STUDIES IN TOPOLOGICAL DYNAMICS WITH EMPHASIS ON CELLULAR AUTOMATA

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by

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CERTIFICATE

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This is to certify that I, T.K. Subrahmonian Moothathu, have carried out the research embodied in the present thesis entitled **STUDIES IN TOPOLOGICAL DYNAMICS WITH EMPHASIS ON CELLULAR AUTOMATA** for the full period prescribed under Ph.D. ordinance of the University.

I declare that, to the best of my knowledge, no part of this thesis was earlier submitted for the award of research degree of any university.

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Dedicated to

the solitary dust-sweeper in the Library.

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ABSTRACT

There are four chapters: two on general Topological Dynamical Systems and two on the Dynamics of one-dimensional cellular automata (CA).

Chapter 1 has some results on Topological Dynamics. They are arranged in such a way that the Chapter serves also as a general introduction. We highlight the main points:

1. For any $n \in \mathbb{N}$, distinct continuous surjections $f_1, f_2, \ldots, f_n : [0,1] \to [0,1]$ are constructed in such a way that any two are factors of each other but no two distinct ones are topologically conjugate.
2. It is shown that for any interval map $f : [0,1] \to [0,1]$, if the set of sensitive points of f is dense, then so are the set of points with finite orbit, and the set of points with infinite orbit.
3. We demonstrate how to construct transitive maps on $\mathbb{R}$, arbitrarily close to the identity map or the reflection map $x \mapsto -x$.
4. A transitive map with the set of visiting times between any two non-empty open sets being syndetic is called syndetically transitive. It is proved that any syndetically transitive map is either minimal or sensitive, from which it is easily deduced that sensitivity is a redundant condition in Devaney's definition of chaos.
5. If $f : X \to X$ is a transitive map of a compact metric space, then some properties of the induced maps on certain hyperspaces (eg: hyperspace of all non-empty compact subsets of X) are investigated. In particular, it is shown that certain induced maps can never be transitive.
6. For $k \in \mathbb{N}$, a rational approximation to $\sqrt{k}$ with exponential rate of convergence is given using a logistic map.

In Chapters 2, CA are studied as Topological Dynamical Systems. Some of the main results on CA that we obtain are: (1) any minimal set is nowhere dense and of zero measure, (2) any orbit is either dense or nowhere dense, (3) transitivity implies weak mixing and hence maximal sensitivity, (4) product of transitive CA is transitive, (5) the set of periodic points of each period is finite if and only if all periodic points are shift-periodic, (6) recurrent points are residual for surjective CA, and (7) all surjective CA are semi-open.

Chapter 3 is devoted to the solution of a single problem: determining the set of periods of additive CA in terms of the coefficients appearing in the linear expression of the CA.

For $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the collection of all additive CA where the addition is done modulo n. Let p be any prime. The highlights are:

(1) For any $F \in \mathcal{F}_p$, the set of periods can be determined using some simple conditions on the coefficients in the linear expression of F.
(2) For any $F \in \mathcal{F}_p$, the set of periods has only four possibilities: $\{1, m\}$ for some m where $1 \leq m < p$, $\mathbb{N} \setminus \{p^m : m \in \mathbb{N}\}$, $\mathbb{N} \setminus \{2p^m : m \in \mathbb{N} \cup \{0\}\}$ or the whole set $\mathbb{N}$.
(3) If $F \in \mathcal{F}_p$, then our proof actually calculates the cardinality of the set $\{x : F^n(x) = x\}$, which is shown to be a power of p except when F is a root of identity.
(3) Using our results, the set of periods of any additive CA, where the addition is done modulo some square-free positive integer, is easily obtained.

In Chapter 4 we go back to the general theory of Topological Dynamics. We show that:

(1) Weak mixing implies mixing for all subshifts of finite type and many sofic shifts.
(2) Any transitive subshift of finite type, which is not a periodic orbit, has the property that one of its powers has the full shift on two symbols as a factor.
(3) If (X, f) is a transitive system on [0,1], or a transitive CA, or a transitive subshift of finite type, then any map which is a limit point of the enveloping semigroup of (X, f), is nowhere continuous on X.

Chapter 4 is concluded by characterizing the ω-limit sets of the shift map in terms of words.

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Contents

Chapter 1

A flavor of Topological Dynamics

1.1 To start with...

Topological Dynamics is mainly concerned with the asymptotic behavior of the orbits of continuous self maps of Hausdorff topological spaces. In olden days the focus was on homeomorphisms. But now, the dynamics of non-invertible continuous maps has well-founded theory [4], [14], [19], [23], [28], [34], [36], [51]. Most of this theory is developed on compact metric spaces as compactness assures the existence of various limiting objects.

In Topological Dynamics, the word **chaos** has become a password. A handful of non-equivalent notions of chaos are available. Analyzing the terms involved in their definitions is one way to get into the theory. Another door of approach is to narrow one's attention to some special class of dynamical systems extensively studied in the literature, such as *interval maps* [60] or *subshifts* [51]. We prefer a third approach, where the concepts are introduced in such a way as to facilitate us to include some of our contributions. Possibly there will be a little bit of awkwardness because of this self-interest, and not all important concepts may find a place in what follows. For example, we do not consider *entropy* [56] and *expansivity* [44], [46]. But we hope that the content of this chapter has enough vitality to provide a few sparks of inspiration for future researchers.

We begin with basic definitions.

Phase space: this is a Hausdorff topological space X where dynamics is thought to be happening. For us, X will be a compact metric space most of the time. Note that Baire Category Theorem has life on compact metric spaces. This Theorem is invariably invoked to get results of the following form: the set of points satisfying such and such dynamical properties is a dense G_δ.

Rule of evolution: this is a continuous map $f : X \to X$. In this case, the pair (X, f) is called a **discrete dynamical system** or simply a **dynamical system**. The role of f as a rule of evolution is to be imagined as follows. If $x \in X$, then $f(x)$ is viewed as the "new position" of x after one unit of time, $f(f(x))$ as the "new position" of x after two units of time, and so on. For brevity, we denote $f \circ f$ by f^2. More generally, for $n \in \mathbb{N}$, we denote by f^n, the n-fold composition of f with itself. If $x \in X$, then (according to the context) the set or sequence $\{x, f(x), f^2(x), f^3(x), \ldots\}$ is called the **orbit** of x under the action of f. Note that time is measured by $\mathbb{N}$ in discrete units (hence the adjective *discrete* before *dynamical systems*), and that the rule f is assumed to be invariant in time.

