mathbb{A}_R \cup S_R$, see Figure 6. Similarly we define

$$\mathfrak{K}_p^{\mathrm{loc}} := \bigcup_{n\geq 0} p^n(\mathfrak{K}_Q).$$

It is immediate that h conjugates $f \mid \mathfrak{K}_f^{\mathrm{loc}}$ and $p \mid \mathfrak{K}_p^{\mathrm{loc}}$. As a consequence, the *local Julia set*

$$\mathfrak{J}_f^{\mathrm{loc}} := \overline{\bigcup_{n\geq 0} \left[f \mid \mathfrak{K}_f^{\mathrm{loc}}\right]^{-n} (\partial Z_f)}$$

is the closure of repelling periodic points because so is $\mathfrak{J}_p^{\mathrm{loc}}$. (Indeed, every $y \in \mathfrak{J}_p^{\mathrm{loc}}$ is the landing point of an external ray R_y because $\mathfrak{J}_p$ is locally connected. Since external rays in a pacman are parametrized by angles in $\mathbb{S}^1$ (see §2.1), p has a periodic external ray R_x landing at $x \in \mathfrak{J}_p^{\mathrm{loc}}$.) Moreover, for every periodic point $y \in \mathfrak{K}_f^{\mathrm{loc}}$ there is a unique periodic bubble chain B_y of $\mathfrak{K}_f^{\mathrm{loc}}$ landing at y.

Lemma 3.9 (External rays)**.** *Let $y \in \mathfrak{J}_f^{\rm loc}$ be a periodic point. Then there is a periodic external ray R_y landing at y with the same period as y.*

Proof. Let $B_y = (Z_1, Z_2, \dots)$ be the bubble chain in $\mathfrak{K}_f^{\rm loc}$ landing at y. Denote by x the unique point in the intersection of $\gamma_1 \cap \partial Z_\star$. By Definition 3.1, the external ray R_1 lands at x. There are two iterated preimages $x_\ell, x_\rho \in \partial Z_1$ of x_1 such that the rays R_ℓ, R_ρ (iterated lifts of R_1) landing at x_ℓ, x_ρ together with Z_1 separate y from $\overline{Z}_f$, see Figure 10. We denote by D the open subdisk of V bounded by R_ℓ, R_ρ, Z_1 and containing y. Let D_p be the (univalent) pullback of D along $f^p\colon \{y\} \to \{y\}$. Then $D_p \Subset D$. By the Schwarz lemma, $f^p\colon D_p \to D$ expands the hyperbolic metric of D.

There is a unique periodic external ray R_y in D with period p. We claim that R_y lands at y. Indeed, parametrize R as $R\colon \mathbb{R}_{>0} \to V$ with $f^p(R_y(t+p)) = R_y(t)$. Since all the points in D away from y escape in finite time under $f^p\colon D_p \to D$, the Euclidean distance between $R(t)$ and y decreases as $t \to +\infty$. □

The next lemma is a preparation for a Pullback Argument

Lemma 3.10 (Rational approximation of γ_1)**.** *For every $\varepsilon > 0$, there are*

- *periodic points $x_\ell, x_\rho \in \mathfrak{J}_f^{\rm loc}$,*
- *external rays R_ℓ and R_ρ landing at x_ℓ, x_ρ respectively,*
- *periodic bubble chains B_ℓ and B_ρ in $\mathfrak{K}_f^{\rm loc}$ landing at x_ℓ, x_ρ respectively, and*
- *internal rays I_ℓ and I_ρ of $\overline{Z}_f$ landing at the points at which B_ℓ and B_ρ are attached*

such that $R_\ell \cup B_\ell \cup I_\ell$ and $R_\rho \cup B_\rho \cup I_\rho$ are in the ε-neighborhood of γ_1 and such that $R_\ell \cup B_\ell \cup I_\ell$ is on the left of γ_1 while $R_\rho \cup B_\rho \cup I_\rho$ is on the right of γ_1.

Proof. Consider a finite set of periodic points $y_1, y_2, \dots, y_p \in \mathfrak{J}_f^{\rm loc}$. By Lemma 3.9 each y_i is the landing point of an external periodic ray, call it $R_{y,i}$, and the landing point of a periodic bubble chain, call it $B_{y,i}$. Let $\{W_1, W_2, \dots, W_p\}$ be the set of connected components of

$$U \setminus \overline{Z_f \bigcup_i (B_{y,i} \cup R_{y,i})};$$

we assume that ∂W_p contains $\partial^{\rm frb} U_f$. By adding more periodic points we can also assume that $c_0 \notin \partial W_p$. Set

$$W \coloneqq W_1 \cup W_2 \cup \dots \cup W_{p-1}.$$

By Schwarz lemma, $f \mid W$ is expanding with respect to the hyperbolic metric of W. Since $c_0 \notin \partial W_p$, there is a sequence of periodic points $x_{\ell,j} \in \mathfrak{J}_f^{\rm loc}$ such that the orbit of x_ℓ is in $\overline{W}$ and such that $x_{\ell,j}$ converges from the left to the unique point x_1 in $\gamma_1 \cap \partial Z_f$.

We claim that the external rays $R_{\ell,j}$ landing at $x_{\ell,j}$ converge to the external ray landing at x_1. Indeed, since $x_{\ell,j} \to x_1$, the external angle of $R_{\ell,j}$ (see §2.1) converges to the external angle of R_1. By continuity, $R_{\ell,j}([0,T])$ converges to $R_1([0,T])$ for any $T \in \mathbb{R}_{>0}$. Since $f \mid W$ is expanding, $R_{\ell,j}([T, +\infty))$ is in a small neighborhood of $x_{\ell,j}$ which converges to x_1.

The bubble chains $B_{\ell,j}$ of $\mathfrak{K}_f^{\rm loc}$ landing at $x_{\ell,j}$ shrink because there are no big limb in a neighborhood of x_1. Define $I_{\ell,j}$ to be the internal ray of Z_f landing at the point where $B_{\ell,j}$ is attached. Then $R_{\ell,j} \cup B_{\ell,j} \cup I_{\ell,j}$ is a required approximation for sufficiently big j. Similarly, $R_\rho \cup B_\rho \cup I_\rho$ is constructed. □

3.5. **Hybrid equivalence.** Recall §2 that a pacman $f\colon U_f \to V_f$ is required to have a locally analytic extension through ∂U_f. By means of the Pullback argument, we will now show the following:

Theorem 3.11. *Let $f\colon U_f \to V_f$ and $g\colon U_g \to V_g$ be two combinatorially equivalent Siegel pacmen and suppose that f and g have the same truncation level. Then f and g are hybrid equivalent.*

Proof. Let p be the unique quadratic polynomial with the same rotation number as f and g. Let h_f and h_g be hybrid conjugacies from neighborhoods of $\overline{Z}_f$ and $\overline{Z}_g$ to a neighborhood of $\overline{Z}_p$ respectively. As in Corollary 3.7, there are prepacmen Q, R, S in the dynamical planes of p, f, and g respectively such that h_f and h_g are conjugacies respecting prepacmen R, Q and S, Q respectively, see Definition 2.4. The composition $h \coloneqq h_g^{-1} \circ h_f$ is a conjugacy respecting R, S.

As in (3.4) we define $\mathfrak{K}^{\text{loc}}_f$, similarly $\mathfrak{K}^{\text{loc}}_g$ and $\mathfrak{K}^{\text{loc}}_p$ are defined. Then h conjugates $f \mid \mathfrak{K}^{\text{loc}}_f$ and $g \mid \mathfrak{K}^{\text{loc}}_g$.

As in Lemma 3.10 let $R_\ell(f) \cup B_\ell(f) \cup I_\ell(f)$ and $R_\rho(f) \cup B_\rho(f) \cup I_\rho(f)$ be approximations of $\gamma_1(f)$ from the left and from the right respectively. Similarly, let $R_\ell(g) \cup B_\ell(g) \cup I_\ell(g)$ and $R_\rho(g) \cup B_\rho(g) \cup I_\rho(g)$ be approximations of $\gamma_1(g)$. We choose the approximations in the compatible ways:

- $B_\ell(g), I_\ell(g), B_\rho(g), I_\rho(g)$ are the image of $B_\ell(f), I_\ell(f), B_\rho(f), I_\rho(f)$ under h;
- $R_\ell(g), R_\rho(g)$ have the same external angles as $R_\ell(f), R_\rho(f)$.

Write

$$T_f \coloneqq \mathfrak{K}^{\text{loc}}_f \bigcup_{n\ge 0} f^n(R_\rho \cup R_\ell) \quad \text{and} \quad T_g \coloneqq \mathfrak{K}^{\text{loc}}_g \bigcup_{n\ge 0} g^n(R_\rho \cup R_\ell).$$

Then T_f and T_g are forward invariant sets such that $V_f \setminus T_f$ and $V_g \setminus T_g$ consist of finitely many connected components. Since $R_\ell(g), R_\rho(g)$ have the same external angles, we can extend h to a qc map $h\colon V_f \to V_g$ such that h is equivariant on $T_f \cup \partial^{\text{ext}} U_f$.

We now slightly increase U_f by moving $\partial^{\text{frb}} U_f$ so that the new disk $\mathfrak{U}_f$ satisfies

$$f(\partial^{\text{frb}}\mathfrak{U}_f) \subset Z_f \cup \overline{B_\ell \cup R_\ell \cup B_\rho \cup R_\rho}.$$

(Indeed, we can slightly move $\gamma_- \subset \partial^{\text{frb}} U_f$ so that its image is within $R_\ell \cup B_\ell \cup Z_f$ and we can slightly move $\gamma_+ \subset \partial^{\text{frb}} U_f$ so that its image is within $R_\ell \cup B_\ell \cup Z_f$.) Similarly, we slightly increase U_g by moving $\partial^{\text{frb}} U_g$ so that the new disk $\mathfrak{U}_g$ satisfies

$$g(\partial^{\text{frb}}\mathfrak{U}_g) \subset Z_g \cup \overline{B_\ell \cup R_\ell \cup B_\rho \cup R_\rho}$$

and such that $h \mid T_f$ lifts to a conjugacy between $f \mid \partial\mathfrak{U}_f$ and $g \mid \partial\mathfrak{U}_g$. This allows us to apply the *Pullback Argument*: we set $h_0 \coloneqq h$ and we construct qc maps

$$h_n : V_f \to V_g \text{ by } h_n(x) \coloneqq \begin{cases} g^{-1} \circ h_{n-1} \circ f(x) & \text{if } x \in \mathfrak{U}_f, \\ h_{n-1}(x) & \text{if } x \notin \mathfrak{U}_f. \end{cases}$$

Then $\lim_n h_n$ is a hybrid conjugacy between f and g. □

3.6. **Standard Siegel pacmen.** We say a Siegel pacman is *standard* if γ_0 passes through the critical value; equivalently if γ_1 passes through the image of the critical value.

A *standard prepacman* R in the dynamical plane of a Siegel maps g is a prepacman around (see §3.2) the critical value of g. Then the pacman r obtained from R is a standard and the renormalization change of variables ψ_R respects the internal ray towards the critical value:

$$\psi_R(I_1(g)) = I_1(r). \tag{3.5}$$

The pacman renormalization associated with R is called a *standard pacman renormalization of* g.

By Theorem 3.11, two standard Siegel pacmen are hybrid equivalent if and only if they have the same rotation number.

Theorem 3.12. *Let f be a standard Siegel pacman. Then $\mathfrak{K}_f$ is locally connected. Moreover, for every $\varepsilon > 0$ there is an $n \geq 0$ such that every connected component of $\mathfrak{K}_f$ minus all of the bubbles with generation at most n is less than ε.*

As a consequence, every periodic point of $\mathfrak{J}_f$ is the landing point of a bubble chain.

Proof. For every $\theta \in \Theta_{\text{bnd}}$, there is a Standard pacman g with rotation number θ such that is g is a renormalization of a quadratic polynomial. The statement now follows from Theorem 3.11 combined with Lemma 3.8. □

3.7. **A fixed point under renormalization.** Consider a Siegel map f with rotation number $\theta \in \Theta_{\text{per}}$ and consider $x \in \partial Z_f$ such that x is neither the critical point nor its iterated preimage. Let $(f^{\mathfrak{a}} \mid X_{-,x}\ ,\ \ f^{\mathbf{b}} \mid X_{+,x})$ be the sector prerenormalization of $f \mid \overline{Z}_f$ as in Definition 3.5. Since $\theta \in \Theta_{\text{per}}$, we can assume (see §A.4) that the renormalization fixes $f \mid \overline{Z}_f$: the gluing map $\psi_x : X_x \to \overline{Z}_f$ projects $(f^{\mathfrak{a}} \mid X_{-,x}\ ,\ \ f^{\mathbf{b}} \mid X_{+,x})$ back to $f \mid \overline{Z}_f$. For $x \in \{c_0, c_1\}$ we write

$$\psi_0 = \psi_{c_0}, \quad X_0 = X_{c_0}, \quad X_{\pm,0} = X_{\pm,c_0}$$

and

$$\psi_1 = \psi_{c_1}, \quad X_1 = X_{c_1}, \quad X_{\pm,1} = X_{\pm,c_1}.$$

Theorem 3.13 ([McM2])**.** *For every $\theta \in \Theta_{\text{per}}$, there is a Siegel map $g_\star \colon U_\star \to V_\star$ with rotational number θ such that for a certain sector renormalization of $g_\star \mid \overline{Z}_{g_\star}$ as above the gluing map ψ_0 extends analytically through $\partial Z_\star \cap \partial X_0$ to a gluing map ψ_0 projecting $(g_\star^{\mathfrak{a}} \mid S_{-,0}\ ,\ \ g_\star^{\mathbf{b}} \mid S_{+,0})$ back to $g_\star \colon U_\star \to V_\star$, where $S_{\pm,0} \subset U_\star$. Moreover, there is an* improvement of the domain*: the forward orbits*

$$\bigcup_{i \in \{0,1,\dots,\mathfrak{a}\}} g_\star^i(S_{-,0}) \qquad \bigcup_{j \in \{0,1,\dots,\mathbf{b}\}} g_\star^j(S_{+,0})$$

are compactly contained in $U_\star \cap V_\star$.

Up to conformal conjugacy, $g_\star$ is unique in a neighborhood of $\overline{Z}_{g_\star}$. The improvement of the domain will allow us in Theorem 3.16 to construct a pacman analytic self-operator.

Corollary 3.14. *The gluing map ψ_1 extends analytically through $\partial Z_{g_\star} \cap \partial X_1$ and, up to replacing ψ_1 with its iterate, satisfies the same properties as ψ_0 in Theorem 3.13; in particular ψ_1 has improvement of the domain.*

Proof. We need to check that $\psi_1 \coloneqq g_\star \circ \psi_0 \circ g_\star^{-1}$ is well defined. Since ψ_0 projects $(g_\star^{\mathfrak{a}}, g_\star^{\mathbf{b}})$ to $g_\star$ and since the maps $g_\star^{\mathfrak{a}}, g_\star^{\mathbf{b}}$ are two-to-one in a neighborhood of c_0, we obtain that for z close to c_1 the gluing map ψ_0 maps $g_\star^{-1}(z)$ to a pair of points that have the same $g_\star$-image. This shows that ψ_1 is well defined. Up to replacing ψ_1 with its iterate we can guarantee that ψ_1 has an improvement of the domain. □

Note that ψ_1 is expanding on $\overline{Z}_{g_\star} \cap \partial X_1$ because $\psi_1 \mid \overline{Z}_{g_\star}$ is conjugate to

$$\overline{\mathbb{D}^1} \to \overline{\mathbb{D}^1}, \quad z \to z^{1/t}, \qquad t > 1.$$

Lemma 3.15 (Fixed Siegel pacman)**.** *For any $\theta \in \Theta_{\text{per}}$ there is a standard Siegel pacman $f_\star \colon U_\star \to V_\star$ that has a standard Siegel prepacman*

$$F_\star = (f_\star^{\mathfrak{a}} \mid U_- \to S_\star, \quad f_\star^{\mathbf{b}} \mid U_+ \to S_\star)$$

together with a gluing map $\psi_\star \colon S_\star \to \overline{V}_\star$ projecting $F_\star$ back to $f_\star$. Moreover, the renormalization has an improvement of the domain: $\Delta_{F_\star} \Subset f_\star^{-1}(U_\star)$. (See §2.3 for the definition of $\Delta_{F_\star}$.)

The pacman $f_\star$ is conformally conjugate to $g_\star$ in a neighborhood of $Z_\star \coloneqq Z_{f_\star}$.

Proof. Consider a Siegel map $g_\star$ from Theorem 3.13 and ψ_1 from Corollary 3.14.

By Corollary 3.7, $g_\star$ has a standard prepacman $Q\colon U_{Q,\pm} \to S_Q$ such that $S_Q \setminus Z_{g_\star}$ is in a small neighborhood of c_1. Since θ is of periodic type, we can prescribe Q to have rotation number θ. Since ψ_1 is expanding on $\partial \overline{Z}_{g_\star}$, for a sufficiently big integer $t \geq 1$ the prepacman

$$(\psi_1^t)^*(Q) \coloneqq (\psi_1^{-t} \circ q_\pm \circ \psi_1^t \colon \psi_1^{-t}(U_{Q,\pm}) \to \psi_1^{-t}(S_Q))$$

has the property that $\psi_1^{-t}(S_Q) \setminus Z_{g_\star}$ is in much smaller neighborhood of c_1.

Let $f_\star \colon U_\star \to V_\star$ be a pacman obtained from Q. The prepacman $(\psi_1^t)_*(Q)$ projects to the standard prepacman, call it

$$F_\star \colon (f_{\star,\pm} \colon U_{\star,\pm} \to S_\star)$$

such that $S_\star \setminus Z_{f_\star}$ is in a small neighborhood of c_1. The map ψ_1^t descents to a gluing map, call it $\psi_\star$, projecting $F_\star$ back to $f_\star$.

If t is sufficiently big, then $\Delta_{F_\star}$ is compactly contained in $f_\star^{-1}(U_\star)$. □

3.8. **Analytic renormalization self-operator.** Applying Theorem 2.7 to $f_\star$ from Lemma 3.15, we obtain

Theorem 3.16 (Analytic operator $\mathcal{R}\colon \mathcal{B} \dashrightarrow \mathcal{B}$)**.** *Let $f_\star \colon U_\star \to V_\star$ be a pacman and $F_\star$ be a prepacman from Lemma 3.15. Then there are small neighborhoods $N_{\widetilde{U}}(f_\star, \varepsilon), N_{\widetilde{U}}(f_\star, \delta)$ of $f_\star$ with $\varepsilon < \delta$ and there is a compact analytic pacman renormalization operator $\mathcal{R}\colon N_{\widetilde{U}}(f_\star, \varepsilon) \to N_{\widetilde{U}}(f_\star, \delta)$ such that $\mathcal{R} f_\star = f_\star$. In the dynamical plane of $f_\star$ the renormalization $\mathcal{R}$ is realized by $F_\star$.*

Proof. Let $f_\star \colon U' \to V'$ be a pacman obtained form $f_\star \colon U_\star \to V_\star$ by slightly decreasing $U_\star$ so that $U' \Subset U_\star$ and $\Delta_{F_\star} \Subset f_\star^{-1}(U')$. By Theorem 2.7, there is a pacman renormalization operator $N_{\widetilde{U}'}(f_\star, \varepsilon) \to N_{\widetilde{U}}(f_\star, \delta)$, where $\widetilde{U}'$ and $\widetilde{U}$ are small neighborhoods of the closures of U' and $U_\star$. Precomposing with the restriction operator $N_{\widetilde{U}}(f_\star, \varepsilon) \to N_{\widetilde{U}'}(f_\star, \varepsilon)$, we obtain the required operator $\mathcal{R}$. □

To simplify notations, we will often write an operator in Theorem 3.16 as $\mathcal{R}\colon \mathcal{B} \dashrightarrow \mathcal{B}$ with $\mathcal{B} = N_{\widetilde{U}}(f_\star, \delta)$. We can assume (by Lemma 3.4) that $f_\star$ has any given truncation level between 0 and 1.

An *indifferent pacman* is a pacman with indifferent α-fixed point. The *rotation number* of an indifferent pacman f is $\theta \in \mathbb{R}/\mathbb{Z}$ so that $\mathbf{e}(\theta)$ is the multiplier at $\alpha(f)$. If, moreover, $\theta \in \mathbb{Q}$, then f is *parabolic*.

We denote by $\theta_\star$ the multiplier of $f_\star$.

Lemma 3.17. *Let* $R_{\text{prm}}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *be the map defined by*

$$R_{\text{prm}}(\theta) = \begin{cases} \frac{\theta}{1-\theta} & \text{if } 0 \le \theta \le \frac{1}{2} \\ \frac{2\theta-1}{\theta} & \text{if } \frac{1}{2} \le \theta \le 1, \end{cases} \tag{3.6}$$

see (A.2). *Then there is a* $\mathfrak{k} \ge 1$ *such that the following holds. Let* $f \in \mathcal{B}$ *be an indifferent pacman with rotation number* θ. *Then* $\mathcal{R}f$ *is again an indifferent pacman with rotation number* $R^{\mathfrak{k}}_{\text{prm}}(\theta)$.

In particular, $R^{\mathfrak{k}}_{\text{prm}}(\theta_\star) = \theta_\star$.

Proof. Recall that the renormalization $\mathcal{R}$ of $f_\star$ is an extension of a sector renormalization of $f \mid \overline{Z}_\star$, see Definition 3.5 and Appendix A. By Lemma A.2, a sector renormalization is an iteration of the prime renormalization. Therefore, $\mathcal{R}$ is an iteration of the prime pacman renormalization $\mathcal{R}_{\text{prm}}$, see Definition 2.3. We need to check that if f is an indifferent pacman with rotation number θ, then $\mathcal{R}_{\text{prm}} f$ is again an indifferent pacman with rotation number $R_{\text{prm}}(\theta)$. By continuity, it is sufficient to assume that f is a parabolic pacman with rotation number $\mathfrak{p}/\mathfrak{q}$. Then f has $\mathfrak{q}$ local attracting petals in a small neighborhood of α. If $\mathfrak{p} \le \mathfrak{q}/2$, then $\mathcal{R}_{\text{prm}}$ deletes $\mathfrak{p}$ local attracting petals; otherwise $\mathcal{R}_{\text{prm}}$ deletes $\mathfrak{q} - \mathfrak{p}$ local attracting petals. We see that $\mathcal{R}_{\text{prm}} f$ has rotation number $R_{\text{prm}}(\mathfrak{p}/\mathfrak{q})$. □

4. Control of pullbacks

Let us fix the renormalization operator

$$\mathcal{R}\colon \mathcal{B} \dashrightarrow \mathcal{B}, \qquad \mathcal{R} f_\star = f_\star$$

from Theorem 3.16 around a fixed Siegel pacman $f_\star$. Since $\mathcal{R}$ is a compact operator, the spectrum of $\mathcal{R}$ is discrete. Therefore, $\mathcal{R}$ has finitely many eigenvalues of absolute value greater than 1, thus there is a well defined *unstable manifold* $\mathcal{W}^u$ of $\mathcal{R}$.

4.1. **Renormalization triangulation.** Suppose that $f_0 \in \mathcal{B}$ is renormalizable $n \ge 0$ times (this is always the case if f_0 is sufficiently close to $f_\star$) and antirenormalizable $-m \ge 0$ times. We write $[f_k\colon U_k \to V] := \mathcal{R}^k f_0$ for the kth (anti-) renormalization of f_0, where $m \le k \le n$. We denote by $\psi_{k-1}\colon S_k \to V$ the renormalization change of variables realizing the renormalization of f_{k-1} (compare with the left side of Figure 17). We write

$$\phi_k := \psi_k^{-1}.$$

Let us cut the dynamical plane of $f_k\colon U_k \to V$, with $k \in \{m, \dots, n\}$, along γ_1; we denote the resulting prepacman by

$$F_k = (f_{k,\pm}\colon U_{k,\pm} \to V \setminus \gamma_1). \tag{4.1}$$

Lemma 4.1. *By restricting $\mathcal{R}$ to a smaller neighborhood of $f_\star$, the following is true. Suppose f_0 is renormalizable $n \geq 1$ times. Then the map*

$$\Phi_n \coloneqq \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n$$

admits a conformal extension from a neighborhood of $c_1(f_n)$ (where Φ_n is defined canonically) to $V \setminus \gamma_1$. The map $\Phi_n \colon V \setminus \gamma_1 \to V$ embeds the prepacman F_n (see (4.1)*) to the dynamical plane of f_0; we denote the embedding by*

$$\begin{aligned} F_n^{(0)} &= \left(f_{n,\pm}^{(0)} \colon U_{n,\pm}^{(0)} \to S_n^{(0)} \right) \\ &= \left(f_{0,-}^{\mathfrak{a}_n} \colon U_{n,-}^{(0)} \to S_n^{(0)}, f_{0,+}^{\mathbf{b}_n} \colon U_{n,+}^{(0)} \to S_n^{(0)} \right), \end{aligned}$$

where the numbers $\mathfrak{a}_n, \mathbf{b}_n$ are the renormalization return times satisfying (A.4)*.*

Let $\mathbb{\Delta}_n$ be the triangulation obtained by spreading around $U_{n,-}^{(0)}$ and $U_{n,+}^{(0)}$, see §2.3 and Figure 6. In the dynamical plane of f_0 we have

$$\mathbb{\Delta}_0 \coloneqq \overline{U}_0 \Supset \mathbb{\Delta}_1 \Supset \mathbb{\Delta}_2 \Supset \cdots \Supset \mathbb{\Delta}_n,$$

$\mathbb{\Delta}_1(f_0)$ is close in Hausdorff topology to $\mathbb{\Delta}_1(f_\star)$, and moreover $f_0(\mathbb{\Delta}_n) \Subset \mathbb{\Delta}_{n-1}$.

We call $\mathbb{\Delta}_n$ the *nth renormalization triangulation.* Examples of $\mathbb{\Delta}_0, \mathbb{\Delta}_1, \mathbb{\Delta}_2$ are shown in Figures 12 and 13. We say that $\mathbb{\Delta}_n(f_0)$ is the *full lift* of $\mathbb{\Delta}_{n-1}(f_1)$. Similarly (i.e. by lifting and then spreading around), a *full lift* will be defined for other objects.

In the proof of Lemma 4.1 we need to deal with the fact that $\psi_1(\gamma_1)$ can spiral around α, see Figure 11 for illustration. Before given the proof, we need to introduce additional notations.

For consistency, we set $\Phi_0 \coloneqq \mathrm{id}$; then $\mathbb{\Delta}_0 = \overline{U}_0$ is a triangulation consisting of two closed triangles – the closures of the connected components of $U_0 \setminus (\gamma_0 \cup \gamma_1)$. We denote these triangles by $\Delta_0(0)$ and $\Delta_0(1)$ so that $\mathrm{int}(\Delta_0(0)), \gamma_0, \mathrm{int}(\Delta_0(1))$ have counterclockwise orientation around α, see Figure 12. Similarly, $\mathbb{\Delta}_0(f_n)$ is defined.

Let $\Delta_n(0, f_0), \Delta_n(1, f_0)$ be the images of $\Delta_0(0, f_n), \Delta_0(1, f_n)$ via the map Φ_n from Lemma 4.1. By definition, $\mathbb{\Delta}_n$ is a triangulated neighborhood of α obtained by spreading around $\Delta_n(0, f_0), \Delta_n(1, f_0)$. We enumerate counterclockwise these triangles as $\Delta_n(i)$ with $i \in \{0, 1, \dots, \mathfrak{q}_n - 1\}$. By construction, $\Delta_n(0) \cup \Delta_n(1) \ni c_1(f_n)$.

We remark that $f_0 \mid \mathbb{\Delta}_n$ is an antirenromalization of $f_n \colon U_n \to V$ in the sense of Appendix A. There is a $\mathfrak{p}_n$ such that

$$f_0 \colon \Delta_n(i) \to \Delta_n(i + \mathfrak{p}_n) \tag{4.2}$$

is conformal for all $i \notin \{-\mathfrak{p}_n, -\mathfrak{p}_n + 1\}$ with index taking modulo $\mathfrak{q}_n$. We have an almost two-to-one map

$$f_0 \colon \Delta_n(-\mathfrak{p}_n) \cup \Delta_n(-\mathfrak{p}_n + 1) \to S_0^{(n)} \supset \Delta_n(0) \cup \Delta_n(1). \tag{4.3}$$

We will show in Theorem 4.6 that if f_0 is close to $f_\star$, then $\mathbb{\Delta}_n = \bigcup_i \Delta_n(i)$ approximates dynamically and geometrically $\overline{Z}_\star$.

By construction, for every triangle $\Delta_n(i, f_0)$ there is a $t \geq 0$ and $j \in \{0, 1\}$ such that a certain branch of f_0^{-t} maps conformally $\Delta_n(i, f_0)$ to $\Delta_n(j, f_0)$. We define $\Psi_{n,i}$ on $\Delta_n(i, f_0)$ by

$$\Psi_{n,i} \coloneqq \Phi_n^{-1} \circ f_0^{-t} \colon \Delta_n(i, f_0) \to \Delta_0(j, f_n). \tag{4.4}$$

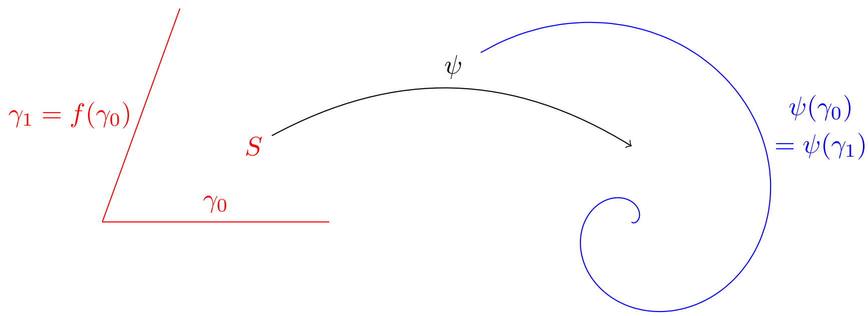


FIGURE 11. Suppose $f(z) = \lambda z$ with $\lambda \notin \mathbb{R}_+$ and let S be the sector between γ_0 and $\gamma_1 = f(\gamma_0)$. Let $\psi\colon S \to \mathbb{C}$ be the gluing map identifying dynamically γ_0 and γ_1. If $|\lambda| \neq 1$, then $\psi(\gamma_0)$ does not land at 0 at a well defined angle.

Let A be an annulus enclosing an open disk O such that $\alpha \in O$. We say A is an N-*wall* if for all $z \in O$ and all j with $|j| \leq N$ we have

$$(f_0 \mid A \cup O)^j(z) \subset O \cup A.$$

If the restricting $f_0 \mid A \cup O$ is univalent, then A is a *univalent wall*.

Fix a small $r > 0$ and denote by $Z^r_\star$ the open subdisk of $Z_\star$ bounded by the equipotential of height r. Set $\mathbb{\Pi}_0 := \overline{U}_0 \setminus Z^r_\star$. It is a closed annulus enclosing α. We decompose $\mathbb{\Pi}_0$ into two closed rectangles $\Pi_0(0) = \mathbb{\Pi}_0 \cap \Delta_0(0)$ and $\Pi_0(1) = \mathbb{\Pi}_0 \cap \Delta_0(1)$; they are the closures of the connected components of $\mathbb{\Pi}_0 \setminus (\gamma_0 \cap \gamma_1)$.

Lemma 4.2 (The wall of $\mathbb{\Delta}_n$)**.** *Suppose all $f_0, f_1, \dots, f_n$ are in a small neighborhood of $f_\star$. Then the following construction of $\mathbb{\Pi}_n(f_0)$, called the* wall *of $\mathbb{\Delta}_n$, holds.*

1. *The map Φ_n extends from a neighborhood of $c_1(f_n)$ to $\mathbb{\Pi}_0 \setminus \gamma_1$;*
2. *Let $\Pi_n(0, f_0)$ and $\Pi_n(1, f_0)$ be the images of $\Pi_0(0, f_n)$ and $\Pi_0(1, f_n)$ under Φ_n. Then, by spreading around $\Pi_0(0, f_n)$ and $\Pi_0(1, f_n)$, we obtain an annulus $\mathbb{\Pi}_n$ enclosing α. We enumerate counterclockwise rectangles in $\mathbb{\Pi}_n$ as $\Pi_n(i)$ with $i \in \{0, 1, \dots, \mathfrak{q}_n - 1\}$.*
3. *We have $\mathbb{\Pi}_0 \Supset \mathbb{\Pi}_1 \Supset \dots \Supset \mathbb{\Pi}_n$ with $\mathbb{\Pi}_0(f_0)$ close to $\mathbb{\Pi}_0(f_\star)$.*
4. *For every $\Pi_n(i)$, there is a $t \geq 0$ such that a certain branch of f_n^{-t} maps $\Pi_n(i)$ to $\Pi_n(j)$ with $j \in \{0, 1\}$. Then*

$$\Psi_{n,i} := \Phi_n^{-1} \circ f_0^{-t}\colon \Pi_n(i, f_0) \to \Pi_0(j, f_n). \tag{4.5}$$

   is conformal. If n is sufficiently big, then all $\Psi_{n,i}$ expand the Euclidean metric and the expanding constant is at least η^n for a fixed $\eta > 1$. In particular, the diameter of rectangles in $\mathbb{\Pi}_n$ tends to 0.
5. *The wall $\mathbb{\Pi}_n(f_0)$ approximates $\partial Z_\star$ in the following sense: $\partial Z_\star$ is a concatenation of arcs $J_0 J_1 \dots J_{\mathfrak{q}_n - 1}$ such that $\Pi_n(i)$ and J_i are close in the Hausdorff topology.*

As with renormalization triangulation, we say that $\mathbb{\Pi}_n(f_0)$ is the *full lift* of $\mathbb{\Pi}_{n-1}(f_1)$.

Proof. If $f_0 = f_\star$, then all the claims follow from the improvement of the domain, see Corollary 3.14. Let us now show that the expansion also holds if $f_0, f_1, \dots, f_n$ are in a small neighborhood of $f_\star$.

Consider still the case $f_0 = \dots = f_n = f_\star$. We can enlarge $\Pi_0(0)$ and $\Pi_0(1)$ to $\widetilde{\Pi}_0(0)$ and $\widetilde{\Pi}_0(1)$ so that

- $\Psi_{n,i}\colon \Pi_n(i) \to \Pi_0(j)$ extends to a conformal map $\Psi_{n,i}\colon \widetilde{\Pi}_n(i) \to \widetilde{\Pi}_0(j)$, where $\widetilde{\Pi}_n(i)$ is the associated enlargement of $\Pi_n(i)$; and
- there is a $k \geq 1$ such that every $\widetilde{\Pi}_k(i)$ is within some $\widetilde{\Pi}_0(j)$ and such that all $\Psi_{k,i} \mid \widetilde{\Pi}_k(i)$ are expanding.

As a consequence, if $n \geq k$, then every $\widetilde{\Pi}_n(i)$ is within some $\widetilde{\Pi}_{n-k}(i_2)$ and we can decompose

$$\Psi_{n,i} \mid \widetilde{\Pi}_n(i) = \Psi_{k,j_1}(f_{n-k}) \circ \Psi_{n-k,i_2}(f_0).$$

(Note that we still have the assumption that $f_0 = f_{n-k} = f_\star$.) Continuing this process we obtain a decomposition

$$\Psi_{n,i} \mid \widetilde{\Pi}_n(i) = \Psi_{k,j_1}(f_{n-k}) \circ \Psi_{k,j_2}(f_{n-2k}) \circ \dots \circ \Psi_{k,j_t}(f_{n-tk}) \circ \Psi_{n-tk,i_t}(f_0), \tag{4.6}$$

with $n - tk \in \{0, 1, \dots, k-1\}$.

Suppose now that $f_0, \dots, f_n$ are close to $f_\star$. By continuity, all $\Psi_{k,j_i}(f_{n-ik})$ are still expanding while $\Psi_{n-tk,i}(f_0)$ is close to $\Psi_{n-tk,i}(f_\star)$ independently on n. This shows Claim (4); other claims are consequences of Claim (4). □

Proof of Lemma 4.1. We will now apply Theorem B.8 to show that the full lift $\mathbb{\Delta}_n(f_0)$ of $\mathbb{\Delta}_0(f_n)$ exists.

Let $Q_0 \subset Z_\star$ be the closed annulus bounded by the equipotentials at heights r and $2r$. Then $Q_0 \subset \mathbb{\Pi}_0$ and we decompose Q_0 into two rectangles $Q_0(0) = \Pi_0(0) \cap Q_0$ and $Q_0(1) = \Pi_0(1) \cap Q_0$. Let $Q_n(0, f_0)$ and $Q_n(1, f_0)$ be the images of $Q_0(0, f_n)$ and $Q_0(1, f_n)$ under Φ_n. By spreading around $Q_n(0, f_n)$ and $Q_n(1, f_n)$, we obtain (by Lemma 4.2) an annulus Q_n enclosing α. We enumerate counterclockwise rectangles in Q_n as $Q_n(i)$ with $i \in \{0, 1, \dots, \mathfrak{q}_n - 1\}$. We have $Q_n(i) \subset \Pi_n(i)$.

Denote by Ω_n the open topological disk enclosed by Q_n. Then $f_0 \mid \Omega_n \cup Q_n$ is an anti-renormalization of $f_1 \mid \Omega_{n-1} \cup Q_{n-1}$ (in the sense of Appendix A) with respect to the dividing pair of curves γ_0, γ_1.

We proceed by induction; suppose the statement is verified for $n-1$. In the dynamical plane of f_1, we denote by $\gamma_0^{(n-1)}$ the lift of $\gamma_0(f_n)$ under the $(n-1)$-anti-renormalization specified so that $\gamma_0^{(n-1)}$ crosses Q_{n-1} at $Q_{n-1}(0) \cap Q_{n-1}(1)$. Note that $Q_n(0) \cup Q_n(1)$ is in a small neighborhood of c_1 because Φ_n is contracting. Therefore, $\gamma_0^{(n-1)} \cap Q_{n-1}$ is close to $\gamma_0 \cap Q_{n-1}$. We can sightly adjust γ_0 in a neighborhood of Q_{n-1}, such that the new γ_0^{new} crosses Q_{n-1} at $Q_{n-1}(0) \cap Q_{n-1}(1)$. Let $\gamma_1^{(n-1)}$ and γ_1^{new} be the images of $\gamma_0^{(n-1)}$ and γ_0^{new} respectively. By Theorem B.8 (note that a wall contains a fence, see Remark B.10) an anti-renormalization of $f_1 \mid \Omega_{n-1} \cup Q_{n-1}$ with respect to $\gamma_0^{(n-1)}, \gamma_1^{(n-1)}$ is naturally conjugate to the corresponding anti-renormalization of $f_1 \mid \Omega_{n-1} \cup Q_{n-1}$ with respect to $\gamma_0^{\text{new}}, \gamma_1^{\text{new}}$. Therefore, the full lift $\mathbb{\Delta}_n(f_0)$ of $\mathbb{\Delta}_{n-1}(f_1)$ exists; $\mathbb{\Delta}_n(f_0)$ is a required triangulation of $\mathbb{\Pi}_n \cup \Omega_n$.

By Lemma 3.15 combined with continuity, we have $f_0(\mathbb{\Delta}_1) \subset \mathbb{\Delta}_0$. Applying induction on n, we obtain $f_0(\mathbb{\Delta}_{n+1}) \Subset \mathbb{\Delta}_n$.