Study of asymptotic behavior: this is the investigation of questions like: what can be said about the orbit $\{x, f(x), f^2(x), f^3(x), \ldots\}$ as $n \to \infty$? Asymptotic analysis is a unique feature of the theory, distinguishing it from many of its siblings, for instance, from the theory of *Group Actions*. It should be mentioned that the term *chaos* first appeared in the literature (in the work of Li and Yorke [50] in 1975) purely in connection with the asymptotic aspects of orbits. In fact we feel like having a few more lines of explanation. Assume that X is a compact metric space with a metric d, and let $f : X \to X$ be continuous. A subset $S \subset X$ with at least two points is called a **scrambled set** (see also [7], [15], [45], [53]) for f if for every pair of distinct points $x, y \in S$, one has that

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

The condition given above roughly means that, the orbits of x and y interact with each other in a complicated way, coming arbitrarily close together and then moving away infinitely often, like a pair of fighting lovers. The presence of such pairs of points certainly indicates some kind of complexity in the system. Originally, Li and Yorke introduced the term *chaos* in the context of their work on interval maps. In modern language, any compact dynamical system (X, f) possessing an uncountable scrambled set is said to have **Li-Yorke chaos**. In Chapter 2, we deduce that any *transitive* cellular automata is Li-Yorke chaotic.

Recurrence and other pointwise notions: the idea of a point x returning or coming arbitrarily close to itself after many iterations by the rule f, is captured by various definitions, see Section 1.4. A significant part of our work on cellular automata deals with such points. *Periodic point* is the simplest where one has that $f^n(x) = x$ for some $n \in \mathbb{N}$. In the same box, one has *minimal points*, *recurrent points* and *non-wandering points* (see also [19]). Even in very complex systems such points may be found in abundance, and this can be perceived as an indication that, in many situations what appears to be utter disorder is perhaps a mixture of various regularities at a deeper level. Sometimes pointwise notions have a global effect also, as is exemplified by the points having dense orbits. The existence of such points is closely related to the indecomposability of the system.

Devaney chaos [28]: it is one of the most popular notions of chaos persisting in the theory. It implies Li-Yorke chaos [38] but not the other way. The ingredients of Devaney chaos are *transitivity, sensitivity* and *denseness of periodic points* - a rigorous definition is given later. Transitivity [42] is basically the indecomposability of the system, equivalent to the existence of a dense orbit in many natural settings, whereas sensitivity [33] roughly means that even small differences between two starting points in the phase space can lead to large deviations under repeated iterations by the rule f. Theory of sensitivity is relevant to experimental scientists, where one has to worry whether small errors in the initial conditions can be ignored or not in the long run. Many systems which are complex in an intuitive sense satisfy the definition of Devaney chaos. Another reason for the popularity of this notion of chaos is the sensational discovery that sensitivity is redundant in the definition [8]. We give a proof for this result in Section 1.7. Also, we have a few results on transitivity and sensitivity, around the corner.

Some classes of dynamical systems widely studied in the literature are:

- Interval maps [4], [19], [60].
- Subshifts [51].
- Rational maps of the complex sphere [11].
- Maps on the torus [5].
- Cellular automata [43], [49], [62].

Among these, the best-understood is perhaps the first one, or more precisely, the class of continuous maps of the closed unit interval [0,1]. Connectedness and the available linear order of the phase space are the keys controlling the dynamical behavior of maps on [0,1]. Even as simple a tool as the Intermediate Value Theorem can do magics on the dynamics there. **Sarkovski's Theorem** (c.f. [19]) which gives the set of periods for interval maps, and the result that transitivity implies Devaney chaos (for interval maps) [13] , are examples of this magic. Among the classes listed above, the least understood is perhaps that of cellular automata. We do not have a characterization even for Devaney chaos in the land of cellular automata.

1.2 An elementary amusement

Before we move onto more involved topics, let us have an elementary amusement concerning the orbits of the **logistic map**. The map $x \mapsto rx(1-x)$ on [0,1] for a fixed parameter $r \in [0,4]$ is called the logistic map. The family of logistic maps has been under scrutiny in connection with the phenomena known as *bifurcation*, see [28], [36].

Exercise: Let $f : [0,1] \to [0,1]$ be $f(x) = rx(1-x)$, where $r \in [2,4] \cap \mathbb{Q}$. Let $a + b\sqrt{k} \in [0,1]$, where $a, b \in \mathbb{Q}$, $a \leq 0 < b$ and $k \in \mathbb{N}$ is square-free. Then all terms in the f-orbit of $a + b\sqrt{k}$ are distinct.

Solution:

$$\begin{aligned} f(a+b\sqrt{k}) &= r(a+b\sqrt{k})(1-a-b\sqrt{k}) \\ &= r(a-a^2-b^2k)+r(b-2ab)\sqrt{k} \\ &= a_1+b_1\sqrt{k}, \text{ say.} \end{aligned}$$

Clearly, $a_1, b_1 \in \mathbb{Q}$. Since $a \leq 0 < b$, we have $a_1 = r(a-a^2-b^2k) < ra$ and $b_1 = rb(1-2a) \geq rb$. Inductively, $f^n(a+b\sqrt{k}) = a_n + b_n\sqrt{k}$ for some $a_n, b_n \in \mathbb{Q}$, where $a_{n+1} < ra_n \leq 0 < rb_n \leq b_{n+1}$. Hence, $f^n(a+b\sqrt{k}) \neq f^m(a+b\sqrt{k})$ for $n \neq m$, since k is square-free.

Amusement: All steps of the solution except the last are true for any $k \in \mathbb{N}$. Therefore, we can explicitly construct a rational approximation to $\sqrt{k}$ for any $k \in \mathbb{N}$ as follows. Let $a_0, b_0 \in \mathbb{Q}$ be such that $a_0 \leq 0 < b_0$ and $a_0 + b_0\sqrt{k} \in [0,1]$. For example, take $a_0 = 0$, $b_0 = \dfrac{1}{k}$ or $a_0 = -[\text{integer part of } \sqrt{k}]$, $b_0 = 1$. Recursively define $a_{n+1} = 4(a_n - a_n^2 - b_n^2 k)$, $b_{n+1} = 4(b_n - 2a_nb_n)$. Then, $a_n, b_n \in \mathbb{Q}$ for every n (if a_0, b_0 are integers, then so are a_n, b_n for all n). Also observe that $a_n + b_n\sqrt{k} = f^n(a_0 + b_0\sqrt{k})$, where $f(x) = 4x(1-x)$. Hence $a_n + b_n\sqrt{k} \in [0,1]$ for every n and therefore $\dfrac{a_n}{b_n} + \sqrt{k} \in [0, \dfrac{1}{b_n}]$ for every n. Since $b_{n+1} \geq 4b_n$, we have that $\dfrac{-a_n}{b_n} \to \sqrt{k}$ exponentially.

Problem: Can our reader think of similar constructions of rational approximations using dynamical systems, say, to roots of polynomials?

1.3 Being factors of each other

In the family of *logistic maps* mentioned above, are all the maps different in their dynamical behavior? This kind of classification problems are often highly non-trivial, and beyond our scope. But we would like to make it precise what it means to say that two dynamical systems are same or that they are different.