Set

$$S_n^{(0)} := f_0\left(\Delta_n(-\mathfrak{p}_n) \cup \Delta_n(-\mathfrak{p}_n + 1)\right)$$

(compare with (4.3)). We can now define $F_n^{(0)}$ as the embedding of F_n from (4.1) to the dynamical plane of f_0. □

In fact, the exact behavior of γ_1 in a small neighborhood of α is irrelevant in the proof of Lemma 4.1. We have

Lemma 4.3 (Changing γ_1)**.** *Let $\gamma_0^{\text{new}}, \gamma_1^{\text{new}} = f_n(\gamma_0^{\text{new}})$ be a new pair of curves in the dynamical plane of f_n such that*

- *$\gamma_0 \setminus Z_\star^r = \gamma_0^{\text{new}} \setminus Z_\star^r$ and $\gamma_1 \setminus Z_\star^r = \gamma_1^{\text{new}} \setminus Z_\star^r$; and*
- *γ_0^{new} and γ_1^{new} are disjoint away from α.*

Then Lemma 4.1 still holds after replacing γ_0, γ_1 with $\gamma_0^{\text{new}}, \gamma_1^{\text{new}}$. More precisely, let $\Delta_0^{\text{new}}(0), \Delta_0^{\text{new}}(1)$ be the closures of the connected components of $U_0 \setminus (\gamma_0^{\text{new}} \cup \gamma_1^{\text{new}})$ in the dynamical plane of f_n. As in Lemma 4.1 the map Φ_n extends from a neighborhood of $c_1(f_n)$ to $V \setminus \gamma_1^{\text{new}}$; let $\Delta_n^{\text{new}}(0, f_0), \Delta_n^{\text{new}}(1, f_0)$ be the images of $\Delta_0^{\text{new}}(0, f_n), \Delta_0^{\text{new}}(1, f_n)$ under the new Φ_n. By spreading around $\Delta_n^{\text{new}}(0, f_0), \Delta_n^{\text{new}}(1, f_0)$ we obtain a new triangulated neighborhood $\mathbb{\Delta}_n^{\text{new}}$ of α.

Note that $\mathbb{\Delta}_n^{\text{new}}$ and $\mathbb{\Delta}_n$ triangulate the same neighborhood of α.

Proof. Since $\gamma_1^{\text{new}}, \gamma_0^{\text{new}}$ coincide with γ_1, γ_0 away from Z^r, the wall $\mathbb{\Pi}_n$ is unaffected; thus we can repeat the proof of Lemma 4.1 for γ_1^{new}. □

4.2. **Siegel triangulations.** We will also consider triangulations that are perturbations of $\mathbb{\Delta}_n$. Let us introduce appropriate notations. Consider a pacman $f \in \mathcal{B}$. A *Siegel triangulation* $\mathbb{\Delta}$ is a triangulated neighborhood of α consisting of closed triangles, each has a vertex at α, such that

- triangles of $\mathbb{\Delta}$ are $\{\Delta(i)\}_{i \in \{0, \dots \mathfrak{q}\}}$ enumerated counterclockwise around α so that $\Delta(i)$ intersects only $\Delta(i-1)$ (on the right) and $\Delta(i+1)$ (on the left); $\Delta(i-1)$ and $\Delta(i+1)$ are disjoint away from α;
- there is a $\mathfrak{p} > 0$ such that f maps $\Delta(i)$ to $\Delta(i + \mathfrak{p})$ for all $i \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$;
- $\mathbb{\Delta}$ has a distinguished 2-wall $\mathbb{\Pi}$ enclosing α and containing $\partial \mathbb{\Delta}$ such that each $\Pi(i) \coloneqq \mathbb{\Pi} \cap \Delta(i)$ is connected and f maps $\Pi(i)$ to $\Pi(i + \mathfrak{p})$ for all $i \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$; and
- $\mathbb{\Pi}$ contains a univalent 2-wall Q such that each $Q(i) \coloneqq Q \cap \Pi(i)$ is connected and f maps $Q(i)$ to $Q(i + \mathfrak{p})$ for all $i \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$.

The nth renormalization triangulation is an example of a Siegel triangulation.

Similar to Lemma 4.2, Part (5), we say that $\mathbb{\Pi}$ *approximates* $\partial Z_\star$ if $\partial Z_\star$ is a concatenation of arcs $J_0 J_1 \dots J_{\mathfrak{q}-1}$ such that $\Pi(i)$ and J_i are close in the Hausdorff topology.

Lemma 4.4. *Let $f \in \mathcal{B}$ be a pacman such that all $f, \mathcal{R}f, \dots, \mathcal{R}^n$ are in a small neighborhood of $f_\star$. Let $\mathbb{\Delta}(\mathcal{R}^n f)$ be a Siegel triangulation in the dynamical plane of $\mathcal{R}^n f$ such that $\mathbb{\Pi}(\mathcal{R}^n f)$ approximates $\partial Z_\star$. Then $\mathbb{\Delta}(\mathcal{R}^n f)$ has a full lift $\mathbb{\Delta}(f)$ which is again a Siegel triangulation. Moreover, $\mathbb{\Pi}(f)$ also approximates $\partial Z_\star$.*

Proof. Follows from the same argument as in the proof of Lemma 4.1. Suppose first $n = 1$. Since all $\Pi(i, \mathcal{R}f)$ are small, the arc γ_0 can be slightly adjusted in a neighborhood of $\mathbb{\Pi}$ so that γ_0 crosses $\mathbb{\Pi}$ at $\Pi(i, \mathcal{R}f) \cap \Pi(i+1, \mathcal{R}f)$ with $i \notin \{-\mathfrak{p}-1, -\mathfrak{p}, -\mathfrak{p}+1\}$. This allows to construct a full lift $\mathbb{\Pi}(f)$ of $\mathbb{\Pi}(\mathcal{R}f)$. Since $\mathfrak{a}_1, \mathbf{b}_1 \geq 2$ (see (A.5)), the annuli $\mathbb{\Pi}(f)$ and $Q(f)$ are again 2-walls, see Lemma B.12. Applying Theorem B.8 from Appendix B we construct a full lift $\mathbb{\Delta}(f)$ of $\mathbb{\Delta}(\mathcal{R}f)$. Lemma 4.2 Part (4) allows to apply induction on n: for big n, the wall $\mathbb{\Pi}(f)$ approximates $\partial Z_\star$ better than $\mathbb{\Pi}(\mathcal{R}^n f)$ approximates $\partial Z_\star$. □

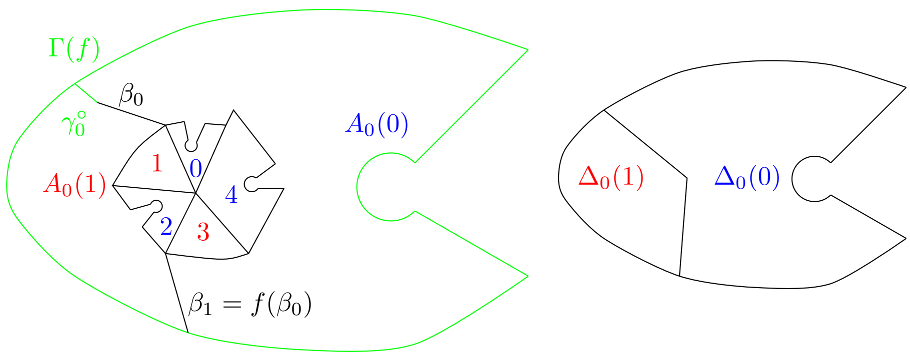


FIGURE 12. Renormalization tiling. Right: triangles $\Delta_0(0), \Delta_0(1)$ are the closures of the connected components of $U_1 \setminus (\gamma_0 \cup \gamma_1)$. They form a renormalization tiling of level 0. Left: the triangles labeled by 0 and 1, i.e. $\Delta_1(0)$ and $\Delta_1(1)$ respectively, are anti-renormalization embeddings of $\Delta_0(0), \Delta_0(1)$; the forward orbit of $\Delta_1(0), \Delta_1(1)$ triangulates a neighborhood of α. Together with $A_0(0) \cup A_0(1)$, this gives a tiling of U_0 of level 1.

We remark that $\mathbb{\Delta}(f)$ is not uniquely defined; it depends on how (the new) γ_0 crosses $\mathbb{\Pi}(\mathcal{R}f)$.

4.3. **Renormalization tiling.** Consider $\partial \mathbb{\Delta}_1(f_0)$ and set (see Figure 12)

$$\Gamma(f_1) \coloneqq \bigcup_i \Psi_{1,i}(\partial \mathbb{\Delta}_1(f_0) \cap \Delta(i, f_0))$$

to be the image of $\partial \mathbb{\Delta}_1(f_0)$ under $\Psi_{1,i}$.

Lemma 4.5. *The set $\Gamma(f_1)$ consists of ∂U_1 and an arc γ_0° such that $\gamma_0^\circ \subset \gamma_0$ and $f_1(\gamma_0^\circ) \not\in \operatorname{int} U_1$. We have $\Gamma(f_1) \cap \partial \mathbb{\Delta}_1(f_1) = \emptyset$.*

There are disjoint arcs β_0 and $\beta_1 = f_1(\beta_0)$ such that

- *the concatenation of γ_0° and β_0 connects $\partial \mathbb{\Delta}_0$ to $\partial \mathbb{\Delta}_1$; and*
- *β_1 connects $\partial \mathbb{\Delta}_0$ to $\partial \mathbb{\Delta}_1$.*

In a small neighborhood of $f_\star$ the curves β_0, β_1 can be chosen so that there is a holomorphic motion of

$$[\mathbb{\Delta}_1 \cup \partial \mathbb{\Delta}_0 \cup \gamma_0^\circ \cup \beta_0 \cup \beta_1]\,(f_0) \tag{4.7}$$

that is equivariant *with the following maps*

1. *$f \colon \beta_0(f_0) \to \beta_1(f_0)$;*
2. *$f_0 \colon \Delta_1(i, f_0) \to \Delta_1(i + \mathfrak{p}_1, f_0)$ for $i \not\in \{-\mathfrak{p}_1, -\mathfrak{p}_1 + 1\}$;*
3. *$\Psi_{1,i} \colon \partial \mathbb{\Delta}_1(f_1) \cap \Delta_1(i, f_1) \to \Gamma(f_0)$.*

Proof. Each triangle $\Delta_1(i)$ has three distinguished closed sides; we denote them by $\lambda(i)$, $\rho(i)$, and $\ell(i)$ such that $\lambda(i)$ and $\rho(i)$ are the left and right sides meeting at the α-fixed point while $\ell(i)$ is the opposite to α side. We have:

$$\partial \mathbb{\Delta}_1 = \bigcup_i \left(\ell_i \bigcup (\lambda(i) \setminus \rho(i+1)) \bigcup (\rho(i+1) \setminus \lambda(i)) \right).$$

Note that $\Psi_{1,i}(\ell(i)) \subset \partial \mathbb{\Delta}_0$ and, moreover, $\bigcup_i \Psi_{1,i}(\ell(i)) = \partial \mathbb{\Delta}_0$.

Let us analyze $(\lambda(i) \setminus \rho(i+1)) \cup (\rho(i+1) \setminus \lambda(i))$. If $\lambda(i) \neq \rho(i+1)$, then one of the curves in $\{\lambda(i), \rho(i+1)\}$ is a preimage of $\gamma_0(f_1)$ while the other is a preimage of $\gamma_1(f_1)$. We have:

$$\Psi_{1,i}(\overline{\lambda(i) \setminus \rho(i+1)}) \bigcup \Psi_{1,i+1}(\overline{\rho(i+1) \setminus \lambda(i)}) = \gamma_0^\circ \setminus .$$

It is clear (see Appendix A) that $\lambda(i) \neq \rho(i+1)$ for at least one i.

The property $\Gamma(f_1) \cap \partial \mathbb{\Delta}_1(f_1) = \emptyset$ follows from $\partial \mathbb{\Delta}_0 \cap f_1(\mathbb{\Delta}_1) = \emptyset$, see Lemma 4.1. Since $\Gamma(f_1) \cap \partial \mathbb{\Delta}_1(f_1) = \emptyset$, we can find β_0 such that $\gamma_0^\circ \cup \beta_0$ is in a small neighborhood of γ_0 and $\gamma_0^\circ \cup \beta_0$ connects $\partial \mathbb{\Delta}_0$ to $\partial(\mathbb{\Delta}_1 \setminus (\Delta_1(-\mathfrak{p}_1) \cup \Delta_1(-\mathfrak{p}_1 - 1)))$. Then $\beta_1 = f_1(\beta_0)$ is disjoint from $\gamma_0^\circ \cup \beta_0$ and β_1 connects $\partial \mathbb{\Delta}_0$ to $\partial \mathbb{\Delta}_1$.

In a small neighborhood of $f_\star$ we have a holomorphic motion of $\partial \mathbb{\Delta}_0(f_0)$. Applying the λ-lemma, we obtain a holomorphic motion of the triangulation $\mathbb{\Delta}_0$ that is equivariant with $f_0 \mid \gamma_0$. Lifting this motion via $\Psi_{1,i}$, we obtain a holomorphic motion of $\partial \mathbb{\Delta}_0 \cup \mathbb{\Delta}_1 \cup \Gamma$ equivariant with (2) and (3). Applying again the λ-lemma, we extend the latter motion to the motion of (4.7) that is also equivariant with (1). □

Let $\mathbb{A}_0$ be the closed annulus between $\partial \mathbb{\Delta}_0$ and $\partial \mathbb{\Delta}_1$. The arcs $\gamma_0^\circ \cup \beta_0, \beta_1$ split $\mathbb{A}_0$ into two closed *rectangles* $A_0(0), A_0(1)$ (see Figure 12) enumerated such that $\operatorname{int}(A_0(0)), \gamma_0^\circ \cup \beta_0, \operatorname{int}(A_0(1)), \beta_1$ have counterclockwise orientation.

Let $\mathbb{A}_n$ be the closed annulus between $\partial \mathbb{\Delta}_n$ and $\partial \mathbb{\Delta}_{n+1}$. Define

$$A_n(0, f_0) := \Phi_n(A_0(0, f_n)) \quad \text{and} \quad A_n(1, f_0) := \Phi_n(A_1(0, f_n))$$

and spread dynamically $A_n(0, f_0), A_n(1, f_0)$ (compare with the definition of $\Delta_n(i)$ in §4.1); we obtain the partition of $\mathbb{A}_n(f_0)$ by rectangles $\{A_n(i, f_0)\}_{0 \le i < \mathfrak{q}_n}$ enumerated counterclockwise. Similar to (4.5) we define the map $\Psi_{n,i} \colon A_n(i, f_0) \to A_0(j, f_n)$ with $j \in \{0, 1\}$.

The nth renormalization tiling is the union of all the triangles of $\mathbb{\Delta}_n$ and the union of all the rectangles of all $\mathbb{A}_m$ for all $m < n$. The nth renormalization tiling is defined as long as all $f_0, \dots, f_n$ are in a small neighborhood of $f_\star$.

A *qc combinatorial pseudo-conjugacy of level n between f_0 and $f_\star$* is a qc map $h \colon \overline{U}_0 \to \overline{U}_\star$ that is compatible with the nth renormalization tilings as follows:

- h maps $\Delta_n(i, f_0)$ to $\Delta_n(i, f_\star)$ for all i;
- h maps $A_m(i, f_0)$ to $A_m(i, f_\star)$ for all i and $m < n$;
- h is equivariant on $\Delta_n(i, f_0)$ for all $i \notin \{-\mathfrak{p}_n, -\mathfrak{p}_n + 1\}$; and
- h is equivariant on $A_m(i, f_0)$ for all $i \notin \{-\mathfrak{p}_m, -\mathfrak{p}_m + 1\}$ and $m < n$.

The following theorem says that $f \mid \mathbb{\Delta}_n(f)$ *approximates* $f_\star \mid \overline{Z}_\star$ both dynamically and geometrically.

Theorem 4.6 (Combinatorial pseudo-conjugacy)**.** *Consider an nth renormalizable pacman f and set*

$$d := \max_{i \in \{0,1,\dots,n\}} \operatorname{dist}(\mathcal{R}^i f, f_\star).$$

If d is sufficiently small, then there is a qc combinatorial pseudo-conjugacy h of level n between f and $f_\star$ and, moreover, the following properties hold. The qc dilatation and the distance between $h \mid \mathbb{\Delta}_n(f)$ and the identity on $\mathbb{\Delta}_n(f)$ are bounded by constants $K(d), M(d)$ respectively with $K(d) \to 1$ and $M(d) \to 0$ as $d \to 0$.

Proof. By Lemma 4.5, the set (4.7) moves holomorphically with f in a small neighborhood of $f_\star$. Applying the λ-lemma, we obtain a holomorphic motion τ of the first renormalization tiling with f in a small neighborhood $\mathcal{U}$ of $f_\star$.

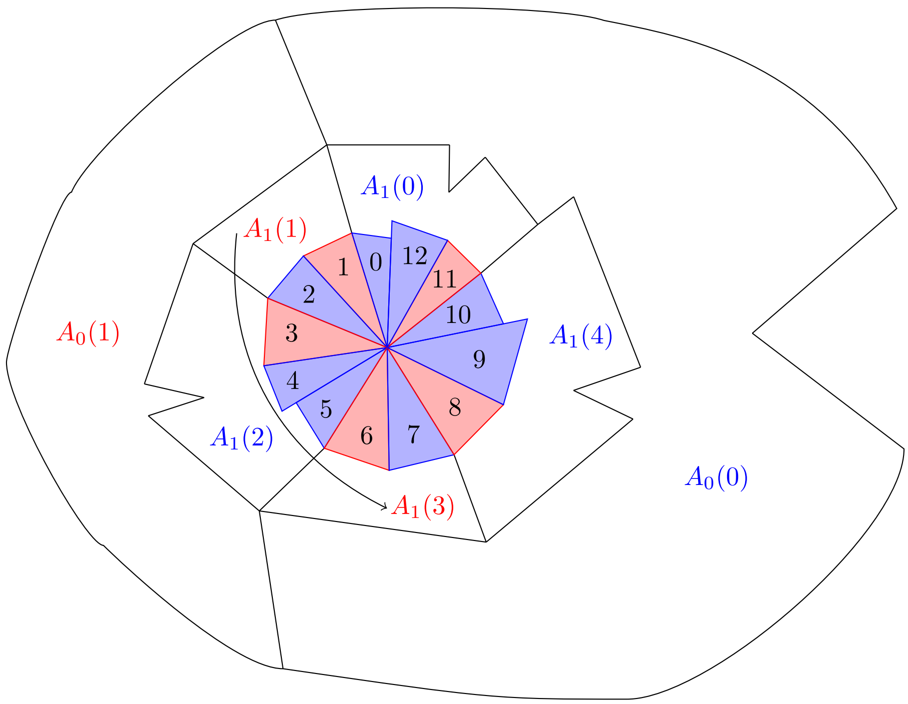


FIGURE 13. Renormalization tiling of level 2; tiling of smaller levels are in Figure 12. There are $\mathfrak{q}_2 = 12$ triangles in $\mathbb{\Delta}_2$ with rotation number $\mathfrak{p}_2/\mathfrak{q}_2 = 5/12$. Geometry of triangles in $\mathbb{\Delta}_2$ is simplified. The image of $\Delta_2(8) \cup \Delta_2(9)$ is a sector slightly bigger than $\Delta_2(0) \cup \Delta_2(1)$ – compare with Figure 6.

Suppose now that d is so small that all $f_i \coloneqq \mathcal{R}^i f$ are in $\mathcal{U}$. For every $\Delta_n(i)$ of f_0 or of $f_\star$ consider the map $\Psi_{n,i} \colon \Delta_n(i) \to \Delta_0(j)$, where $\Delta_0(j)$ is the corresponding triangle of f_n or of $f_\star$. Then h on $\Delta_n(i)$ is defined by applying first $\Psi_{n,i} \colon \Delta_n(i, f_0) \to \Delta_0(j, f_n)$ (see (4.4)), then applying the motion τ from $\Delta_0(j, f_n)$ to $\Delta_0(j, f_\star)$, and then applying $\Psi^{-1}_{n,i} \colon \Delta_0(j, f_\star) \to \Delta_n(i, f_\star)$.

Similarly, for every $A_m(i)$ of f_0 or of $f_\star$ consider the map $\Psi_{m,i} \colon A_m(i) \to A_0(j)$, where $A_0(j)$ is the corresponding rectangle of f_m or of $f_\star$. Then h on $A_m(i)$ is defined by applying first $\Psi_{m,i} \colon A_m(i, f_0) \to A_0(j, f_m)$, then applying the motion τ from $A_m(i, f_0)$ to $A_m(i, f_\star)$, and then applying $\Psi^{-1}_{m,i} \colon A_0(j, f_\star) \to A_m(i, f_\star)$.

Observe now that h is well defined for all points on the boundaries of all the rectangles and all the triangles because τ is equivariant with (1), (2), (3) – see Lemma 4.5. Therefore, all points have well defined images under h.

The qc dilatation of h is bounded by the qc dilatation of τ at f_i with $i \in \{0, 1, \dots, n\}$. This bounds the qc dilatation of h by $K(d)$ as above with $K(d) \to 1$ as $d \to 0$.

If $n = 1$, then since τ is continuous, the distance between $h \mid \mathbb{\Delta}_1(f_0)$ and the identity on $\mathbb{\Delta}_1(f_0)$ is bounded by $M(d)$ as required. If $n > 1$, then $\mathbb{\Delta}_n(f_0) \Subset U_0$ and the claim follows from the compactness of qc maps with bounded dilatation □

Corollary 4.7. *There is an $\varepsilon > 0$ with the following property. Suppose that $f \in \mathcal{B}$ is infinitely renormalizable and that all $\mathcal{R}^n f$ for $n \geq 0$ are in the ε-neighborhood of $f_\star$. Then there is a qc map $h\colon U_f \to U_\star$ such that h^{-1} is a conjugacy on $\overline{Z}_\star$. Therefore, a certain restriction of f is a Siegel map and $f, f_\star$ are hybrid conjugate on neighborhoods of their Siegel disks.*

Proof. If ε is sufficiently small, then by Theorem 4.6, for every $n \geq 0$ there exists qc combinatorial pseudo-conjugacy h_n of level n between f and $f_\star$ such that the dilation of h_n is uniformly bounded for all n. By compactness of qc map, we may pass to the limit and construct a qc map $h\colon U_f \to U_\star$ such that h^{-1} is a conjugacy on $\overline{Z}_\star$. It follows, in particular, that f is a Siegel map. By Theorem 3.6, the maps $f, f_\star$ are hybrid conjugate on neighborhoods of their Siegel disks. □

4.4. **Control of pullbacks.** Recall from Lemma 4.1 that $\mathfrak{a}_n, \mathbf{b}_n$ denote the closest renormalization return times computed by (A.4). By definition, $\mathfrak{a}_n + \mathbf{b}_n = \mathfrak{q}_n$. We now restrict out attention to $f \in \mathcal{W}^u$.

Key Lemma 4.8. *There is a small open topological disk D around $c_1(f_\star)$ and there is a small neighborhood $\mathcal{U} \subset \mathcal{W}^u$ of $f_\star$ such that the following property holds. For every sufficiently big $n \geq 1$, for each $\mathfrak{t} \in \{\mathfrak{a}_n, \mathbf{b}_n\}$, and for all $f \in \mathcal{R}^{-n}(\mathcal{U})$, we have $c_{1+\mathfrak{t}}(f) \coloneqq f^{\mathfrak{t}}(c_1) \in D$ and D pullbacks along the orbit $c_1(f), c_2(f), \dots, c_{1+\mathfrak{t}}(f) \in D$ to a disk D_0 such that $f^{\mathfrak{t}}\colon D_0 \to D$ is a branched covering; moreover, $D_0 \subset U_f \setminus \gamma_1$.*

Proof. The main idea of the proof is to block the forbidden part of the boundary $\partial^{\mathrm{frb}} U_f$ from the backward orbit of D. The proof is split into short subsections. We start the proof by introducing conventions and additional terminology. The central argument will be in Claim 10, Part (4).

4.4.1. *The triangulated disk $\mathbb{\Delta}$ approximates $\overline{Z}_\star$.* Throughout the proof we will often say that a certain object is *small* if it has a small size independently of n. Choose a big $s \gg 0$ and choose a small neighborhood $\mathcal{U}$ of $f_\star$ such that every $f \in \mathcal{R}^{-n}(\mathcal{U})$ is at least $m \coloneqq n + s$ renormalizable and each $f_i \coloneqq \mathcal{R}^i f$ with $i \in \{0, 1, \dots, m\}$ is close to $f_\star$.

Consider the m-th renormalization triangulation $\Delta_m(i)$ of f. Let h be a qc combinatorial pseudo-conjugacy of level m as in Theorem 4.6. To keep notation simple, we sometimes drop the subindex m and write $\Delta(i), \mathbb{\Delta}, \mathfrak{q}, \mathfrak{p}$ for $\Delta_m(i), \mathbb{\Delta}_m, \mathfrak{q}_m, \mathfrak{p}_m$.

Since f_i with $i \in \{0, 1, \dots, m\}$ are close to $f_\star$, the map $h \mid \mathbb{\Delta}$ is close (by Theorem 4.6) to the identity. In particular, $\mathbb{\Delta}(f) = h^{-1}(\mathbb{\Delta}(f_\star))$ approximates $\overline{Z}_\star$ in the sense of Theorem 4.6. Since s is big and since $\mathfrak{a}_i, \mathbf{b}_i$ have exponential growth with the same exponent (A.4), we have

$$t/\mathfrak{q}_m \in \{\mathfrak{a}_n/\mathfrak{q}_{n+s}\ ,\quad \mathbf{b}_n/\mathfrak{q}_{n+s}\} \quad \text{is sufficiently small.} \tag{4.8}$$

4.4.2. *Disks $D_k \ni f^k(c_1)$.* Let us argue that $D \ni f^{\mathfrak{t}}(c_1)$. Consider first the dynamical plane of $f_\star$. Since n is big, we see that $f_\star^{\mathfrak{a}_n}(c_1), f_\star^{\mathbf{b}_n}(c_1)$ are sufficiently close to $c_1(f_\star)$; i.e. $D \ni f_\star^{\mathfrak{t}}(c_1)$. It follows from (4.8) that

$$\min\{\mathfrak{a}_m, \mathbf{b}_m\} - 1 > \max\{\mathfrak{a}_n, \mathbf{b}_n\} \geq \mathfrak{t}. \tag{4.9}$$

This shows that $c_1, \dots, f_\star^{\mathfrak{t}}(c_1)$ do not visit triangles $\Delta(-\mathfrak{p}_m, f_\star) \cup \Delta(-\mathfrak{p}_m + 1, f_\star)$ as it takes either $\mathfrak{a}_m - 1$ or $\mathbf{b}_m - 1$ iterations for a point in $\Delta(0, f_\star) \cup \Delta(1, f_\star)$ to visit them. Since h is a conjugacy away from $\Delta(-\mathfrak{p}) \cup \Delta(-\mathfrak{p} + 1)$, we obtain that

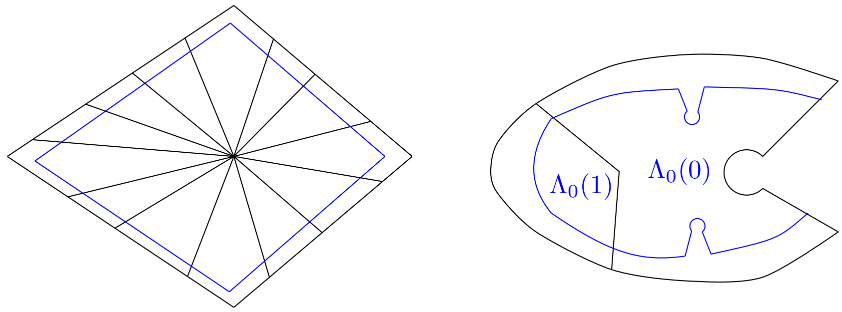


FIGURE 14. Right: $\Lambda_0(0, f_m)$ and $\Lambda_0(1, f_m)$ are shrunk versions of $\Delta_0(0, f_m)$ and $\Delta_0(1, f_m)$. Left: by transferring $\Lambda_0(0, f_m)$ and $\Lambda_0(1, f_m)$ to $\Lambda_m(0, f_0)$ and $\Lambda_m(1, f_0)$ by $\Phi_{m,0}$, and spreading these triangles dynamically, we obtain the triangulated neighborhood $\mathbb{A}_m$ of α such that $\mathbb{A}_m$ is a slightly shrunk version of $\mathbb{\Delta}_m$; compare with Figures 12 and 13.

h^{-1} maps $c_1, \dots, f_\star^{\mathfrak{t}}(c_1)$ to $c_1, \dots, f^{\mathfrak{t}}(c_1)$. Since h is close to the identity, $f^{\mathfrak{t}}(c_1)$ is close to $f_\star^{\mathfrak{t}}(c_1)$; thus $f^{\mathfrak{t}}(c_1) \in D$.

Let $D_0, D_1, \dots D_{\mathfrak{t}} = D$ be the pullbacks of D along the orbit $c_1, \dots, f^{\mathfrak{t}}(c_1) \in D$; i.e. D_i is the connected component of $f^{-1}(D_{i+1})$ containing $f^i(c_1)$. Our main objective is to show that D_i does not intersect $\partial^{\text{frb}} U_f$; this will imply that $f\colon D_i \to D_{i+1}$ is a branched covering for all $i \in \{1, \dots, \mathfrak{t}\}$.

4.4.3. *Sectors* $\Delta(I)$ *and* $\Lambda(I)$. An *interval* I of $\mathbb{Z}/\mathfrak{q}\mathbb{Z}$ is a set of consecutive numbers $i, i+1, \dots, i+j$ taking modulo $\mathfrak{q}$. We define the *sector parametrized by* I as $\Delta(I) \coloneqq \bigcup_{i \in I} \Delta(i)$. Furthermore, we set

$$f^{-1}(I) \coloneqq \begin{cases} I - \mathfrak{p} & \text{if } I \cap \{\mathfrak{p}, \mathfrak{p}+1\} = \emptyset \\ (I - \mathfrak{p}) \cup \{0, 1\} & \text{otherwise.} \end{cases} \tag{4.10}$$

In other words, we require that if $I - \mathfrak{p}$ contains one of $0, 1$, then it also contains another number. By (4.2) and (4.3)

Claim 1. *The preimage of* $\Delta(I)$ *under* $f \mid \mathbb{\Delta}$ *is within* $\Delta(f^{-1}(I))$. □

Unfortunately, we do not have the property that

$$\text{if } D_j \cap \mathbb{\Delta} \subset \Delta(I), \quad \text{then } D_{j+1} \cap \mathbb{\Delta} \subset \Delta(f^{-1}(I))$$

because the image of $\Delta(-\mathfrak{p}) \cup \Delta(-\mathfrak{p}+1)$ is slightly bigger than $\Delta(0) \cup \Delta(1)$, see (4.3). To handle this issue, we will play with a slightly shrunk version of $\mathbb{\Delta}$. We will define a triangulated neighborhood $\mathbb{A}$ such that

$$\mathbb{A} \subseteq f^i(\mathbb{A}) \subseteq \mathbb{\Delta} \tag{4.11}$$

for all $i \in \{0, 1, \dots, \min\{\mathfrak{a}_m, \mathbf{b}_m\}\}$.

Consider the dynamical plane of f_m and let $\Lambda_0(0, f_m)$ and $\Lambda_0(1, f_m)$ be (see Figure 14) the closures of the connected components of $f_m^{-1}(U_m) \setminus (\gamma_1 \cup \gamma_0)$ such that $\alpha \in \Lambda_0(0, f_m) \subset \Delta_0(0, f_m)$ and $\alpha \in \Lambda_0(1, f_m) \subset \Delta_0(1, f_m)$. Writing $\mathbb{A}_0(f_m) = \Lambda_0(0, f_m) \cup \Lambda_1(1, f_m)$ we obtain a shrank version of $\mathbb{\Delta}_0(f_m)$. The map $\Phi_{m,0}$ embeds $\Lambda_0(0, f_m)$ and $\Lambda_1(1, f_m)$ to the dynamical plane of f_0; spreading around the embedded triangles, we obtain a triangulated neighborhood $\mathbb{\Lambda}$ of α such that (4.11) holds.

Let us also give a slightly different description of $\mathbb{\Lambda}$. Recall (4.4) that $\Psi_{0,m}$ maps each $\Delta_m(i, f_0)$ conformally to some $\Delta_0(j, f_m)$. Then $\Lambda(i) = \Lambda_m(i, f_0) \subset \Delta_m(i, f_0)$ is the preimage of $\Lambda_0(j, f_m)$ under $\Psi_{0,m} \colon \Delta_m(i, f_0) \to \Delta_0(j, f_m)$. We define

$$\mathbb{\Lambda} \coloneqq \bigcup_{0 \le i < \mathfrak{q}} \Lambda(i) \quad \text{and} \quad \Lambda(I) = \bigcup_{i \in I} \Lambda(i).$$

Since $h \mid \mathbb{\Delta}$ is close to the identity, $\mathbb{\Lambda}$ also approximates $Z_\star$ in the sense of Theorem 4.6. By definition, we have

Claim 2. *We have $\Lambda(i) = \mathbb{\Lambda} \cap \Delta(i)$ for every i. The preimage of $\Lambda(I)$ under $f \mid \mathbb{\Lambda}$ is within $\Lambda(f^{-1}(I))$.* □

The following claim is a refinement of (4.11). This will help us to control the intersections of D_k with $\mathbb{\Lambda}$.

Claim 3. *Let I be an interval. Consider $z \in \mathbb{\Lambda}$. If*

$$f^i(z) \in \Delta(I)$$

for $i < \min\{\mathfrak{a}, \mathbf{b}\}$, then

$$z \in \Lambda(f^{-i}(I)).$$

As a consequence, if $T \cap \mathbb{\Delta} \subset \Delta(I)$ for an interval I and a set $T \subset V$, then

$$f^{-i}(T) \cap \mathbb{\Lambda} \subset \Lambda(f^{-i}(I))$$

for all $i < \min\{\mathfrak{a}, \mathbf{b}\}$.

Proof. Since $f^i(z) \in \Delta(I)$, every preimage of $f^i(z)$ under the i-th iterate of $f \mid \mathbb{\Delta}$ is within $\Delta(f^{-1}(I))$ by Claim 1. This shows that $z \in \Lambda(f^{-i}(I)) = \Delta(f^{-i}(I)) \cap \mathbb{\Lambda}$.

The second statement follows from the first because points in $\mathbb{\Lambda}$ do not escape $\mathbb{\Delta}$ under f^i. □

4.4.4. *Truncated sectors S_k and disks $\mathfrak{D}_k \supset \mathfrak{D}'_k \supset D_k$.* Let $I_\mathfrak{t}$ be the smallest interval containing $\{0, 1\}$ such that $\Delta(I_\mathfrak{t}, f) \supset D_\mathfrak{t} \cap \mathbb{\Delta}(f)$ for all f subject to the condition of Key Lemma. Set $I_{\mathfrak{t}-j} \coloneqq f^{-j}(I_\mathfrak{t})$. By Claim 3 we have $D_k \cap \mathbb{\Lambda} \subset \Lambda(I_k)$.

Recall that the intersection of each $\Delta(i, f_\star)$ with $\overline{Z}_\star$ is a closed sector of $\overline{Z}_\star$ bounded by two closed internal rays of $Z_\star$. Since D_0 is small, we obtain:

Claim 4. *All $|I_k|/\mathfrak{q}$ are small. All $\Delta(I_k, f_\star)$ have a small angle at the α-fixed point.*

For every $j \le \mathfrak{t} - 3 - p$, the intervals $I_j, I_{j+1}, \dots, I_{j+p+3}$ are pairwise disjoint. Moreover, intervals $I_0, I_1, \dots, I_{p+1}$ are disjoint from $\{-\mathfrak{p}, -\mathfrak{p} + 1\}$.

Proof. It is sufficient to prove the statement for $f_\star$; the map h transfers the result to the dynamical plane of f.

All $\Delta(i, f_\star)$ have comparable angles (see Lemma A.3): there are $x < y$ independent on n such that the angle of $\Delta(i)$ at α is between $x/\mathfrak{q}$ and $y/\mathfrak{q}$.

Let χ be the angle of $\Delta(I_\mathfrak{t})$ at α. The angle χ is small because $D = D_\mathfrak{t}$ is small. By definition of $I_k = f^{-1}(I_{k+1})$ (see (4.10)) the angle of $\Delta(I_{k+1})$ at α is bounded by the angle of $\Delta(I_k)$ at α plus $y/\mathfrak{q}$. Therefore, the angle at α of every $\Delta(I_k)$ is bounded by $\chi + (2 + \mathfrak{t})y/\mathfrak{q}$, where the number $(2 + \mathfrak{t})y/\mathfrak{q}$ is also small by (4.8). We obtain that all $\Delta(I_k)$ have small angles.

Since $f_\star \mid \overline{Z}_\star$ is an irrational rotation and $|I_k|\mathfrak{q}$ are small, we see that $I_j, I_{j+1}, \dots, I_{j+p+3}$ are disjoint. Since I_0 contains $\{0,1\}$ we see that $I_0, I_1, \dots, I_{p+1}$ do not intersect $\{-\mathfrak{p}, -\mathfrak{p}+1\} \subset f^{-1}(I_0)$. □

Recall §3 that the Siegel disk $Z_\star$ of $f_\star$ is foliated by equipotentials parametrized by their heights ranging from 0 (the height of α) to 1 (the height of $\partial Z_\star$). We denote by $Z^r_\star$ to be the open subdisk of $Z_\star$ bounded by the equipotential of height r.

Next we define S_k to be $\Lambda(I_k)$ truncated by a curve in $h^{-1}(Z^r_\star \setminus Z^{r-\varepsilon}_\star)$ such that S_k are backward invariant under $f \mid \mathbb{A}$. Assume that $r < 1$ is close to 1 and choose $\varepsilon > 0$ such that $1 - r$ is much bigger than ε. Consider an interval I_k for $k \le \mathfrak{t}$ and consider $i \in I_k$.

- If for all $\ell \in \{k, k+1, \dots, \mathfrak{t}\}$ we have $i + \mathfrak{p}\ell \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$, then define
$$S_k(i) \coloneqq \Lambda(i) \setminus h^{-1}(Z^r_\star);$$
- otherwise define
$$S_k(i) \coloneqq \Lambda(i) \setminus h^{-1}(Z^{r-\varepsilon}_\star).$$

Set $S_k \coloneqq \bigcup_{i \in I_k} S_k(i)$. Since $\Lambda(I_k, f_\star)$ has a small angle at α (see Claim 4) and the truncation level r is close to 1, we have

Claim 5. *All S_k are small.* □

Claim 6. *For every $k \le \mathfrak{t}$, the preimage of S_{k+1} under $f \mid \mathbb{A}$ is within S_k.*

Proof. By Claim 2 we only need to check that the truncation is respected by backward dynamics. If $i \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$, then
$$f\colon S_k(i) \to S_{k+1}(i+\mathfrak{p})$$
is a homeomorphism. Suppose $i \in \{-\mathfrak{p}, -\mathfrak{p}+1\}$. Then $S_{k+1}(i+\mathfrak{p}) = \Lambda(i) \setminus h^{-1}(Z^r_\star)$ because $i+\mathfrak{p}, \dots, i+\mathfrak{p}(\mathfrak{t}-k)$ are disjoint from $\{-\mathfrak{p}, -\mathfrak{p}+1\}$ by (4.9). On the other hand, by definition of S_k,
$$S_k \supset h^{-1}\left((\Lambda(-\mathfrak{p}, f_\star) \cup \Lambda(-\mathfrak{p}+1, f_\star)) \setminus Z^{r-\varepsilon}_\star \right).$$
Since h is close to identity, the preimage of $S_{k+1}(i+\mathfrak{p})$ under $f \mid \mathbb{A}$ is within S_k. □

We can assume that $D_\mathfrak{t}$ is so small that it does not intersect $h^{-1}(Z^r_\star)$. Then $D_\mathfrak{t} \cap \mathbb{A} \subset S_\mathfrak{t}$; using Claims 3 and 6 we obtain $D_k \cap \mathbb{A} \subset S_k$.

Next let us inductively enlarge D_k as $\mathfrak{D}_k \supset \mathfrak{D}'_k \supset D_k$. Set
$$\mathfrak{D}_\mathfrak{t} = \mathfrak{D}'_\mathfrak{t} \coloneqq D_\mathfrak{t}$$
and define $\mathfrak{D}'_k$ to be the connected component of $f^{-1}(\mathfrak{D}_{k+1})$ intersecting D_k. We define $\mathfrak{D}_k$ to be the filled-in $\mathfrak{D}'_k \cup \operatorname{int} S_k$; i.e. $\mathfrak{D}_k$ is $\mathfrak{D}'_k \cup \operatorname{int} S_k$ plus all of the bounded components of $\mathbb{C} \setminus (\mathfrak{D}'_k \cup \operatorname{int} S_k)$.

Claim 7. *For all $k \le p$ we have $S_k = \overline{\mathfrak{D}_k} \cap \mathbb{A}$.*

Proof. Follows from $D_k \cap \mathbb{A} \subset S_k$, Claim 6, and the definition of $\mathfrak{D}_k$. □

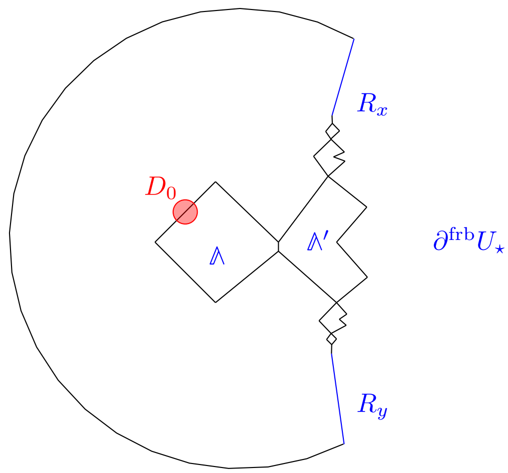


FIGURE 15. Separation of $\partial^{\rm frb} U_f$ from α. Disks $\mathbb{A}$ and $\mathbb{A}'$ approximate $\overline{Z}_\star$ and $\overline{Z}'_*$, iterated lifts of $\mathbb{A}'$ construct periodic bubble chains B_x and B_y landing at periodic points x and y, together with external rays R_x, R_y the bubble chains B_x, B_y separate $\partial^{\rm frb} U_f$ from the critical value. The configuration is stable because of the stability of local dynamics at x and y. Disks D_k may intersect $\mathbb{A}'$ but, by Claim 10, they do not intersect $B_x \cup B_y \setminus \mathbb{A}'$.

4.4.5. *Bubble chains.* Below we will separate the forbidden part of the boundary $\partial^{\rm frb} U_f$ from all $\mathfrak{D}_j$ by external rays and bubble chains (see Figure 15). Recall §3.1 that for $f_\star$ a bubble chain is a sequence of iterated lifts of $\overline{Z}_\star$; for f the role of $\overline{Z}_\star$ will be played by $\mathbb{A}$.

Consider first the dynamical plane of $f_\star$. Let $x, y \in \mathfrak{K}_*$ be two periodic points close to $\partial^{\rm frb} U_\star$ such that the external periodic rays R_x, R_y landing at x and y together with the bubble chains $B_x, B_y \subset \mathfrak{K}_*$ starting at the critical point and landing at x, y separate $\partial^{\rm frb} U_\star$ from the critical value as well as from all the remaining points in the forward orbit of x, y. We recall that B_x, B_y exist by Theorem 3.12. By definition (see §3.1), the ray B_x is a sequence

$$Z_1 = \overline{Z}'_*, Z_2, Z_3, \dots \tag{4.12}$$

such that Z_i is an iterated lift of Z'_* attached to Z_{i-1} and such that Z_n shrink to x. A similar description holds for B_y. Let p be a common period of x and y. Then p is also a common period of R_x, R_y as well as of B_x, B_y. The latter means that there are sub-chains $B'_x \subset B_x$ and $B'_y \subset B_y$ such that

$$B_x = f_*^p(B'_x) \quad \text{and} \quad B_y = f_*^p(B'_y). \tag{4.13}$$

Since f is close to $f_\star$, by Lemma 2.6 rays R_x, R_y exist in the dynamical plane of f and are close to those that are in the dynamical plane of $f_\star$.

Set $\mathbb{A}'$ to be the closure of the connected component of $f^{-1}(\mathbb{A}) \setminus \mathbb{A}$ that has a non-empty intersection with $\mathbb{A}$. Then $\mathbb{A}'$ is connected and

$$\mathbb{A} \cap \mathbb{A}' \subset \Lambda(-\mathfrak{p}) \cup \Lambda(-\mathfrak{p}+1).$$

We say that $\mathbb{\Lambda}'$ is *attached* to $\mathbb{\Lambda}$, or more specifically that $\mathbb{\Lambda}'$ is *attached* to $\Lambda(-\mathfrak{p}) \cup \Lambda(-\mathfrak{p}+1)$. Observe also that $\mathbb{\Lambda}'$ approximates $\overline{Z}'_*$ because $\mathbb{\Lambda}$ is close to $\overline{Z}_*$ and f is close to $f_\star$.

A *bubble of generation* $e+1 \geq 1$ for f is an f^e-lift of $\mathbb{\Lambda}'$. Fix a big $M \gg 1$. Since $\mathbb{\Lambda}'$ is close to $\overline{Z}'_*$, the map f is close to $f_\star$, and $\partial\mathbb{\Lambda} \cap (\Lambda(0) \cup \Lambda(1))$ is small, we have

Claim 8. *Every bubble Z_δ of $f_\star$ of generation up to M is approximated by a bubble $\mathbb{\Lambda}_\delta$ of f such that*

- *$\mathbb{\Lambda}_\delta$ is close to Z_δ and $f \mid \mathbb{\Lambda}_\delta$ is close to $f_\star \mid Z_\delta$;*
- *if Z_δ is attached to Z_γ, then $\mathbb{\Lambda}_\delta$ is attached to $\mathbb{\Lambda}_\gamma$; and*
- *if Z_δ is attached to $Z_\star$, then $\mathbb{\Lambda}_\delta$ is attached to $\mathbb{\Lambda} \setminus (\Lambda(0) \cup \Lambda(1))$.* □.

Using Claim 8, we approximate the bubbles Z_k in B_x with $k \leq M$ (see (4.12)) by the corresponding bubbles $\mathbb{\Lambda}_k$. We can assume that the remaining Z_{M+j} are within the linearization domain of x. Taking pullbacks within the linearization domain of x, we construct the *bubble chain* $B_x(f)$ landing at x as a sequence $\mathbb{\Lambda}' = \mathbb{\Lambda}_1, \mathbb{\Lambda}_2, \dots$ Similarly, $B_y(f)$ is constructed. Equation (4.13) holds in the dynamical planes of f. Thus we constructed $(R_x \cup B_x \cup B_y \cup R_y)\,(f)$ that is close to $(R_x \cup B_x \cup B_y \cup R_y)\,(f_\star)$.

Assume that D is so small that it is disjoint from the forward orbit of $R_x \cup R_y$. As a consequence, we obtain:

Claim 9. *All $\mathfrak{D}_k$ are disjoint from $R_x \cup R_y$.* □

4.4.6. *Control of $\mathfrak{D}_k$.*

Claim 10. *For all $k \in \{0, 1, \dots, \mathfrak{t}\}$ the following holds*

1. *$\mathfrak{D}_k$ intersects $\mathbb{\Lambda}'$ if and only if $I_k \supset \{-\mathfrak{p}, -\mathfrak{p}+1\}$;*
2. *if $\mathfrak{D}_k$ intersects $\mathbb{\Lambda}'$, then*
$$\mathfrak{D}_k \cap \mathbb{\Lambda}' \subset f^{-1}(S_{k+1})$$
is in a small neighborhood of c_0;
3. *if $\mathfrak{D}_k$ intersects $\mathbb{\Lambda}'$, then $\mathfrak{D}_{k+1}, \mathfrak{D}_{k+2}, \dots, \mathfrak{D}_{k+p+1}$ are disjoint from $\mathbb{\Lambda}'$;*
4. *if $\mathfrak{D}_k$ intersects $B_x \cup B_y$, then the intersection is within $\mathbb{\Lambda}'$ and, in particular, $I_k \supset \{-\mathfrak{p}, -\mathfrak{p}+1\}$;*
5. *$\mathfrak{D}_k$ is an open disk disjoint from $\partial^{\mathrm{frb}} U_f$; in particular, $f \colon \mathfrak{D}'_k \to \mathfrak{D}_{k-1}$ is a branched covering.*

Proof. We proceed by induction. Suppose that all of the statements are proven for moments $\{k+1, k+2, \dots, \mathfrak{t}\}$; let us prove them for k.

If $I_k \supset \{-\mathfrak{p}, -\mathfrak{p}+1\}$, then $D_{k+1} \supset \Lambda(0) \cup \Lambda(1) \ni c_1$ and we see that $\mathfrak{D}'_k = f^{-1}(\mathfrak{D}_{k+1})$ intersects $\mathbb{\Lambda}'$.

Suppose $I_k \cap \{-\mathfrak{p}, -\mathfrak{p}+1\} = \emptyset$. Then $\mathfrak{D}_{k+1}$ does not contain c_1. Thus every point in $\mathfrak{D}_{k+1}$ has at most one preimage under $f \mid \mathfrak{D}'_k$. By Claim 6, we have

$$\mathfrak{D}'_k \cap (\mathbb{\Lambda} \cup \mathbb{\Lambda}') \subset S_k, \quad \text{and thus} \quad \mathfrak{D}'_k \cap \mathbb{\Lambda}' = \emptyset.$$

This proves Part (1).

Part (2) follows from $\mathfrak{D}_{k+1} \cap \mathbb{\Lambda} \subset S_{k+1}$ (see Claim 7) combined with the fact that S_{k+1} is a neighborhood of c_1, see Claim 5. Part (3) follows from Part (1) combined with Claim 4.

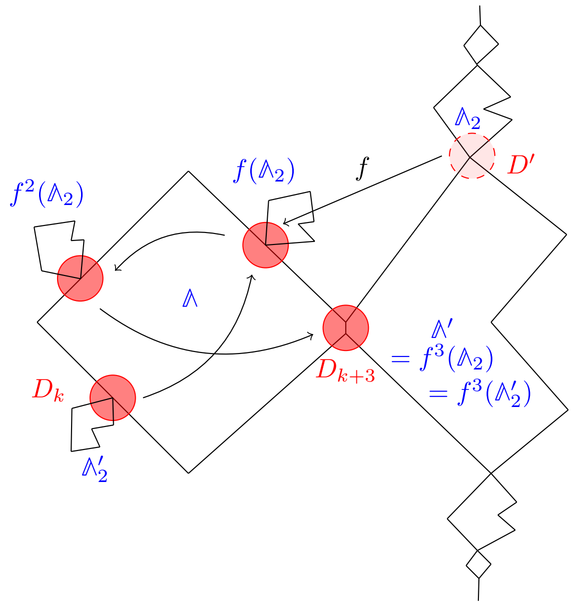


FIGURE 16. Illustration to the proof of Claim 10, Part (4) in case $e = 3$. Suppose that f^3 maps the bubble $\mathbb{A}_2$ (in B_x) to $\mathbb{A}'$ and suppose that $T \coloneqq D_k \cap \mathbb{A}_2 \neq \emptyset$. Let $\mathbb{A}'_2$ be the lift of $f(\mathbb{A}_2)$ attached to $\mathbb{A}$. Since $f(\mathbb{A}_2)$ is attached to $S_{k+1} = \overline{D}_{k+1} \cap \mathbb{A}$, we obtain that the pullback of $f(\mathbb{A}_2)$ along $f\colon D_k \to D_{k+1}$ is attached to S_k. This shows that $T \subset \mathbb{A}'_2$ contradicting $T \subset \mathbb{A}_2$.

Let us now prove Part (4), see Figure 16 for illustration. Assume that Part (4) does not hold; we assume that $\mathfrak{D}_k \cap (B_x \setminus \Lambda') \neq \emptyset$; the case $\mathfrak{D}_k \cap (B_y \setminus \Lambda') \neq \emptyset$ is similar. Write

$$B_x = (\mathbb{A}' = \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \dots),$$

where $\mathbb{A}_i$ is a bubble attached to $\mathbb{A}_{i-1}$. Then there is a $\mathbb{A}_i$ with $i \geq 2$ such that

$$T \coloneqq \mathfrak{D}_k \cap \mathbb{A}_i \neq \emptyset.$$

We assume that $i \geq 2$ is minimal. Recall that $\mathbb{A}_i$ is an iterated lift of $\mathbb{A}'$. Therefore, there is a minimal e such that f^e maps $\mathbb{A}_i$ to $\mathbb{A}'$. Observe first that $e \leq p$ because, otherwise, by periodicity of B_x we have

$$f^p(T) \subset \mathfrak{D}_{k+p} \cap f^p(\mathbb{A}_i) \subset B_x \setminus \mathbb{A}'$$

contradicting the assumption that Part (4) holds for $k + p$.

Consider the bubbles

$$f(\mathbb{A}_i), f^2(\mathbb{A}_i), \dots, f^e(\mathbb{A}_i) = \mathbb{A}'.$$

By Claim 8 they are attached to $\mathbb{A} \setminus (\Lambda(0) \cup \Lambda(1))$. More precisely, each $f^j(\mathbb{A}_i)$ with $j \in \{1, \dots, e\}$ is attached to $S_{k+j} \subset \overline{\mathfrak{D}_{k+j}}$. We also have $f^e(T) \subset \mathbb{A}'$.

Let $\mathbb{A}'_i$ be the lift of $f(\mathbb{A}_i)$ attached to S_k. We note that $\mathbb{A}'_i \neq \mathbb{A}_i$. Observe that $\mathfrak{D}_{k+1}$ does not contain the critical value. Indeed, $I_{k+e} \supset \{-\mathfrak{p}, -\mathfrak{p}+1\}$ by Part 1, thus $I_{k+1} \not\supset \{0, 1\}$ by Claim 4. Therefore, T is the preimage of $f(T)$ under $f\colon \mathbb{A}'_i \to f(\mathbb{A}_i)$; i.e. $T \subset \mathbb{A}'_i$. This contradicts to $T \subset \mathbb{A}_i$.

Part (5) holds because $\partial^{\rm frb} U_f$ is separated from $\mathbb{A}$ by $B_x \cup B_y \cup R_x \cup R_y$ and because the intersection of $\mathfrak{D}_k$ with $\mathbb{A}'$ is either small or empty, Part (2). $\square$

This shows $f^{\mathfrak{t}} \colon D_0 \to D_{\mathfrak{t}}$ is a branched covering. Observe next that $D_0 \cap \mathbb{A} \subset S_0$ is a small neighborhood of c_1 that is disjoint from γ_1. We can easily separate $D_0 \setminus \mathbb{A}$ from $\gamma_1 \setminus \mathbb{A}$ using $\mathbb{A}$ and finitely many backward iterated lifts of $B_x \cup B_y \cup R_x \cup R_y$. This finishes the proof of the Key Lemma. $\square$

5. Maximal prepacmen

Let $g : X \to Y$ be a holomorphic map between Riemann surfaces. Recall that g is:

- proper, if $g^{-1}(K)$ is compact for each compact $K \subset Y$;
- σ-proper (see [McM2, §8]) if each component of $g^{-1}(K)$ is compact for each compact $K \subset Y$; or equivalently if X and Y can be expressed as increasing unions of subsurfaces X_i, Y_i such that $g : X_i \to Y_i$ is proper.

A proper map is clearly σ-proper.

A prepacman $\mathbf{F} = (\mathbf{f}_- , \mathbf{f}_+)$ of a pacman f is called *maximal* if both $\mathbf{f}_-$ and $\mathbf{f}_+$ extend to σ-proper maps $\mathbf{f}_- : \mathbf{X}_- \to \mathbb{C}$ and $\mathbf{f}_+ : \mathbf{X}_+ \to \mathbb{C}$. We will usually normalize $\mathbf{F}$ such that $0 = \psi_{\mathbf{F}}^{-1}(\text{critical value})$, where $\psi_{\mathbf{F}}$ is a quotient map from $\mathbf{F}$ to $\mathbf{f}$, see §2.3. Under this assumption $\mathbf{F}$ is defined uniquely up to rescaling.

Theorem 5.1 (Existence of maximal prepacmen)**.** *Every $f \in \mathcal{W}^u$ sufficiently close to $f_\star$ has a maximal prepacman $\mathbf{F}$ that depends analytically on f.*

A refined statement will be proven as Theorem 5.5.

5.1. **Linearization of ψ-coordinates.** Consider again $[f_0 \colon U_0 \to V] \in \mathcal{W}^u$ close to $f_\star$. By definition of $\mathcal{W}^u$, the map f_0 can be antirenormalized infinitely many times. We define the *tower of anti-renormalizations* as

$$\mathcal{T}(f_0) = (F_k)_{k\le 0}\,.$$

Each f_k embeds to the dynamical pane of f_{k-1} as a prepacman $F_k^{(k-1)}$ such that $f_{k,\pm}^{(k-1)}$ are iterates of f_{k-1}.

Let us specify the following translation

$$T_k \colon z \to z - c_1(f_k).$$

Let us now translate each f_k so that $c_1(f_k) = 0$. We mark the translated objects with "$\bullet$." For $k \le 0$, set

$$\phi_k^\bullet(z) := T_{k-1} \circ \phi_k \circ T_k^{-1}$$

so that $\phi_k^\bullet(0) = 0$. Similarly, define $U_k^\bullet := T_k(U_k)$ and $V_k^\bullet := T_k(V)$; and conjugate all f_k and all $F_k = F_k^{(k)}$ by T_k; the resulting maps are denoted by $f_k^\bullet \colon U_k^\bullet \to V_k^\bullet$ and by

$$F_k^\bullet = (f_{k,\pm}^\bullet \colon U_{k,\pm}^\bullet \to S_k^\bullet).$$

We also write $\gamma_1^\bullet(f_k) := T_k(\gamma_1)$. The tower $(F_k^\bullet)_{k\le 0}$ is illustrated on Figure 17.

Denote by

$$\mu_* := \phi_*'(c_1(f_\star)) = (\phi_*^\bullet)'(0), \qquad |\mu_\star| < 1$$

the self-similarity coefficient of $\overline{Z}_\star$.

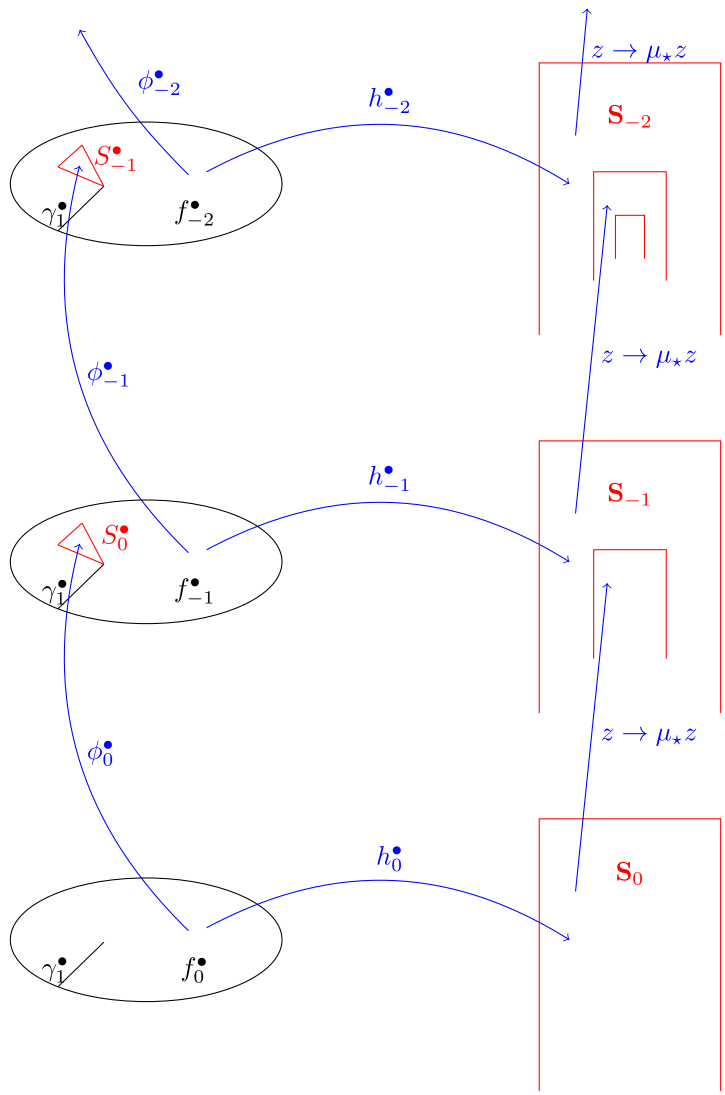


FIGURE 17. Left: each pacman $f_i^\bullet$ embeds as a prepacman to the dynamical plane of $f_{i-1}^\bullet$ via $\phi_i^\bullet$. Right: sectors $S_i^\bullet$ after linearization of ψ-coordinates. Note that $S_i^\bullet$ can intersect $\gamma_1^\bullet$ in a small neighborhood of $\alpha^\bullet = T_i(\alpha)$.

Lemma 5.2 (Linearization)**.** *For every $f_0 \in \mathcal{W}^u$ sufficiently close to $f_\star$, the limit*

$$h_0^\bullet(z) = h_{f_0}^\bullet := \lim_{i\to-\infty} \frac{\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)}{\mu_*^{-i}} \tag{5.1}$$

is a univalent map on a certain neighborhood of 0 *(independent on f_0).*

We remark that the linearization is normalized in such a way that $h_0'(0) = 1$ if $f_0 = f_\star$.

Proof. Follows from a standard linearization argument. Write $\phi_i^\bullet(z) = \mu_i z + O(z^2)$; since $\phi_i^\bullet$ tends exponentially fast to $\phi_*^\bullet$ we see that μ_i tends exponentially fast to μ_* and that the constant in the error term does not depend on i. For z in a small neighborhood of 0, we have

$$\left|\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)\right| \le C(|\mu_*| + \varepsilon)^i |z|$$

for some constants C and $\varepsilon > 0$ such that $|\mu_*| + 2\varepsilon < 1$. Write

$$h^{(i)}(z) := \frac{\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)}{\mu_*^{-i}}.$$

Then

$$\frac{h^{(i-1)}(z)}{h^{(i)}(z)} = \frac{\phi_i^\bullet\left(\phi_{i+1}^\bullet \circ \ldots \phi_0^\bullet(z)\right)}{\mu_*\phi_{i+1}^\bullet \circ \ldots \phi_0^\bullet(z)} = \frac{\mu_{-i+1}}{\mu_*} + O\left(\phi_{i+1}^\bullet \circ \ldots \phi_0^\bullet(z)\right).$$

tends exponentially fast to 1 in some neighborhood of 0. This implies that $h^{(i)}(z)$ converges to a univalent map in some neighborhood of 0. □

Let us write $h_i^\bullet = h_{f_i}$ and we set $\mathbf{S}_i := h_i^\bullet(S_i)$. We will usually use bold symbols for object in linear coordinates. By construction (5.1), the maps $h_i^\bullet$ satisfy the linearization equation (see Figure 17)

$$h_{i-1}^\bullet \circ \phi_i^\bullet = [z \to \mu_\star z] \circ h_i^\bullet. \tag{5.2}$$

For $i \le 0$, set

$$h_i^\#(z) := \mu_*^{-i} h_0^\bullet(z). \tag{5.3}$$

It follows from (5.2) that

$$h_0^\bullet(z) = h_{-1}^\#\left(\phi_0^\bullet(z)\right) = \cdots = h_i^\#\left(\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)\right). \tag{5.4}$$

We will usually use "#" to mark linearized objects rescaled by $\mu_\star^{-i}$.

Lemma 5.3 (Extension of $h_0^\bullet$)**.** *Under the above assumptions $h_0^\bullet$ extends to a univalent map $h_0^\bullet\colon \operatorname{int}(V_0^\bullet \setminus \gamma_1^\bullet) \to \mathbb{C}$.*

Proof. By Lemma 4.1 the map $\phi_i^\bullet \circ \cdots \circ \phi_{-1}^\bullet$ extends to a conformal map defined on $\operatorname{int}(V_0^\bullet \setminus \gamma_1^\bullet)$. Since $\phi_i^\bullet \circ \cdots \circ \phi_{-1}^\bullet$ is contracting, for every $z \in \operatorname{int}(V_0^\bullet \setminus \gamma_1^\bullet)$ there is an $i < 0$ such that $\phi_i^\bullet \circ \cdots \circ \phi_{-1}^\bullet(z)$ is within a neighborhood of 0 where $h_i^\bullet$ is defined (this is easily true if $f_0^\bullet = f_\star^\bullet$; applying Theorem 4.6 we obtain this property for all $f_0^\bullet$). Therefore, (5.4) extends dynamically $h_0^\bullet$ to $\operatorname{int}(V_0^\bullet \setminus \gamma_1^\bullet)$. □

Let us now conjugate every map $F_k^\bullet$ by $h_k^\#$; we define $\mathbf{F}_k^\# := h_k^\# \circ F_k^\bullet \circ \left(h_k^\#\right)^{-1}$. We construct the *tower in linear coordinates*

$$\mathcal{T}^\#(\mathbf{F}_0) = \left(\mathbf{F}_k^\#\right)_{k\le 0} = \left(\mathbf{f}_{k,\pm}^\#\colon \mathbf{U}_{k,\pm}^\# \to \mathbf{S}_k^\#\right)_{k\le 0}, \tag{5.5}$$

where

$$\operatorname{int}\left(\mathbf{S}_k^{\#}\right) = h_k^{\#}\left(V_k^{\bullet} \setminus \gamma_1^{\bullet}\right) = h_k^{\#} \circ T_k\left(V_k \setminus \gamma_1\right), \tag{5.6}$$

and similarly other objects marked by "#" are defined.

It follows from (A.4) that

Lemma 5.4. *There are numbers $m_{1,1}, m_{1,2}, m_{2,1}, m_{2,2}$ such that for $k < 0$ we have*

$$\mathbf{f}_{k+1,-}^{\#} = (\mathbf{f}_{k,-}^{\#})^{m_{1,1}} \circ (\mathbf{f}_{k,+}^{\#})^{m_{1,2}},$$
$$\mathbf{f}_{k+1,+}^{\#} = (\mathbf{f}_{k,-}^{\#})^{m_{2,1}} \circ (\mathbf{f}_{k,+}^{\#})^{m_{2,2}}.$$

□

Note also that

$$\mathbf{f}_{k,\pm}^{\#} = \frac{1}{\mu_*^k}\mathbf{f}_{k,\pm}\left(\mu_*^k z\right). \tag{5.7}$$

5.2. **Global extension of prepacmen in $\mathcal{W}^u$.** Using Key Lemma 4.8 we deduce

Theorem 5.5 (Existence of a maximal prepacman)**.** *If $f_0 \in \mathcal{W}^u$ is sufficiently close to $f_\star$, then every pair $\mathbf{F}_i^{\#} = \left(\mathbf{f}_{k,\pm}^{\#}\right)$ in the tower $\mathcal{T}^{\#}(\mathbf{F}_0)$ (see (5.5)) extends to σ-proper branched coverings*

$$\mathbf{f}_{k,\pm}^{\#} \colon \mathbf{X}_{k,\pm}^{\#} \to \mathbb{C}.$$

Proof. Let

$$\mathfrak{F}_0 = (f_{0,\pm} \colon \mathfrak{U}_{0,\pm} \to \mathfrak{S} \coloneqq V \setminus (\gamma_1 \cup O))$$

be a commuting pair obtained from $F_0 = (f_{0,\pm} \colon U_{0,\pm} \to V \setminus \gamma_1)$ by removing a small neighborhood O of α from $V \setminus \gamma_1$ and by removing $f_{0,\pm}^{-1}(O)$ from $U_{0,\pm}$. By Lemma 4.1 the map $\phi_k \circ \cdots \circ \phi_{-1}$ embeds $\mathfrak{F}_0$ to the dynamical plane of f_k as commuting pair denoted by

$$\mathfrak{F}_0^{(k)} = \left(f_k^{\mathfrak{a}_k}, f_k^{\mathbf{b}_k}\right) \colon \mathfrak{U}_{0,-}^{(k)} \cup \mathfrak{U}_{0,+}^{(k)} \to \mathfrak{S}_0^{(k)}. \tag{5.8}$$

Since ϕ_k is contracting at the critical value the diameter of $U_{0,-}^{(k)} \cup U_{0,+}^{(k)} \cup \mathfrak{S}_0^{(k)} \ni c_1(f_n)$ tends to 0. By Key Lemma 4.8, for a sufficiently big $k < 0$ there is a small open topological disk D around the critical value of f_k such that the pair (5.8) extends into a pair of commuting branched coverings

$$F_0^{(k)} = \left(f_k^{\mathfrak{a}_k}, f_k^{\mathbf{b}_k}\right) \colon W_-^{(k)} \cup W_+^{(k)} \to D, \tag{5.9}$$

with $W_-^{(k)} \cup W_+^{(k)} \cup D \subset V \setminus \gamma_1$.

Conjugating (5.9) by $h_k^{\#} \circ T_k$ we obtain the commuting pair

$$(\mathbf{f}_{0,-}, \mathbf{f}_{0,+}) \colon \mathbf{W}_-^{(k)} \cup \mathbf{W}_+^{(k)} \to \mathbf{D}^{(k)}.$$

Since for a sufficiently big t and all $m \le 0$ the modulus of the annulus $\mathbf{D}^{(tm-t)} \setminus \mathbf{D}^{(tm)}$ is uniformly bounded from 0 we obtain $\bigcup_{k \ll 0} \mathbf{D}^{(k)} = \mathbb{C}$. Setting

$$\mathbf{X}_{0,-} \coloneqq \bigcup_{k \ll 0} \mathbf{W}_-^{(k)}, \quad \mathbf{X}_{0,+} \coloneqq \bigcup_{n \ll 0} \mathbf{W}_+^{(k)} \tag{5.10}$$

we obtain σ-proper maps $\mathbf{f}_{0,\pm} \colon \mathbf{X}_{0,\pm} \to \mathbb{C}$. Similarly, $(\mathbf{f}_{k,\pm}^{\#})$ extends to a pair of σ-proper maps. □

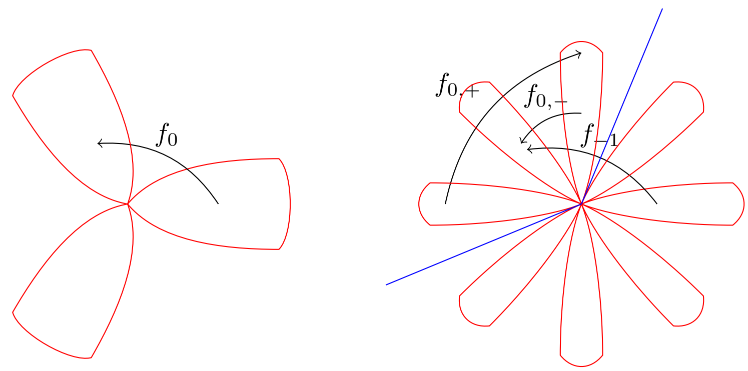


FIGURE 18. A parabolic pacman f_0 with rotation number $1/3$ embeds as a prepacman into the dynamical plane of a parabolic pacman f_{-1} with rotationa number $3/8$. We have $f_{0,-} = f_{-1}^3$ and $f_{0,+} = f_{-1}^2$.

6. MAXIMAL PARABOLIC PREPACMEN

Consider a parabolic pacman $f_0 \in \mathcal{W}^u$ close to $f_\star$ such that Theorem 5.5 applies for f_0. As in §5 we denote by $\mathbf{F}_n = (\mathbf{f}_{n,\pm})$ the maximal prepacmen of $f_n = \mathcal{R}^n f_0$ with $n \leq 0$ and by $\mathbf{F}_n^\#$ the rescaled version of $\mathbf{F}_n$ so that $\mathbf{F}_0$ is an iteration of $\mathbf{F}_n^\#$, see Lemma 5.4.

6.1. **The post-critical set of a maximal prepacman.** The *forward orbit of* $z \in \mathbb{C}$ under $\mathbf{F}_n$ is

$$\operatorname{orb}_z(\mathbf{F}_n) := \{\mathbf{f}_{n,-}^s \circ \mathbf{f}_{n,+}^r(z) \mid s, r \geq 0\};$$

we do not require that $\mathbf{f}_{n,-}^s \circ \mathbf{f}_{n,+}^r(z)$ is defined for all pairs s, r. A *finite orbit of* z is

$$\operatorname{orb}_z^{\leq q}(\mathbf{F}_n) := \left\{\mathbf{f}_{n,-}^s \circ \mathbf{f}_{n,+}^r(z) \mid s, r \in \{0, 1, \dots, q\}\right\}.$$

Similarly, $\operatorname{orb}_z(\mathbf{F}_n^\#)$ and $\operatorname{orb}_z^{\leq q}(\mathbf{F}_n^\#)$ are defined. Since $\mathbf{F}_0$ is an iteration of $\mathbf{F}_n^\#$, there is a $k > 1$ such that

$$\operatorname{orb}_z^{\leq q}(\mathbf{F}_0) \subseteq \operatorname{orb}_z^{\leq k^{-n} q}\left(\mathbf{F}_n^\#\right)$$

for all $n \leq 0$ and $z \in \mathbb{C}$.

An *orbit path* of $\mathbf{F}_m$ is a sequence $x_0, x_1, \dots, x_n$ such that either $x_{i+1} = \mathbf{f}_{m,-}(x_i)$ or $x_{i+1} = \mathbf{f}_{m,+}(x_i)$. Since $\mathbf{F}_0$ is an iteration of $\mathbf{F}_n^\#$, an orbit path of $\mathbf{F}_0$ is a "sub-orbit" path of $\mathbf{F}_n^\#$.

Let us denote by

$$C(\mathbf{F}_k) := \{z \in \mathbb{C} \mid \mathbf{f}_{k,-}'(z) = 0 \text{ or } \mathbf{f}_{k,+}'(z) = 0\}$$

the set of critical points of $\mathbf{F}_k$; its *post-critical set* is

$$P(\mathbf{F}_k) = \bigcup_{\substack{n+m>0 \\ n,m\geq 0}} \mathbf{f}_{k,-}^n \circ \mathbf{f}_{k,+}^m(C_i).$$

Similarly $P(\mathbf{F}_n^\#)$ is defined. Clearly,

$$P(\mathbf{F}_0) \subset P\left(\mathbf{F}_n^\#\right) = \mu_*^n P(\mathbf{F}_n).$$

Recall that 0 is a critical value of $\mathbf{F}_n^\#$ for all $n \le 0$; we denote by $\mathfrak{o}_n^\#$ the critical point of $\mathbf{F}_n^\#$ such that $\mathfrak{o}_n^\#$ is identified with the critical point $c_0(f_n)$ under $\operatorname{int} \mathbf{S}_n^\# \simeq V \setminus \gamma_1$, see (5.6).

Lemma 6.1 (Every critical orbit "passes" through 0)**.** *For any critical point x_0 of $\mathbf{f}_{0,\iota}$ with $\iota \in \{- \,, +\}$ the following holds. For all sufficiently big $n < 0$ there is an orbit path of $\mathbf{F}_n^\#$*

$$x_0, x_1, x_2, \dots x_k; \qquad x_i = \mathbf{f}^\#_{n,j(i)}(x_{i-1}) \tag{6.1}$$

such that

- $\mathbf{f}_{0,\iota} = \mathbf{f}^\#_{n,j(k)} \circ f^\#_{n,j(k-1)} \circ \dots \circ \mathbf{f}^\#_{n,j(1)}$*, in particular* $x_k = \mathbf{f}_{0,\iota}(x_0)$*;*
- $x_i = \mathfrak{o}_n^\#$ *and* $x_{i+1} = 0$ *for some* i*.*

Therefore,

$$P(\mathbf{F}_0) \subset \bigcup_{n \le 0} \operatorname{orb}_0\left(\mathbf{F}_n^\#\right).$$

Proof. Clearly, the second statement follows from the first. We will use notations from the proof of Theorem 5.5. Suppose for definiteness $\iota =$ "$-''$. Recall (5.10) that $\operatorname{Dom} \mathbf{f}_{0,-} = \bigcup_{n \ll 0} \mathbf{W}_-^{(n)}$; thus $x_0 \in \mathbf{W}_-^{(n)}$ for some $n < 0$. The map $\mathbf{f}_{0,-} \mid \mathbf{W}_-^{(n)}$ is conformally conjugate to $f_n^{\mathfrak{a}_n} \mid W_-^{(n)} \to D$ (see (5.9)) after identifying $\mathbf{W}_-^{(n)}$ with $W_-^{(n)}$. This shows that $x_0, \mathbf{f}_{0,-}(x_0)$ us within an actual orbit $x_0, x_1, \dots, x_k$ of

$$(\mathbf{f}^\#_{n,\pm} \colon \mathbf{U}^\#_{n,\pm} \to \mathbf{S}^\#_n).$$

which is a prepacman of f_n. We deduce that one of x_i is $\mathfrak{o}_n^\#$ and $x_{i+1} = 0$. □

6.2. **Global attracting basin of a parabolic pacman.** Since $\mathbf{f}_{0,\pm} \colon \operatorname{Dom} \mathbf{f}_{0,\pm} \to \mathbb{C}$ are σ-proper commuting maps with maximal domain we have

$$\operatorname{Dom}(\mathbf{f}_{0,-} \circ \mathbf{f}_{0,+}) = \operatorname{Dom}(\mathbf{f}_{0,+} \circ \mathbf{f}_{0,-}) \subset \operatorname{Dom} \mathbf{F}_0 \coloneqq \operatorname{Dom} \mathbf{f}_{0,-} \cap \operatorname{Dom} \mathbf{f}_{0,+}. \tag{6.2}$$

There is a small open attracting parabolic flower H_0 around the α-fixed point of f_0. Each petal of H_0 lands at α at a well-defined angle. Assume H_0 is small enough so that $H_0 \subset V \setminus \gamma_1$, possibly up to a slight rotation of γ_1. By Lemma 4.3 the flower H_0 lifts to the dynamical plane of $\mathbf{F}_0$ via the identification $V \setminus \gamma_1 \simeq \operatorname{int} \mathbf{S}_0$; we denote by $\mathbf{H}_0$ the lift.

Let $\mathbf{e}(\mathfrak{p}_0/\mathfrak{q}_0) \neq 1$ be the multiplier of the α-fixed point of f_0. Since f_0 is close to $f_\star$, we have $\mathfrak{q}_0 > 1$. By replacing H_0 with its sub-flower we can assume that there are exactly $\mathfrak{q}_0$ connected components of H_0 with combinatorial rotation number $\mathfrak{p}_0/\mathfrak{q}_0$. We enumerate them counterclockwise as $H_0^0, H_0^1, \dots, H_0^{\mathfrak{q}_0 - 1}$. Then f_0 maps H_0^i to $H_0^{i+\mathfrak{p}_0}$. We will show in Corollary 6.4 that H_0 is in fact unique; i.e. f_0 has exactly $\mathfrak{q}_0$ attracting direction at α. Denote by $\mathbf{H}_0^i$ the lift of H_0^i to the dynamical plane of $\mathbf{F}_0$.