Topological conjugacy: this is the notion of equivalence for two dynamical systems. Let (X, f), (Y, g) be dynamical systems. If there is a continuous surjection $h : X \to Y$ such that $h \circ f = g \circ h$, then we say that (Y, g) is a **factor** of (X, f), or simply that g

is a factor of f. If in addition, h is a homeomorphism, then the two systems (or simply f and g) are said to be **topologically conjugate**. Being topologically conjugate is an equivalence relation on the class of all dynamical systems, and systems in the same equivalence class have the same dynamical behavior and for all practical purposes they are considered to be the same.

Topological Dynamics has certain similarities to Ergodic Theory [67]. In the setting of Ergodic Theory, examples of non-equivalent "dynamical systems" which are "factors" of each other are given in the recent book ([32], p.157) using the polished tool "Rudolf's counterexamples machine". Below we consider a similar task in Topological Dynamics.

Let (X, f) and (Y, g) be two dynamical systems. When (Y, g) is only a factor of (X, f), not all properties of f are inherited by g. Now, consider the situation where f and g are factors of each other. Does it follow that f and g have the same dynamical nature? In other words, does it follow that f and g are topologically conjugate? We show that the answer is no.

Here, our careful reader might ask: does one come across the situation where two dynamical systems are factors of each other, often? We point out how abundant such pairs are. Let $f, g : X \to X$ be continuous. Then, trivially, $g \circ (f \circ g) = (g \circ f) \circ g$ and $f \circ (g \circ f) = (f \circ g) \circ f$. Therefore, if f and g are surjective, then $f \circ g$ and $g \circ f$ are factors of each other! Inductively one has the following:

Lemma 1.3.1. Let $g_1, g_2, \ldots, g_n : X \to X$ be continuous surjections and let α, β permutations on $\{1, 2, \ldots, n\}$. If α and β differ by a cyclic permutation, then $g_{\alpha(1)} \circ g_{\alpha(2)} \circ \cdots \circ g_{\alpha(n)}$ and $g_{\beta(1)} \circ g_{\beta(2)} \circ \cdots \circ g_{\beta(n)}$ are factors of each other.

It follows that if $f, g : X \to X$ are homeomorphisms, then $f \circ g$ and $g \circ f$ are dynamically the same (topologically conjugate), even though $f \circ g \neq g \circ f$ in general. That is, in a "dynamical" sense, the group of self-homeomorphisms of X is "abelian", which looks pleasing.

Now, we show that for continuous self-maps of closed interval [0,1], the property of 'being factors of each other' is indeed far from equivalent to that of 'being topologically conjugate', by constructing examples. Observe the following two simple facts which we will be using:

Lemma 1.3.2. (a) For a self-homeomorphism h of [0,1], one must have $h(\{0,1\}) = \{0,1\}$.
(b) If (X, f) and (Y, g) are topologically conjugate via the homeomorphism $h : X \to Y$, and if $x \in X$ has k pre-images under f, then $h(x) \in Y$ has k pre-images under g.

Theorem 1.3.3. *For each natural number $n \geq 2$, there exist distinct continuous surjections $f_1, f_2, \dots, f_n : [0,1] \to [0,1]$ such that any two are factors of each other but no two distinct ones are topologically conjugate.*

Proof. Fix $n \geq 2$. Choose points $\{a_{ij} : 1 \leq j \leq i \leq n\}$ in $(0,1)$ such that $0 < a_{11} < a_{21} < a_{22} < \cdots < a_{n1} < a_{n2} < \cdots < a_{nn} < 1$. One can find continuous functions $g_1, g_2, \dots, g_n : [0,1] \to [0,1]$ which satisfy:

(i) $g_i(1) = 1$,
(ii) $g_i^{-1}(0) = \{a_{i1}, a_{i2}, \dots, a_{ii}\}$,
(iii) $g_i(x) < a_{11}$ for $x \in [0, a_{ii}]$,
(iv) the graph of g_i is linear on $[a_{ii}, 1]$.

Note that, these conditions imply that if A is a finite subset of $[a_{11}, 1]$, then $|g_i^{-1}(A)| = |A|$ for every i. Let $f_1 = g_1 \circ g_2 \circ \cdots \circ g_n$, $f_2 = g_2 \circ \cdots \circ g_n \circ g_1$, $\dots$, $f_n = g_n \circ g_1 \circ g_2 \circ \cdots \circ g_{n-1}$. Then, by Lemma 1.3.1, any two of $f_1, \dots, f_n$ are factors of each other. Now, note that $f_i^{-1}(1) = \{1\}$ and $|f_i^{-1}(0)| = i$ for every i. Hence, it follows from Lemma 1.3.2 that f_i and f_k are not topologically conjugate for $1 \leq i < k \leq n$. □

Problem: Is it possible to produce an infinite/uncountable family of continuous self maps of [0,1] such that any two of them are factors of each other but no two distinct members are topologically conjugate?

1.4 Definitions and other basic material

Below we list some more definitions, and a few known results that we need to proceed further. In this section, if a result is left alone without any reference or proof, it means the result is elementary and can easily be proved. In other cases, we provide either a reference or a proof, choosing the easier option.

Let (X, f) be a dynamical system. A subset $Y \subset X$ is said to be **f-invariant** if $f(Y) \subset Y$. Note that for any $x \in X$, the f-orbit of x, which we denote by $O_f(x)$, is f-invariant. For $Y \subset X$ and $n \in \mathbb{N}$, $f^{-n}(Y)$ is the set $\{x \in X : f^n(x) \in Y\}$.

Let (X, f) be a dynamical system. A point $x \in X$ is said to be:

1. **periodic** if $f^n(x) = x$ for some $n \in \mathbb{N}$ [the smallest such n is the **period** of x].
2. **eventually periodic** if $f^{n+k}(x) = f^k(x)$ for some $n, k \in \mathbb{N}$ [note that this is equivalent to saying that $O_f(x)$ is finite].
3. **recurrent** if $f^{n_k}(x) \to x$ for some increasing sequence (n_k) of natural numbers.
4. **non-wandering** if for any open set U containing x, there is $n \in \mathbb{N}$ such that $U \cap f^{-n}(U) \neq \emptyset$.

Let $P(f)$, $E(f)$, $R(f)$, and $\Omega(f)$ denote respectively the sets of all periodic points, all eventually periodic points, all recurrent points and all non-wandering points of f. Note that among these sets, $\Omega(f)$ is a closed set whereas the other sets are not closed in general [19]. In any dynamical system (X, f) we have $P(f) \subset E(f)$ and $P(f) \subset R(f) \subset \Omega(f)$.

Let (X, f) be a dynamical system. We say f is:

1. **transitive** if for any two non-empty open sets $U, V \subset X$, there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.
2. **totally transitive** if f^n is transitive for every $n \in \mathbb{N}$.
3. **weak mixing** if $f \times f$ on $X \times X$ is transitive.
4. **mixing** if for any two non-empty open sets $U, V \subset X$, there is $n \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$ for every $k \geq n$.

Proposition 1.4.1. *[31] Let (X, f) be a dynamical system. Then, f is weak mixing if and only if for any two non-empty open sets $U, V \subset X$, the set $\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$ is **thick** (that is, it contains arbitrarily large blocks of consecutive numbers).*

Proposition 1.4.2. *For all systems, mixing $\Rightarrow$ weak mixing $\Rightarrow$ total transitivity $\Rightarrow$ transitivity.*

Proposition 1.4.3. *For a dynamical system (X, f), the following are equivalent:*

(1) f is transitive.

(2) For any non-empty open set $V \subset X$, $\bigcup_{n=0}^{\infty} f^{-n}(V)$ is dense in X.

(3) Any f-invariant set $Y \subset X$ is either dense or nowhere dense in X.

For a dynamical system (X, f), let $D(f) = \{x \in X :$ the f-orbit of x is dense in $X\}$. The relation between $D(f)$ and transitivity is as follows:

Proposition 1.4.4. *If $D(f) \neq \emptyset$ and if X has no isolated points, then f is transitive. In the other direction, if f is transitive and if X is a compact metric space, then $D(f)$ is a dense G_δ subset of X.*

Proof. The first part is left as an exercise. Now, suppose that f is transitive and X is a compact metric space. Let $\{B_k : k \in \mathbb{N}\}$ be a countable base of (non-empty) open sets. Let $U_k = \bigcup_{n=0}^{\infty} f^{-n}(B_k)$. Then, U_k is open. Since, f is transitive, U_k is dense in X. Put $U = \bigcap_{k \in \mathbb{N}} U_k$. Then U is a G_δ set. By Baire Category Theorem, U is dense in X. It is a matter of direct checking that $x \in U$ if and only if the orbit of x visits every B_k. Since $\{B_k : k \in \mathbb{N}\}$ is a base, it follows that $U = D(f)$. $\square$

Proposition 1.4.5. *Let (X, f) be totally transitive. Then, $D(f^n) = D(f)$ for every $n \in \mathbb{N}$.*

Sets which are minimal in a natural sense among f-invariant sets are important in the theory, see [20] and references therein. The formal definition runs as follows. Let (X, f) be a dynamical system. A subset $M \subset X$ is called a **minimal set** for f if M is

(i) non-empty,

(ii) closed,

(iii) f-invariant, and

(iv) there is no proper subset of M satisfying (i), (ii) and (iii).

It may be noted that (iv) could be replaced by (iv'): for any $x \in M$, the f-orbit of x is dense in M.

In a dynamical system (X, f), if the closure of the f-orbit of a point $x \in X$ is a minimal set, then x is called a **minimal point**. It is a folklore result that $x \in X$ is a minimal point if and only if for every neighbourhood U of x, the set $\{n \in \mathbb{N} : f^n(x) \in U\}$ is **syndetic** (which means, an infinite set with bounded gaps). If the closure of the f-orbit of a minimal point is the whole of X, that is if X itself is a minimal set for f, then (X, f) is called a **minimal system** or f is called a **minimal map**. It is easy to see that every minimal map is **syndetically transitive**, where f is said to be syndetically transitive if for every non-empty open sets $U, V \subset X$, the set $\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$ is syndetic.

Let (X, f) be a dynamical system, where assume that X is a compact metric space. By an abuse of terminology, we say f is **equicontinuous** [2], [54] if the family $\{f^n : n \in \mathbb{N}\}$ is equicontinuous in the usual sense. On the other extreme we have the notion of *sensitivity.* An element $x \in X$ called a **point of sensitivity** for f if x fails to be a point of equicontinuity for the family $\{f^n : n \in \mathbb{N}\}$. That is, x is a point of sensitivity if there is some $\delta > 0$ with the property that for any neighbourhood U of x, there exists $n \in \mathbb{N}$ such that $diam[f^n(U)] > \delta$.

Let $S(f)$ and $S_\delta(f)$ denote respectively, the set of all sensitive points of f and the set of all δ-sensitive points of f. Note that $S(f) = \bigcup_{\delta > 0} S_\delta(f)$. If $S_\delta(f) = X$ for some $\delta > 0$, then f is said to be **sensitive**. We remark that it can happen that $S(f) = X$ but $S_\delta(f) \neq X$ for any $\delta > 0$. We say f is **maximally sensitive** if for every positive $\delta < diam[X]$, $S_\delta(f) = X$. Also observe that $S(f^n) = S(f)$ for every $n \in \mathbb{N}$.

Proposition 1.4.6. *For any dynamical system (X, f), where X is a compact metric space, weak mixing implies maximal sensitivity.*

We have two main notions of *chaos* when (X, d) is a compact metric space and $f : X \to X$ is continuous: *Li-Yorke chaos* and *Devaney chaos*, the first bieng defined already.

(X, f) is said to be **Devaney chaotic** if f is transitive, sensitive and has a dense set of periodic points.

Theorem 1.4.7. *[8] Let (X, f) be a dynamical system, where X is an infinite compact metric space. If f is transitive and has a dense set of periodic points, then f is sensitive and hence Devaney chaotic.*

A more general version of this result is given in Section 1.7.

Theorem 1.4.8. *[13] Any transitive $f : [0,1] \to [0,1]$ is Devaney chaotic.*

Theorem 1.4.9 (c.f. [42], [60]). *Any transitive $f : [0,1] \to [0,1]$ possesses a periodic point of period 6.*

We conclude this section by stating the classic Theorem of Sarkovski (c.f. [19]). The **Sarkovski ordering** of natural numbers is as below:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2.3 \rhd 2.5 \rhd 2.7 \rhd \cdots \rhd 2^2.3 \rhd 2^2.5 \rhd 2^2.7 \rhd \cdots$$
$$\rhd 2^n.3 \rhd 2^n.5 \rhd 2^n.7 \rhd \cdots \rhd 2^{n+1}.3 \rhd 2^{n+1}.5 \rhd 2^{n+1}.7 \rhd \cdots \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

Theorem 1.4.10 (Sarkovski's Theorem). *Let $f : [0,1] \to [0,1]$ be continuous. Let $n, m \in \mathbb{N}$ such that $n \rhd m$ in the ordering above. If f has a periodic point of period n, then f has a periodic point of period m.*

For instance, the Theorem says that the existence of period 3 implies the existence of all other periods for an interval map. Moreover, since Sarkovski's ordering is a total order, it follows that only countably many subsets of $\mathbb{N}$ can arise as the set of periods of interval maps.

1.5 Sensitivity on [0,1]

Let $f : [0,1] \to [0,1]$ be continuous throughout this section. Recall from the previous section that $D(f)$, $S(f)$, $P(f)$ and $E(f)$ denote respectively, the set of points with dense orbit, the set of sensitive points, the set of periodic points and the set of eventually periodic points. It is known (for interval maps) that the denseness of $D(f)$ implies that of $P(f)$, and that $D(f) \subset S(f)$ [$\because$ Theorem 1.4.8]. Below we show, among other things, that the denseness of $S(f)$ implies that of $E(f)$.