Lemma 6.2. *There are $\mathfrak{r}, \mathfrak{s} \ge 1$ with $\mathfrak{r} + \mathfrak{s} = \mathfrak{q}_0$ such that*

$$\mathbf{f}_{0,-}^{\mathfrak{r}} \circ \mathbf{f}_{0,+}^{\mathfrak{s}}(\mathbf{H}_0^i) \subset \mathbf{H}_0^i.$$

The set $\mathbf{H}_0$ is in $\operatorname{Dom}(\mathbf{f}_{0,-}^a \circ \mathbf{f}_{0,+}^b)$ for all $a, b \ge 0$.

It will follow from Proposition 6.5 that $\mathbf{f}_{0,-}^{\mathfrak{r}} \circ \mathbf{f}_{0,+}^{\mathfrak{s}} \colon \mathbf{H}_0^i \to \mathbf{H}_0^i$ is the first return map.

Proof. We have $f_0^{\mathfrak{q}_0}(H_0^i) \subset H_0^i$. Cutting the prepacman f_0 along γ_1 we see that there are $\mathfrak{r}, \mathfrak{s} \geq 1$ with $\mathfrak{r} + \mathfrak{s} = \mathfrak{q}_0$ such that $f_{0,-}^{\mathfrak{r}} \circ f_{0,+}^{\mathfrak{s}}(H_0^i) \subset H_0^i$. This implies the first claim. As a consequence $\mathbf{H}_0$ is in $\operatorname{Dom}(\mathbf{f}_{0,-}^{\mathfrak{r}j} \circ \mathbf{f}_{0,+}^{\mathfrak{s}j})$ for all $j \geq 0$. Combined with (6.2), we obtain the second claim. □

As a consequence, all of the branches of $\mathbf{f}_{0,-}^{a} \circ \mathbf{f}_{0,+}^{b}$ with $a, b \in \mathbb{Z}$ are well defined for points in $\mathbf{H}_0$. Set

$$\mathbf{H} \coloneqq \bigcup_{a,b\in\mathbb{Z}} (\mathbf{f}_{0,-})^a \circ (\mathbf{f}_{0,+})^b \,(\mathbf{H}_0)$$

to be the full orbit of $\mathbf{H}_0$. Since $\mathbf{f}_{0,-}$, $\mathbf{f}_{0,+}$ commute, $\mathbf{H}$ is an open fully invariant subset of $\mathbb{C}$ within $\operatorname{Dom} \mathbf{f}_{0,-} \cap \operatorname{Dom} \mathbf{f}_{0,+}$. We call $\mathbf{H}$ the *global attracting basin* of the α-fixed point.

A connected component $\mathbf{H}'$ of $\mathbf{H}$ is *periodic* if there are $s, r \in \mathbb{N}_{>0}$ such that $\mathbf{f}_{0,-}^{s} \circ \mathbf{f}_{0,+}^{r}(\mathbf{H}') = \mathbf{H}'$. A pair (s, r) is called a period of H'. We will show in Corollary 6.6 that there is no component $\mathbf{H}'$ of $\mathbf{H}$ such that $\mathbf{f}_{0,-}^{r}(\mathbf{H}') = \mathbf{H}'$ or $\mathbf{f}_{0,+}^{r}(\mathbf{H}') = \mathbf{H}'$ for some $r > 0$.

By Lemma 6.2, the components of $\mathbf{H}$ intersecting $\mathbf{H}_0$ are $(\mathfrak{r}, \mathfrak{s})$-periodic. Observe next that for any periodic component $\mathbf{H}'$ and any component $\mathbf{H}''$ of $\mathbf{H}$ there are $a, b \geq 1$ with $\mathbf{f}_{0,-}^{a} \circ \mathbf{f}_{0,+}^{b}(\mathbf{H}'') = \mathbf{H}'$; i.e. $\mathbf{H}'$ and $\mathbf{H}''$ are dynamically related. Indeed, by definition there are $a', b' \in \mathbb{Z}$ such that a certain branch of $\mathbf{f}_{0,-}^{a'} \circ \mathbf{f}_{0,+}^{b'}$ maps $\mathbf{H}''$ to $\mathbf{H}'$. Applying $\mathbf{f}_{0,-}^{\mathfrak{s}t} \circ \mathbf{f}_{0,+}^{\mathfrak{r}t}$ with $t \gg 1$, we obtain required $a, b \geq 1$. As consequence, all the periodic components have the same periods; in particular they are $(\mathfrak{r}, \mathfrak{s})$-periodic.

6.3. **Attracting Fatou coordinates.** It is classical that $f_0^{\mathfrak{q}_i} : H_0^0 \to H_0^0$ admits *attracting Fatou coordinates:* a univalent map $h : H_0^0 \to \mathbb{C}$ such that

- $h \circ f_0^{\mathfrak{q}_0}(z) = h(z) + 1$; and
- there is an $L > 1$ such that

$$h(H_0^0) \supset \{z \in \mathbb{C} \mid \operatorname{Re}(z) > L\}. \tag{6.3}$$

There is a unique dynamical extension $h \colon H_0 \to \mathbb{C}$ such that

$$h \circ f_0(z) = h(z) + 1/\mathfrak{q}_0. \tag{6.4}$$

Lifting h to the dynamical plane of $\mathbf{F}_0$ we obtain $\mathbf{h} \colon \mathbf{H}_0 \to \mathbb{C}$.

Lemma 6.3 (Fatou coordinates of $\mathbf{H}$)**.** *The map $\mathbf{h} \colon \mathbf{H}_0 \to \mathbb{C}$ extends uniquely to a map $\mathbf{h} \colon \mathbf{H} \to \mathbb{C}$ satisfying*

$$\mathbf{h} \circ \mathbf{f}_{0,\pm}(z) = \mathbf{h}(z) + 1/\mathfrak{q}_0. \tag{6.5}$$

for any choice of "±". For every component $\mathbf{H}'$ of $\mathbf{H}$, the map $\mathbf{h} \mid \mathbf{H}'$ is σ-proper. The singular values of $\mathbf{h}$ are exactly the $\mathbf{h}$-images of the critical points of $\mathbf{F}_0$ and their iterated preimages.

Moreover, components of $\mathbf{H}_0$ are in different components of $\mathbf{H}$. The set $\mathbf{H}$ is a proper subset of $\mathbb{C}$. By postcomposing $\mathbf{h}$ with a translation we can assume that

$$\mathbf{h}(0) = 0. \tag{6.6}$$

Proof. On $\mathbf{H}_0$ Equation (6.5) is just a lift of(6.4). Applying $\mathbf{f}_{0,\pm}^{-1}$ and using commutativity of $\mathbf{f}_{0,-}, \mathbf{f}_{0,+}$, we obtain a unique extension of $\mathbf{h}$ to $\mathbf{H}$ such that (6.5) holds.

Since $\mathbf{f}_{0,-}, \mathbf{f}_{0,+}$ are σ-proper maps, so is $\mathbf{h} \mid \mathbf{H}$. Indeed, suppose that $\mathbf{H}' \subset \mathbf{H}$ is a periodic component intersecting $\mathbf{H}_0$; the other cases follow by applying a certain branch of $\mathbf{f}^a_{0,-} \circ \mathbf{f}^b_{0,+}$, where $a, b \in \mathbb{Z}$. Recall from Lemma 6.2 that $\mathbf{H}'$ is $(\mathfrak{r}, \mathfrak{s})$-periodic. Consider a compact set $K \subset \mathbb{C}$. We denote by $\mathbf{K}$ a connected component of the preimage of K under $\mathbf{h} \mid \mathbf{H}'$. Then for a sufficiently big $i \gg 1$ we have $\operatorname{Re}(K + i) > L$ and $\mathbf{K}_2 := \mathbf{f}^{\mathfrak{r}i}_{0,-} \circ \mathbf{f}^{\mathfrak{s}i}_{0,+}(\mathbf{K})$ intersects $\mathbf{H}_0$, where L is a constant from (6.3). Then $\mathbf{K}_2$ is compact as a connected component of the preimage of $K + i$ under $\mathbf{h} \mid \mathbf{H}' \cap \mathbf{H}_0$. We obtain that $\mathbf{K} \subset \mathbf{f}^{-\mathfrak{r}i}_{0,-} \circ \mathbf{f}^{-\mathfrak{s}i}_{0,+}(\mathbf{K}_2)$ is compact because $\mathbf{f}^{\mathfrak{r}i}_{0,-} \circ \mathbf{f}^{\mathfrak{s}i}_{0,+}$ is σ-proper. This also shows that singular values of $\mathbf{h}$ are the $\mathbf{h}$-images of either critical points of $\mathbf{F}_0$ or their iterated preimages. (We recall a σ-proper map has no asymptotic values.)

Let $\mathbf{H}^x_0$ and $\mathbf{H}^y_0$ be two different components of $\mathbf{H}_0$ and let $\mathbf{H}^x$ and $\mathbf{H}^y$ be the periodic components of $\mathbf{H}$ containing $\mathbf{H}^x_0$ and $\mathbf{H}^y_0$. Since all points in $\mathbf{H}^x$ and $\mathbf{H}^y$, escape eventually to $\mathbf{H}^x_0$ and $\mathbf{H}^y_0$ under the iteration of $\mathbf{f}^{\mathfrak{r}}_{0,-} \circ \mathbf{f}^{\mathfrak{s}}_{0,+}$ we have $\mathbf{H}^x \neq \mathbf{H}^y$. As a consequence $\mathbf{H} \neq \mathbb{C}$. The claim concerning (6.6) is immediate. □

From now on we assume that (6.6) holds. Denote by $\mathbf{H}^{\text{per}} \subset \mathbf{H}$ the union of periodic components of $\mathbf{H}$.

Corollary 6.4 (Critical point). *The set $\mathbf{H}^{\text{per}}$ contains $P(\mathbf{F}_0)$ and at least one critical point. In particular, $0 \in \mathbf{H}^{\text{per}}$. All the critical points of $\mathbf{F}_0$ are within $\mathbf{H}$. n the dynamical plane of f_0 the flower H_0 is unique: f_0 has exactly $\mathfrak{q}_0$ attracting direction at α cyclically permuted by f_0.*

Proof. Since $\mathbf{h}\colon \mathbf{H}^{\text{per}} \to \mathbb{C}$ is not a covering map, $\mathbf{H}^{\text{per}}$ contains at least one critical point of $\mathbf{F}_0$. Since $\mathbf{H}^{\text{per}}$ is forward invariant, $\mathbf{H}^{\text{per}}$ contains $\mathfrak{o}^{\#}_n$ for all sufficiently big $n < 0$, see Lemma 6.1. Therefore, $\mathbf{H}^{\text{per}}$ contains all of the critical values of $\mathbf{F}_0$. Since $\mathbf{H}$ is fully invariant, it contains all of the critical points. As a consequence, H_0 is unique because the global attracting basin of another flower would also contain 0. □

6.4. **Dynamics of periodic components.** It follows from Lemma 6.2 that

$$\mathbf{H} = \bigcup_{a,b \in \mathbb{Z}} (\mathbf{f}^{\#}_{n,-})^a \circ (\mathbf{f}^{\#}_{n,+})^b \, (\mathbf{H}_0)$$

for all $n \le 0$. It is also clear that $\mathbf{H}^{\text{per}}$ is the union of $\mathbf{F}^{\#}_n$-periodic components.

Let H_n be a small parabolic attracting flower of f_n admitting a lift to the dynamical plane of $\mathbf{F}^{\#}_n$; we denote this lift by $\mathbf{H}^{\#}_n \to H_n$. We denote by $\mathfrak{p}_n/\mathfrak{q}_n$ the combinatorial rotation number of f_n.

Let I_n be an index set enumerating clockwise the connected components of H_n starting with the component closest to γ_1. Since H_n embeds naturally to the dynamical plane of f_{n-1} (see Figure 18), we have a natural embedding of I_n to I_{n-1}.

Let us write

$$I_0 = \{-a_0, -a_0 + 1, \dots b_0 - 1, b_0\}$$

with $a_0, b_0 > 0$ and $a_0 + b_0 + 1 = \mathfrak{q}_0$. The component of H_0 indexed by $i + 1$ follows in the clockwise order the component of H_0 indexed by i. Then f_0 maps the component of H_0 indexed by i to the component of H_0 indexed by either $i - \mathfrak{p}_0$ or $i + \mathfrak{q}_0 - \mathfrak{p}_0$ depending on whether $i - \mathfrak{p}_0 \ge -a_0$.

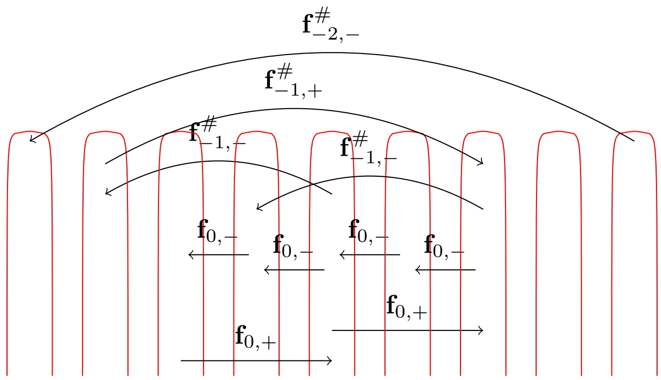


FIGURE 19. The maximal prepacman $\mathbf{F}_0 = (\mathbf{f}_{0,\pm})$ of a parabolic pacman f_0 with rotation number $1/3$, see Figure 18. The map $\mathbf{f}_{0,-}$ shifts periodic components of $\mathbf{H}$ to the left while $\mathbf{f}_{0,+}$ shifts the periodic components of $\mathbf{H}$ to the right. We have $\mathbf{f}_{n,-} = \mathbf{f}^2_{n-1,-} \circ \mathbf{f}_{n-1,+}$ and $\mathbf{f}_{n,+} = \mathbf{f}_{n-1,-} \circ \mathbf{f}_{n-1,+}$ for all n (obtained from $f_{0,-} = f_1^3$ and $f_{0,+} = f_1^2$ in Figure 18).

For every $n < 0$, choose a parameterization $I_n = \{-a_n, -a_n+1, \dots b_n - 1, b_n\}$ so that the natural embedding of I_n to I_{n-1} is viewed as $I_n \subset I_{n-1}$. Set $I_{-\infty} := \cup_{n\le 0} I_n = \mathbb{Z}$.

Recall (see §6.2) that a connected component $\mathbf{H}'$ of $\mathbf{H}$ is periodic if $\mathbf{f}^s_{0,-} \circ \mathbf{f}^r_{0,+}(\mathbf{H}') = \mathbf{H}'$ for some $s, r \in \mathbb{N}_{>0}$.

Proposition 6.5 (Parameterization of $\mathbf{H}^{\text{per}}$)**.** *The connected components of* $\mathbf{H}^{\text{per}}$ *are uniquely enumerated as* $(\mathbf{H}^i)_{i\in\mathbb{Z}}$ *so that for every sufficiently big* $n \ll 0$ *the component* $\mathbf{H}^i$ *contains the image of the component of* H_n *indexed by* i *under* $H_n \simeq \mathbf{H}^{\#}_n \subset \mathbf{H}^{\text{per}}$.

The actions of $\mathbf{f}^{\#}_{n,\pm}$ *on* $(\mathbf{H}^i)_{i\in\mathbb{Z}}$ *are given (up to interchanging* $\mathbf{f}^{\#}_{n,-}$ *and* $\mathbf{f}^{\#}_{n,+}$*) by*

$$\mathbf{f}^{\#}_{n,-}(\mathbf{H}^i) = (\mathbf{H}^{i-\mathfrak{p}_n}) \text{ and } \mathbf{f}^{\#}_{n,+}(\mathbf{H}^i) = \mathbf{H}^{i+\mathfrak{q}_n-\mathfrak{p}_n}. \tag{6.7}$$

Moreover, by re-enumerating components of $\mathbf{H}_0$ *we can assume that* $\mathbf{H}^0$ *contains* 0.

Proof. By construction, $I_{-\infty} \simeq \mathbb{Z}$ enumerates all of the periodic components of $\mathbf{H}$ intersecting $\cup_{n\le 0}\mathbf{H}^{\#}_n$ with actions given by (6.7). Since $\bigcup_{i\in\mathbb{Z}} \mathbf{H}^i$ is forward invariant and since every periodic component is in the forward orbit of $\mathbf{H}^0$ (see §6.2), we obtain $\bigcup_{i\in\mathbb{Z}} \mathbf{H}^i = \mathbf{H}^{\text{per}}$. We can re-enumerate $(\mathbf{H}^i)_{i\in\mathbb{Z}}$ in a unique way so that $\mathbf{H}^0 \ni 0$. □

Corollary 6.6. *There is no component* $\mathbf{H}'$ *of* $\mathbf{H}$ *such that* $\mathbf{f}^r_{0,-}(\mathbf{H}') = \mathbf{H}'$ *or* $\mathbf{f}^r_{0,+}(\mathbf{H}') = \mathbf{H}'$ *for some* $r > 0$.

Proof. Suppose converse and consider such $\mathbf{H}'$, say $\mathbf{f}^r_{0,-}(\mathbf{H}') = \mathbf{H}'$. Choose $a, b \in \mathbb{Z}$ such that a certain branch of $\mathbf{f}^a_{0,-} \circ \mathbf{f}^b_{0,+}$ maps $\mathbf{H}'$ to $\mathbf{H}^0$. Recall that $(\mathfrak{r}, \mathfrak{s})$ is a period of $\mathbf{H}^0$. By postcomposing $\mathbf{f}^a_{0,-} \circ \mathbf{f}^b_{0,+}$ with an iterate of $\mathbf{f}^{\mathfrak{r}}_{0,-} \circ \mathbf{f}^{\mathfrak{s}}_{0,+}$ we can assume that $a, b \ge 0$. It now follows from Proposition 6.5 that applying first $\mathbf{f}^r_{0,-} \mid \mathbf{H}'$ and then $\mathbf{f}^a_{0,-}\circ \mathbf{f}^b_{0,+}$ is different from applying $\mathbf{f}^a_{0,-}\circ \mathbf{f}^b_{0,+} \mid \mathbf{H}'$ and then $\mathbf{f}^r_{0,-}$. This is a contradiction. □

Corollary 6.7. *For $a,b,c,d\geq 0$ and $n\leq 0$,*

$$\left(\mathbf{f}^{\#}_{n,-}\right)^a \circ \left(\mathbf{f}^{\#}_{n,+}\right)^b (0) = \left(\mathbf{f}^{\#}_{n,-}\right)^c \circ \left(\mathbf{f}^{\#}_{n,+}\right)^d (0)$$

if and only if $a=c$ and $b=d$.

Proof. It is sufficient to prove it for $n=0$. Suppose $\mathbf{f}^a_{0,-}\circ \mathbf{f}^b_{0,+}(0) = \mathbf{f}^c_{0,-}\circ \mathbf{f}^d_{0,+}(0)$. It follows from (6.5) that $a+b=c+d$. If (a,b) is not proportional to (c,d), then $\mathbf{f}^a_{0,-}\circ \mathbf{f}^b_{0,+}(0)$, $\mathbf{f}^c_{0,-}\circ \mathbf{f}^d_{0,+}(0)$ are in different connected components of $\mathbf{H}^{\text{per}}$, see (6.7). Therefore, $a=c$ and $b=d$. □

6.5. **Valuable flowers of parabolic pacmen.** This subsection is a preparation for proving the scaling theorem (§8); it will not be used in proving the hyperbolicity theorem (§7).

Definition 6.8 (Valuable flowers)**.** Let f be a parabolic pacman with rotational number $\mathfrak{p}/\mathfrak{q}$. A *valuable flower* is an open forward invariant set $\mathbb{H}$ such that

(A) $\mathbb{H}\cup\{\alpha(f)\}$ is connected;
(B) $\mathbb{H}$ has $\mathfrak{q}$ connected components $\mathbb{H}^0, \mathbb{H}^1 \dots, \mathbb{H}^{\mathfrak{q}-1}$, called *petals*, enumerated counterclockwise at α; every $\mathbb{H}^i$ is an open topological disk;
(C) $f(\mathbb{H}^i)\subset \mathbb{H}^{i+\mathfrak{p}}$;
(D) all of the points in $\mathbb{H}$ are attracted by α;
(E) $\mathbb{H}^{-\mathfrak{p}}$ contains the critical point of f.

We remark that a local flower (see §6.2) satisfies (A)–(D).

We say a Siegel triangulation (see §4.2) $\mathbb{\Delta}$ *respects* a flower $\mathbb{H}$ if different petals of $\mathbb{H}$ are in different triangles of $\mathbb{\Delta}$.

Theorem 6.9 (Parabolic valuable flowers)**.** *Let $f_0\in\mathcal{W}^u$ be a parabolic pacman. Then for all sufficiently big $n\ll 0$ the pacman $f_n=\mathcal{R}^{-n}f_0$ has a valuable flower $\mathbb{H}_n$ and a Siegel triangulation $\mathbb{\Delta}(f_n)$ respecting $\mathbb{H}_n$ such that*

- *$\mathbb{\Delta}(f_n)$ has a wall $\sqcap(f_n)$ approximating $\partial Z_\star$;*
- *$\mathbb{\Delta}(f_{n-1})$ and $\mathbb{H}_{n-1}$ are full lifts of $\mathbb{\Delta}(f_n)$ and $\mathbb{H}_n$.*

Moreover, for a given closed disk $\mathbf{D}\subset\mathbf{H}^0$ the flower $\mathbb{H}_n$ with $n\ll 0$ can be constructed in such a way that $\mathbf{D}$ projects via $\operatorname{int}\left(\mathbf{S}^{\#}_n\right)\simeq V\setminus\gamma_1$ (see (5.6)) to $\mathbb{H}^0_n$.

Proof. Let us recall (see §6.2) that a local flower H_0 was chosen sufficiently small such that $H_0\subset V\setminus\gamma_1$, possibly up to a slight rotation of γ_1 in a small neighborhood of α. We denote by $\mathbb{\Delta}^{\text{new}}_0$ the triangulation obtained from $\mathbb{\Delta}_0$ by this slight adjustment of γ_1. By Lemma 4.3, the triangulation $\mathbb{\Delta}^{\text{new}}_0$ admits a full lift $\mathbb{\Delta}^{\text{new}}_{-n}$ to the dynamical plane of f_n for all $n\leq 0$. Since H_0 is respected by $\mathbb{\Delta}^{\text{new}}_0$, the flower H_0 also admits a full lift H_n to the dynamical plane of f_n such that H_n is respected by $\mathbb{\Delta}^{\text{new}}_{-n}$.

6.5.1. *Valuable petals.* Recall that $\mathfrak{p}_n/\mathfrak{q}_n$ denotes the rotation number of f_n. A *valuable petal* $\mathbb{H}^j_n$ of f_n is an open connected set attached to α such that

- $f^{\mathfrak{q}_n}_n$ extends analytically from a neighborhood of α to $f^{\mathfrak{q}_n}_n\colon \mathbb{H}^j_n\to\mathbb{H}^j_n$; (in particular, $\mathbb{H}^j_n$ is $f^{\mathfrak{q}_n}_n$-invariant)
- $f^{\mathfrak{q}_n}_n\colon \mathbb{H}^j_n\to\mathbb{H}^j_n$ has a critical point; and
- all points in $\mathbb{H}^j_n$ are attracted to α.

Claim 1. *There is an $n \ll 0$ such that f_n has a valuable petal $\mathbb{H}_n^0$ containing the critical value* 0 *such that $\mathbb{H}_n^0 = H_n^0 \cup D$, where H_n^0 is a petal of H_n and D is a small neighborhood of c_1 containing the projection of $\mathbf{D}$ via* (5.6)*. Moreover, there is an $M > 0$ such that $f_n^{\mathfrak{q}_n M}(\mathbb{H}_n^0) \subset H_n$.*

Proof. In the dynamical plane of $\mathbf{F}_0$ consider the petal $\mathbf{H}^0 \ni 0$. Recall from §6.4 that $\mathbf{H}_n^{\#}$ denotes the lift of H_n to the dynamical plane of $\mathbf{F}_n^{\#}$. If $n \ll 0$ is sufficiently big, then $\mathbf{H}^0$ contains a unique connected component of $\mathbf{H}_n^{\#}$, call it $(\mathbf{H}_n^{\#})^0$. Note also that $(\mathbf{H}_n^{\#})^0 = (\mathbf{H}_m^{\#})^0$ for all sufficiently big $n, m \ll 0$, see Proposition 6.5.

Enlarge $\mathbf{D}$ to a bigger closed disk $\mathbf{D} \subset \mathbf{H}^0$ such that

- $(\mathbf{H}_n^{\#})^0 \cup \mathbf{D}$ is forward invariant under the first return map $\mathbf{f}_{0,-}^{\mathfrak{r}} \circ \mathbf{f}_{0,+}^{\mathfrak{s}}$, see Lemma 6.2; and
- $\mathbf{f}_{0,-}^{\mathfrak{r}} \circ \mathbf{f}_{0,+}^{\mathfrak{s}} \left((\mathbf{H}_n^{\#})^0 \cup \mathbf{D}\right) \ni 0$.

Since $\mathbf{D}$ is compact, we have $\mathbf{D} \subset \mathbf{S}_n^{\#}$ for all sufficiently big $n \ll 0$. For such n we can project $\mathbf{D}$ to the dynamical plane of f_n; we denote this projection by $D \ni c_1$. By construction, $D \cup H_n^0$ is $f_n^{\mathfrak{q}_n}$-invariant: $f_n^{\mathfrak{q}_n} \colon H_n^0 \to H_n^0$ has an analytic extension to $f_n^{\mathfrak{q}_n} \colon D \cup H_n^0 \to D \cup H_n^0$. For $n \ll 0$, the disk D is a small neighborhood of c_1. □

For $n \ll 0$, we enumerate petals of H_n counterclockwise so that $H_n^0 \subset \mathbb{H}_n^0$. Choose a big K (we will specify K in §6.5.3). For $k \in \{0, 1, \dots, K\}$ we define D_k to be the image of $D_0 = D$ under $f_n^{\mathfrak{q}_n k}$, and for $k \in \{-K, -K+1, \dots, -1\}$ we define D_k to be the lift of D_0 along the orbit of $f_n^{-\mathfrak{q}_n k} \colon H_n^{\mathfrak{q}_n k} \to H_n^0$. Then

$$\mathbb{H}_n^{\mathfrak{q}_n k} := H_n^{\mathfrak{q}_n k} \cup D_k; \tag{6.8}$$

is a valuable petal extending $H_n^{\mathfrak{q}_n k}$ for all $k \in \{-K, \dots, K\}$. For $n \ll 0$, all $\mathbb{H}_n^{\mathfrak{q}_n k}$ are in a small neighborhood of $\overline{Z}_\star$.

6.5.2. *Walls respecting H_n.* Set $N := M + 3$, where M is defined in Claim 1. Let us consider the dynamical plane of f_0. In a small neighborhood of α we can choose a univalent $(N+1)\mathfrak{q}_0$-wall A_0 *respecting* H_0 in the following way:

(a) α is in the bounded component O_0 of $\mathbb{C} \setminus A_0$ while the critical point and the critical value of f_0 are in the unbounded component of $\mathbb{C} \setminus A_0$;
(b) each petal H_0^i intersects A_0 at a connected set;

and by enlarging H_0, we can also guarantee:

(c) H_0 contains all $z \in A_0 \cup O_0$ with forward orbits in $A_0 \cup O_0$.

We can also assume that the intersection of A_0 with each triangle of $\mathbb{\Delta}_0^{\text{new}}$ is a closed topological rectangle. Lifting these rectangles to the dynamical plane of f_n and spreading around them, we obtain a *full lift* A_n of A_0. Then A_n is a univalent $N\mathfrak{q}_n$-wall (see Lemma B.12) enclosing an open topological disk $O_n \ni \alpha$ such that A_n respects H_n as above (see (a)–(c)). Naturally, A_n consists of closed topological rectangles: each rectangle is in a certain triangle of $\mathbb{\Delta}_n^{\text{new}}$.

Claim 2. *For $n \ll 0$, the wall A_n approximates $\partial Z_\star$ in the sense of Lemma 4.2, Part* (5)*: $\partial Z_\star$ is a concatenation of arcs $J_0 J_1 \dots J_{m-1}$ such that J_i is close to the i-th rectangle of A_n counting counterclockwise.*

Proof. By Theorem 4.6, it is sufficient to prove such statement in the dynamical plane of $f_\star$: if A_0 is an annulus bounded by two equipotentials of $Z_\star$, then a full

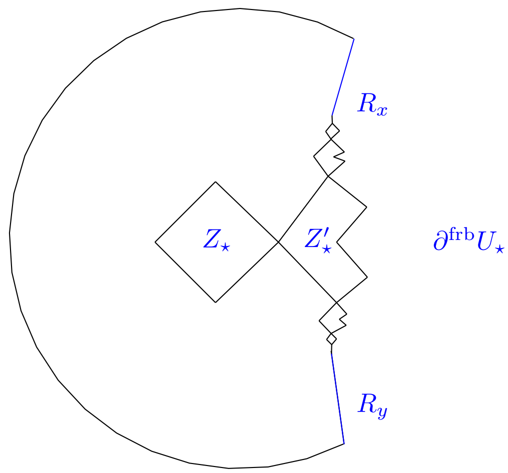


FIGURE 20. Separation of $\partial^{\text{frb}} U_\star$. Co-Siegel disk $Z'_\star$ together with its iterated lifts form two periodic bubble chains landing at periodic points x and y. The bubble chains together with external rays R_x and R_y separate $\partial^{\text{frb}} U_\star$ from α.

lift A_n approximates $\partial Z_\star$ for a big n. Since the renormalization change of variables for $f_\star$ is conjugate to $z \to z^t$ with $t < 1$, the claim follows. □

Consider the dynamical plane of $f_\star$. Recall that $f_\star \mid \overline{Z}_\star$ is a homeomorphism. For $k \in \mathbb{Z}$, we define

$$c_k := \left(f_\star \mid \overline{Z}_\star\right)^k (c_0).$$

Consider now the dynamical plane of f_n. For $k \in \{-K, -K+1, \dots, K\}$, we define $c_k(f_n) \in f_n^k\{c_0\}$ to be the closest point to $c_k(f_\star)$. The point $c_k(f_n)$ is well defined as long as f_n is in a small neighborhood of $f_\star$.

Claim 3. *For $k \in \{-K, -K+1, \dots, K\}$, we have*

- $c_{k-1}(f_n) \in \mathbb{H}_n^{\mathfrak{q}_n k}$*; and*
- $\mathbb{H}_n^{\mathfrak{q}_n k} \setminus O_n$ *is in a small neighborhood of* c_{k-1}

Proof. The first statement follows from $c_{k-1}(f_n) \in D_k \subset \mathbb{H}_n^{\mathfrak{q}_n k}$, see (6.8). The second statement follows from the improvement of the domain. □

Claim 4. *Let P be a connected component of $O_n \setminus H_n$. Then $f_n^{\mathfrak{q}_n i} \mid P$ is univalent for all $i \in \{1, \dots, N\}$. Moreover,*

$$f_n^{\mathfrak{q}_n i}(P) \subset f_n^{\mathfrak{q}_n j}(P) \quad \textit{for all } i < j \textit{ in } \{0, 1, \dots, N\}.$$

Proof. The first claim follows from the assertion that A_n is an $N\mathfrak{q}_n$-wall. The second claim follows from (c). □

6.5.3. *Julia rays in $\partial\mathfrak{J}_\star$.* Consider the dynamical plane of $f_\star : U_\star \to V$. By Theorem 3.12, we can choose (see Figure 20) two periodic points $x, y \in \mathfrak{J}_\star$ together with two periodic external rays R_x, R_y landing at x, y and two periodic bubble chains B_x, B_y landing at x, y so that x and y are close to $\partial^{\text{frb}} U_\star$ and $R_x \cup B_x \cup B_y \cup R_y$

separates $\partial^{\text{frb}} U_\star$ from c_1 as well as from all the remaining points in the forward orbit of x, y. Let p be a common period of x, y. Set $K \coloneqq 4p$.

A *Julia ray* J of $\mathfrak{J}_\star$ is a simple arc in $\mathfrak{J}_\star$ starting at a point in $\partial Z_\star$.

Claim 5. *There are Julia rays $J_x \subset B_x$ and $J_y \subset B_y$ such that J_x and J_y start at the critical point c_0 and land at x and y respectively. Moreover, J_x and J_y are periodic with period p: the rays J_x and J_y decompose as concatenations $J^1_x J^2_x J^3_x \dots$ and $J^1_y J^2_y J^3_y \dots$ such that $f^p_\star$ maps J^k_x and J^k_y to J^{k-1}_x and J^{k-1}_y respectively.*

Proof. Write $B_x = (Z_1, Z_2, \dots)$; since x is close to $\partial^{\text{frb}} U_\star$ we see that $Z_1 = \overline{Z}'_\star$. Since x is periodic with period p, there is an $a > 0$ such that f^p maps Z_{a+i} to Z_i for all i.

Let $J^1_x \subset \mathfrak{J}_\star$ be a simple arc in $\partial Z_1 \cup \partial Z_2 \cup \dots \cup \partial Z_a$ connecting the critical point c_0 to the point where ∂Z_{a+1} is attached to ∂Z_a. We inductively define J^j_x to be the iterated lift of J^{j-1}_x such that J^j_x starts where J^{j-1}_x terminates. This constructs $J_x = J^1_x J^2_x J^3_x \dots$; similarly $J_y = J^1_y J^2_y J^3_y \dots$ is constructed. □

6.5.4. *Julia rays for f_n.* Recall that in Claim 5 we specified Julia rays $J_x(f_\star)$ and $J_y(f_\star)$. Since f_0 is sufficiently close to $f_\star$, the periodic points x, y exist in the dynamical plane of f_0 and are close to that of $f_\star$. For $n \ll 0$ let us now construct *Julia rays* $J_x(f_n) = J^1_x J^2_x J^3_x \dots$ and $J_y(f_n) = J^1_y J^2_y J^3_y \dots$ such that

(1) f^p_n maps J^k_x to J^{k-1}_x and J^k_y to J^{k-1}_y (compare with Claim 5);
(2) $J^k_x(f_n)$ and $J^k_y(f_n)$ are in small neighborhoods of $J^k_x(f_*)$ and $J^k_y(f_*)$ respectively;
(3) for $z \in J^1_x \cup J^2_x \cup J^1_y \cup J^2_y$ there is a $q \le 2p$ such that either $f^q_n(z) \in O_n$ or $f^q_n(z) \in \bigcup_{|k| \le 2p} \mathbb{H}^{k\mathfrak{q}_n}_n$. In the former case we can assume that $f^\ell_n(z) \notin A_n \cup O_n$ for $\ell \in \{0, 1, \dots, q-1\}$.

Construction of J_x and J_y. We will use notations from the proof of Claim 5. By stability of periodic points, x, y exist for f_n and are close to $x(f_\star), y(f_\star)$. The curve J^1_x is a simple arc in $\partial Z_1 \cup \partial Z_2 \cup \dots \cup \partial Z_a$. We split J_1 as the concatenation $\ell_1 \cup \ell_2 \dots \cup \ell_a$ with $\ell_j = J^1_x \cap \partial Z_j$. Let $f^{d(j)}_\star$ be the smallest iterate mapping Z_j to $\overline{Z}_\star$. Since $J_x \subset \mathfrak{J}_\star$, the curve

$$\widetilde{\ell}_j \coloneqq f^{d(j)}_\star(\ell_j)$$

is a simple arc in $\partial Z_\star$ connecting c_1 and a certain $c_{t(j)}$.

Using Claims 2 and 3, we approximate each $\widetilde{\ell}_j(f_\star)$ by a curve $\widetilde{\ell}_j(f_n)$ within $O_n \cup \mathbb{H}^{t(j)+1}_n \cup \mathbb{H}^0_n$. Lifting $\widetilde{\ell}_j(f_n)$ along the branch of $f^{d(j)}_n$ that is close to $f^{d(j)}_\star \mid \ell_j(f_\star)$, we construct $\ell_j(f_n)$ that is close to $\ell_j(f_\star)$. Assembling all ℓ_j, we construct $J^1_x(f_n)$. By continuity, pulling back $J^1_x(f_n)$ we construct finitely many $J^k_x(f_n)$ approximating $J^k_x(f_\star)$ such that the remaining curves $J^k_x(f_\star)$ are within the linearization domain of x. Taking pullbacks within the linearization domain of x, we construct a ray $J_x(f_n)$ landing at x. Similarly, J_y is constructed. Property (3) follows from $|t(j)| \le p$. □

6.5.5. *Blocking $\partial^{\text{frb}} U_n$.* Recall from Claim (1) that $f^{\mathfrak{q}_n M}_n(\mathbb{H}^0_n) \subset H^0_n$. For $t \in \{M, M-1, M-2, \dots, 0\}$ we set $H^{(t)}_n$ to be the forward f_n-orbit of $f^{\mathfrak{q}_n k}_n(\mathbb{H}^0_n)$.

Claim 6. *The flower $H^{(t)}_n$ does not intersect $\partial^{\text{frb}} U_n$ for all $t \in \{M, \dots, 0\}$.*

As a consequence, H_n extends to a required $\mathbb{H}_n$ for $n \ll 0$.

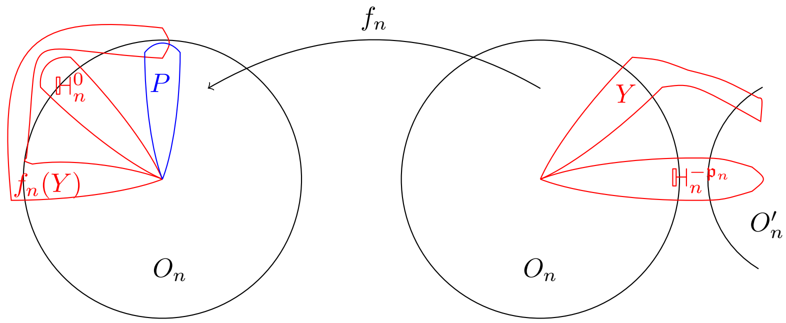


FIGURE 21. Illustration to the proof of Claim 6. If Y intersects O_n', then applying f_n we obtain that $P \cup f_n(Y)$ encloses $\mathbb{H}_n^0$. Since P is surrounded by the wall A_n, the set $f_n^{\mathfrak{q}_n M}(P \cup f_n(Y))$ also encloses $\mathbb{H}_n^0$. Then $f_n^{\mathfrak{q}_n} \mid f_n^{\mathfrak{q}_n M}(P \cup f_n(Y))$ has degree one while $f_n^{\mathfrak{q}_n} \mid \mathbb{H}_n^0$ has degree 2; this is a contradiction.

Proof. Recall that valuable petals $\mathbb{H}_n^{k\mathfrak{p}_n} \subset U_n$ with $|k| \le K$ are already constructed. Set

$$H_n'^{(t)} \coloneqq H_n^{(t)} \setminus \bigcup_{|k| \le K} \mathbb{H}_n^{k\mathfrak{p}_n}. \tag{6.9}$$

Let us show that $H_n'^{(t)}$ does not hit $R_x \cup J_x \cup J_y \cup R_y$; this would imply that $H_n^{(t)}$ does not intersect $\partial^{\text{frb}} U_n$. Suppose converse; let t be the first moment (i.e. t is the closest to M) when $H_n'^{(t)}$ hits $J_x \cup J_y$. Denote by X a petal of $H_n'^{(t)}$ intersecting $J_x \cup J_y$. Choose $z \in X \bigcap (J_x \cup J_y)$; we can assume that $z \in J_x^1 \cup J_x^2 \cup J_y^1 \cup J_y^2$, otherwise t is not the first moment when $H_n'^{(t)}$ hits $J_x \cup J_y$. By Property (3) from §6.5.4, there is a $q \le 2p$ such that either $f_n^q(z) \in O_n$ or $f_n^q(z) \in \bigcup_{|k| \le 2p} \mathbb{H}_n^{k\mathfrak{q}_n}$. The latter would imply that X is a petal in $\bigcup_{|k| \le 4p} \mathbb{H}_n^{k\mathfrak{q}_n}$; this contradicts to (6.9). Therefore, $f_n^q(z) \in O_n$.