Theorem 1.5.1. *If $S(f)$ is dense in $[0,1]$ (in particular, if f is sensitive), then both $E(f)$ and its complement, $E(f)^c$, are dense in $[0,1]$.*

Proof. Recall that $S(f^n) = S(f)$ for every $n \in \mathbb{N}$. We show $E(f)$ is dense. Let $J \subset [0,1]$ be an open interval. Choose $x \in J \cap S(f)$. Then, there exist $\delta > 0$ and infinitely many n such that $diam[f^n(J)] > \delta$. Since $f^n(J)$'s are intervals, we can find $r, s \in \mathbb{N}$ such that

$$f^r(J) \cap f^{r+s}(J) \neq \emptyset.$$

Since $S(f)$ is dense, f cannot collapse an interval to a singleton. Therefore by induction,

$$f^{(k-1)s}(f^r(J)) \cap f^{ks}(f^r(J)) \neq \emptyset \text{ for } k = 1, 2, \ldots.$$

Put $g = f^s$ and $L = \bigcup_{k=0}^{\infty} g^k[f^r(J)]$. Then L is an interval and $g(L) \subset L$.

Case 1: $g^n(y) = y$ for some $y \in L$ and $n \in \mathbb{N}$. Then, $y = f^{ks+r}(z)$ for some $z \in J$ and $z \in E(f)$.

Case 2: $g^n(y) \neq y$ for every $y \in L$ and $n \in \mathbb{N}$. Then, either $g(y) > y$ for every $y \in L$ or $g(y) < y$ for every $y \in L$. It follows that the g-orbit of any point of L is a monotone sequence and hence converges to a point of $\overline{L} \setminus L$, which must be a fixed point of g. Hence L cannot contain any sensitive point, contradicting our assumption that $S(f)$ is dense.

Now we show $E(f)^c$ is dense. If $E(f)^c$ is not dense, there is a closed interval J contained in $E(f)$. For $m, n \in \mathbb{N}$, define $A_{m,n} = \{x \in J : f^{m+n}(x) = f^m(x)\}$, which is a closed set. Therefore, each $A_{m,n}$ is closed and $J = \bigcup_{m,n \in \mathbb{N}} A_{m,n}$. By Baire Category Theorem, for at least one pair (m,n), $A_{m,n}$ contains a non-trivial interval, say L. Then, $f^m(L)$ is also a non-trivial interval (f cannot collapse intervals since $S(f)$ is dense) and f^n restricted to L is the identity map. Therefore, f^n has no point of sensitivity in the interior of L, and hence $S(f^n) = S(f)$ cannot be dense. Thus we have a contradiction. $\square$

Question: Let $f : [0,1] \to [0,1]$ be continuous. If both $E(f)$ and $E(f)^c$ are dense, does it follow that $S(f)$ is also dense?

On a general compact space the analogue of the above Theorem need not hold:

Example 1.5.2. Consider $X = [0,1] \times S^1$, where S^1 is the unit circle. Let $t : [0,1] \to [0,1]$ be the *tent map* $t(x) = 1 - |1 - 2x|$ and $g : S^1 \to S^1$ be an *irrational rotation* $g(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)}$ for some fixed irrational α. Since $E(g) = \emptyset$, we get that $E(t \times g) = \emptyset$. But $t \times g$ is sensitive because t is sensitive.

By imitating the second part of the proof of Theorem 1.5.1, it can be shown that if $S(f)$ is dense for a dynamical system (X, f), where X is any compact metric space, then $E(f)^c$ is dense.

$S(f)$ can be dense even when f is not surjective.

Example 1.5.3. Consider the map $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{4} \\ 1 - 2x, & \text{if } \frac{1}{4} < x \leq \frac{1}{2} \\ x - \frac{1}{2}, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Any non-trivial interval is mapped into $[0, \frac{1}{2}]$ as a non-trivial interval. And f restricted to $[0, \frac{1}{2}]$ is a copy of the tent map $x \mapsto 1 - |1 - 2x|$ (which is known to be sensitive). Therefore f is sensitive on $[0,1]$.

Surjectivity of f is a necessary condition for $D(f)$ to be dense. Even with surjectivity, the denseness of $S(f)$ does not imply the denseness of $D(f)$. This can be seen by slightly modifying the last example. On the region $\frac{1}{2} < x \leq 1$ redefine f as $f(x) = 2x - 1$. Then f is surjective and sensitive but it can be easily argued that $D(f) = \emptyset$.

For interval maps we know that, period 3 implies Li-Yorke chaos [50], and that transitivity of f implies the existence of period 3 for f^2 [$\because$ Theorem 1.4.9]. It may be interesting to investigate the possible relation between sensitivity and period 3. A combination of existing results yields the following answer.

Proposition 1.5.4. *(i) If $S_\delta(f)$ has non-empty interior for some $\delta > 0$, then f^n has a periodic point of period 3 for some $n \in \mathbb{N}$.*

(ii) Even if f has a periodic point of period 3, it can happen that $S_\delta(f)$ has empty interior for every $\delta > 0$.

Proof. (i) Since $S_\delta(f)$ has non-empty interior, by the result in ([60], p.22), there exist closed subintervals $I_0, I_1, \ldots, I_{n-1}$ of $[0,1]$ such that $f(I_j) = I_{j+1(mod\, n)}$ and f restricted to $\bigcup_{j=0}^{n-1} I_j$ is transitive. It follows that $f^n : I_0 \to I_0$ is transitive. Then, by Theorem 1.4.9, f^{2n} has a point of period 3.

(ii) We give a continuous map f such that f has a periodic point of period 3, but $S_\delta(f)$ has empty interior for every $\delta > 0$. Our map f is piecewise linear having three linear pieces with values at the turning points specified by $f(0) = \frac{1}{2}$, $f(\frac{1}{4}) = 1$, $f(\frac{3}{4}) = 1$ and $f(1) = 0$. That is,

$$f(x) = \begin{cases} 2x + \frac{1}{2}, \text{ if } x \in [0, \frac{1}{4}], \\ 1, \text{ if } x \in [\frac{1}{4}, \frac{3}{4}], \\ 4(1-x), \text{ if } x \in [\frac{3}{4}, 1]. \end{cases}$$

Clearly, $0 \mapsto \frac{1}{2} \mapsto 1 \mapsto 0$ is a periodic orbit of period 3.

Now, f is a constant map on $[\frac{1}{4}, \frac{3}{4}]$. Therefore, any sensitive point of f must be contained in $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$. Let if possible, $\delta > 0$ be such that $S_\delta(f)$ has non-empty interior. Then, by ([60], p.22), there exist closed subintervals $I_0, I_1, \ldots, I_{n-1}$ of $[0,1]$ such that $f(I_j) \subset I_{j+1(mod\, n)}$ and f restricted to $\bigcup_{j=0}^{n-1} I_j$ is δ-sensitive. By the observation above, each I_j is contained in $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$. Next, note that the slope of the graph of f is 2 on $[0, \frac{1}{4}]$ and -4 on $[\frac{3}{4}, 1]$. Therefore, for any I_j, we have $|f(I_j)| \geq 2|I_j|$. Thus we get $|f^n(I_0)| \geq 2^n |I_0|$, and hence $f^n(I_0)$ is not contained in I_0, a contradiction. Therefore, for any $\delta > 0$, the set of δ-sensitive points of f must have empty interior. □

1.6 Transitive maps on $\mathbb{R}$

In 1937, Besicovitch gave a construction of a transitive homeomorphism of the plane $\mathbb{R}^2$ (c.f. [59]). Other constructions of transitive maps on non-compact manifolds are available in [3], [24] and [57]. Still, such constructions are either not very common or not very easy. Below we demonstrate how to construct transitive maps on $\mathbb{R}$, arbitrarily close to the identity map or the reflection map $x \mapsto -x$.

Lemma 1.6.1. A continuous map $f : \mathbb{R} \to \mathbb{R}$ satisfying the following five properties is transitive:

(1) The set $C = \{\cdots < c_{-1} < c_0 < c_1 < \cdots\}$ of critical points of f is bounded neither above nor below,

(2) $\sup\{c_{k+1} - c_k : k \in \mathbb{Z}\}$ is finite,

(3) The set of periodic points of f has a <u>subset</u> $B = \{\cdots < b_{-1} < b_0 < b_1 < \cdots\}$ such that $f(B) = B$ and $c_k < b_k < c_{k+1}$ for every $k \in \mathbb{Z}$,

(4) there is $\alpha > 2$ such that $|f'(x)| \geq \alpha$ for every $x \in \mathbb{R} \setminus (B \cup C)$,

(5) $[c_{i-1}, c_{i+2}] \subset f([c_k, c_{k+1}])$ whenever $f(b_k) = b_i$.

Proof. If $J \subset [c_k, c_{k+1}]$ is a non-degenerate interval, then (2) and (4) imply that there is a natural number n such that $f^n(J)$ contains at least two critical points. Hence, to prove transitivity, it is enough to show the following: given $k \in \mathbb{Z}$, there are $n_1, n_2 \in \mathbb{N}$ such that $[c_{k-1}, c_k] \subset f^{n_1}([c_k, c_{k+1}])$ and $[c_{k+1}, c_{k+2}] \subset f^{n_2}([c_k, c_{k+1}])$.

Consider $[c_k, c_{k+1}]$ for some $k \in \mathbb{Z}$. If b_k is of period l, then by the choice of B and the property (5), we see that $[c_{k-1}, c_{k+2}] \subset f^l([c_k, c_{k+1}])$. Take $n_1 = n_2 = l$. $\square$

Example 1.6.2. Consider the two functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by:

$$f_1(n) = \begin{cases} n, & \text{if } n \text{ is even} \\ n+4, & \text{if } n \equiv 1 (\text{mod } 4) \\ n-4, & \text{if } n \equiv 3 (\text{mod } 4) \end{cases}$$

and the graph of f_1 is linear on $[n, n+1]$ for each $n \in \mathbb{Z}$;

$$f_2(n) = \begin{cases} -n, & \text{if } n \text{ is even} \\ -n+4, & \text{if } n \equiv 1 (\text{mod } 4) \\ -n-4, & \text{if } n \equiv 3 (\text{mod } 4) \end{cases}$$

and the graph of f_2 is linear on $[n, n+1]$ for each $n \in \mathbb{Z}$.

It is easy to see that f_1 and f_2 satisfy the conditions of the Lemma ($C = 2\mathbb{Z} + 1$ and take $B = 2\mathbb{Z}$). Hence both are transitive.

The "scaled" versions of f_1 and f_2 approximate the identity map and the reflection map respectively.

1.7 More on transitivity

Some recent works (see [2], [37], [39], and references therein) on transitive maps involve associating interesting subsets of natural numbers (*IP sets, syndetic sets, sets with positive upper density*, etc.) to dynamical systems. If (X, f) is a transitive dynamical system and if U, V are non-empty open subsets of X, one considers the set $N(U, V) = N_f(U, V) := \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$. The basic idea is that certain stronger forms of transitivity can be characterized or distinguished in terms of these $N(U, V)$'s. In fact, an old result of Furstenberg [31] already says that f is weak mixing if and only if $N(U, V)$ contains arbitrarily large blocks of consecutive numbers for every non-empty open $U, V \subset X$.

Some simple observations about $N(U, V)$'s are given in the Proposition below - probably they are already known, but we could not find a reference.

Proposition 1.7.1. *Let $f : X \to X$ be transitive. Then,*
(1) For every non-empty open $U \subset X$ and every $k \in N(U, U)$, there exist infinitely many arithmetic progressions of length 3 and common difference k, contained in $N(U, U)$.
(2) For every non-empty open $U \subset X$ and every $k \in N(U, U)$, there exists a sequence $(a_n)_{n\in\mathbb{N}} \subset N(U, U)$ such that $a_1 = k$ and $a_1 + a_2 + \cdots + a_n \in N(U, U)$ for every $n \in \mathbb{N}$.
(3) For every non-empty open $U_0, U_1, \ldots, U_n \subset X$ there exists non-empty open $W \subset X$ such that $N(W, W) \subset \bigcap_{i=0}^{n} N(U_i, U_i)$. In particular, $\bigcap_{i=0}^{n} N(U_i, U_i) \neq \emptyset$.

Proof. Let $U \subset X$, $k \in N(U, U)$ and let $V = U \cap f^{-k}(U)$. Then, $N(V, V) - k$, $N(V, V)$ and $N(V, V) + k$ are contained in $N(U, U)$. This proves (1). A repeated application of this idea gives (2). To obtain (3), observe that using transitivity, inductively we can find $j_1, j_2, \ldots j_n \in \mathbb{N}$ such that $U_0 \cap f^{-j_1}(U_1) \cap \cdots \cap f^{-j_n}(U_n) \neq \emptyset$. Take $W = U_0 \cap f^{-j_1}(U_1) \cap \cdots \cap f^{-j_n}(U_n)$. $\square$

The third statement of the Proposition says that if U is varied over all non-empty open subsets of X, then $N(U, U)$'s form a *filter-base* [see some Topology book for the definition of a *filter*]. Hence we can talk about *the* filter associated to a transitive system. It can be seen that if (Y, g) is a factor of (X, f), then (g is also transitive, and) the filter associated to g is contained in the filter associated to f. Therefore the filter associated to a transitive system is a dynamical invariant. To our knowledge these filters have not been studied in the literature.

Question: Is there anything interesting about these filters?