Write

$$O_n' \coloneqq f_n^{-1}(O_n) \setminus (A_n \cup O_n) \ni f_n^{q-1}(z)$$

and set $Y \coloneqq f^{q-1}(X)$. We have $O_n' \cap Y \ni f_n^{q-1}(z)$, see Figure 21. Since $\mathbb{H}_n^{-\mathfrak{p}_n}$ contains a critical point, we see that $f_n^q(z)$ is within a connected components P of $O_n \setminus (H_n \cup \{\alpha\})$ and, moreover, $P \cup f_n(Y)$ surrounds $\mathbb{H}_n^0$.

Let us apply $f_n^{\mathfrak{q}_n M}$ to $f_n(Y) \cup P$. By Claim 4 (recall that $N > M + 1$, see §6.5.2), we have $f_n^{\mathfrak{q}_n M}(P) \subset (A_n \cup O_n) \setminus H_n$ and $f_n^{\mathfrak{q}_n M}(P)$ does not contain a critical point of $f_n^{\mathfrak{q}_n}$. On the other hand, $f_n^{\mathfrak{q}_n M+1}(Y)$ does not contain a critical point of $f_n^{\mathfrak{q}_n}$ as a subset of H_n. Note that $f_n^{\mathfrak{q}_n M}(P \cup f_n(Y))$ still surrounds $\mathbb{H}_n^0$. This is a contradiction: $f_n^{\mathfrak{q}_n} \mid f_n^{\mathfrak{q}_n M}(P \cup f_n(Y))$ has degree one while $f_n^{\mathfrak{q}_n} \mid \mathbb{H}_n^0$ has degree 2. □

6.5.6. *Siegel triangulation.* It remains to construct a Siegel triangulation $\mathbb{A}(f_n)$ respecting $\mathbb{H}_n$ for $n \ll 0$. In the dynamical plane of f_n, let us choose a curve $\ell_1 \subset V$ connecting ∂V to α such that ℓ_1 enters U_n in O_n', then reaches $\partial \mathbb{H}_n^{-\mathfrak{p}_n}$, then travels to α within $\partial \mathbb{H}_n^{-\mathfrak{p}_n}$. We can assume that $\ell_1 \setminus O_n$ is disjoint from $\gamma_1 \setminus O_n$. Observe that ℓ_1 is liftable to the dynamical planes f_m for all $m \le n$. Indeed, $\ell_1 \cap \partial \mathbb{H}_n^{-\mathfrak{p}_n}$ is liftable because so is $\partial \mathbb{H}_n^{-\mathfrak{p}_n}$, while $\ell_1 \setminus \partial \mathbb{H}_n^{-\mathfrak{p}_n}$ is liftable because it is disjoint from γ_1.

Let us slightly perturb ℓ_1 so that the new ℓ_1 is disjoint from $\mathbb{H}_n$. Define ℓ_0 to the preimage of ℓ_1 connecting ∂U_n to α. Then $\ell_1 \cup \ell_0$ splits U_n into two closed sectors; they form the triangulation denoted by $\mathbb{\Delta}(f_n)$.We can assume that ℓ_1 was chosen so that $\ell_1 \setminus O_n$ and $\ell_0 \setminus O_n$ are connected. We define the wall $\mathbb{\Pi}(f_n)$ to be the closures of two connected components of $U_n \setminus (O_n \cup \ell_0 \cup \ell_1)$.

For $m \leq n$ we define $\mathbb{\Delta}(f_m)$ and $\mathbb{\Pi}(f_m)$ to be the full lifts of $\mathbb{\Delta}(f_n)$ and $\mathbb{\Pi}(f_n)$. Then $\mathbb{\Delta}(f_m)$ is a required triangulation for $m \ll n$. □

7. Hyperbolicity Theorem

Recall that by $\lambda_\star$ we denote the multiplier of the α-fixed point of $f_\star$. For λ close to $\lambda_\star$ set

$$\mathcal{F}(\lambda) := \{f \in \mathcal{W}^u \mid \text{the multiplier of } \alpha \text{ is } \lambda\}$$

the analytic sub-manifold of $\mathcal{W}^u$ parametrized by fixing the multiplier at α. Then $\mathcal{F}(\lambda)$ forms a foliation of a neighborhood of $f_\star$.

7.1. Holomorphic motion of $P(\mathbf{F}_0)$.

Let $\mathcal{U} \subset \mathcal{W}^u$ be a small neighborhood of $f_\star$ such that for every $f \in \mathcal{U}$ has a maximal prepacmen, see Theorem 5.5.

Lemma 7.1 (Holomorphic motion of the critical orbits)**.** *For every $\mathfrak{p}/\mathfrak{q}$, the set*

$$\bigcup_{n \leq 0} \operatorname{orb}_0(\mathbf{F}_n^\#)$$

moves holomorphically with $f_0 \in \mathcal{F}(\mathbf{e}(\mathfrak{p}/\mathfrak{q})) \cap \mathcal{U}$.

Recall from Lemma 6.1 that $P(\mathbf{F}_0) \subset \bigcup_{n \leq 0} \operatorname{orb}_0(\mathbf{F}_n^\#)$ thus $P(\mathbf{F}_0)$ also moves holomorphically with $f_0 \in \mathcal{F}(\mathbf{e}(\mathfrak{p}/\mathfrak{q})) \cap \mathcal{U}$.

Proof. By Corollary 6.7, points in $\operatorname{orb}_0(\mathbf{F}_n^\#)$ do not collide with each other when $f_0 \in \mathcal{F}(\mathbf{e}(\mathfrak{p}/\mathfrak{q})) \cap \mathcal{U}$ is deformed. This gives a holomorphic motion of $\operatorname{orb}_0(\mathbf{F}_0) \subset \operatorname{orb}_0(\mathbf{F}_1^\#) \subset \operatorname{orb}_0(\mathbf{F}_2^\#) \subset \dots$ and we can take the union. □

Let $\mathcal{U}' \subset \mathcal{U}$ be a neighborhood of $f_\star$ such that every non-empty $\mathcal{F}(\lambda) \cap \mathcal{U}'$ has radius at least three times less than those of $\mathcal{F}(\lambda) \cap \mathcal{U}$.

Corollary 7.2 (Extended holomorphic motions)**.** *For $f_0 \in \mathcal{F}(\mathbf{e}(\mathfrak{p}/\mathfrak{q})) \cap \mathcal{U}'$ there is a holomorphic motion $\tau(f_0)$ of $\widehat{\mathbb{C}}$ such that $\tau(f_0)$ is equivariant (with the dynamics of $(\mathbf{F}_n^\#)_n$) on*

$$\bigcup_{n \leq 0} \operatorname{orb}_0(\mathbf{F}_n^\#).$$

Proof. Follows by applying the λ-lemma to the holomorphic motion from Lemma 7.1. □

Corollary 7.3 (Passing to the limit of holomorphic motions)**.** *For $f_0 \in \mathcal{F}(\lambda_\star) \cap \mathcal{U}'$ there is a holomorphic motion $\tau(f_0)$ of $\widehat{\mathbb{C}}$ such that $\tau(f_0)$ is equivariant on*

$$\bigcup_{n \leq 0} \operatorname{orb}_0(\mathbf{F}_n^\#).$$

Proof. Choose a sequence $\mathfrak{p}_n/\mathfrak{q}_n$ such that $\mathbf{e}(\mathfrak{p}_n/\mathfrak{q}_n) \to \mathbf{e}(\theta_\star)$. By passing to the limit in Corollary 7.2 we obtain the required property. □

Corollary 7.4. *The dimension of $\mathcal{F}(\lambda_\star)$ is* 0.

Proof. Suppose the dimension of $\mathcal{F}(\lambda_\star)$ is greater than 0. Consider the space $\mathcal{F}(\lambda_\star) \cap \mathcal{U}'$. By Corollary 7.3 the set $\overline{P(\mathbf{F}_0)} \subset \bigcup_{n\le 0} \operatorname{orb}_0(\mathbf{F}_n^\#)$ moves holomorphically with $f_0 \in \mathcal{F}(\lambda_\star) \cap \mathcal{U}'$. Projecting this holomorphic motion to the dynamical plane of f_0, we obtain a holomorphic motion of the post-critical set of $f_0 \in \mathcal{F}(\lambda_\star) \cap \mathcal{U}'$. Therefore, there is a small neighborhood of $f_\star$ in $\mathcal{F}(\lambda_\star) \cap \mathcal{U}'$ consisting of Siegel maps. But all such maps must be in the stable manifold of $f_\star$ by Theorem 7.5. □

7.2. **The exponential convergence.** The following theorem follows from [McM2, Theorem 8.1].

Theorem 7.5. *Suppose that a pacman $f \in \mathcal{B}$ is Siegel of the same rotation number as $f_\star$ such that f is sufficiently close to $f_\star$. Then $\mathcal{R}^n f$ converges exponentially fast to $f_\star$.*

Remark 7.6. *The proof of* [McM2, Theorem 8.1] *is based on a "deep point argument". Alternatively, the exponential convergence follows from a variation of the Schwarz lemma following the lines of* [L1, AL1].

7.3. **The hyperbolicity theorem.**

Theorem 7.7 (Hyperbolicity of $\mathcal{R}$). *The renormalization operator $\mathcal{R}\colon \mathcal{B} \to \mathcal{B}$ is hyperbolic at $f_\star$ with one-dimensional unstable manifold $\mathcal{W}^u$ and codimension-one stable manifold $\mathcal{W}^s$.*

In a small neighborhood of $f_\star$ the stable manifold $\mathcal{W}^s$ coincide with the set of pacmen in $\mathcal{B}$ that have the same multiplier at the α-fixed point as $f_\star$. Every pacman in $\mathcal{W}^s$ is Siegel.

In a small neighborhood of $f_\star$ the unstable manifold $\mathcal{W}^u$ is parametrized by the multipliers of the α-fixed points of $f \in \mathcal{W}^u$.

Proof. It was already shown in Corollary 7.4 that the dimension of $\mathcal{W}^u$ is one. Let us show that $\mathcal{W}^s$ has codimension one. Denote by $\mathcal{B}^*$ the submanifold of $\mathcal{B}$ consisting of all the pacmen with the same multiplier at the α-fixed point as $f_\star$. Then $\mathcal{R}$ naturally restricts to $\mathcal{R}\colon \mathcal{B}^* \to \mathcal{B}^*$. Consider the derivative $\operatorname{Diff}(\mathcal{R} \mid \mathcal{B}^*)$; by Corollary 7.4 the spectrum of $\operatorname{Diff}(\mathcal{R} \mid \mathcal{B}^*)$ is within the closed unit disk. Suppose that the spectrum of $\operatorname{Diff}(\mathcal{B}^*)$ intersects the unit circle. By [L1, Small orbits theorem] $\mathcal{R} \mid \mathcal{B}^*$ has a small slow orbit: there is an $f \in \mathcal{B}^*$ such that f is infinitely many times renormalizable but

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\mathcal{R}^n f)\| = 0.$$

Moreover, it can be assumed that $\{\mathcal{R}^n f\}_{n\ge 0}$ is in a sufficiently small neighborhood of $f_\star$. By Corollary 4.7, f is Siegel pacman and by Theorem 7.5, $\mathcal{R}^n f$ converges exponentially fast to $f_\star$. This is a contradiction. Therefore, the spectrum of $\mathcal{R}$ is compactly contained in the unit disk, all of the pacmen in $\mathcal{B}^*$ are infinitely renormalizable and thus are Siegel (Corollary 4.7). The submanifold $\mathcal{B}^*$ coincides with $\mathcal{W}^s$ in a small neighborhood of $f_\star$. □

7.4. **Control of Siegel disks.** The following lemma follows from [McM2, Theorem 8.1] combined with Theorem 3.6 and Lemma 3.4.

Lemma 7.8. *Every Siegel map f has a pacman renormalization $\mathcal{R}_2 f$ such that $\mathcal{R}_2 f$ is in $\mathcal{B}$ and is sufficiently close to $f_\star$.* □

We say a holomorphic map $f\colon U \to V$ is *locally Siegel* if it has a distinguished Siegel fixed point. The following corollary follows from Theorem 7.7 combined with Lemma 7.8

Corollary 7.9. *Let $f\colon U \to W$ be a Siegel map with rotation number $\theta \in \Theta_{\text{per}}$ and let $N(f)$ be a small Banach neighborhood of f. Then every locally Siegel map $g \in N(f)$ with rotation number θ is a Siegel map. The Siegel disk $\overline{Z}_g$ is contained in a small neighborhood of $\overline{Z}_f$.* □

8. Scaling Theorem

In this section we prove a refined version of Theorem 1.2. Consider $\theta_\star \in \Theta_{\text{per}}$ and let f be a Siegel map with rotation number $\theta_\star$. Let $\mathcal{U} \ni f$ be a small Banach neighborhood of f and let $\mathcal{W} \subset \mathcal{U}$ be a one-dimensional slice containing f such that $\mathcal{W}$ is transverse to the hybrid class of f; i.e. in a small neighborhood of $f \in \mathcal{W}$ all maps have different multipliers at their α-fixed points.

We say a map $g \in \mathcal{U}$ is *satellite* if it has a satellite valuable flower:

Definition 8.1 (Satellite valuable flowers)**.** A *satellite valuable flower* of g is an open forward invariant set $\mathbb{H}$ such that

(A) $\mathbb{H} \cup \{\alpha(g)\}$ is connected;
(B) $\mathbb{H}$ has $\mathfrak{q}$ connected components $\mathbb{H}^0, \mathbb{H}^1 \dots, \mathbb{H}^{\mathfrak{q}-1}$, called *petals*, enumerated counterclockwise at α; every $\mathbb{H}^i$ is an open topological disk;
(C) $g(\mathbb{H}^i) \subset \mathbb{H}^{i+\mathfrak{p}}$, where $\mathfrak{p}$ is coprime to $\mathfrak{q}$;
(D) there is an attracting periodic cycle $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{\mathfrak{q}-1})$ with $\gamma_i \in \mathbb{H}^i$ attracting all points in $\mathbb{H}$;
(E) $\mathbb{H}^{-\mathfrak{p}}$ contains the critical point of g.

The number $\mathfrak{p}/\mathfrak{q}$ is called the *combinatorial rotation number* of $\mathbb{H}$. The *multiplier* of $\mathbb{H}$ is the multiplier of γ.

For convenience, let us say that a parabolic valuable flower (see Definition 6.8) with rotation number $\mathfrak{p}/\mathfrak{q}$ is a satellite valuable flower with rotation number $\mathfrak{p}/\mathfrak{q}$ and multiplier 1.

Since $\theta_\star$ is periodic, there exists $\mathfrak{k} \geq 0$ with $R^{\mathfrak{k}}_{\text{prm}}(\theta_\star) = \theta_\star$, where R_{prm} is (1.1); see also Appendix A and, in particular, (A.2).

Theorem 8.2. *Suppose a sequence $(\mathfrak{p}_n/\mathfrak{q}_n)_{n=0}^{-\infty}$ converges to $\theta_\star$ so that $R^{\mathfrak{k}}_{\text{prm}}(\mathfrak{p}_n/\mathfrak{q}_n) = \mathfrak{p}_{n+1}/\mathfrak{q}_{n+1}$. Fix $\lambda_1 \in \mathbb{D}^1$ and a small neighborhood of $\overline{Z}_f$. Then there is a continuous path $\lambda_t \in \mathbb{D}^1$ with $t \in (0,1]$ emerging from $1 = \lambda_0$ such that for every sufficiently big $n \ll 0$ there is a unique path $g_{n,t} \in \mathcal{W}$, where $t \in [0,1]$, with the following properties*

- *$g_{n,t}$ has a satellite valuable flower $\mathbb{H}_{n,t}$ with rotation number $\mathfrak{p}_n/\mathfrak{q}_n$ and multiplier λ_t;*
- *all $\mathbb{H}_{n,t}$ are in the given small neighborhood of $\overline{Z}_f$; and*
- $\operatorname{dist}(f, g_{n,t}) \sim \left((R^{\mathfrak{k}}_{\text{prm}})'(\theta_\star)\right)^n$ *for every t.*

Note that the path $g_{n,t}$ starts at a unique parabolic map in $\mathcal{W}$ with rotation number $\mathfrak{p}_n/\mathfrak{q}_n$.

Proof. The proof is split into short subsections. Consider a pacman hyperbolic renormalization operator $\mathcal{R}\colon \mathcal{B} \dashrightarrow \mathcal{B}$ around a fixed point $f_\star = \mathcal{R}(f_\star)$ with rotation number $\theta_\star$. By passing to iterates, we can assume that $\mathcal{R}$ acts on the rotation numbers as $R^{\mathfrak{t}}_{\text{prm}}$, see (3.6). As before, $\mathcal{W}^u$ denotes the unstable manifold of $\mathcal{R}$ at $f_\star$.

8.0.1. *Perturbation of parabolic pacmen.* By shifting the sequence $(\mathfrak{p}_n/\mathfrak{q}_n)_n$ we can assume that $\mathfrak{p}_0/\mathfrak{q}_0$ is close to $\theta_\star$. Then there is a unique parabolic pacman $f_0 \in \mathcal{W}^u$ with rotation number $\mathfrak{p}_0/\mathfrak{q}_0$. Then $f_n \coloneqq \mathcal{R}^n f_0,\quad n \le 0$ has rotation number $\mathfrak{p}_n/\mathfrak{q}_n$. By Theorem 6.9 and possibly by further shifting $(\mathfrak{p}_n/\mathfrak{q}_n)_n$, we can assume that:

- each f_n has a valuable flower $\mathbb{H}(f_n)$ at the α-fixed;
- each f_n has a triangulation $\mathbb{\Delta}(f_n)$ respecting $\mathbb{H}(f_n)$: different petals of $\mathbb{H}(f_n)$ are in different triangles of $\mathbb{\Delta}(f_n)$;
- $\mathbb{\Delta}(f_n)$ has a wall $\mathbb{\Pi}(f_n)$ approximating $\partial Z_\star$;
- $\mathbb{\Delta}(f_n)$ and $\mathbb{H}(f_n)$ are the full lifts of $\mathbb{\Delta}(f_{n+1})$ and $\mathbb{H}(f_{n+1})$.

Let $g_0 \in \mathcal{W}^u$ be a slight perturbation of f_0 that splits α into a repelling fixed point α and an attracting cycle $\gamma(g_0)$ such that α is on the boundary of the immediate attracting basin of $\gamma(g_0)$. Then $\mathbb{\Delta}(f_0)$, $\mathbb{\Pi}(g_0)$, $\mathbb{H}(f_0)$ are perturbed to $\mathbb{\Delta}(g_0)$, $\mathbb{\Pi}(g_0)$, $\mathbb{H}(g_0)$ such that all points in $\mathbb{H}(g_0)$ are attracted by $\gamma(g_0)$. We can assume that the perturbation is sufficiently small such that $\mathbb{\Pi}(g_0)$ still approximates $\partial Z_\star$. By Lemma 4.4, there are full lifts $\mathbb{\Delta}(g_n)$, $\mathbb{H}(g_n)$ of $\mathbb{\Delta}(g_0)$, $\mathbb{H}(g_0)$.

As before, we denote by $\mathbf{F}_n$ and $\mathbf{G}_n$ the maximal prepacmen of f_n and g_n and we denote $\mathbf{G}^\#_n$ the the rescaled version of $\mathbf{G}_n$ such that $\mathbf{G}_0 = \mathbf{G}^\#_0$ is an iteration of $\mathbf{G}^\#_n$. Recall from §6.2 that $\mathbb{H}(f_0)$ admits a global extension $\mathbf{H}(\mathbf{F}_0)$ in the dynamical plane of $\mathbf{F}_0$. Similarly, we now define the maximal extension $\mathbf{H}(\mathbf{G}_n)$ of $\mathbb{H}(g_n)$.

Each $\mathbb{H}(g_n)$ lifts to the dynamical plane of $\mathbf{G}^\#_n$; denote by $\mathbf{H}(g_0)$ the lift of $\mathbb{H}(g_0)$. Similar to (6.2), we set

$$\mathbf{H}(\mathbf{G}_0) \coloneqq \bigcup_{a,b\in\mathbb{Z}} (\mathbf{g}_{0,-})^a \circ (\mathbf{g}_{0,+})^b \left(\mathbf{H}(g_0)\right)$$

to be the full orbit of $\mathbf{H}(g_0)$. The same argument as in the proof of Lemma 6.2 shows that $\mathbf{H}(\mathbf{G}_0)$ is fully invariant and is within $\operatorname{Dom} \mathbf{G}_{0,-} \cap \operatorname{Dom} \mathbf{G}_{0,+}$.

Denote by $\mathbf{H}^{\text{per}}(\mathbf{G}_0)$ the union of periodic components of $\mathbf{H}(\mathbf{G}_0)$. The same argument as in the proof of Proposition 6.5 shows:

Proposition 8.3 (Parameterization of $\mathbf{H}^{\text{per}}(\mathbf{G}_0)$)**.** *The connected components of $\mathbf{H}^{\text{per}}(\mathbf{G}_0)$ are uniquely enumerated as $(\mathbf{H}^i)_{i\in\mathbb{Z}}$ such that $\mathbf{H}^0 \ni 0$ and such that the actions of $\mathbf{g}^{\#0}_{n,\pm}$ on $(\mathbf{H}^i)_{i\in\mathbb{Z}}$ are given (up to interchanging $\mathbf{g}_{n,-}$ and $\mathbf{g}_{n,+}$) by*

$$\mathbf{g}^\#_{n,-}(\mathbf{H}^i) = (\mathbf{H}^{i-\mathfrak{p}_n}) \text{ and } \mathbf{g}^\#_{n,+}(\mathbf{H}^i) = \mathbf{H}^{i+\mathfrak{q}_n-\mathfrak{p}_n}. \quad \square \tag{8.1}$$

8.0.2. *QC-deformation of g_n.* Suppose first that $\lambda_1 \neq 0$. Denote by λ_0 the multiplier of $\gamma(g_0)$. Let $\mathbf{g}^{\mathfrak{r}}_{0,-} \circ \mathbf{g}^{\mathfrak{s}}_{0,+} \colon \mathbf{H}^0(\mathbf{G}_0) \to \mathbf{H}^0(\mathbf{G}_0)$ be the first return map (compare with Lemma 6.2). There is a semiconjugacy $\mathbf{h} \colon \mathbf{H}^0(\mathbf{G}_0) \to \mathbb{D}^1$ from $\mathbf{g}^{\mathfrak{r}}_{0,-} \circ \mathbf{g}^{\mathfrak{s}}_{0,+}$ to the linear map $z \to \lambda_0 z$. Choose a continuous path of qc maps $\tau_t \colon \mathbb{D}^1 \to \mathbb{D}^1$ with $t \in [0,1]$ such that $\tau_0 = \text{id}$ and τ_t conjugates $z \to \lambda_0 z$ to $z \to \lambda_t z$.

Lifting τ_t under $\mathbf{h}$ and spreading dynamically the associated Beltrami form, we obtain a qc map $\boldsymbol{\tau}_t \colon \mathbb{C} \to \mathbb{C}$ conjugating $\mathbf{G}_0$ to a maximal prepacman $\mathbf{G}_{0,t}$; similarly $\boldsymbol{\tau}_t$ conjugates $\mathbf{G}^\#_{n,t}$ to a maximal prepacman $\mathbf{G}^\#_{n,t}$ for $n \le 0$. Note that $\boldsymbol{\tau}_t$, as well as $\mathbf{G}^\#_{n,t}$, is defined up to affine rescaling. We can assume that $\boldsymbol{\tau}_t$ is a continuous path starting at $\text{id} = \boldsymbol{\tau}_0$.

Define now $\tau_{n,t}$ to be the projection of $\boldsymbol{\tau}$ to the dynamical plane of g_n via $\operatorname{int} \mathbf{S}^\#_k \simeq V \setminus \gamma_1$ (see (5.6)); recall that the last identification is a composition (see (5.6) and (5.3)) of the rescaling of $\mathbf{S}_k^{\#}$ under $z \to \mu_\star^n z$ and the identification $\operatorname{int} \mathbf{S}_k \simeq V \setminus \gamma_1$. Since $|\mu_\star| < 1$ the family $\tau_{n,t}$ is a equicontinuous on n. By construction, $\tau_{n,t}$ conjugates g_n to a pacman $g_{n,t}$.

Since the family $\tau_{n,t}$ is equicontinuous on n, there is a small $T > 0$ such that all $g_{n,t}$ are in $\mathcal{B}$ for $t \le T$. For $m \le 0$ consider the sequence $\mathcal{R}^{-n+m}(g_{n,t})$. All pacmen in this sequence are qc-conjugate with uniform dilatation. By compactness of qc-maps, $\mathcal{R}^{-n+m}(g_{n,t})$ has an accumulated point $q_{m,t} \in \mathcal{B}$, and moreover, we can assume that $\mathcal{R} q_{m,t} = q_{m+1,t}$; i.e. $q_{m,t} \in \mathcal{W}^u$ and $q_{m,t}$ tends to $f_\star$ as m tends to $-\infty$. We define $\mathbb{\Delta}(q_{n,t}), \mathbb{\Pi}(q_{n,t}), \mathbb{H}(q_{n,t})$ to be the images of $\mathbb{\Delta}(g_{n,t}), \mathbb{\Pi}(g_{n,t}), \mathbb{H}(g_{n,t})$ via the qc-conjugacy from $g_{n,t}$ to $q_{n,t}$. By improvement of the domain, $\mathbb{\Delta}(q_{n,t})$ is in a small neighborhood of $\overline{Z}_\star$ and $\mathbb{\Pi}(q_{n,t})$ approximates $\partial Z_\star$ for $n \ll 0$. By shifting the sequence $(\mathfrak{p}_n/\mathfrak{q}_n)_n$ we can assume that this already occurs for $n = 0$. We can now repeat the above argument and construct $q_{n,t}$ for $t \in [T, 2T]$. After finitely many repetitions, we construct $q_{n,t}$ for all t in $[0, 1]$.

8.0.3. *QC-surgery towards the center.* Suppose now $\lambda_1 = 0$. In this case we apply a qc-surgery. As in §8.0.2 we denote by λ_0 the multiplier of $\gamma(g_0)$.

Consider the first return map

$$\mathbf{w}_0 \coloneqq \mathbf{g}_{0,-}^{\mathfrak{r}} \circ \mathbf{g}_{0,+}^{\mathfrak{s}} \colon \mathbf{H}^0(\mathbf{G}_0) \to \mathbf{H}^0(\mathbf{G}_0).$$

It has a unique attracting fixed point $\boldsymbol{\gamma}^0$ and a unique critical value at 0. Thus $\mathbf{w}_0$ has also a unique critical point. We can choose a small disk $\mathbf{D}$ around $\boldsymbol{\gamma}^0$ such that

- $0 \in \mathbf{w}_0(\mathbf{D}) \Subset \mathbf{D}$;
- $\mathbf{w}_0 \colon \mathbf{H}^0(\mathbf{G}_0) \setminus \mathbf{D} \to \mathbf{H}^0(\mathbf{G}_0) \setminus \mathbf{w}_0(\mathbf{D})$ is 2-to-1 covering map.

By Theorem 6.9, we can project $\mathbf{D}$ to a disk within $\mathbb{H}(g_0)$. We claim that there is a continuous path of qc maps $\boldsymbol{\tau}'_t \colon \mathbf{H}^0(\mathbf{G}_0) \to \mathbf{H}^0(\mathbf{G}_0)$ and a continuous path $\mathbf{w}_t \colon \mathbf{H}^0(\mathbf{G}_0) \to \mathbf{H}^0(\mathbf{G}_0)$ such that

- $\boldsymbol{\tau}'_t$ is equivariant on $\mathbf{H}^0(\mathbf{G}_0) \setminus \mathbf{D}$;
- $\mathbf{w}_t$ has a unique critical value at 0 and a unique attracting fixed point at $\boldsymbol{\gamma}_{0,t}$;
- $\boldsymbol{\gamma}_{0,1} = 0$; i.e. 0 is supperattracting fixed point of $\mathbf{w}_1$.

Indeed, it is sufficient to construct $\mathbf{w}_t \mid \mathbf{D}$ and $\boldsymbol{\tau}'_t \mid \mathbf{D}$ equivariant on $\partial \mathbf{D}$; pulling back the Beltrami differential of $\boldsymbol{\tau}'_t \mid \mathbf{D}$ via the covering map $\mathbf{w}_0 \mid \mathbf{H}^0(\mathbf{G}_0) \setminus \mathbf{D}$ gives the Beltrami differential for $\boldsymbol{\tau}'_t \mid \mathbf{H}^0(\mathbf{G}_0)$.

Applying $\mathbf{G}_0$, we spread dynamically the Beltrami form of $\boldsymbol{\tau}'$ to obtain a global qc map $\boldsymbol{\tau}_t \colon \mathbb{C} \to \mathbb{C}$ which is unique up to affine rescaling. Spreading dynamically the surgery, we obtain a continuous path of maximal prepacmen $\mathbf{G}_{n,t}^{\#}$. Define now $\tau_{n,t}$ to be the projection of $\boldsymbol{\tau}_t$ to the dynamical plane of g_n via $\operatorname{int} \mathbf{S}_n^{\#} \simeq V \setminus \gamma_1$; similarly, $g_{n,t}$ is the projection of $\mathbf{G}_{n,t}^{\#}$. The argument now continues in the same way as in §8.0.2.

8.0.4. *Lamination around $f_\star$.* In §8.0.1, §8.0.2, §8.0.3 we constructed continuous paths $q_{n,t} \in \mathcal{W}^u$, $n \ll 0$ with $\mathcal{R}(q_{n,t}) = q_{n+1,t}$ so that each $q_{n,t}$ has a valuable flower $\mathbb{H}(q_{n,t})$ with multiplier λ_t, where $\lambda_0 = 1$. Moreover, $\mathbb{H}(q_{n,t})$ is within a triangulation $\mathbb{\Delta}(q_n)$ respecting $\mathbb{H}(q_{n,t})$ such that the wall $\mathbb{H}(f_n)$ approximate $\partial Z_\star$.

For a big $m \ll 0$, we define $\mathcal{F}_{m,t}$ to be the set of all pacmen close to $q_{m,t}$ such that the multiplier of $\gamma(q_{m,t})$ is λ_t. Locally $(\mathcal{F}_{m,t})_t$ is a codimension-one lamination of $\mathcal{B}$. Since $\mathcal{F}_{m,t}$ is in a small neighborhood of $q_{m,t}$, every pacman $g \in \mathcal{F}_{m,t}$ has a valuable flower $\mathbb{H}(g)$ and a triangulation $\mathbb{A}(g)$ respecting $\mathbb{H}(g)$ such that $\mathbb{A}(g)$ and $\mathbb{H}(g)$ depend continuously on g. The wall $\sqcap(g)$ approximates $\partial Z_\star$.

For $n \le m$, we define

$$\mathcal{F}_{n,t} \coloneqq \{g \in \mathcal{B} \mid \mathcal{R}^{m-n}(g) \in \mathcal{F}_{m,t}\}.$$

Since $\mathcal{R}$ is hyperbolic,

$$\boldsymbol{\mathcal{F}} \coloneqq \{\mathcal{F}_{n,t}\}_{n,t} \cup \{\mathcal{W}^s\} \tag{8.2}$$

forms a codimension-one lamination in a neighborhood of $f_\star$. A pacman $g \in \mathcal{F}_{n,t}$ has $\mathbb{H}(g)$ and $\mathbb{A}(g)$ satisfying the same conditions as above. In particular, all the pacmen in $\mathcal{F}_{n,t}$ are hybrid conjugate in neighborhoods of their valuable flowers.

8.0.5. *Scaling.* By Corollary 3.7, the Siegel map f can be renormalized to a pacman. By Lemma 7.8 we can assume that the renormalization of f is within a small neighborhood of $f_\star$. This allows us to define an analytic renormalization operator $\mathcal{R}_2 \colon \mathcal{U} \dashrightarrow \mathcal{B}$ from a small neighborhood of f to a small neighborhood of $f_\star$. Since maps in $\mathcal{W}$ have different multipliers, the image of $\mathcal{W}$ under $\mathcal{R}_2$ is transverse to the lamination $\boldsymbol{\mathcal{F}}$, see (8.2).

We define $f_{n,t}$ to be the unique intersection of $\mathcal{F}_{n,t}$ with the image of $\mathcal{W}$ under $\mathcal{R}_2$, and we define $g_{n,t} \in \mathcal{W}$ to be the preimage of $f_{n,t}$ via $\mathcal{R}_2$. Since $\sqcap(f_{n,t})$ approximates $\partial Z_\star$, the triangulation $\mathbb{A}(f_{n,t})$ and the valuable flower $\mathbb{H}(f_{n,t})$ have full lifts $\mathbb{A}(g_{n,t})$ and $\mathbb{H}(g_{n,t})$, see Lemma 4.4. Since the holonomy along $\boldsymbol{\mathcal{F}}$ is asymptotically conformal [L1, Appendix 2, The λ-lemma (quasi- conformality)], we obtain the scaling result for $g_{n,t}$.

8.0.6. *Uniqueness of $g_{n,t}$.* Recall (Theorem 7.7) that $\mathcal{W}^u$ is parametrized by the multipliers of the α-fixed points. Therefore, parabolic pacmen with rotation numbers $\mathfrak{p}_n/\mathfrak{q}_n,\ \ n \ll 0$ are unique. As a consequence the paths of satellite pacmen emerging from these parabolic pacmen are unique. Similarly, parabolic maps $g_{n,0} \in W$ with rotation numbers $\mathfrak{p}_n/\mathfrak{q}_n$ are unique; thus the paths $g_{n,t}$ are unique. □

Appendix A. Sector renormalizations of a rotation

Consider $\theta \in \mathbb{R}/\mathbb{Z}$ and let

$$\mathbb{L}_\theta \colon : \overline{\mathbb{D}^1} \to \overline{\mathbb{D}^1}, \quad z \to \mathbf{e}(\theta) z$$

be the corresponding rotation of the closed unit disk by angle θ.

A.1. **Prime renormalization of a rotation.** Assume that $\theta \neq 0$ and consider a closed internal ray $\mathbb{I}$ of $\overline{\mathbb{D}^1}$. A *fundamental sector* $\mathbb{Y} \subset \overline{\mathbb{D}^1}$ of $\mathbb{L}_\theta$ is the smallest closed sector bounded by $\mathbb{I}$ and $\mathbb{L}_\theta(\mathbb{I})$. If $\theta = 1/2$, then $\mathbb{I} \cup \mathbb{L}_\theta(\mathbb{I})$ is a diameter and both sectors of $\overline{\mathbb{D}^1}$ bounded by $\mathbb{I} \cup \mathbb{L}_\theta(\mathbb{I})$ are fundamental. The angle ω at the vertex of $\mathbb{Y}$ is θ if $\theta \in [0, 1/2]$ or $1-\theta$ if $1-\theta \in [0,1/2]$.

A fundamental sector is defined uniquely up to rotation; let us first rotate it such that $1 \in \overline{\mathbb{D}^1} \setminus \mathbb{Y}$. Set $\mathbb{Y}_- \coloneqq \mathbb{L}_\theta^{-1}(\mathbb{Y})$ and set $\mathbb{Y}_+$ to be the closure of $\mathbb{D}^1 \setminus (\mathbb{Y} \cup \mathbb{Y}_-)$, see Figure 22. Then

$$(\mathbb{L}_\theta \mid \mathbb{Y}_+, \ \ \mathbb{L}_\theta^2 \mid \mathbb{Y}_-) \tag{A.1}$$

is the first return of points in $\mathbb{Y}_- \cup \mathbb{Y}_+$ back to $\mathbb{Y}_- \cup \mathbb{Y}_+$. The *prime renormalization of* $\mathbb{L}_\theta$ is the rotation $\mathbb{L}_{R_{\text{prm}}(\theta)} \colon \overline{\mathbb{D}^1} \to \overline{\mathbb{D}^1}$ obtained from (A.1) by applying the gluing map

$$\psi_{\text{prm}} \colon \mathbb{Y}_- \cup \mathbb{Y}_+ \to \mathbb{D}^1, \qquad z \to z^{1/(1-\omega)}.$$

Lemma A.1. *We have*

$$R_{\mathrm{prm}}(\theta) = \begin{cases} \frac{\theta}{1-\theta} & \text{if } 0 \le \theta \le \frac{1}{2}, \\ \frac{2\theta-1}{\theta} & \text{if } \frac{1}{2} \le \theta \le 1. \end{cases} \tag{A.2}$$

Present θ using continued fractions in the following ways

$$\theta = [0; a_1, a_2, \dots] = 1 - [0; b_1, b_2, \dots].$$

with $a_i, b_i \in \mathbb{N}_{>0}$. Then

$$R_{\mathrm{prm}}([0; a_1, a_2, \dots]) = \begin{cases} [0; a_1 - 1, a_2, \dots] & \text{if } a_1 > 1, \\ 1 - [0; a_2, a_3 \dots] & \text{if } a_1 = 1, \end{cases}$$

and

$$R_{\mathrm{prm}}(1 - [0; b_1, b_2, \dots]) = \begin{cases} 1 - [0; b_1 - 1, b_2, \dots] & \text{if } b_1 > 1, \\ [0; b_2, b_3 \dots] & \text{if } b_1 = 1. \end{cases}$$

As a consequence, θ is periodic under R_{prm} if and only if there is a θ' with periodic continued fraction expansion such that $\theta = R^n_{\mathrm{prm}}(\theta')$ for some $n \ge 0$.

Proof. Follows by routine calculations. If $\theta \in [0, 1/2]$, then projecting $z \to \mathbf{e}(\theta)z$ by ψ_{prm} we obtain

$$z \to \left(\mathbf{e}(\theta) z^{1-\theta}\right)^{1/1-\theta} = \mathbf{e}\left(\frac{\theta}{1-\theta}\right) z.$$

If $\theta \in [1/2, 1]$, then projecting $z \to \mathbf{e}(\theta - 1)z$ by ψ_{prm} we obtain

$$z \to \left(\mathbf{e}(\theta - 1) z^{\theta}\right)^{1/\theta} = \mathbf{e}\left(\frac{\theta - 1}{\theta}\right) z.$$

Observe that $\frac{\theta-1}{\theta} \in [-1, 0]$; adding $+1$ we obtain $\frac{2\theta-1}{\theta}$.