For $n \in \mathbb{N}$, let $X_n = \underbrace{X \times \cdots \times X}_{n-\text{times}}$ and $f_n = \underbrace{f \times \cdots \times f}_{n-\text{times}}$.

Proposition 1.7.2. *Let (X, f) be a transitive system. Then, for every $n \in \mathbb{N}$,*
(1) The filter associated to (X_n, f_n) is the same as the filter associated to (X, f).
(2) (X_n, f_n) is non-wandering (that is, all points in X_n are non-wandering).
(3) (X_n, f_n) has a dense G_δ set of recurrent points, whenever X is a complete metric space.

Proof. The first two statements follow from part (3) of the previous Proposition. To get (3), use Proposition 2.8.2 from Chapter 2. □

Now, let (X, d) be a compact metric space and let X_H be the space of all non-empty compact subsets of X, with the Hausdorff metric d_H. If $f : X \to X$ is continuous, then it induces a continuous map $f_H : X_H \to X_H$ in the natural way. Comparing the dynamics of f and f_H has become fashionable among some researchers recently. One interesting thing is that the transitivity of f need not imply that of f_H. In fact, f_H is transitive if and only if f_H is weak mixing if and only if f is weak mixing [10]. But we have the following:

Proposition 1.7.3. *If $f : X \to X$ is transitive, then f_H is non-wandering and hence has a dense G_δ set of recurrent points.*

Proof. Consider a collection $\mathbb{U} = \{U_0, U_1, \ldots, U_n\}$ of non-empty open subsets X. Let $\mathbb{U}_H = \{K \in X_H : K \subset \bigcup_{i=0}^{n} U_i$ and $K \cap U_i \neq \emptyset$ for every $i\}$. Sets of the form $\mathbb{U}_H$ form a base for the topology on X_H. Given a basic open set $\mathbb{U}_H$, by part (3) of Proposition 1.7.1, we can find $n \in \mathbb{N}$ such that $U \cap f^{-n}(U) \neq \emptyset$ for every $U \in \mathbb{U}$. Let $x_U \in U$ be such that $f^n(x_U) \in U$. Let $A = \{x_U : U \in \mathbb{U}\}$. Then, $A, f_H^n(A) \in \mathbb{U}_H$ and hence f_H is non-wandering. Now, appeal to Proposition 2.8.2. $\square$

Question: What are the similarities between the dynamics of (X_n, f_n) and that of (X_H, f_H)?

Question: Is there any use in considering induced maps of still higher levels, such as $(f_H)_H$? [here, it may be noted that f_H is a factor of $(f_H)_H$ via the set union map from $(X_H)_H$ to X_H].

Now, let (X, d) be a compact, connected metric space. Consider the following closed subspaces of X_H:

$$C_kX := \{A \in X_H : A \text{ has at most } k \text{ connected components}\}, \quad k \in \mathbb{N}.$$

These spaces have been objects of interest in the hyperspace theory [52]. For a continuous map $f : X \to X$, let $f_k : C_kX \to C_kX$ be the induced map given by $f_k(A) = f(A)$. Note that this is well-defined. We show that it is impossible to induce transitive maps on certain C_kX's.

Proposition 1.7.4. *Let X be a compact subspace of $\mathbb{R}^2$ such that X intersection the closed unit disc equals $[-1, 1]$. Let $f : X \to X$ be continuous. Then for every $k \in \mathbb{N}$, $f_k : C_kX \to C_kX$ fails to be transitive.*

Proof. Fix k. Suppose that f_k is transitive and we will find a contradiction. Let $A \in C_kX$ be such that the f_k-orbit of A is dense. Then, it is easily argued that A must have exactly k components and that no component of A is a singleton. Let $\epsilon > 0$ be very small and let $B = \bigcup_{i=1}^{k} J_i$, where $J_i = [4i\epsilon, (4i + 2)\epsilon]$. Then, $B \in C_kX$. Find $n < m$ such that $d_H(B, f^n(A)) < \epsilon$ and $d_H(B, f^m(A)) < \epsilon$. Then, by the choice of B, we must have $f^n(A) \cap f^m(A) \neq \emptyset$. Therefore we can make the following assumption: there are $A \in C_kX$ and $n \in \mathbb{N}$ such that the f_k-orbit of A is dense and $A \cap f^n(A) \neq \emptyset$.

Choose an element $b \in X$ such that $f^n(b) \neq b$. Let $\alpha = \frac{1}{3}d(b, f^n(b))$. Let $\beta \in (0, \alpha)$ be such that $d(b, c) \leq \beta$ implies $d(f^n(b), f^n(c)) < \alpha$ for every $c \in X$. Now for some $m \in \mathbb{N}$, $d_H(\{b\}, f_k^m(A)) < \beta$. Then, for every $a \in A$, $d(b, f^m(a)) < \beta < \alpha$ and

$d(f^n(b), f^{m+n}(a)) < \alpha$. Hence $f^m(A) \cap f^{m+n} = \emptyset$. This contradicts the previous assumption that $A \cap f^n(A) \neq \emptyset$. □

Note that $[0, 1]$, S^1, or more generally *trees* and *graphs* are homeomorphic to the space mentioned in the hypothesis of the Proposition. Also, in the proof we used only the fact that $[-1, 1]$ sits inside the space in a particular way. So the above result extends to more spaces, not necessarily embedable in $\mathbb{R}^2$. We stop our discussion on hyperspace here.

Next, recall the definition of syndetically transitive map from Section 1.4, where we noted that any minimal map is syndetically transitive.

Proposition 1.7.5. *Let (X, f) be a dynamical system where X is a compact metric space. If f is syndetically transitive, then f is minimal or sensitive (may be both).*

Proof. Let d be an admissible metric for X. Assume f is not minimal and we will show that f is sensitive. Let p be a point whose orbit is not dense. Find a δ- ball $B(q, \delta)$ for some $q \in X$ and some $\delta > 0$ such that the orbit of p does not enter $B(q, \delta)$. Let $V = B(q, \delta/4)$. Now, given any non-empty open set $U \subset X$, since $\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$ is syndetic, there exists $k \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $f^{n+j}(U) \cap V \neq \emptyset$ for some $j \in \{1, 2, \ldots, k\}$. By transitivity and continuity of f, find $w \in U$ and $n \in \mathbb{N}$ such that $d(f^{n+j}(w), f^j(p)) < \delta/4$ for every $j \in \{1, 2, \ldots, k\}$. Let $y \in U$ and $j \in \{1, 2, \ldots, k\}$ be such that $f^{n+j}(y) \in V$. Then it follows from triangle inequality that $d(f^{n+j}(y), f^{n+j}(w)) > \delta/2$. Since $y, w \in U$, this proves sensitivity. □

Recall that f is said to be equicontinuous when the family $\{f^n : n \in \mathbb{N}\}$ is equicontinuous.