Write

$$\theta = \frac{1}{a_1 + [0; a_2, a_3, \dots]}.$$

and observe that $\theta \in [0, 1/2]$ if and only if $a_1 > 1$. (With the exception $\theta = [0; 1, 1]$.) If $a_1 > 1$, then

$$\frac{\theta}{1-\theta} = \frac{1}{a_1 + [0; a_2, a_3, \dots] - 1} = R_{\mathrm{prm}}(\theta).$$

If $a_1 = 1$, then

$$\frac{2\theta - 1}{\theta} = 2 - a_1 - [0; a_2, a_3, \dots] = R_{\mathrm{prm}}(\theta).$$

Similarly $R_{\mathrm{prm}}(1 - [0; b_1, b_2, \dots])$ is verified. □

A.2. **Sector renormalization.** A *sector renormalization* $\mathcal{R}$ of $\mathbb{L}_\theta$ is

- a *renormalization* sector $\mathbb{X}$ presented as a union of two subsectors $\mathbb{X}_- \cup \mathbb{X}_+$ normalized so that $1 \in \mathbb{X}_- \cap \mathbb{X}_+$;
- a pair of iterates, called a sector *pre-renormalization*,

$$\left(\mathbb{L}^{\mathfrak{a}}_\theta \mid \mathbb{X}_-, \quad \mathbb{L}^{\mathbf{b}}_\theta \mid \mathbb{X}_+\right) \tag{A.3}$$

  realizing the first return of points in $\mathbb{X}_- \cup \mathbb{X}_+$ back to $\mathbb{X}$; and

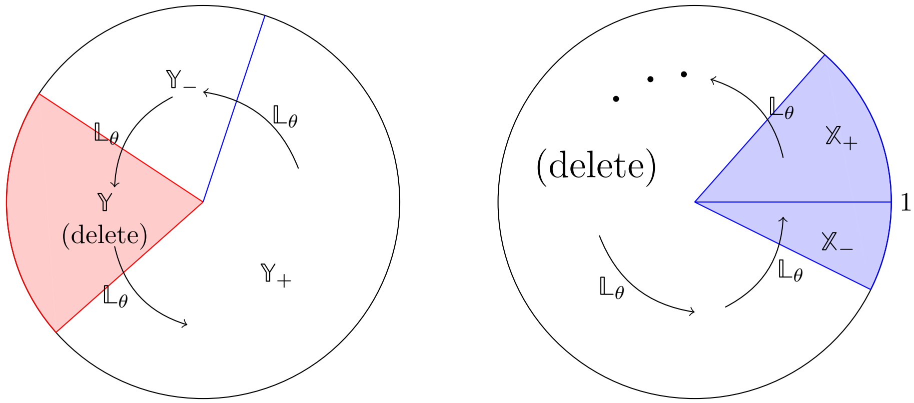


FIGURE 22. Left: the prime renormalization deletes a fundamental sector $\mathbb{Y}$ and projects $(\mathbb{L}_\theta^2 \mid \mathbb{Y}_- \,,\quad \mathbb{L}_\theta \mid \mathbb{Y}_+)$ to a new rotation. Right: $\left(\mathbb{L}_\theta^{\mathfrak{q}+1} \mid \mathbb{X}_- \,,\quad \mathbb{L}_\theta^{\mathfrak{q}} \mid \mathbb{X}_+\right)$ is the first return map to a fundamental sector $\mathbb{Y} = \mathbb{X}_- \cup \mathbb{X}_+$

.

- the gluing map
$$\psi\colon \mathbb{X}_- \cup \mathbb{X}_+ \to \overline{\mathbb{D}^1}, \qquad z \to z^{1/\omega},$$
projecting (A.3) to a new rotation $\mathbb{L}_\mu$, where ω is the angle of $\mathbb{X}$ at 0.

We write $\mathcal{R}\mathbb{L}_\theta = \mathbb{L}_\mu$, and we call $\mathfrak{a}$ and $\mathbf{b}$ the *renormalization return times.* We allow one of the sectors $\mathbb{X}_\pm$ to degenerate, but not both. Note that the assumption $1 \in \mathbb{X}_- \cap \mathbb{X}_+$; can always be achieved using rotation.

The prime renormalization is an example of a sector renormalization.

Suppose two sector renormalizations $\mathcal{R}_1(\mathbb{L}_\theta) = \mathbb{L}_\mu$ and $\mathcal{R}_2(\mathbb{L}_\mu) = \mathbb{L}_\nu$ are given. The *composition* $\mathcal{R}_2 \circ \mathcal{R}_1(\mathbb{L}_\theta) = \mathbb{L}_\nu$ is obtained by lifting the pre-renormalization of $\mathcal{R}_2$ to the dynamical plane of $\mathbb{L}_\theta$.

Lemma A.2. *A sector renormalization is an iteration of the prime renormalization.*

Proof. Suppose $\mathcal{R}$ is a sector renormalization with renormalization return times $\mathfrak{a}$ and $\mathbf{b}$ as above. Proceed by induction on $\mathfrak{a} + \mathbf{b}$. If $\mathfrak{a} + \mathbf{b} = 3$, then $\mathcal{R}$ is the prime renormalization. Otherwise, we project the pre-renormalization of $\mathcal{R}$ to the dynamical plane of $\mathcal{R}_{\text{prm}}(\mathbb{L}_\theta)$ and obtain the new sector renormalization $\mathcal{R}'$ of $\mathcal{R}_{\text{prm}}(\mathbb{L}_\theta)$ so that
$$\mathcal{R}' \circ \mathcal{R}_{\text{prm}}(\mathbb{L}_\theta) = \mathcal{R}(\mathbb{L}_\theta).$$
The renormalization return times a', b' of $\mathcal{R}'$ satisfy $a' + b' < a + b$. □

Consider again the fundamental sector $\mathbb{Y}$ bounded by $\mathbb{I}$ and $\mathbb{L}_\theta(\mathbb{I})$. There is a unique $\mathfrak{a} > 0$ such that $\mathbb{L}^{-\mathfrak{a}}(\mathbb{I}) \subset \mathbb{Y}$. Up to rotation, we can assume that $\mathbb{L}^{-\mathfrak{a}}(\mathbb{I})$ lands at 1. We define $\mathbb{X}_+$ to be the subsector of $\mathbb{Y}$ bounded $\mathbb{I}$ and $\mathbb{L}^{-\mathfrak{a}}(\mathbb{I})$ and we define $\mathbb{X}_-$ to be the subsector of $\mathbb{Y}$ bounded $\mathbb{L}(\mathbb{I})$ and $\mathbb{L}^{-\mathfrak{a}}(\mathbb{I})$. Then
$$\left(\mathbb{L}_\theta^{\mathfrak{a}} \mid \mathbb{X}_-, \quad \mathbb{L}_\theta^{\mathfrak{a}+1} \mid \mathbb{X}_+\right)$$

is a sector pre-renormalization, called the *first return to the fundamental sector*, see Figure 22. We denote by $\mathcal{R}_{\text{fast}}$ the associated sector renormalization and we write $\mu = R_{\text{fast}}(\theta)$ if $\mathcal{R}_{\text{fast}}(\mathbb{L}_\theta) = \mathbb{L}_\mu$.

By Lemma A.2, for every $\theta \neq 0$ there is a unique $\mathfrak{n}(\theta)$ such that $R_{\text{fast}}(\theta) = R_{\text{prm}}^{\mathfrak{n}(\theta)}(\theta)$. We note that if $\theta \in \{1/m, 1-1/m\}$ with $m > 1$, then $\mathfrak{n}(\theta) = m-1$. (In this case the sector $\mathbb{X}_-$ is degenerate.)

A.3. **Renormalization triangulation.** Given a sector pre-renormalization (A.3), the set of sectors

$$\bigcup_{i=0}^{\mathfrak{a}-1} \mathbb{L}_\theta(\mathbb{X}_-) \bigcup_{i=0}^{\mathbf{b}-1} \mathbb{L}_\theta(\mathbb{X}_+)$$

is called a *renormalization triangulation* of $\mathbb{D}^1$. Alternatively, consider the associated renormalization $\mathbb{L}_\mu = \mathcal{R}(\mathbb{L}_\theta)$. The internal rays towards 1 and $\mathbb{L}_\mu(1)$ split $\mathbb{D}^1$ into two closed sectors $\mathbb{T}_0$ and $\mathbb{T}_1$. We call $\{\mathbb{T}_-, \mathbb{T}_+\}$ the *basic triangulation of* $\mathbb{L}_\mu$. Lifting the sectors $\mathbb{T}_-, \mathbb{T}_+$ via the gluing map, and spreading them dynamically we obtain the renormalization triangulation. We also say that the renormalization triangulation is the *full lift* of the basic triangulation.

Let Θ_N be the set of angles θ such that $\theta = [0; a_1, a_2, \dots]$ with $|a_i| \le N$ or $\theta = 1 - [0; a_1, a_2, \dots]$ with $|a_i| \le N$. By Lemmas A.1 and A.2, the set Θ_N is invariant under any sector renormalization.

Lemma A.3. *For every N there is a $t > 1$ with the following property. Consider the renormalization triangulation associated with some sector renormalization of $\mathbb{L}_\theta$, where $\theta \in \Theta_N$. Then any two triangles have comparable angles at 0: the ratio of the angles is between $1/t$ and t.*

Proof. There is a neighborhood U of 1 such that for all $\theta \in \Theta_N$ we have $\mathbb{L}_\theta(1) \notin U$. Therefore, both sectors in the basic triangulation have comparable angles at 0 uniformly on $\theta \in \Theta_N$. Since a renormalization triangulation is the full lift of a basic triangulation, the lemma is proven. □

A.4. **Periodic case.** It follows from Lemmas A.1 and A.2 that $\mathbb{L}_\theta$ is a fixed point of some sector renormalization if and only if $\theta \in \Theta_{\text{per}}$. Suppose $\theta \in \Theta_{\text{per}}$ and choose a sector renormalization $\mathcal{R}_1$ such that $\mathcal{R}_1(\mathbb{L}_\theta) = \mathbb{L}_\theta$. Write $\mathcal{R}_n \coloneqq \mathcal{R}_1^n$ and denote by $\mathfrak{a}_n, \mathbf{b}_n$ and ψ_n the renormalization return times and the gluing map of $\mathcal{R}_n$. Then $\psi_n = \psi_1^n$ and there is a matrix $\mathbb{M}$ with positive entries such that

$$\begin{pmatrix} \mathfrak{a}_n \\ \mathbf{b}_n \end{pmatrix} = \mathbb{M}^{n-1} \begin{pmatrix} \mathfrak{a}_1 \\ \mathbf{b}_1 \end{pmatrix}. \tag{A.4}$$

As a consequence, $\mathfrak{a}_n, \mathbf{b}_n$ have exponentially growth with the same exponent.

We also note that

$$\mathfrak{a}_1, \mathbf{b}_1 \ge 2 \tag{A.5}$$

because $\mathcal{R}_1 = \mathcal{R}_{\text{prm}}^t$ with $t > 1$.

Appendix B. Lifting of curves under anti-renormalization

In this appendix we give a sufficient condition for liftability of arcs under a sector anti-renormalization. This implies that the sector antirenormalization is robust with respect to a particular choice of cutting arcs, see Theorem B.8.

B.1. **Robustness of anti-renormalization.** Consider a closed pointed topological disk $(W,0)$ and let U,V be two closed topological subdisks of W such that $0\in \operatorname{int}(U\cap V)$. A homeomorphism $f\colon U\to V$ fixing 0 is called a *partial homeomorphism* of $(W,0)$ and is denoted by $f\colon W \dashrightarrow W$ or $f\colon (W,0)\dashrightarrow (W,0)$. If $U=V=W$, then f is an actual self-homeomorphism of $(W,0)$.

B.1.1. *Leaves over* $\boldsymbol{f\colon (W,0)\dashrightarrow (W,0)}$. Let γ_0,γ_1 be two simple arcs connecting 0 to points in ∂W such that γ_0 and γ_1 are disjoint except for 0 and such that γ_1 is the image of γ_0 in the following sense: $\gamma_0' \coloneqq \gamma_0\cap U$ and $\gamma_1'\coloneqq \gamma_1\cap V$ are simple closed curves such that f maps γ_0' to γ_1'. Such pair γ_0,γ_1 is called *dividing*. Then $\gamma_0\cup\gamma_1$ splits W into two closed sectors $\mathbf{A}$ and $\mathbf{B}$ denoted so that $\operatorname{int}\mathbf{A},\gamma_1,\operatorname{int}\mathbf{B},\gamma_0$ are clockwise oriented around 0, see the left-hand of Figure 23. We say that $\gamma_0=\ell(\mathbf{A})=\rho(\mathbf{B})$ is the *left boundary of* $\mathbf{A}$ and the *right boundary of* $\mathbf{B}$ and we say that $\gamma_1=\rho(\mathbf{A})=\ell(\mathbf{B})$ is the *right boundary of* $\mathbf{A}$ and *the left boundary of* $\mathbf{B}$.

Let X,Y be topological spaces and let $g\colon X\dashrightarrow Y$ be a partially defined continuous map. We define

$$X\sqcup_g Y \coloneqq X\sqcup Y/(\operatorname{Dom} g\ni x\sim g(x)\in \operatorname{Im} g).$$

Consider a (finite or infinite) sequence $(S_k)_k$, where each S_k is a copy of either $\mathbf{A}$ or $\mathbf{B}$. Define the partial map $g_k\colon \rho(S_k)\dashrightarrow \ell(S_{k+1})$ by

$$g_k \coloneqq \begin{cases} \mathrm{id}\colon \gamma_1'\to\gamma_1' & \text{if } (S_k,S_{k+1})\cong(\mathbf{A},\mathbf{B}),\\ \mathrm{id}\colon \gamma_0'\to\gamma_0' & \text{if } (S_k,S_{k+1})\cong(\mathbf{B},\mathbf{A}),\\ f^{-1}\colon \gamma_1'\to\gamma_0' & \text{if } (S_k,S_{k+1})\cong(\mathbf{A},\mathbf{A}),\\ f\colon \gamma_0'\to\gamma_1' & \text{if } (S_k,S_{k+1})\cong(\mathbf{B},\mathbf{B}).\end{cases} \tag{B.1}$$

The *dynamical gluing of* $(S_k)_k$ is

$$\dots S_{k-1}\sqcup_{g_{k-1}} S_k\sqcup_{g_k} S_{k+1}\sqcup_{g_{k+1}}\dots.$$

The *jump* $\iota(k)$ *from* S_k *to* S_{k+1} is set to be $0,0,-1,1$ if (S_k,S_{k+1}) is a copy of $(\mathbf{A},\mathbf{B})$, $(\mathbf{B},\mathbf{A})$, $(\mathbf{A},\mathbf{A})$, $(\mathbf{B},\mathbf{B})$ respectively.

For a sequence $\mathbf{s}=(a_i)_{i\in I}$ we denote by $\mathbf{s}[i]$ the i-th element in $\mathbf{s}$; i.e. $\mathbf{s}[i]=a_i$.

Definition B.1 (Leaves of $f\colon W\dashrightarrow W$)**.** Suppose $\mathbf{s}\in\{\mathbf{A},\mathbf{B}\}^{\mathbb{Z}}$. Set $W_{\mathbf{s}}[i]$ to be a copy of the closed sector $\mathbf{s}[i]$. The *leaf* $W_{\mathbf{s}}$ is the surface obtained by the dynamical gluing of $(W_{\mathbf{s}}[i])_{i\in\mathbb{Z}}$.

The *projection* $\pi\colon W_{\mathbf{s}}\to W$ maps each $\operatorname{int}W_{\mathbf{s}}[i]\cup\rho(W_{\mathbf{s}}[i])$ to $\operatorname{int}\mathbf{s}[i]\cup\rho(\mathbf{s}[i])$. By $\pi^{-1}_{\mathbf{s},i}\colon \operatorname{int}\mathbf{s}[i]\cup\rho(\mathbf{s}[i])\to\operatorname{int}W_{\mathbf{s}}[i]\cup\rho(W_{\mathbf{s}}[i])$ we denote the corresponding inverse branch.

Note that if $\mathbf{s}[i]=\mathbf{s}[i+1]$, then π is discontinuous at $W_{\mathbf{s}}[i]\cap W_{\mathbf{s}}[i+1]$. As z approaches $W_{\mathbf{s}}[i]\cap W_{\mathbf{s}}[i+1]$ from $\operatorname{int}W_{\mathbf{s}}[i]$, respectively $\operatorname{int}W_{\mathbf{s}}[i+1]$, its image $\pi(z)$ approaches $\rho(\mathbf{s}[i])$, respectively $\ell(\mathbf{s}[i+1])\neq\rho(\mathbf{s}[i])$.

For every $\mathbf{s}$, there is a unique point $\widetilde{0}\in W_{\mathbf{s}}$ such that $\pi(\widetilde{0})=0$. By construction, $W_{\mathbf{s}}\setminus\{\widetilde{0}\}$ is topologically a closed half-plane.

For $J\subseteq\mathbb{Z}$ we write $W_{\mathbf{s}}[J]=\bigcup_{j\in J}W_{\mathbf{s}}[j]$. To keep notation simple, we write $W_{\mathbf{s}}[\geq i]=W_{\mathbf{s}}[\{k\mid k\geq i\}]$ and similar for “$>$”,“$\leq$”, “$<$”.

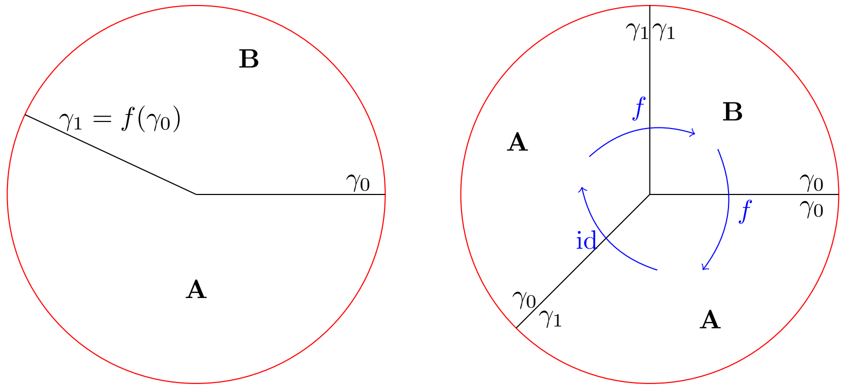


FIGURE 23. Left: a homeomorphism $f\colon W \to W$ and a diving pair γ_0, γ_1. Right: the $1/3$ anti-renormalization of f (with respect to the clockwise orientation).

B.1.2. *Lifts of curves.* Let $\alpha\colon [0,1] \to W \setminus \{0\}$ be a curve in W. A *lift of α to $W_{\mathbf{s}}$* is a curve $\widetilde{\alpha}\colon [0,1] \to W_{\mathbf{s}}$ such that

- for every $t \in [0,1]$, there is an $n(t) \in \mathbb{Z}$ such that $\pi(\widetilde{\alpha}(t)) = f^{n(t)}(\alpha(t))$;
- $n(0) = 0$;
- $n(t)$ is constant for all t for which $\widetilde{\alpha}(t)$ is within some $\operatorname{int} W_{\mathbf{s}}[i]$ or its right boundary; and
- if $\widetilde{\alpha}(t') \in \operatorname{int} W_{\mathbf{s}}[i]$ while $\widetilde{\alpha}(t) \in \operatorname{int} W_{\mathbf{s}}[i+1]$, then $n(t) - n(t')$ is equal to the jump from $W_{\mathbf{s}}[i]$ to $W_{\mathbf{s}}[i+1]$.

In other words, whenever α crosses the boundary of $\mathbf{s}[i]$, the lift of α is adjusted to respect the dynamical gluing. Similarly is defined a lift of a curve parametrized by $[0,1)$. Note that $\pi(\widetilde{\alpha})$ is, in general, discontinuous.

For every curve α as above, there is at most one lift of α starting at a given preimage of $\alpha(0)$ under $\pi\colon W_{\mathbf{s}} \to W$. It is easy to see that there is an $\varepsilon > 0$ such that all lifts (specified by starting points) of $\alpha\colon [0,\varepsilon] \to W$ exist, and thus unique. The main question we address is the existence of global lifts.

If $\alpha\colon [0,1) \to W \setminus \{0\}$ is such that $\alpha(1) = \lim_{t\to 1} \alpha(t) = 0$, then we say that a lift $\widetilde{\alpha}$ of α *lands at* $\widetilde{0}$ if $\pi(\widetilde{\alpha}(t)) \to 0$ as $t \to 1$.

B.1.3. *Anti-renormalizations.*

Lemma B.2. *For every $\mathfrak{q} \in \mathbb{N}_{>2}$ and every $\mathfrak{p} \in \{1, 2, \dots, \mathfrak{q}-1\}$ coprime with $\mathfrak{q}$ there exists a unique $\mathfrak{q}$-periodic sequence $\mathbf{s} \in \{\mathbf{A}, \mathbf{B}\}^{\mathbb{Z}}$ such that*

- *$\mathbf{s}[0] = \mathbf{A}$ and $\mathbf{s}[-1] = \mathbf{B}$;*
- *for every $j \in \mathbb{Z}$ with $(j \bmod \mathfrak{q}) \notin \{-\mathfrak{p}, -\mathfrak{p}-1\}$, we have $\mathbf{s}[j+\mathfrak{p}] = \mathbf{s}[j]$;*
- *$\mathbf{s}[-\mathfrak{p}-1] = \mathbf{A}$ and $\mathbf{s}[-\mathfrak{p}] = \mathbf{B}$.*

Since $\mathfrak{p}$ and $\mathfrak{q}$ are coprime, there are unique $\mathfrak{a}, \mathfrak{b} \in \{1, 2, \dots, \mathfrak{q}-1\}$, called the *renormalization return times*, such that

$$\begin{aligned} \mathfrak{p}\mathfrak{a} &= -1 \mod \mathfrak{q}, \\ \mathfrak{p}\mathfrak{b} &= 1 \mod \mathfrak{q}. \end{aligned}$$

Note that $\mathfrak{a} + \mathfrak{b} = \mathfrak{q}$.

Proof of Lemma B.2. We have:

- $\mathbf{s}[i\mathfrak{p}+j\mathfrak{q}] = \mathbf{A}$ for all $i \in \{0,1,\dots,\mathfrak{a}-1\}$ and all $j \in \mathbb{Z}$; and
- $\mathbf{s}[-1+i\mathfrak{p}+j\mathfrak{q}] = \mathbf{B}$ for all $i \in \{0,\dots,\mathbf{b}-1\}$ and all $j \in \mathbb{Z}$.

□

For a sequence $\mathbf{s}$ as in Lemma B.2, let

$$\mathbf{s}/\mathfrak{q} \in \{\mathbf{A},\mathbf{B}\}^{\mathbb{Z}/\mathfrak{q}\mathbb{Z}}, \qquad (\mathbf{s}/\mathfrak{q})[i] := \mathbf{s}[i+\mathbb{Z}\mathfrak{q}]$$

be the quotient sequence, and let $W_{\mathbf{s}/\mathfrak{q}}$ be the quotient of the leaf $W_{\mathbf{s}}$ by identifying each $W_{\mathbf{s}}[k]$ with $W_{\mathbf{s}}[k+\mathfrak{q}]$. We denote by $\pi\colon W_{\mathbf{s}/\mathfrak{q}} \to W$ the natural projection. Then the $\mathfrak{p}/\mathfrak{q}$-*anti-renormalization* $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}} \dashrightarrow W_{\mathbf{s}/\mathfrak{q}}$ is defined as follows (see Figure 23):

- for every $j \notin \{-\mathfrak{p}-1,-\mathfrak{p}\}$, the map $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}}[j] \to W_{\mathbf{s}/\mathfrak{q}}[j+\mathfrak{p}]$ is the natural isomorphism;
- the map $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}}[-\mathfrak{p}-1,-\mathfrak{p}] \dashrightarrow W_{\mathbf{s}/\mathfrak{q}}[-1,0]$ is $f\colon W\setminus\gamma_0 \dashrightarrow W\setminus\gamma_1$.

Note that $\mathfrak{p}/\mathfrak{q}$ is the clockwise rotation number.

By construction, $(f_{-1}^{\mathfrak{a}} \mid W_{\mathbf{s}/\mathfrak{q}}[0],\ \ f_{-1}^{\mathbf{b}} \mid W_{\mathbf{s}/\mathfrak{q}}[-1])$ is the first return of f_{-1} back to $W_{\mathbf{s}/\mathfrak{q}}[-1,0]$. After appropriate gluing of arcs in $\partial\big(W_{\mathbf{s}/\mathfrak{q}}[-1,0]\big)$, the map $(f_{-1}^{\mathfrak{a}} \mid W_{\mathbf{s}/\mathfrak{q}}[0],\ \ f_{-1}^{\mathbf{b}} \mid W_{\mathbf{s}/\mathfrak{q}}[-1])$ is $f\colon W \dashrightarrow W$.

Denote by

$$\gamma_0^{\mathbf{s}/\mathfrak{q}}, \qquad \gamma_p^{\mathbf{s}/\mathfrak{q}} \tag{B.2}$$

the left boundaries of $W_{\mathbf{s}/\mathfrak{q}}[0]$ and $W_{\mathbf{s}/\mathfrak{q}}[p]$ respectively. Then $\gamma_0^{\mathbf{s}/\mathfrak{q}}, \gamma_p^{\mathbf{s}/\mathfrak{q}}$ is a dividing pair for $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}} \dashrightarrow W_{\mathbf{s}/\mathfrak{q}}$ and the anti-renormalization procedure can be iterated.

Let β be a curve in W and let $\widetilde{\beta}$ be a lift of β to $W_{\mathbf{s}}$. The image of $\widetilde{\beta}$ in $W_{\mathbf{s}/\mathfrak{q}} \simeq W_{\mathbf{s}}/\sim$ is called a *lift of β to $W_{\mathbf{s}/\mathfrak{q}}$*. For example, $\gamma_0^{\mathbf{s}/\mathfrak{q}}$ is a lift of γ_0.

B.1.4. *Prime anti-renormalization.* The $1/3$ and $2/3$-anti-renormalizations are called *prime*. It is easy to check that

- if $\mathfrak{p}/\mathfrak{q} = 1/3$, then $\mathbf{s}/\mathfrak{q} = (\mathbf{A},\mathbf{A},\mathbf{B})$;
- if $\mathfrak{p}/\mathfrak{q} = 2/3$, then $\mathbf{s}/\mathfrak{q} = (\mathbf{A},\mathbf{B},\mathbf{B})$.

Lemma B.3 (Compare with Lemma A.2)**.** *An anti-renormalization is an iteration of prime anti-renormalizations.*

Proof. We proceed by induction on $\mathfrak{q}$. Assume $\mathfrak{q} > 3$, define $\mathfrak{p}'/\mathfrak{q}' := R_{\text{prm}}(\mathfrak{p}/\mathfrak{q})$ (see (A.2)), and observe that $\mathfrak{q}' < \mathfrak{q}$.

- If $0 < \mathfrak{p} < \mathfrak{q}/2$, then the $\mathfrak{p}/\mathfrak{q}$-anti-renormalization is the $1/3$-anti-renormalization of the $\mathfrak{p}'/\mathfrak{q}'$-anti-renormalization.
- If $\mathfrak{q}/2 < \mathfrak{p} < \mathfrak{q}$, then the $\mathfrak{p}/\mathfrak{q}$ anti-renormalization is the $2/3$-anti-renormalization of the $\mathfrak{p}'/\mathfrak{q}'$-anti-renormalization.

□

Denote by $\mathbf{s}_\bullet := (\mathbf{A},\mathbf{B})^{\mathbb{Z}}$ the sequence in $\{\mathbf{A},\mathbf{B}\}^{\mathbb{Z}}$ with even entries equal to $\mathbf{A}$ and odd entries equal to $\mathbf{B}$. Simplifying notations, we write $W_{\mathbf{s}_\bullet} = W_\bullet$.

Suppose that $f\colon W \to W$ is a homeomorphism. Then

$$\pi\colon W_\bullet \setminus \{\widetilde{0}\} \to W \setminus \{0\} \tag{B.3}$$

is a universal cover. Let $(\widetilde{\gamma}_i^\bullet \subset W_\bullet)_{i\in\mathbb{Z}}$ be all of the lifts of γ_0 and γ_1 enumerated from left to right such that $W_\bullet[i]$ is between $\widetilde{\gamma}_i^\bullet$ and $\widetilde{\gamma}_{i+1}^\bullet$; in particular, $\widetilde{\gamma}_i^\bullet$ is a lift of $\gamma_{i \bmod 2}$. Let

$$f_- \ , \ f_+ \colon W_\bullet \to W_\bullet$$

be the lifts of $f\colon W \to W$ specified so that

$$f_-(\widetilde{\gamma}_{2i}^\bullet) = \widetilde{\gamma}_{2i-1}^\bullet \quad \text{and} \quad f_+(\widetilde{\gamma}_{2i}^\bullet) = \widetilde{\gamma}_{2i+1}^\bullet.$$

Observe also that f_- and f_+ commute. Write

$$\tau \coloneqq f_-^{-1} \circ f_+ \colon W_\bullet \to W_\bullet;$$

then $\tau \mid (W_\bullet \setminus \{\widetilde{0}\})$ is a deck transformation of $\widetilde{W}$ and we can rewrite (B.3) as

$$W \setminus \{0\} \simeq \left(W_\bullet \setminus \{\widetilde{0}\}\right)/\langle\tau\rangle.$$

We also write $W \simeq W_\bullet/\langle\tau\rangle$.

Lemma B.4 (The 1/3-anti-renormalization)**.** *Suppose $f\colon W \to W$ is a self-homeomorphism. Set*

$$\begin{aligned} f_{-1,+} \coloneqq& f_+, \\ f_{-1,-} \coloneqq& \tau^{-1} = f_- \circ f_+^{-1}, \\ \tau_{-1} \coloneqq& f_{-1,-}^{-1} \circ f_{-1,+} = f_-^{-1} \circ f_+^2, \end{aligned}$$

Then τ_{-1} acts properly discontinuously on $W_\bullet \setminus \{\widetilde{0}\}$; we view

$$\left(W_\bullet \setminus \{\widetilde{0}\}\right)/\langle\tau_{-1}\rangle$$

as a punctured closed topological disk and we view $W_\bullet/\langle\tau_{-1}\rangle$ as a closed topological disk.

Let $f_{-1}\colon W_{-1} \to W_{-1}$ be the 1/3*-anti-renormalization of f. Then f_{-1} is conjugate to*

$$f_{-1,-}/\langle\tau_{-1}\rangle = f_{-1,+}/\langle\tau_{-1}\rangle \colon W_\bullet/\langle\tau_{-1}\rangle \to W_\bullet/\langle\tau_{-1}\rangle$$

by the conjugacy

$$h\colon W_{-1} \to W_\bullet[0,1,2]/\langle\tau_{-1}\rangle$$

mapping

$$W_{-1}[0], \quad W_{-1}[1], \quad W_{-1}[2]$$

respectively to

$$W_\bullet[2]/\langle\tau_{-1}\rangle, \quad W_\bullet[0]/\langle\tau_{-1}\rangle, \quad W_\bullet[1]/\langle\tau_{-1}\rangle$$

which are copies of $\mathbf{A}, \mathbf{A}, \mathbf{B}$.

Remark B.5. *The conjugacy h is uniquely characterized by the following properties:*

- *h maps $\widetilde{\gamma}_2^\bullet/\langle\tau\rangle$ to $\gamma_0^{\mathtt{s/q}}$ (see* (B.2)*);*
- *if $\ell \subset W \setminus \{0\}$ is a curve starting at γ_0 and $\widetilde{\ell} \subset W_\bullet$ is the unique lift of ℓ starting at $\widetilde{\gamma}_2^\bullet$, then $h(\widetilde{\ell}/\langle\tau_{-1}\rangle) \subset W_{-1}$ is the unique lift of ℓ starting at $\gamma_0^{\mathtt{s/q}}$.*

Proof of Lemma B.4. Clearly, $W_\bullet[0,1,2]$ is a fundamental domain for τ_{-1}. It is easy to see (see Figure 24) that h identifies

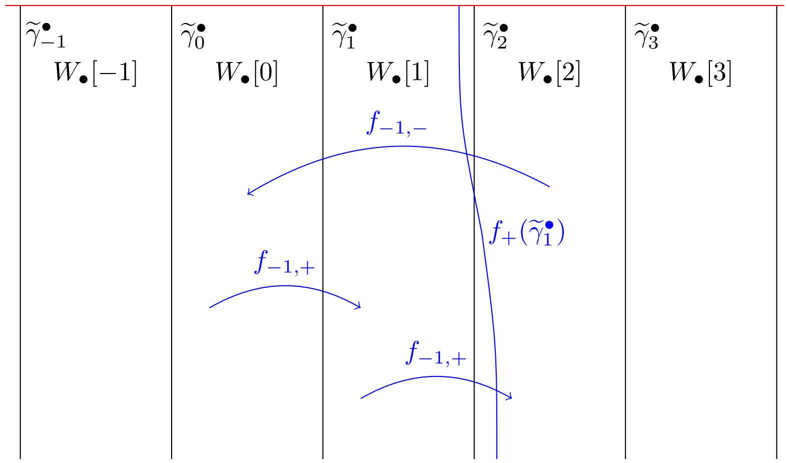


FIGURE 24. Illustration to Lemma B.3: $f_{-1,-}\colon W_\bullet[2] \to W_\bullet[0]$ and $f_{-1,+}\colon W_\bullet[0,1] \to W_\bullet[1,2]$ become the 1/3-anti-renormalization of f after gluing $\widetilde{\gamma}^\bullet_0$ and $\widetilde{\gamma}^\bullet_3$; see also Figure 23.

- $f_{-1}\colon W_{-1}[0] \to W_{-1}[1]$ (which is $\mathrm{id}\colon \mathbf{A} \to \mathbf{A}$) with
$$f_{-1,-} = \tau^{-1}\colon W_\bullet[2] \to W_\bullet[0].$$
- $f_{-1}\colon W_{-1}[1,2] \to W_{-1}[2,0]$ (which is $f\colon W \setminus \gamma_0 \to W \setminus \gamma_1$) with
$$f_{-1,+}\colon W_\bullet[0,1] \to W_\bullet[1,2].$$

□

Similar to Lemma B.4 we have

Lemma B.6 (The 2/3-anti-renormalization)**.** *Suppose $f\colon W \to W$ is a self-homeomorphism. Set*
$$\begin{aligned} f_{-1,+} &\coloneqq \tau = f_+ \circ f_-^{-1},\\ f_{-1,-} &\coloneqq f_-,\\ \tau_{-1} &\coloneqq f_{-1,-}^{-1} \circ f_{-1,+} = f_-^{-2} \circ f_+, \end{aligned}$$
Then τ_{-1} acts properly discontinuously on $W_\bullet \setminus \{\widetilde{0}\}$; we view
$$\left(W_\bullet \setminus \{\widetilde{0}\}\right)/\langle \tau_{-1}\rangle$$
as a punctured closed topological disk and we view $W_\bullet/\langle\tau_{-1}\rangle$ as a closed topological disk.

Let $f_{-1}\colon W_{-1} \to W_{-1}$ be the 2/3-anti-renormalization of f. Then f_{-1} is conjugate to
$$f_{-1,-}/\langle\tau_{-1}\rangle = f_{-1,+}/\langle\tau_{-1}\rangle\colon W_\bullet/\langle\tau_{-1}\rangle \to W_\bullet/\langle\tau_{-1}\rangle$$
by the conjugacy
$$h\colon W_{-1} \to W_\bullet[-1,0,1]/\langle\tau_{-1}\rangle$$
mapping
$$W_{-1}[0],\quad W_{-1}[1],\quad W_{-1}[2]$$

respectively to

$$W_\bullet[0]/\langle\tau_{-1}\rangle,\quad W_\bullet[1]/\langle\tau_{-1}\rangle,\quad W_\bullet[-1]/\langle\tau_{-1}\rangle$$

which are the copies of $\mathbf{A},\mathbf{B},\mathbf{B}$. □

B.1.5. *Fences.* Consider again a partial homeomorphism $f\colon W \dashrightarrow W$ and let $\mathbf{s}$ be an anti-renormalization sequence from Lemma B.2. We view W as a subset of $\mathbb{C}$.

A *fence* is a simple closed curve $Q \subset \operatorname{Dom} f$ such that

- 0 is in the bounded component Ω of $\mathbb{C}\setminus Q$; and
- Q intersects γ_0 at a single point x and Q intersects γ_1 at $f(x)$.

Let $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}} \dashrightarrow W_{\mathbf{s}/\mathfrak{q}}$ be an anti-renormalization of f_{-1} as in §B.1.3. We denote by $Q_{\mathbf{s}}$ the lift of Q to $W_{\mathbf{s}}$ and we denote by $Q_{\mathbf{s}/\mathfrak{q}}$ the projection of $Q_{\mathbf{s}}$ to $W_{\mathbf{s}/\mathfrak{q}}$.

Lemma B.7. *The curve* $Q_{\mathbf{s}/\mathfrak{q}}$ *is again a fence respecting* $\gamma_0^{\mathbf{s}/\mathfrak{q}},\gamma_p^{\mathbf{s}/\mathfrak{q}}$*; see* (B.2).

Proof. Every $Q_{\mathbf{s}/\mathfrak{q}}\cap W_{\mathbf{s}/\mathfrak{q}}[i]$ is an arc connecting a point on the left boundary of $W_{\mathbf{s}/\mathfrak{q}}[i]$ to a point on the right boundary of $W_{\mathbf{s}/\mathfrak{q}}[i]$. Moreover, $Q_{\mathbf{s}/\mathfrak{q}}\cap W_{\mathbf{s}/\mathfrak{q}}[i]$ meets $Q_{\mathbf{s}/\mathfrak{q}}\cap W_{\mathbf{s}/\mathfrak{q}}[i+1]$ because $g_k\colon \rho(S_i) \dashrightarrow \ell(S_{i+1})$ (see (B.1)) respects the intersection of Q with γ_0,γ_1. □

B.1.6. *Robustness of anti-renormalization.*

Theorem B.8. *Let* $f\colon W \dashrightarrow W$ *be a partial homeomorphism,* $\gamma_0,\gamma_1\subset W$ *be a dividing pair of arcs,* $Q\subset \operatorname{Dom} f$ *be a fence respecting* γ_0,γ_1 *and enclosing* $\Omega\ni 0$*, and let*

$$f_{-1}\colon W_{-1}\dashrightarrow W_{-1}$$

be the $\mathfrak{p}/\mathfrak{q}$*-anti-renormalization of* f*; see §B.1.3.*

Assume that $\gamma_0^{\text{new}},\gamma_1^{\text{new}}$ *is another pair of dividing arcs such that* $\gamma_0^{\text{new}}\setminus\Omega,\gamma_1^{\text{new}}\setminus\Omega$ *coincides with* $\gamma_0\setminus\Omega,\gamma_1\setminus\Omega$*. Denote by*

$$f_{-1,\text{new}}\colon W_{-1,\text{new}}\dashrightarrow W_{-1,\text{new}}$$

the $\mathfrak{p}/\mathfrak{q}$*-anti-renormalization of* f *relative to the pair* $\gamma_0^{\text{new}},\gamma_1^{\text{new}}$*. Then* f_{-1} *and* $f_{-1,\text{new}}$ *are naturally conjugate by* $h\colon W_{-1}\to W_{-1,\text{new}}$ *uniquely specified by the following properties:*

1. $\pi\circ h(z)=\pi(z)$ *for every* $z\in W_{-1}\setminus\Omega_{-1}$*, where* Ω_{-1} *is the topological disk enclosed by* Q_{-1}*, see Lemma B.7; and*
2. *if* $\widetilde{\beta}\subset W_{-1}$ *is a lift of a curve* $\beta\subset W$*, then* $h(\widetilde{\beta})$ *is a lift of* β *to* $W_{-1,\text{new}}$*.*

Proof. Since the pair $\gamma_0^{\text{new}}\setminus\Omega,\gamma_1^{\text{new}}\setminus\Omega$ coincides with $\gamma_0\setminus\Omega,\gamma_1\setminus\Omega$, Condition (1) uniquely specifies $h \mid W_{-1}\setminus\Omega_{-1}$.

Let us now extend $f\colon W\dashrightarrow W$ to a homeomorphism $f\colon W\to W$ mapping γ_0 to γ_1. The extension changes $f_{-1}\mid W_{-1}\setminus\Omega_{-1}$ and $f_{-1,\text{new}}\mid W_{-1,\text{new}}\setminus\Omega_{-1,\text{new}}$ but does not affect $f_{-1}\mid\Omega_{-1}$ and $f_{-1,\text{new}}\mid\Omega_{-1,\text{new}}$. Therefore, it is sufficient to prove the theorem under the assumption that $f\colon W\to W$ is a homeomorphism.

Since every anti-renormalization is an iteration of prime anti-renormalizations (see Lemma B.3), we can further assume that f_{-1} and $f_{-1,\text{new}}$ are prime anti-renormalizations. By Lemmas B.4 and B.6 both f_{-1} and $f_{-1,\text{new}}$ are naturally conjugate to

$$f_{-1,-}/\langle\tau_{-1}\rangle = f_{-1,+}/\langle\tau_{-1}\rangle\colon W_\bullet/\langle\tau_{-1}\rangle\to W_\bullet/\langle\tau_{-1}\rangle,$$

which is independent on the choice of γ_0,γ_1. It remains to observe that the conjugacy between f_{-1} and $f_{-1,\text{new}}$ satisfies Condition 2 – see Remark B.5. □

Corollary B.9 (Lifting condition)**.** *The curves $\gamma_{0,\text{new}}$ and $\gamma_{1,\text{new}}$ have unique lifts*

$$h^{-1}\left(\gamma_0^{\mathbf{s}/\mathfrak{q},\text{new}}\right), \qquad h^{-1}\left(\gamma_p^{\mathbf{s}/\mathfrak{q},\text{new}}\right) \subset W_{-1} \tag{B.4}$$

(see (B.2)*) such that the pair* (B.4) *coincide with $\gamma_0^{\mathbf{s}/\mathfrak{q}}, \gamma_p^{\mathbf{s}/\mathfrak{q}}$ in $W_{-1} \setminus \Omega$. Moreover,* (B.4) *is a dividing pair.* □

B.2. **Lifting theorem.** In this subsection we give a sufficient condition for liftability of arcs to leaves. This gives an alternative "manual" proof of Theorem B.8. Theorem B.14 is not used directly elsewhere in the paper.

As before, we consider a partial homeomorphism $f\colon (W,0) \dashrightarrow (W,0)$ and we assume that γ_0, γ_1 is a dividing pair of curves. We also view W as a subset of $\mathbb{C}$.

A *wall around* 0 *respecting* γ_0, γ_1 is either a closed annulus or a simple closed curve $Q \subset U \cap V$ such that

(1) $\mathbb{C} \setminus Q$ has two connected components. Moreover, denoting by Ω the bounded component of $\mathbb{C} \setminus Q$, we have $0 \in \Omega$.
(2) $\gamma_0 \cap Q$ and $\gamma_1 \cap Q$ are connected.
(3) if $x \in \Omega$, then $f^{\pm 1}(x) \in Q \cup \Omega$.

In other words, points in W do not jump over Q under the iteration of f. If Q is a simple closed curve, then f restricts to an actual homeomorphism $f\colon \Omega \to \Omega$.

Remark B.10. *Note that a wall contains a fence, see §B.1.5. Therefore, in the statement of Theorem B.8 we can replace a fence with a wall.*

For a sequence $\mathbf{s} \in \{\mathbf{A}, \mathbf{B}\}^{\mathbb{Z}}$ we denote by $Q_{\mathbf{s}}$ and $\Omega_{\mathbf{s}}$ the closures of the preimages of Q and Ω under $\pi\colon W_{\mathbf{s}} \to W$. We denote by $Q_{\mathbf{s}}[i]$ and $\Omega_{\mathbf{s}}[i]$ the intersections of $Q_{\mathbf{s}}$ and $\Omega_{\mathbf{s}}$ with $W_{\mathbf{s}}[i]$.

Lemma B.11. *The set $Q_{\mathbf{s}}$ is connected. The closure of the connected component of $W_{\mathbf{s}} \setminus Q_{\mathbf{s}}$ containing $\widetilde{0}$ is $\Omega_{\mathbf{s}}$.*

Proof. Follows from the definition: since points in Ω do not jump over Q every $Q_{\mathbf{s}}[i]$ intersects $Q_{\mathbf{s}}[i+1]$, therefore $Q_{\mathbf{s}}$ is connected and the claim follows. □

Suppose $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}} \to W_{\mathbf{s}/\mathfrak{q}}$ is an anti-renormalization of f and suppose W has a wall Q (respected by γ_0, γ_1, f) enclosing Ω. The image of $Q_{\mathbf{s}}$ in $W_{\mathbf{s}/\mathfrak{q}}$ is called the *full lift $Q_{\mathbf{s}/\mathfrak{q}}$ of Q*. Similarly, we denote by $\Omega_{\mathbf{s}/\mathfrak{q}}$ the image of $\Omega_{\mathbf{s}}$ in $W_{\mathbf{s}/\mathfrak{q}}$. We say that Q is an *N-wall* if it take at least N iterates of $f^{\pm 1}$ for points in Ω to cross Q. It follows by definition that:

Lemma B.12. *If Q is an N*-wall*, then $Q_{\mathbf{s}/\mathfrak{q}}$ is an $(N-1)\min\{\mathfrak{a}, \mathbf{b}\}$-wall.* □

Let β_0, $\beta_1 \coloneqq f(\beta_0) \subset W$ be two simple curves ending at 0 such that they are disjoint away from 0. We say that β_0, β_1 *respect* (Q, γ_0, γ_1) if

(1) β_0, β_1 start outside of $\Omega \cup Q$;
(2) β_0, β_1 do not intersect $(\gamma_0 \cup \gamma_1) \setminus \Omega$; and
(3) $\beta_0 \cap Q$ and $\beta_1 \cap Q$ are connected subset of different connected components of $Q \setminus (\gamma_0 \cap \gamma_1)$.

If we think that the components of $Q \setminus (\gamma_0 \cap \gamma_1)$ are gates of the wall Q, then (3) says that β_0, β_1 enter Ω through different gates.

Remark B.13. *We can slightly relax Conditions (2) and (3) to allow* β_0, β_1 *to touch (but not cross-intersect)* γ_0, γ_1 *in* $W \setminus \Omega$*. For example, we can allow* $\beta_0 \setminus \Omega = \gamma_0 \setminus \Omega$ *and* $\beta_1 \setminus \Omega = \gamma_1 \setminus \Omega$*.*

We say that a sequence $\mathbf{s}$ is *mixed* if $(\mathbf{A}, \mathbf{B})$ or $(\mathbf{B}, \mathbf{A})$ appears infinitely many times in both $\mathbf{s}[\geq 0]$ and $\mathbf{s}[\leq 0]$.

Theorem B.14 (Lifting of curves)**.** *Let* $f \colon W \dashrightarrow W$ *be a partial homeomorphism, let* $Q \subset W$ *be a wall respecting* γ_0, γ_1*, and let* $\beta_0, \beta_1 = f(\beta_0)$ *be a pair of curves respecting* (Q, γ_0, γ_1)*. Let* $\mathbf{s} \in \{\mathbf{A}, \mathbf{B}\}^{\mathbb{Z}}$ *be a sequence of a mixed type. Then all lifts of* β_0, β_1 *in* $W_{\mathbf{s}}$ *exist, are pairwise disjoint, and land at* $\widetilde{0}$*.*

Proof. We split the proof into short subsections.

B.2.1. *Notations and Conventions.* As in §B.1.4 we denote by $\mathbf{s}_\bullet := (\mathbf{A}, \mathbf{B})^{\mathbb{Z}}$ the sequence in $\{\mathbf{A}, \mathbf{B}\}^{\mathbb{Z}}$ with even entries equal to $\mathbf{A}$ and odd entries equal to $\mathbf{B}$. Simplifying notations, we write $W_{\mathbf{s}_\bullet} = W_\bullet$, $\Omega_{\mathbf{s}_\bullet} = \Omega_\bullet$, and $Q_{\mathbf{s}_\bullet} = Q_\bullet$. We note that $\pi \colon \Omega_\bullet \setminus \{\widetilde{0}\} \to \Omega \setminus \{0\}$ and $\pi \colon Q_\bullet \to Q$ are universal coverings. However, $\pi \colon W_\bullet \setminus \{\widetilde{0}\} \to W \setminus \{0\}$ needs not be a covering map: the sectors of $W_\bullet$ are glued through γ'_i and not through γ_i.

Denote by $U_\bullet$ and $V_\bullet$ the preimages of U and V under $\pi \colon W_\bullet \to W$. The map $f \colon U \setminus \{0\} \to V \setminus \{0\}$ admits a lift $\widetilde{f} \colon U_\bullet \setminus \{\widetilde{0}\} \to V_\bullet \setminus \{\widetilde{0}\}$ unique up to the action of the group of decks transformation, which is isomorphic to $\mathbb{Z}$. We always set $\widetilde{f}(\widetilde{0}) := \widetilde{0}$ – this is a continuous extension. We often write $\widetilde{f}$ as a partial homeomorphism $W_\bullet \dashrightarrow W_\bullet$, and call it a *lift* of $f \colon W \dashrightarrow W$.

We specify two lifts $\widetilde{f}_-$, $\widetilde{f}_+ \colon W_\bullet \dashrightarrow W_\bullet$ of f as follows

- $\widetilde{f}_-$ maps $\rho(W_\bullet[1])$ (which is a copy of γ_0) to $\ell(W_\bullet[1])$;
- $\widetilde{f}_+$ maps $\rho(W_\bullet[1])$ to $\rho(W_\bullet[2])$.

To simplify notation, we omit the tilde: $f_- = \widetilde{f}_-$ and $f_+ = \widetilde{f}_+$. Note that f_-, f_+^{-1} move points slightly to the left, while f_+, f_-^{-1} move points slightly to the right.

We make the following assumptions. We assume that β_0 starts and thus crosses the wall in $\mathbf{A}$ while β_1 starts and thus crosses the wall in $\mathbf{B}$. We also assume that $\mathbf{s}[0] = \mathbf{A}$. All other cases are completely analogous.

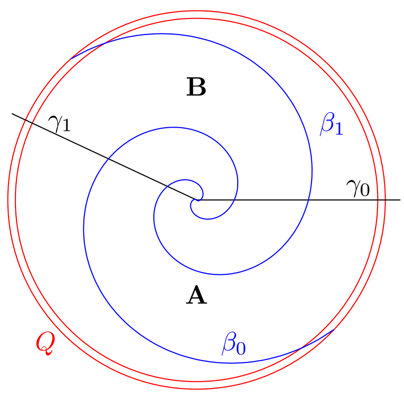
We parametrize all the lifts of β_0, β_1 in $W_{\mathbf{s}}$ by starting points: for $i \in \mathbb{Z}$ we denote by $\widetilde{\beta}_i = \widetilde{\beta}_i^{\mathbf{s}}$ the lift of β_0 (if $\mathbf{s}[i] = \mathbf{A}$) or of β_1 (if $\mathbf{s}[i] = \mathbf{B}$) starting in $W_{\mathbf{s}}[i]$. Recall that every lift $\widetilde{\beta}_i$ exists locally around its starting point. We will show that $\widetilde{\beta}_0$ exists and lands at $\widetilde{0}$, by a completely analogous argument all $\widetilde{\beta}_i$ exist and land at $\widetilde{0}$.

Similarly we parametrize all the lifts of β_0, β_1 in $W_\bullet$ by starting points: we denote by $\widetilde{\beta}_i^\bullet$ the lift of β_0 (if i is even) or of β_1 (if i is odd) starting at a point in $W_\bullet[i]$. Since $\pi \colon \Omega_\bullet \setminus \{\widetilde{0}\} \to \Omega \setminus \{0\}$ is a universal cover, all $\widetilde{\beta}_i^\bullet$ exist, pairwise disjoint, and land at $\widetilde{0}$.

We also write $\widetilde{\gamma}_i = \rho(W_{\mathbf{s}}[i-1])$ and $\widetilde{\gamma}_i^\bullet = \rho(W_\bullet[i-1])$. (By construction, $\widetilde{\gamma}_i^\bullet$ is a lift of γ_0 or of γ_1 under $\pi \colon W_\bullet \to W$.)

B.2.2. *Example: clockwise spiraling.* Let us illustrate the idea of the proof in the case when β_0 and β_1 spiral clockwise around 0, see the left-hand of Figure 25. Take the following 5-periodic sequence:

$$\mathbf{s}[5k, 5k+1, 5k+2, 5k+3, 5k+4] = (\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{B}, \mathbf{B}) \quad \forall k \in \mathbb{Z}.$$



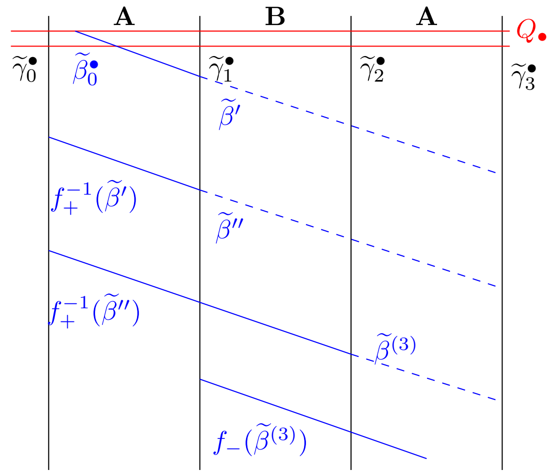


FIGURE 25. Left: β_0 and β_1 spiral clockwise around 0. Right: the curve $\widetilde{\beta}^{(k+1)}$ is $f_\pm^{\mp 1}(\widetilde{\beta}^{(k)})$ truncated by an appropriate $\widetilde{\gamma}_\ell^\bullet$.

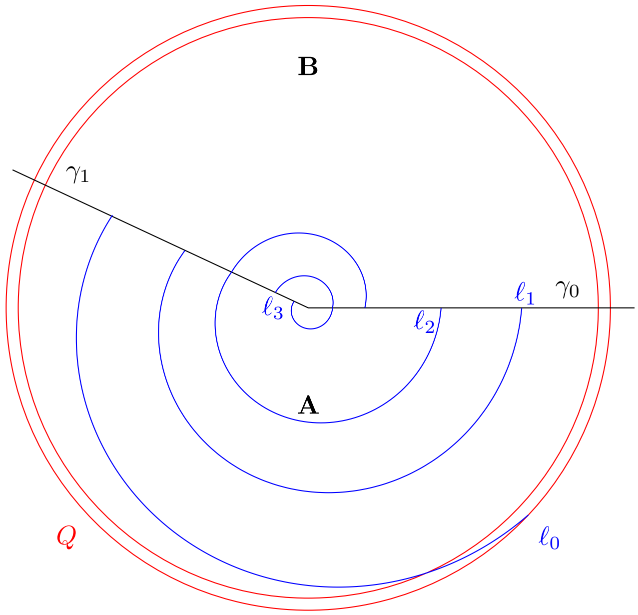
Let us inductively construct the curves $\widetilde{\beta}^{(k)}$, as it is illustrated on the right part of Figure 25:

- $\widetilde{\beta}'$ is the sub-curve of $\widetilde{\beta}_0^\bullet$ (a lift of β_0) on the right of $\widetilde{\gamma}_1^\bullet$. Since $\mathbf{s}[0,1] = (\mathbf{A},\mathbf{A})$ consider $f_+^{-1}(\widetilde{\beta}')$. (The curve $\widetilde{\beta}'$ is in the domain of f_+^{-1} because $\widetilde{\beta}'$ is below the wall.)
- $\widetilde{\beta}''$ is the sub-curve of $f_+^{-1}(\widetilde{\beta}')$ on the right of $\widetilde{\gamma}_1^\bullet$. Since $\mathbf{s}[1,2] = (\mathbf{A},\mathbf{A})$ consider $f_+^{-1}(\widetilde{\beta}'')$. (The curve $\widetilde{\beta}''$ is in the domain of f_+^{-1} because $\widetilde{\beta}''$ is below $\widetilde{\beta}'$.)
- $\widetilde{\beta}^{(3)}$ is the sub-curve of $f_+^{-1}(\widetilde{\beta}'')$ on the right of $\widetilde{\gamma}_2^\bullet$ – the subindex is 2 because $\mathbf{s}[2,3] = (\mathbf{A},\mathbf{B})$ but $\mathbf{s}[3,4] = (\mathbf{B},\mathbf{B})$.
- Since $\mathbf{s}[3,4] = (\mathbf{B},\mathbf{B})$ we consider next $f_-(\widetilde{\beta}^{(3)})$. (The curve $\widetilde{\beta}^{(3)}$ is in the domain of f_- because $\widetilde{\beta}^{(3)}$ is below $\widetilde{\beta}''$.)
- $\widetilde{\beta}^{(4)}$ is the subcurve of $f_-(\widetilde{\beta}^{(3)})$ on the right of $\widetilde{\gamma}_3^\bullet$ – the subindex is 3 because $\mathbf{s}[4,5] = (\mathbf{B},\mathbf{A})$ but $\mathbf{s}[5,6] = (\mathbf{A},\mathbf{A})$.
- Since $\mathbf{s}[5,6] = (\mathbf{A},\mathbf{A})$, we consider next $f_+^-(\widetilde{\beta}^{(4)})$.

The construction continues by periodicity.

Define now $\widetilde{\ell}_0 := \widetilde{\beta}_0^\bullet \setminus \widetilde{\beta}'$, $\widetilde{\ell}_1 := f_+^{-1}(\widetilde{\beta}') \setminus \widetilde{\beta}''$, $\widetilde{\ell}_2 := f_+^{-1}(\widetilde{\beta}'') \setminus \widetilde{\beta}^{(3)}$, $\widetilde{\ell}_3 := f_-(\widetilde{\beta}^{(3)}) \setminus \widetilde{\beta}^{(4)}, \dots$, and $\ell_i := \pi(\widetilde{\ell}_i)$, see the upper part of Figure 26. Then the curve $\widetilde{\beta}_0$ is the concatenation (see the bottom part of Figure 26) of

- $\pi_{\mathbf{s},0}^{-1}(\ell_0)$; followed by
- $\pi_{\mathbf{s},1}^{-1}(\ell_1)$ – because f^{-1} maps the end point of ℓ_0 to the starting point of ℓ_1 (recall that $\mathbf{s}[0,1] = (\mathbf{A},\mathbf{A})$); followed by
- $\pi_{\mathbf{s},2}^{-1}(\ell_2 \cap \mathbf{A})$ – because f^{-1} maps the end point of ℓ_1 to the starting point of ℓ_2; followed by
- $\pi_{\mathbf{s},3}^{-1}(\ell_2 \cap \mathbf{B})$ – because $\mathbf{s}[2,3] = (\mathbf{A},\mathbf{B})$; followed by
- $\pi_{\mathbf{s},4}^{-1}(\ell_3 \cap \mathbf{B})$ – because f maps the end point of ℓ_2 to the starting point of ℓ_3; followed by



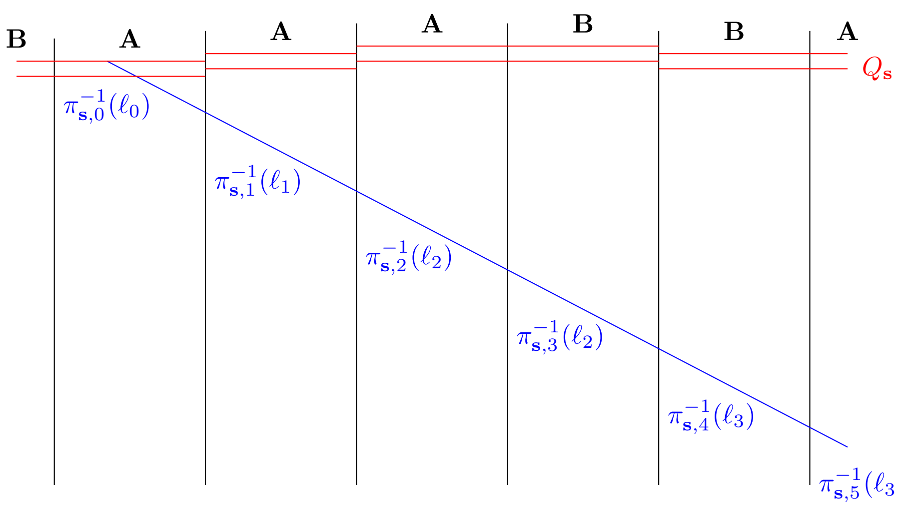


FIGURE 26. Top: the curves ℓ_i are disjoint and within $\Omega \cap Q$. Bottom: construction of $\widetilde{\beta}_0$ as the concatentaion of appropriate lifts of ℓ_i.

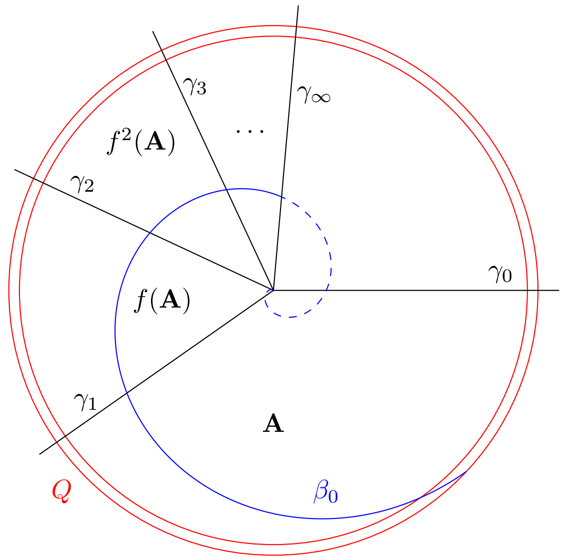


FIGURE 27. Only $\beta_0 \cap (\mathbf{A} \cup f(\mathbf{A}) \cup f^2(\mathbf{A}) \cup \dots)$ is liftable to $W_{\mathbf{s}}$ if $\mathbf{s} = (\dots, \mathbf{A}, \mathbf{A}, \mathbf{A}, \dots)$.

- $\pi^{-1}_{\mathbf{s},4}(\ell_3 \cap \mathbf{A})$ – because $\mathbf{s}[4,5] = (\mathbf{B}, \mathbf{A})$.

The construction continues by periodicity.

Note that $\ell_1 \subset \Omega \cup Q$ (ℓ_1 can intersect Q) and ℓ_1 is disjoint from ℓ_0. The curve $\ell_2 \cap \mathbf{A}$ is separated by ℓ_1 from Q while $\ell_2 \cap \mathbf{B}$ is separated by the continuation of ℓ_1 from Q. By induction, all ℓ_i are well defined.

Let $\Omega_\bullet[\kappa(j)] \cup Q_\bullet[\kappa(j)]$ be the strip where $\widetilde{\ell}_j$ starts. Since $\mathbf{s}$ is mixing, we have $\kappa(j) \to +\infty$. Therefore, every $z \in \widetilde{\beta}^\bullet_0$ eventually escapes to a certain $\widetilde{\ell}_j$ under the iteration of

$$f_+^{-1},\ \ f_+^{-1},\ \ f_-,\ \ f_+^{-1},\ \ f_+^{-1},\ \ f_-, \dots$$

This shows that $\widetilde{\beta}_0$ is a complete lift of $\beta_0 \setminus \{0\}$. Since the curves $\widetilde{\ell}_j$ tend to the right and they are all below $\widetilde{\beta}^\bullet_0$, the curves $\widetilde{\ell}_j$ tend to $\widetilde{0}$. Therefore, ℓ_i tend to 0; i.e. $\widetilde{\beta}_0$ lands at $\widetilde{0}$.

B.2.3. *Example: no mixing condition.* Let us illustrate that Theorem B.14 fails if $\mathbf{s}$ is not mixing. Suppose $\mathbf{s} = (\dots, \mathbf{A}, \mathbf{A}, \mathbf{A}, \dots)$. Choose $f\colon W \dashrightarrow W$ as it shown on Figure 27: the curves $\gamma_i = f^i(\gamma_0)$ accumulate at γ_∞ and the sector between γ_0 and γ_∞ (counting clockwise) is the concatenation

$$\mathbf{A} \cup f(\mathbf{A}) \cup f^2(\mathbf{A}) \cup \dots =: \mathbf{X}$$

Then only $\beta_0 \cap \mathbf{X}$ is liftable to $W_{\mathbf{s}}$.

B.2.4. *Combinatorics of jumps.* We now adapt the argument form §B.2.2 to a possibility that $\widetilde{\beta}_0$ oscillates at $\widetilde{0}$. We start by introducing additional notations.

We define the following quantities. Recall first that for $j \in \mathbb{Z}$ the jump is defined by

$$\iota(j) := \begin{cases} 0 & \text{if } \mathbf{s}[j-1,j] \in \{(\mathbf{A},\mathbf{B}),(\mathbf{B},\mathbf{A})\}, \\ 1 & \text{if } \mathbf{s}[j-1,j] = (\mathbf{B},\mathbf{B}), \\ -1 & \text{if } \mathbf{s}[j-1,j] = (\mathbf{A},\mathbf{A}). \end{cases}$$

For $j > 0$ define

$$\begin{aligned} \nu(j) =& \#\{k \in \{1,\dots,j\} \mid \iota(k) = 1\}, \\ \mu(j) =& \#\{k \in \{1,\dots,j\} \mid \iota(k) = -1\}, \\ \kappa(j) =& \#\{k \in \{1,\dots,j\} \mid \iota(k) = 0\}, \end{aligned} \tag{B.5}$$

while for $j < 0$ define

$$\begin{aligned} \nu(j) =& -\#\{k \in \{j+1,\dots,0\} \mid \iota(k) = 1\}, \\ \mu(j) =& -\#\{k \in \{j+1,\dots,0\} \mid \iota(k) = -1\}, \\ \kappa(j) =& -\#\{k \in \{j+1,\dots,0\} \mid \iota(k) = 0\}. \end{aligned} \tag{B.6}$$

In particular, $\mu(j) + \nu(j) + \kappa(j) = j$ for all $j \neq 0$. We also write $\mu(0) = \nu(0) = \kappa(0) = 0$.

For $i < j$ we define the *jump from $W_{\mathbf{s}}[i]$ to $W_{\mathbf{s}}[j]$* to be the sum of jumps from $W_{\mathbf{s}}[i+k]$ to $W_{\mathbf{s}}[i+k+1]$ with k ranging from 0 to $j-i-1$. The *jump from $W_{\mathbf{s}}[j]$ to $W_{\mathbf{s}}[i]$* is defined to be the negative of the jump from $W_{\mathbf{s}}[i]$ to $W_{\mathbf{s}}[j]$. It follows from definitions:

Claim 1. *The jump from $W_{\mathbf{s}}[0]$ to $W_{\mathbf{s}}[k]$ is $\nu(k) - \mu(k)$.* □

Claim 2. *Suppose that the lift $\widetilde{\beta}_0$ exists for all $t \in [0, \bar{t}]$. If $\widetilde{\beta}_0(t) \in \operatorname{int} W_{\mathbf{s}}[k] \cup \rho(W_{\mathbf{s}}[k])$, then*

$$\widetilde{\beta}_0(t) = \pi^{-1}_{\mathbf{s},k}\left(f^{\nu(k)-\mu(k)} \circ \beta_0(t)\right) = \pi^{-1}_{\mathbf{s},k} \circ \pi\left(f_-^{\nu(k)} \circ f_+^{-\mu(k)} \circ \widetilde{\beta}_0^\bullet(t)\right) \tag{B.7}$$

and all maps in this equation are well defined.

Proof. If $\widetilde{\beta}_0(t) \in \operatorname{int} W_{\mathbf{s}}[k] \cup \rho(W_{\mathbf{s}}[k])$, then by definition of the lift of a curve and by Claim 1 we have $\pi(\widetilde{\beta}_0(t)) = f^{\nu(k)-\mu(k)}(\beta_0(t))$. This is the first equality in (B.7). The second equality holds because $f_-^{\nu(k)} \circ f_+^{-\mu(k)}$ is a lift of $f^{\nu(k)-\mu(k)}$. □

Since $\mathbf{s}$ is of mixed type, we obviously have:

Claim 3. *If $j \to \pm\infty$, then $\kappa(j) \to \pm\infty$ respectively.* □

B.2.5. *Basic dynamical properties.* For $X \subset W$ and $n \in \mathbb{Z}$, we write

$$f^n(X) = f^n\left(X \cap \operatorname{Dom} f^n\right).$$

Claim 4. *For $n \in \mathbb{Z}$ we have*

$$f^n(\gamma_0) \cap f^{n+1}(\gamma_0) = \{0\} \quad \textit{and} \quad f^n(\beta_0) \cap f^{n+1}(\beta_0) = \{0\}.$$

Proof. Follows from $\gamma_0 \cap \gamma_1 = \{0\} = \beta_0 \cap \beta_1$ and the assumption that f is a partial homeomorphism. □

Claim 5. *The curve $\widetilde{\gamma}_0^\bullet \cap \Omega_\bullet$ is in the domains of $f_\pm^{\pm 1}$ (for any choice of "+" and "−"). Moreover, we have:*

1. $f_+^{-1}\left(\widetilde{\gamma}_1^\bullet \cap \Omega_\bullet\right) \subset \widetilde{\gamma}_0^\bullet$,
2. $f_-\left(\widetilde{\gamma}_1^\bullet \cap \Omega_\bullet\right) \subset W_\bullet[-1,0] \cup \Omega_\bullet[-1,0]$,

(3) $f_+\left(\widetilde{\gamma}_0^\bullet \cap \Omega_\bullet\right) \subset \widetilde{\gamma}_1^\bullet$,
(4) $f_-^{-1}\left(\widetilde{\gamma}_0^\bullet \cap \Omega_\bullet\right) \subset W_\bullet[0,1] \cup \Omega_\bullet[0,1]$.

Proof. Recall that the lifts f_+ and f_- are specified so that f_+^{-1} and f_- move points slightly to the left while f_+ and f_-^{-1} move points slightly to the right. Therefore,

- (1) follows from $f^{-1}(\gamma_1) \subset \gamma_0$;
- (2) follows from $f^{-1}(\gamma_0) \cap \gamma_0 = \{0\}$, see Claim 4.
- (3) follows from $f(\gamma_0) \subset \gamma_1$;
- (4) follows from $f(\gamma_1) \cap \gamma_1 = \{0\}$, see Claim 4.

□

B.2.6. *Gulfs $D_{>0}$ and $D_{<0}$.* Let us define the *gulf* $D_{>0}$ to be the closed region in $\Omega_\bullet[>0]$ located on the right of $\widetilde{\gamma}_1^\bullet = \rho(W_\bullet[0])$ and on the left of $\widetilde{\beta}_0^\bullet$. We recall that both $\widetilde{\gamma}_1^\bullet$ (and similarly $\widetilde{\beta}_0^\bullet$) decomposes $\Omega_\bullet$ into two connected components; thus $D_{>0}$ is well defined. Similarly, the *gulf* $D_{<0}$ is the closed region in $\Omega_\bullet[<0]$ located on the left of $\widetilde{\gamma}_0^\bullet$ and on the right of $\widetilde{\beta}_0^\bullet$.

Claim 6. *Both $D_{>0}$ and $D_{<0}$ are in the domains of $f_\pm^{\pm 1}$ (for any choice of "+" and "−"). Moreover, we have:*
(1) $f_+^{-1}\left(D_{>0}\right) \subset D_{>0} \cup \Omega_\bullet[0] \cup Q_\bullet[0]$,
(2) $f_-\left(D_{>0}\right) \subset D_{>0} \cup \Omega_\bullet[-1,0] \cup Q_\bullet[-1,0]$,
(3) $f_+\left(D_{<0}\right) \subset D_{<0} \cup \Omega_\bullet[0] \cup Q_\bullet[0]$,
(4) $f_-^{-1}\left(D_{>0}\right) \subset D_{<0} \cup \Omega_\bullet[0,1] \cup Q_\bullet[0,1]$.

Proof. The first claim follows from $D_{>0} \cup D_{<0} \subset \Omega_\bullet$. Statements (1)–(4) follow from Statements (1)–(4) of Claim 5 respectively. Indeed, by definition, $f_+^{-1}(D_{>0})$ is bounded by $f_+^{-1}(\widetilde{\gamma}_1^\bullet \cap D_{>0}) \subset \widetilde{\gamma}_0^\bullet$ and by $f_+^{-1}(\widetilde{\beta}_0^\bullet \cap D_{>0})$; this implies (1). Other Statements are analogous. □

B.2.7. *Channels D_k, D_{-k}, T_k, T_{-k}.* Let us now apply inductively $f_+^{-\mu(k)} \circ f_-^{\nu(k)}$ to $D_{>0}$; on each step we define D_k as the set of points that escape to $W_\bullet[\kappa(k)]$, see Figure 28 and its caption.

Consider first the case $k=1$. If $\mathbf{s}[1] = \mathbf{A}$, then $\nu(1) = 0, \quad \mu(1) = 1$, and $\kappa(1) = 0$. In this case (see Figure 28) we define

$$
\begin{aligned}
T_{>0} &\coloneqq f_+^{-1}(D_{>0},)\\
T_1 &\coloneqq T_{>0} \cap W[0],\\
D_1 &\coloneqq f_+(T_1),\\
T_{>1} &\coloneqq \overline{T_{>0} \setminus T_1},\\
D_{>1} &\coloneqq f_+(T_{>1}).
\end{aligned}
$$

If $\mathbf{s}[1] = \mathbf{B}$, then $\nu(1) = 0, \quad \mu(k) = 0$, and $\kappa(1) = 1$. In this case we define

$$
\begin{aligned}
T_{>0} &\coloneqq D_{>0},\\
T_1 = D_1 &\coloneqq T_{>0} \cap \Omega[1],\\
T_{>1} = D_{>1} &\coloneqq \overline{T_{>0} \setminus T_1}.
\end{aligned}
$$

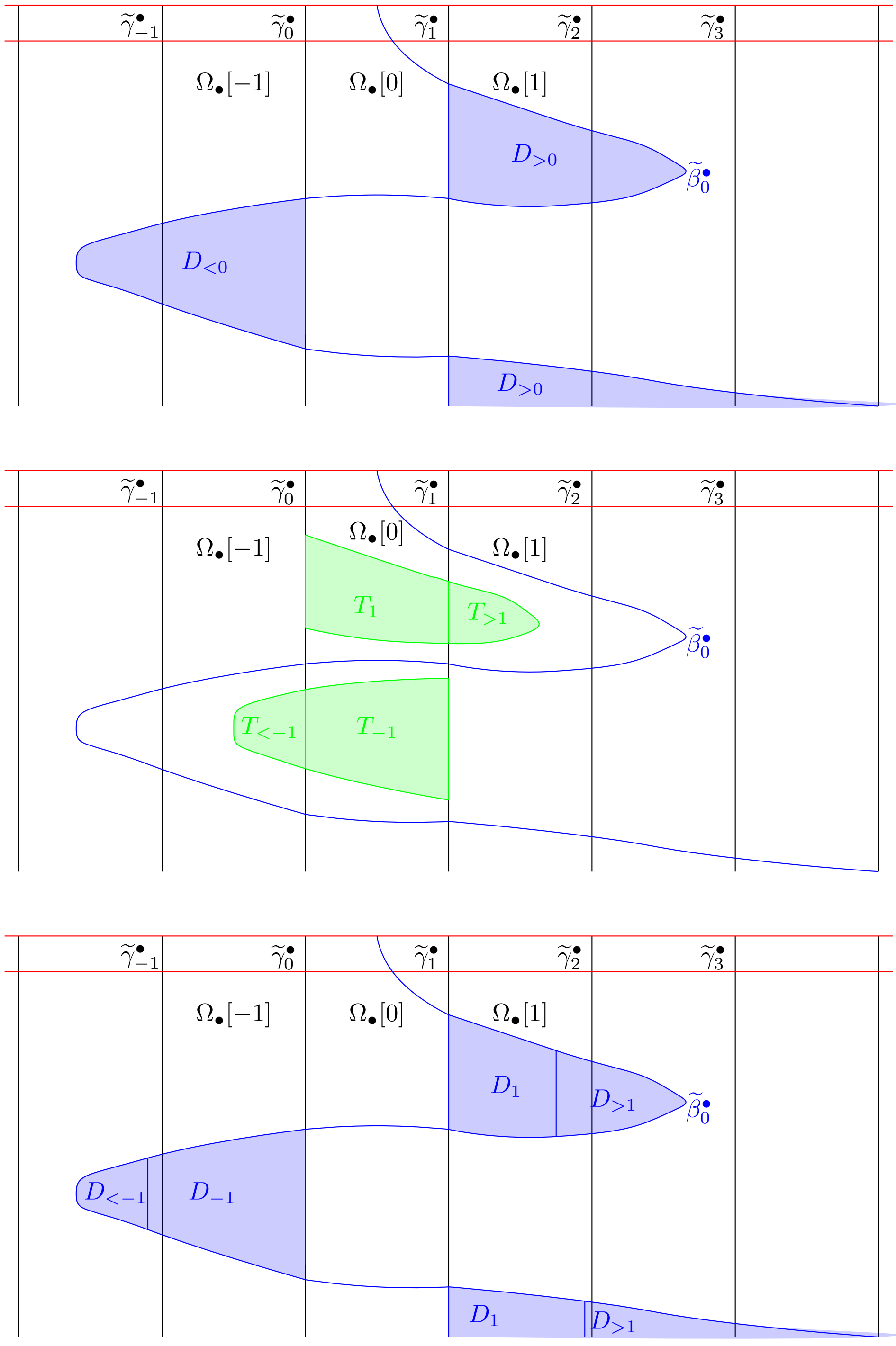


FIGURE 28. Top: closed regions $D_{>0}$ and $D_{<0}$. Middle: assuming that $\mathbf{s}[-1,0,1] = (\mathbf{A},\mathbf{A},\mathbf{A})$, we have $f_+^{-1}(D_{>0}) = T_{>0} = T_1 \cup T_{>1}$ and $f_-(D_{<0}) = T_{<0} = T_{-1} \cup T_{<-1}$. Bottom: decompositions $D_1 \cup D_{>1} = D_{>0}$ and $D_{-1} \cup D_{<-1} = D_{<0}$.

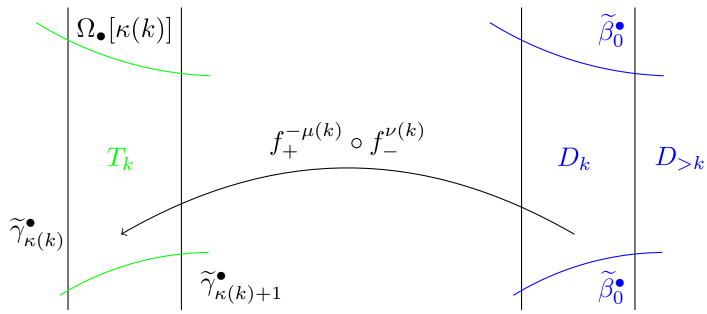


FIGURE 29. The region D_k is the set of points in $D_{>0}$ that escape to $\Omega_\bullet[\kappa(k)]$ under $f_+^{-\mu(k)} \circ f_-^{\nu(k)}$. The region $D_{>k}$ is the set of points in $D_{>0}$ on the right of D_k.

In general, for $k > 0$ we set inductively

$$\begin{aligned} T_k &:= f_+^{-\mu(k)} \circ f_-^{\nu(k)}(D_{>k-1}) \cap W_\bullet[\kappa(k)], \\ D_k &:= f_+^{\mu(k)} \circ f_-^{-\nu(k)}\left(T_k\right), \\ T_{>k+1} &:= \overline{T_{>k} \setminus T_{k+1}}, \\ D_{>k+1} &:= \overline{D_{>k} \setminus D_{k+1}}, \end{aligned}$$

and similarly, for $k < 0$, we set

$$\begin{aligned} T_{-k} &:= f_+^{-\mu(-k)} \circ f_-^{\nu(-k)}(D_{<-k+1}) \cap W_\bullet[\kappa(-k)], \\ D_{-k} &:= f_+^{\mu(-k)} \circ f_-^{-\nu(-k)}\left(T_{-k}\right), \\ T_{<-k-1} &:= \overline{T_{<-k} \setminus T_{-k-1}}, \\ D_{<-k-1} &:= \overline{D_{<-k} \setminus D_{-k-1}}. \end{aligned}$$

The case $\mathbf{s}[-1,0,1] = (\mathbf{A},\mathbf{A},\mathbf{A})$ is in Figure 28. We call $D_{>k}, D_{<-k}$ *gulfs*, and we say that T_k, D_k are *channels*. The channels T_k play the role of the curves $\widetilde{\ell}_k$ from §B.2.2.

By an easy induction $f_+^{-\mu(k)} \circ f_-^{\nu(k)}(D_{>k}) \subset D_{>0}$ for $k > 0$; thus by Claim 6 the gulf $D_{>k}$ is in the domain of $f_+^{-\mu(k+1)} \circ f_-^{\nu(k+1)}$. Similarly, $D_{<-k}$ is in the domain of $f_+^{-\mu(-k-1)} \circ f_-^{\nu(-k-1)}$.

For $k \neq 0$ write $\ell(T_k) := \ell(W_\bullet[\kappa(k)]) \cap T_k$ and $\rho(T_k) := \rho(W_\bullet[\kappa(k)]) \cap T_k$.

Claim 7. *The gulfs $D_{>0}$ and $D_{<0}$ are the unions $D_1 \cup D_2 \cup D_3 \cup \dots$ and $D_{-1} \cup D_{-2} \cup D_{-3} \cup \dots$ respectively. Moreover, $D_i \cap D_j = \emptyset$ if $|i-j| \geq 0$. Write $\delta := D_k \cap D_{k+1}$ for $k \notin \{-1, 0\}$. Then*

$$\begin{aligned} f_+^{-\mu(k)} \circ f_-^{\nu(k)}(\delta) \subset \rho(T_k) \subset \widetilde{\gamma}^\bullet_{\kappa(k)+1}, \\ f_+^{-\mu(k+1)} \circ f_-^{\nu(k+1)}(\delta) \subset \ell(T_{k+1}) \subset \widetilde{\gamma}^\bullet_{\kappa(k+1)}. \end{aligned}$$

Proof. We will verify the claim for $D_{>0}$; the case of $D_{<0}$ is similar.

By induction, if for $z \in D_{>0}$ the point $f_+^{-\mu(k)} \circ f_-^{\nu(k)}(z)$ is on the right of $W_\bullet[\kappa(k)]$, then $f_+^{-\mu(k+1)} \circ f_-^{\nu(k+1)}(z)$ is ether on the right of $W_\bullet[\kappa(k+1)]$ or

$f_+^{-\mu(k+1)} \circ f_-^{\nu(k+1)}(z) \in W_\bullet[\kappa(k+1)]$. Thus points in $D_{>k}$ do not jump over T_{k+1} under one iteration. Recall that f_+^{-1} and f_- move points to the left. By Claim 3, $\kappa(k) \to +\infty$. Therefore, every point in $D_{>0}$ eventually escapes to some $T_k \subset W_\bullet[\kappa(k)]$. Let us now show that a point in $D_{>0}$ escapes to at most two (neighboring) T_kth.

For $k > 0$ the channel $T_k = f_+^{-\mu(k)} \circ f_-^{\nu(k)}(D_k)$ is on the left of $f_+^{-\mu(k)} \circ f_-^{\nu(k)}(\widetilde{\beta}_0^\bullet)$, on the left of $\widetilde{\gamma}^\bullet_{\kappa(k)+1}$, and on the right of $\widetilde{\gamma}^\bullet_{\kappa(k)}$. On the other hand, $f_+^{-\mu(k)} \circ f_-^{\nu(k)}(D_{>k})$ is on the right of $\widetilde{\gamma}^\bullet_{\kappa(k)+1}$. It is now easy to see that

$$T_k \cap f_+^{-\mu(k)} \circ f_-^{\nu(k)}(D_{>k}) = T_k \cap f_+^{-\mu(k)} \circ f_-^{\nu(k)}(D_{k+1}) = f_+^{-\mu(k)} \circ f_-^{\nu(k)}(\delta) \subset \widetilde{\gamma}^\bullet_{\kappa(k)+1}.$$

This proves that a point in $D_{>0}$ escapes to at most two T_kth; it also verifies the first identity. Similarly, the second identity is verified. □

B.2.8. *The natural map from $D_{<0} \cup \Omega_\bullet[0] \cup Q_\bullet \cup D_{>0}$ to $W_{\mathbf{s}}$.* For convenience, let us extend by continuity the map $\pi_{\mathbf{s},i}^{-1}\colon \operatorname{int} \mathbf{s}[i] \cup \rho(\mathbf{s}[i]) \to \operatorname{int} W_{\mathbf{s}}[i] \cup \rho(W_{\mathbf{s}}[i])$ to $\pi_{\mathbf{s},i}^{-1}\colon \mathbf{s}[i] \to W_{\mathbf{s}}[i]$.

We define the map $\theta_0\colon \Omega_\bullet[0] \cup Q_\bullet[0] \to \Omega_{\mathbf{s}}[0] \cup Q_{\mathbf{s}}[0]$ to be $\pi_{\mathbf{s},0}^{-1} \circ \pi$. For $k \neq 0$ we define the map $\theta_k\colon D_k \to W_{\mathbf{s}}[k]$ as $f_+^{-\mu(k)} \circ f_-^{\nu(k)}\colon D_k \to T_k$, followed by $\pi\colon T_k \to \mathbf{s}[k] \subset W$, and followed by $\pi_{\mathbf{s},k}^{-1}\colon \mathbf{s}[k] \to W_{\mathbf{s}}[k]$. Combining θ_k and θ_0 we obtain the map

$$\theta\colon D_{<0} \cup \Omega_\bullet[0] \cup Q_\bullet[0] \cup D_{>0} \to W_{\mathbf{s}}$$

such that $\theta \mid D_k = \theta_k$ and $\theta \mid \Omega_\bullet[0] \cup Q_{\mathbf{s}}[0] = \theta_0$. We note that there is no ambiguity on $D_k \cap D_{k+1}$:

Claim 8. *The map θ is a homeomorphism on its image. Furthermore, for every curve*

$$\alpha\colon [0,1] \to D_{<0} \cup \Omega_\bullet[0] \cup Q_\bullet[0] \cup D_{>0}$$

starting in $\Omega_\bullet[0] \cup Q_\bullet[0]$, the lift of $\pi(\alpha)$ starting in $\Omega_{\mathbf{s}}[0] \cup Q_{\mathbf{s}}[0]$ exists and is equal to $\theta(\alpha)$.

Proof. It is routine to check that θ_k and θ_j agree on $\operatorname{Dom}\theta_k \cap \operatorname{Dom}\theta_j = \emptyset$. Indeed, if $|k-j| > 1$, then $\operatorname{Dom}\theta_k \cap \operatorname{Dom}\theta_j = \emptyset$ by Claim 7. If $j = k+1$, then writing $\delta \coloneqq \operatorname{Dom}\theta_k \cap \operatorname{Dom}\theta_{k+1}$, we check (see again Claim 7) that $\theta_k \mid \delta = \theta_{k+1} \mid \delta$ and $\theta_k(\delta) \subset W_{\mathbf{s}}[k] \cap W_{\mathbf{s}}[k+1]$. Since θ_k and θ_j have disjoint images away from $\operatorname{Dom}\theta_k \cap \operatorname{Dom}\theta_j$, we obtain that θ is a homeomorphism.

By the definition, $\operatorname{Im}(\theta_k) \subset W_{\mathbf{s}}[k]$ and $f_+^{\mu(k)} \circ f_-^{-\nu(k)}$ transfers D_k to $W_{\mathbf{s}}[k]$ – this is the correct number of iterations, see Claim 2 and (B.7). Therefore, $\theta(\alpha)$ is the lift of $\pi(\alpha)$ starting in $Q\bullet[0] \cup \Omega\bullet[0]$.

□

As a corollary, we obtain

Claim 9. *All $\widetilde{\beta}_m$ exist.*

Proof. By Claim 8, $\widetilde{\beta}_0 = \theta(\widetilde{\beta}_0^\bullet)$ exists. By a similar argument all $\widetilde{\beta}_m$ exist; let us give a brief sketch. For every $\widetilde{\beta}_m^\bullet$ define $D'_{>m}$ to be the closed region on the left of $\widetilde{\beta}_m^\bullet$ and on the right of $\widetilde{\gamma}^\bullet_{m+1}$, define $D'_{<m}$ to be the closed region on the right of $\widetilde{\beta}_m^\bullet$ and on the left of $\widetilde{\gamma}^\bullet_m$. As in §B.2.4 specify the quantities $\nu(k), \mu(k), \kappa(k)$ with

the only difference is that the count in (B.5) starts from $m+1$ instead of 1 (and similar in (B.6)). By the same argument as for $\widetilde{\beta}_0$ we construct

$$\theta' \colon D_{<m} \cup \Omega_\bullet[m] \cup Q_\bullet[m] \cup D_{>m} \to W_{\mathbf{s}}$$

such that Claim 8 (with necessary adjustments) holds for $\widetilde{\beta}_m$. □

Claim 10. *All $\widetilde{\beta}_i$ land at $\widetilde{0}$.*

Proof. Let us show that $\widetilde{\beta}_0$ lands at $\widetilde{0}$; other cases are completely analogous. By Claim 8 we have $\widetilde{\beta}_0 = \theta(\widetilde{\beta}_0^\bullet)$. Parametrize β_0 as $\beta_0 \colon [0,1] \to W$ with $\beta_0(1) = 0$.

Choose a big $M > 0$. Since $\theta \mid \operatorname{Dom}\theta \cap W_\bullet[-M, -M+1, \dots, M]$ is continuous we have

- if $\widetilde{\beta}_0^\bullet(t_n) \in W_\bullet[-M, -M+1, \dots, M]$ and $t_n \to 1-0$, then $\pi(\theta(\widetilde{\beta}_0^\bullet(t_n)) \to 0$.

It remains to show that if $t_n \to 1-0$ such that $\widetilde{\beta}_0^\bullet(t) \in W_\bullet[> M_n] \cup W_\bullet[< -M_n]$ with $M_n \to +\infty$, then $\pi(\theta(\widetilde{\beta}_0^\bullet(t_n)) \to 0$.

Write $\widetilde{\beta}_0^\bullet(t_n) \in D_{k(n)}$; then $k(n) \to \pm\infty$. By Claim 3 $\kappa \circ k(n) \to \pm\infty$. Recall that $T_{\kappa\circ k(n)} \subset \Omega_\bullet[\kappa \circ k(n)]$ (see Figure 29) and that $T_{\kappa\circ k(n)} \subset \Omega_\bullet[\kappa \circ k(n)]$ is separated by $\widetilde{\beta}_0^\bullet$ from $Q_\bullet[\kappa \circ k(n)]$.

Since $\pi(\widetilde{\beta}_0^\bullet \cap W_\bullet[\kappa \circ k(n)]) \to 0$, we obtain that $\pi(T_{\kappa\circ k(n)}) \to 0$. Since $\pi(T_{\kappa\circ k(n)}) \ni \pi(\theta(\widetilde{\beta}_0^\bullet(t_n))$, we obtain $\pi(\theta(\widetilde{\beta}_0^\bullet(t_n)) \to 0$. □

Claim 11. *All lifts of β_0, β_1 are pairwise disjoint.*

Proof. By Claim 10 all $\widetilde{\beta}_i$ land at $\widetilde{0}$; therefore, every $\widetilde{\beta}_i$ disconnects $\Omega_{\mathbf{s}}$ into two connected components. It follows from Claims 2 and 4 that $\widetilde{\beta}_i, \widetilde{\beta}_{i+1}$ are disjoint. Since $\widetilde{\beta}_{i-1}$ is on the left from $\widetilde{\beta}_i$ while $\widetilde{\beta}_{i+1}$ is on the right from $\widetilde{\beta}_i$, we obtain that $\widetilde{\beta}_{i-1}$ and $\widetilde{\beta}_{i+1}$ are also disjoint. Repeating the argument, we obtain that all $\widetilde{\beta}_i$ are pairwise disjoint. □

□

B.2.9. *Proof of Theorem B.8.* Let $D \subset W \setminus \{0\}$ be a topological disk. A *lift* of D to $W_{\mathbf{s}}$ is defined in the same way as a lift of a curve, see §B.1.2. Alternatively, a lift $\iota \colon D \to W_{\mathbf{s}}$ of D is characterized by the property that if $\alpha \subset D$ is a curve, then $\iota(\alpha)$ is the lift of $\alpha \subset D$ starting in D. A *lift* of D to $W_{\mathbf{s}/\mathfrak{q}}$ is the projection of a lift of D to $W_{\mathbf{s}}$.

Since $\gamma_0^{\text{new}} \setminus \Omega, \gamma_1^{\text{new}} \setminus \Omega$ coincide with $\gamma_0 \setminus \Omega, \gamma_1 \setminus \Omega$, Condition (1) uniquely specifies $h \mid W_{\mathbf{s}/\mathfrak{q}} \setminus \Omega_{\mathbf{s}/\mathfrak{q}}$.

Since the pair $\gamma_0^{\text{new}}, \gamma_1^{\text{new}}$ is dividing, it splits W into two closed sectors, call them $\mathbf{A}_{\text{new}}$ and $\mathbf{B}_{\text{new}}$ specified so that $\mathbf{A}_{\text{new}} \setminus \Omega = \mathbf{A} \setminus \Omega$ and $\mathbf{B}_{\text{new}} \setminus \Omega = \mathbf{B} \setminus \Omega$. We need to show that all lifts of $\mathbf{A}_{\text{new}}$ and $\mathbf{B}_{\text{new}}$ to $W_{\mathbf{s}/\mathfrak{q}}$ exist. By Theorem B.14 (see also Remark B.13), all lifts of $\gamma_0^{\text{new}}, \gamma_1^{\text{new}}$ to $W_{\mathbf{s}}$ exist, pairwise disjoint, and land at $\widetilde{0}$. Projecting to $W_{\mathbf{s}/\mathfrak{q}}$, we obtain that all lifts of $\gamma_0^{\text{new}}, \gamma_1^{\text{new}}$ to $W_{\mathbf{s}/\mathfrak{q}}$ exist, pairwise disjoint, and land at 0. The lifts of $\gamma_0^{\text{new}}, \gamma_1^{\text{new}}$ split $W_{\mathbf{s}/\mathfrak{q}}$ into $\mathfrak{q}$ closed sectors; each of them is a lift of either $\mathbf{A}_{\text{new}}$ or $\mathbf{B}_{\text{new}}$. Mapping this lifts of $\mathbf{A}_{\text{new}}$ or $\mathbf{B}_{\text{new}}$ to the corresponding sectors of $W_{\mathbf{s}/\mathfrak{q},\text{new}}$, we obtain a required h.

Since a lift of a curve (if it exists) is uniquely specified by a starting point, the conjugacy h is unique. □

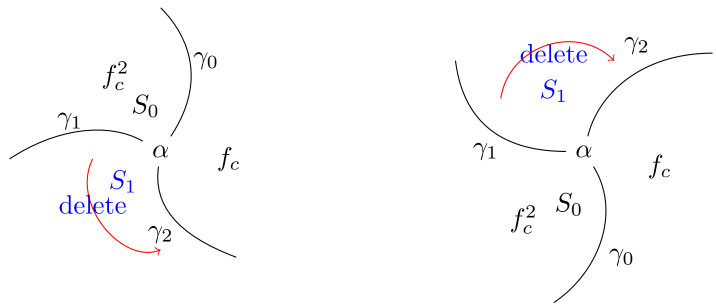


FIGURE 30. Possible local dynamics at the α-fixed point.

Remark B.15. *Anti-renormalization can easily be defined for a partial branched covering $f_0\colon (W,0) \dashrightarrow (W,0)$ of any degree. In this case it is natural to assume that γ_0 does not contain a critical point of f. To apply Theorem B.8, it is sufficient to assume that there is a* univalent *fence Q (respected by γ_0, γ_1, f) enclosing Ω such that $f \mid Q \cup \Omega$ has degree one. The anti-renormalization is robust with respect to a replacement γ_0, γ_1 with a new pair $\gamma_0^{\text{new}}, \gamma_1^{\text{new}}$ as above.*

APPENDIX C. THE MOLECULE RENORMALIZATION

Let us denote by $\boldsymbol{\mathcal{M}ol}$ the main molecule of the Mandelbrot set; i.e. $\boldsymbol{\mathcal{M}ol}$ is the smallest closed subset of $\mathcal{M}$ containing the main hyperbolic component as well as all hyperbolic components obtained from the main component via parabolic bifurcations; see [DH1, L2] for the background on the Mandelbrot set. In this appendix we write $f_c(z) = z^2 + c$.

C.1. **Branner-Douady maps.** Let us denote by $\mathcal{L}_{\mathfrak{p}/\mathfrak{q}}$ the primary $\mathfrak{p}/\mathfrak{q}$-limb of the Mandelbrot set and let us denote by $\mathcal{M}_{\mathfrak{p}/\mathfrak{q}} \subset \mathcal{L}_{\mathfrak{p}/\mathfrak{q}}$ the $\mathfrak{p}/\mathfrak{q}$-satellite small copy of $\mathcal{M}$. We also write $\mathcal{L}_{0/1} = \mathcal{M}_{0/1} = \mathcal{M}$.

In [BD] Branner and Douady constructed a partial surjective continuous map $\boldsymbol{R}_{\text{prm}} : \mathcal{L}_{1/3} \dashrightarrow \mathcal{L}_{1/2}$ such that its inverse $\boldsymbol{R}_{\text{prm}}^{-1}\colon \mathcal{L}_{1/2} \to L_{1/3}$ is an embedding. This construction could be easily generalized to a continuous map $\boldsymbol{R}_{\text{prm}} : \mathcal{L}_{\mathfrak{p}/\mathfrak{q}} \dashrightarrow \mathcal{L}_{\boldsymbol{R}_{\text{prm}}(\mathfrak{p}/\mathfrak{q})}$, where (compare to (A.2))

$$R_{\text{prm}}(\mathfrak{p}/\mathfrak{q}) = \begin{cases} \frac{\mathfrak{p}}{\mathfrak{q}-\mathfrak{p}} & \text{if } 0 < \frac{\mathfrak{p}}{\mathfrak{q}} \le \frac{1}{2} \\ \frac{2\mathfrak{p}-\mathfrak{q}}{\mathfrak{p}} & \text{if } \frac{1}{2} \le \frac{\mathfrak{p}}{\mathfrak{q}} < 1, \end{cases}$$

as follows. Recall that $c \in \mathcal{L}_{\mathfrak{p}/\mathfrak{q}}$ if and only if in the dynamical plane of f_c there are exactly $\mathfrak{q}$ external rays landing at the α-fixed point and the rotation number of these rays is $\mathfrak{p}/\mathfrak{q}$; i.e. if γ is a ray landing at α, then there are $\mathfrak{p}-1$ rays landing at α between γ and $f_c(\gamma)$ counting counterclockwise.

Choose an external ray γ_0 landing at α in the dynamical plane of f_c with $c \in \mathcal{L}_{\mathfrak{p}/\mathfrak{q}}$. Define $\gamma_1 = f_c(\gamma_0)$ and $\gamma_2 = f_c(\gamma_1)$. Denote by S_0 the open sector between γ_0 and γ_1 not containing γ_2, see Figure 30. Similarly, let S_1 be the open sector between γ_1 and γ_2 not containing γ_0. We assume that γ_0 is chosen such that S_1 does not contain the critical value, thus S_1 has two conformal lifts, one of them is S_0, we denote by S_0' the other. If $S_1 \supset S_0'$, then replace S_0' by its unique lift in $\mathbb{C} \setminus S_1$.

Let us delete S_1, glue γ_1 and γ_2 dynamically $\gamma_1 \ni x \sim f(x) \in \gamma_2$, and iterate f_c twice on S_0. We obtain a new map denoted by $\bar{f}_c\colon \mathbb{C} \setminus S_0' \to \mathbb{C}$. The *filled-in Julia set* $\overline{K}_c$ of $\bar{f}_c$ is the set of points with bounded orbits that do not escape to S_0'. The set $\overline{K}_c$ is connected if and only if 0 does not escape to S_0'; in this case the new local dynamics of $\bar{f}_c$ at α has rotation number $R_{\text{prm}}(\mathfrak{p}/\mathfrak{q})$ and, moreover, $\bar{f}_c$ is hybrid equivalent to a quadratic polynomial $f_{\boldsymbol{R}_{\text{prm}}(c)}$ with $c \in \mathcal{L}_{R_{\text{prm}}(\mathfrak{p}/\mathfrak{q})}$. This defines the map $\boldsymbol{R}_{\text{prm}} : \mathcal{L}_{\mathfrak{p}/\mathfrak{q}} \dashrightarrow \mathcal{L}_{\mathcal{R}_{\text{prm}}(\mathfrak{p}/\mathfrak{q})}$.

In general, $\boldsymbol{R}_{\text{prm}} : \mathcal{L}_{\mathfrak{p}/\mathfrak{q}} \dashrightarrow \mathcal{L}_{R_{\text{prm}}(\mathfrak{p}/\mathfrak{q})}$ depends on the choice of γ_0. However, if $c \in \mathcal{M}_{\mathfrak{p}/\mathfrak{q}}$, then $\boldsymbol{R}_{\text{prm}}(c) \in \mathcal{M}_{R_{\text{prm}}(\mathfrak{p}/\mathfrak{q})}$ and $\boldsymbol{R}_{\text{prm}} : \mathcal{M}_{\mathfrak{p}/\mathfrak{q}} \to \mathcal{M}_{R_{\text{prm}}(\mathfrak{p}/\mathfrak{q})}$ coincides with the canonical homeomorphism between small copies of the Mandelbrot set.

C.2. **The molecule and the fast molecule maps.** Denote by $\boldsymbol{\Delta}$ the main hyperbolic component of $\mathcal{M}$. Recall that a parameter $c \in \partial\boldsymbol{\Delta}$ is parametrized by the multiplier $\mathbf{e}(\theta(c))$ of its non-repelling fixed point. We define *the molecule map* $\boldsymbol{R}_{\text{prm}}\colon \mathcal{M} \dashrightarrow \mathcal{M}$ such that

- $\boldsymbol{R}_{\text{prm}}\colon \mathcal{L}_{\mathfrak{p}/\mathfrak{q}} \dashrightarrow \mathcal{L}_{R_{\text{prm}}(\mathfrak{p}/\mathfrak{q})}$ is the Branner–Douady renormalization map for $\mathfrak{p}/\mathfrak{q} \neq 0/1$ and for some choice of γ_0; and
- if $c \in \partial\boldsymbol{\Delta}$, then $\boldsymbol{R}_{\text{prm}}(c)$ is so that

$$\theta(\boldsymbol{R}_{\text{prm}}(c)) = \begin{cases} \frac{\theta(c)}{1-\theta(c)} & \text{if } 0 \le \theta(c) \le \frac{1}{2}, \\ \\ \frac{2\theta(c)-1}{\theta(c)} & \text{if } \frac{1}{2} \le \theta(c) \le 1. \end{cases}$$

Siegel parameters of periodic type are exactly periodic points of $\boldsymbol{R}_{\text{prm}} \mid \partial\boldsymbol{\Delta}$ (Lemma A.2). Furthermore, for a satellite copy of the Mandelbrot set $\mathcal{M}_s$, there is an $n \geq 1$ such that $\boldsymbol{R}^n_{\text{prm}}\colon \mathcal{M}_s \to \mathcal{M}$ is the Douady-Hubbard straightening map.

The map $\boldsymbol{R}_{\text{prm}}\colon \mathcal{M} \dashrightarrow \mathcal{M}$ is combinatorially modeled by $Q(z) \coloneqq z(z+1)^2$, see Figure 31. The latter map has a unique parabolic fixed point as 0. The attracting basin of 0 contains exactly one critical point of Q. The second critical point is a preimage of 0. Denote by F the invariant Fatou component of Q. We can extend $\boldsymbol{R}_{\text{prm}}$ to $\boldsymbol{\Delta}$ so that $\boldsymbol{R}_{\text{prm}} \mid \overline{\boldsymbol{\Delta}}$ is conjugate, say by π, to $Q \mid \overline{F}$. Then π extends uniquely to a monotone continuous map $\pi : \boldsymbol{\mathcal{M}ol} \to K_Q$ semi-conjugating $\boldsymbol{R}_{\text{prm}} \mid \boldsymbol{\mathcal{M}ol}$ and $Q \mid K_Q$, where K_Q is the filled-in Julia set of Q:

$$\begin{array}{ccc} \boldsymbol{\mathcal{M}ol} & \xrightarrow{\ \boldsymbol{R}_{\text{prm}}\ } & \boldsymbol{\mathcal{M}ol} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ K_Q & \xrightarrow{\ \ Q\ \ } & K_Q \end{array}$$

If the MLC-conjecture holds, then π is a homeomorphism.

For every $c \in \partial\Delta \setminus \{\text{cusp}\}$ define $\mathfrak{n}(c) \coloneqq \mathfrak{n}(\theta_c)$, where θ_c is the rotation number of f_c and $\mathfrak{n}(\theta)$ is specified by $R_{\text{fast}}(\theta) = R^{\mathfrak{n}(\theta)}_{\text{prm}}(\theta)$, see §A.2. For every $c \in \mathcal{L}_{\mathfrak{p}/\mathfrak{q}}$ define $\mathfrak{n}(c) \coloneqq \mathfrak{n}(c_{\mathfrak{p}/\mathfrak{q}})$, where $c_{\mathfrak{p}/\mathfrak{q}}$ is the root of $\mathcal{L}_{\mathfrak{p}/\mathfrak{q}}$. The *fast Molecule map* is a partial map on $\mathcal{M}$ defined by

$$\boldsymbol{R}_{\text{fast}}(c) = \boldsymbol{R}^{\mathfrak{n}(c)}_{\text{prm}}(c).$$

The restriction $\boldsymbol{R}_{\text{fast}} \mid \partial\boldsymbol{\mathcal{M}ol} \setminus \{\text{cusp}\}$ is continuous but it does not extend continuously to the cusp: $\mathcal{R}_{\text{fast}}(\partial\mathcal{M}_{1/n}) = \partial\mathcal{M}$.

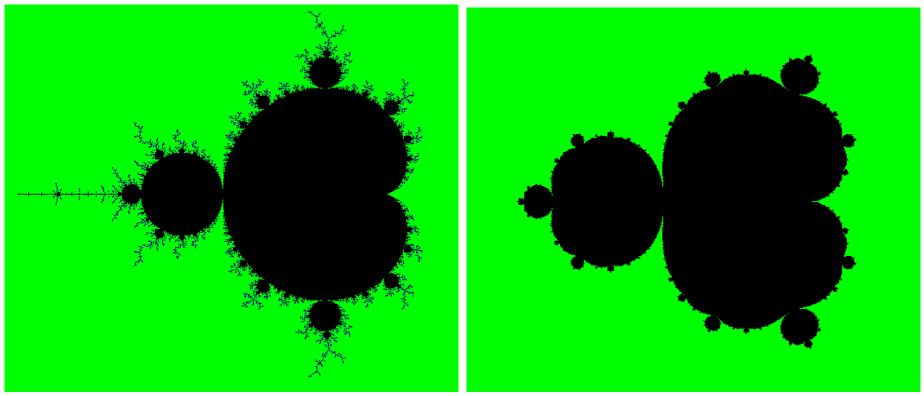


FIGURE 31. Left: the Mandelbrot set. Right: the filled Julia set of $Q(z) = z(z+1)^2$.

C.3. **Hyperbolicity theorem for the bounded type.** Given a renormalization operator $\mathcal{R}\colon \mathcal{B} \dashrightarrow \mathcal{B}$, its *renormalization horseshoe* is the set of points in $\mathcal{B}$ with bi-infinite pre-compact orbits.

Recall that for a map $f\colon X \to X$ its *natural extension* is the set of orbits:

$$\varprojlim_{f} X \coloneqq \{(x_i)_{i\in\mathbb{Z}} \mid f(x_i) = x_{i+1}\}$$

endowed with induced topology from $X^{\mathbb{Z}}$. We denote by

$$\widehat{R}_{\text{prm}}\colon \varprojlim_{R_{\text{prm}}} \Theta_N \to \varprojlim_{R_{\text{prm}}} \Theta_N$$

the natural extension of $R_{\text{prm}} \mid \Theta_N$. The latter map corresponds to the set of parameters in $\partial\boldsymbol{\Delta}$ that do not visit a certain neighborhood of the cusp under $\boldsymbol{R}_{\text{prm}} \mid \partial\boldsymbol{\Delta}$.

Theorem C.1 (Horseshoe of bounded type)**.** *For every $N > 1$ there is*

- *a space of pacmen $\mathcal{B}_N$ endowed with complex structures modeled on families of Banach spaces, see* [L1, Appendix 2]*;*
- *a compact hyperbolic analytic pacman renormalization operator $\mathcal{R}_{\text{prm}}\colon \mathcal{B}_N \dashrightarrow \mathcal{B}_N$;*

such that the renormalization horseshoe $\mathcal{R}_{\text{prm}}\colon \mathcal{H}_N \to \mathcal{H}_N$ has the following property

- *$\mathcal{H}_N$ is compact and consists of Siegel Pacmen with rotation numbers in Θ_N;*
- *$\mathcal{R}_{\text{prm}} \mid \mathcal{H}_N$ is topologically conjugate to $\widehat{R}_{\text{prm}} \mid \varprojlim_{R_c} \Theta_N$ via the map evaluating the rotation number of pacmen;*
- *at every $f \in \mathcal{H}_N$, there is a stable codimension-one manifold $\mathcal{W}^s_f$ and an unstable one-dimensional manifold $\mathcal{W}^u_f$; moreover $(\mathcal{W}^s_f)_{f\in\mathcal{H}_N}$ and $(\mathcal{W}^u_f)_{f\in\mathcal{H}_N}$ form invariant laminations;*
- *for every $f \in \mathcal{H}_N$, the stable manifold $\mathcal{W}^s_f$ coincides with the set of pacmen in $\mathcal{B}$ that have the same multiplier at the α-fixed point as f; every pacman in $\mathcal{W}^s_f$ is Siegel; all of the pacmen in $\mathcal{W}^s_f$ are hybrid conjugate;*
- *in a small neighborhood of $f \in \mathcal{H}_N$ the unstable manifold $\mathcal{W}^u_f$ is parametrized by the multipliers of the α-fixed points of pacmen in $\mathcal{W}^u_f$.*

Outline of the proof. Techniques of [McM2] imply the existence of a horseshoe $\mathcal{H}_N$ of Siegel maps parametrized by $\widehat{R}_{\text{prm}} \mid \varprojlim_{R_c} \Theta_N$ and endowed with some renormalization operator $\mathcal{R}$. Moreover, Siegel maps with rotation numbers in Θ_N converge to $\mathcal{H}_N$ exponentially fast (see §7.2).

Applying Corollary 3.7, we promote each map in $\mathcal{H}_N$ to a standard pacman. Covering $\mathcal{H}_N$ by finitely many Banach balls and applying Theorem 2.7, we extend $\mathcal{R}$ to a compact analytic operator $\mathcal{R}\colon \mathcal{B}_N \dashrightarrow \mathcal{B}_N$. The action of $\mathcal{R}$ on the rotation numbers of pacmen in $\mathcal{B}$ is some iterate of R_{prm}.

Since $\mathcal{R}$ is compact, at each $f \in \mathcal{H}_N$ there is an unstable finite-dimensional manifold $\mathcal{W}^u_f$; the operator $\mathcal{R}$ restricts to an expanding map $\mathcal{R}\colon \mathcal{W}^u_f \dashrightarrow \mathcal{W}^u_{\mathcal{R}f}$.

Every map $g \in \mathcal{W}^u_f$ has a maximal prepacman $\mathbf{G} = (\mathbf{g}_\pm)$ unique up to affine rescaling. We normalize $\mathbf{G}$ so that 0 and 1 project to the critical value and the critical point of g respectively. Both $\mathbf{g}_\pm$ are σ-proper.

As in §7, define

$$\mathcal{F}(\lambda) := \{g \in \mathcal{W}^u_f \mid \text{the multiplier of } \alpha \text{ is } \lambda\}$$

and consider a sequence $\mathfrak{p}_n/\mathfrak{q}_n \to \theta_f$, where θ_f is the rotation number of f. The orbit $\text{orb}_0(\mathbf{G})$ moves holomorphically with $g \in \mathcal{F}(\mathbf{e}(\mathfrak{p}_n/\mathfrak{q}_n))$; passing to the limit we obtain that $\text{orb}_0(\mathbf{G})$ moves holomorphically with $g \in \mathcal{F}(\mathbf{e}(\theta_f))$. By rigidity of Siegel pacmen, $\dim(\mathcal{W}^u_f) = 1$. The small orbit argument implies that $\text{codim}(\mathcal{W}^s_f) = 1$.

Applying Theorem 2.7, we factorize $\mathcal{R}$ as an iterate $\mathcal{R}_{\text{prm}}$. By shrinking $\mathcal{B}_N$ we can guarantee that $\mathcal{H}_N$ is the set of all pacmen in $\mathcal{B}_N$ with bi-infinite orbit. The assertion that unstable manifolds are parametrized by the multipliers of the α-fixed points is straightforward. By shrinking $\mathcal{B}_N$, we can guarantee that stable manifolds consist of Siegel pacmen. □

Theorem C.1 implies a general version of the scaling theorem (Theorem 8.2): the centers of all the satellite hyperbolic components of $\mathcal{M}$ of bounded type (i.e. with rotation numbers in Θ_N) scale uniformly around all the Siegel polynomials of bounded type (i.e. with rotation numbers in Θ_N) with the rate determined by the approximation rate of the continued fraction expression of $\theta \in \Theta_N$.

C.4. **The Molecule Conjecture.** We conjecture that there is a pacman renormalization operator $\mathcal{R}_{\text{fast}}\colon \mathcal{B}_{\boldsymbol{\mathcal{M}ol}} \to \mathcal{B}_{\boldsymbol{\mathcal{M}ol}}$ with the following properties. The operator $\mathcal{R}_{\text{fast}}$ is hyperbolic and piecewise analytic with one-dimensional unstable direction such that its renormalization horseshoe $\mathcal{R}_{\text{fast}}\colon \mathcal{H}_{\boldsymbol{\mathcal{M}ol}} \to \mathcal{H}_{\boldsymbol{\mathcal{M}ol}}$ is compact and combinatorially associated with $\mathcal{R}_{\text{fast}} \mid \boldsymbol{\mathcal{M}ol} \setminus \{\text{cusp}\}$ as follows.

There is a continuous surjective map $\rho\colon \mathcal{H}_{\boldsymbol{\mathcal{M}ol}} \to \boldsymbol{\mathcal{M}ol}$ that is a semi-conjugacy away from the cusp:

$$\begin{array}{ccc} \mathcal{H}_{\boldsymbol{\mathcal{M}ol}} \setminus \rho^{-1}(\text{cusp}) & \xrightarrow{\quad \mathcal{R}_{\text{fast}} \quad} & \mathcal{H}_{\boldsymbol{\mathcal{M}ol}} \setminus \rho^{-1}(\text{cusp}) \\ \Big\downarrow \rho & & \Big\downarrow \rho \\ \partial \boldsymbol{\mathcal{M}ol} \setminus \{\text{cusp}\} & \xrightarrow{\quad \boldsymbol{R}_{\text{fast}} \quad} & \partial \boldsymbol{\mathcal{M}ol} \setminus \{\text{cusp}\} \end{array}.$$

Denote by $\partial^{\text{irr}}\boldsymbol{\mathcal{M}ol}$ the set of non-parabolic parameters in $\partial\boldsymbol{\mathcal{M}ol}$. Conjecturally, $\mathcal{R}_{\text{fast}} \mid \mathcal{H}_{\boldsymbol{\mathcal{M}ol}}$ is the natural extension of $\boldsymbol{R}_{\text{fast}} \mid \partial\boldsymbol{\mathcal{M}ol} \setminus \{\text{cusp}\}$ compactified by adding limits to parabolic parameters at all possible directions. Such construction is known as a parabolic enrichment, see [La, D2].

The space $\mathcal{B}_{\boldsymbol{\mathcal{M}ol}}$ has a codimension-one stable lamination $(\mathcal{F}^s_c)_{c\in\boldsymbol{\mathcal{M}ol}}$ such that all pacmen in $\mathcal{F}^s_c$ are hybrid conjugate to f_c in neighborhoods of their "mother hedgehogs", see §C.5. For every $f \in \mathcal{H}_{\boldsymbol{\mathcal{M}ol}}$, the leaf $\mathcal{F}^s_{\rho(f)}$ is a stable manifold of $\mathcal{R}_{\text{fast}}$ at f. The unstable manifold of $\mathcal{R}_{\text{fast}}$ at f is parametrized by a neighborhood of $\rho(f)$. Locally, $\mathcal{R}_{\text{fast}}$ can be factorize as an iterate of $\mathcal{R}_{\text{prm}}\colon \mathcal{B}_{\boldsymbol{\mathcal{M}ol}} \to \mathcal{B}_{\boldsymbol{\mathcal{M}ol}}$; however the latter operator has parabolic behavior at $\rho^{-1}(\text{cusp})$.

The Molecule Conjecture contains both Theorem C.1 (for bounded type parameters from $\partial\boldsymbol{\Delta}$) and the Inou-Shishikura theory [IS] (for high type parameters from $\partial\boldsymbol{\Delta}$). It also implies the local connectivity of the Mandelbrot set for all parameters on the main (and thus any) molecule.

C.5. **Conjecture on the upper semicontinuity of the mother hedgehog.** A closely related conjecture is the upper semicontinuity of the mother hedgehog. For a non-parabolic parameter $c \in \partial\boldsymbol{\Delta}$, consider the closed Siegel disk $\overline{Z}_c$ of f_c; if f_c has a Cremer point, then $\overline{Z}_c \coloneqq \{\alpha\}$. If $\overline{Z}_c$ contains a critical point, then we set $H_c \coloneqq \overline{Z}_c$. Otherwise, f_c has a *hedgehog* (see [PM]): a compact closed connected filled-in forward invariant set $H' \supsetneqq \overline{Z}_c$ such that $f_c\colon H' \to H'$ is a homeomorphism. We define H_c to be the *mother hedgehog* (see [Ch]): the closure of the union of all of the hedgehogs of f_c.

Recall that the filled-in Julia set K_g of a polynomial depends upper semicontinuously on g. Thinking of H_c as an indifferent-dynamical analogue of K_g, we conjecture:

Conjecture C.2. The mother hedgehog H_c depends upper semicontinuously on c.

For bounded type parameters (i.e. when H_c is a Siegel quasidisk) Conjecture C.2 follows from the continuity of the Douady-Ghys surgery. In fact, in the bounded case H_c depends continuously on c. Theorem C.1 implies a general version of Corollary 7.9: for bounded combinatorics the Siegel quasidisk of a Siegel map depends continuously on the map.

Conjecture C.2 can be adjusted for parabolic parameters $c \in \boldsymbol{\Delta}$ as follows. Let A_c be the immediate attracting basin of the parabolic fixed point α. Then there is a choice of a valuable flower H_c with $\overline{H}_c \subset A_c \cup \{\alpha\}$ such that H_c depends upper semicontinuously on $c \in \partial\boldsymbol{\Delta}$. For example, H_c is the union of all limiting mother hedgehogs for perturbations of f_c.

Similarly, Conjecture C.2 can be adjusted for all parameters in $\partial\boldsymbol{\mathcal{M}ol}$. Our result on the control of the valuable flower (see Theorem 8.2) can be thought as a partial case of this general conjecture.

Conjecture C.2 and its generalizations describe in a convenient way how an attracting fixed point bifurcates into repelling. An important consequence is control of the post-critical set: if a perturbation of f_c is within $\boldsymbol{\mathcal{M}ol}$, then the new post-critical set is within a small neighborhood of H_c. A statement of this sort (for parabolic parameters approximating a Siegel polynomial) was proven by Buff and Chéritat, see [BC, Corollary 4]. This was a necessary ingratiate in constructing a Julia set with positive measure.

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