Corollary 1.7.6. *If X is a compact metric space and $f : X \to X$ syndetically transitive, then f is either sensitive or equicontinuous.*

Proof. It is known that a minimal system on a compact metric space is either sensitive or equicontinuous [6]. □

Many familiar systems such as irrational rotations, shift map, tent map, are indeed syndetically transitive. It is easy to see that transitivity together with denseness of periodic points gives syndetical transitivity. Also, a minimal system containing a periodic orbit must be finite. Hence from Proposition 1.7.5, we are able to conclude that in Devaney's definition of chaos, sensitivity is a redundant condition:

Theorem 1.7.7. *Let $f : X \to X$ be continuous, where X is an infinite compact metric space. If f is transitive and has a dense set of periodic points, then f is sensitive.*

1.8 Topological properties of classes of maps

We conclude this chapter by making a few observations on the topological status of certain classes of maps. Let X be a compact metric space, $\{B_n\}_{n\in\mathbb{N}}$ be a countable base of open balls for X and let $C(X) = \{\text{continuous maps } f : X \to X\}$ with supremum metric. Then,

1. $\{\text{transitive maps in } C(X)\}$

$$= \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{f \in C(X) : f^k(B_n) \cap B_m \neq \emptyset\},$$

   a G_δ set.

2. $\{\text{totally transitive maps in } C(X)\}$

$$= \bigcap_{p\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{f \in C(X) : f^{pk}(B_n) \cap B_m \neq \emptyset\},$$

   a G_δ set.

3. $\{\text{weakly mixing maps in } C(X)\}$

$$= \bigcap_{n_1,n_2\in\mathbb{N}} \bigcap_{m_1,m_2\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{f \in C(X) : f^k(B_{n_j}) \cap B_{m_j} \neq \emptyset \text{ for } j = 1, 2\},$$

   a G_δ set.

4. $\{\text{mixing maps in } C(X)\}$

$$= \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcap_{j=k}^{\infty} \{f \in C(X) : f^j(\overline{B_n}) \cap \overline{B_m} \neq \emptyset\},$$

   an $F_{\sigma\delta}$ set.

5. {minimal maps in $C(X)$}

$$= \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \{f \in C(X) : b_k \in \bigcup_{j=0}^{m} f^{-j}(\overline{B_n})\},$$

an $F_{\sigma\delta}$ set, where b_k is the centre of the ball B_k.

6. {sensitive maps in $C(X)$}

$$= \bigcup_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{f \in C(X) : diam[f^k(B_n)] > \frac{1}{m}\},$$

a $G_{\delta\sigma}$ set.

7. {equicontinuous maps in $C(X)$}

$$= \bigcap_{m\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} \bigcap_{B_n\in\mathbb{B}_p} \bigcap_{k\in\mathbb{N}} \{f \in C(X) : diam[f^k(B_n)] \leq \frac{1}{m}\},$$

an $F_{\sigma\delta}$ set, where $\mathbb{B}_p = \{B_n : diam[B_n] < \frac{1}{p}\}$.

8. {maps in $C(X)$ with all points non-wandering}

$$= \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{f \in C(X) : f^k(B_n) \cap B_n \neq \emptyset\},$$

a G_δ set.

9. {maps in $C(X)$ with periodic points dense}

$$= \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{f \in C(X) : f^k \text{ has a fixed point in } \overline{B_n}\},$$

an $F_{\sigma\delta}$ set.

10. {maps in $C(X)$ with eventually periodic points dense}

$$= \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{f \in C(X) : f^k \text{ has a fixed point in } f^m(\overline{B_n})\},$$

an $F_{\sigma\delta}$ set.

Chapter 2

Dynamics of cellular automata

"This elephant is like a pillar", said the one who had hugged its leg.
"No, the elephant is like a broom", said the one who had caught hold of its tail.

Cellular automata (CA) are described by different people using different languages. So, it may be necessary to state in the beginning itself what is our point of view and what we do not include in our perception. We work in a purely mathematical set up, considering one-dimensional cellular automata as Topological Dynamical Systems.

We do not consider the computational aspects of CA. Nor do we consider the relation of CA to formal language theory. Application of CA to other fields like Physics, Biology, is also not our present concern. We shall not ponder over the visual patterns emerging in the space-time diagram of the evolution of states. And, there will not be any empirical study involving numerical parameters.

To get a flavor of certain aspects of CA that we exclude here, search for the works of K.Culik, J.Kari, A.Salomaa, K.Sutner, S.Wolfram, etc.. About the history of CA, we content ourselves by making two remarks: it all started with von Neumann in 1950s, with some of his adventures on self-reproducing machines, and a solid mathematical foundation for CA theory was laid down by G.A.Hedlund [35]. Details of historical and mathematical developments of CA theory are described in the excellent surveys [43], [49] and [62], which are, by the way, available on the internet.

We confess that this dissertation does not contain any profound theory which might advance the pace of CA research. We were less ambitious in our endeavors and the nature of our work could be better phrased as "pushing forward inch by inch". Almost all our results are on well-established concepts already existing in the literature. For the better or worse, formulating new definitions in order to have better perspectives and new discoveries, is something we have refrained from in general.

By definition (one-dimensional) CA are maps on sequence spaces, expressible in terms of local rules. Because of Hedlund's characterization, CA can also be viewed as continuous maps commuting with the shift map acting on sequences. Thus we have two approaches for attacking problems. The *local rule based approach* is elementary, but requires a lot of combinatorial skills and often gets out of hand. The *shift based approach* is suitable for applying topological and measure theoretical techniques, but it is rather like catching fish with a single sweep and is not always delicate enough to discover the intricate structures. A combination of the two approaches seems to be profitable in some situations.

As one starts investigating the Topological Dynamics of CA, sooner or later one realizes that there is a lack of sufficient tools. One difficulty in CA theory is the complicatedness created by the overlapping of the arguments when the *local rule* is being applied. Due to this overlapping phenomena, generally CA do not yield to techniques involving *morphisms*. One way to avoid this difficulty is to restrict one's attention to a subclass such as *additive CA* which are group homomorphisms. A second difficulty is that the phase space of CA is homeomorphic to the Cantor set, a space with so much freedom, and therefore this phase space (unlike as in the case of interval maps) puts almost no restriction on the dynamical behavior.

Such difficulties are highly challenging and they constitute the intractability of CA. Researchers have been striving for decades hoping to unveil the mysteries in the field. Those who get attracted by the apparent simplicity of CA are soon forced to face a universe of utter complexity often beyond their measure. At present, our knowledge of CA is limited in many ways that even a simple question such as, whether every surjective CA has a dense set of periodic points or not, is open.

Our principle was to work things out from the scratch, and somehow we could extract a number of significant results from very basic definitions. Some of the main results on CA that we obtain in this chapter are: (1) any minimal set is nowhere dense and of zero measure, (2) any orbit is either dense or nowhere dense, (3) transitivity implies weak mixing and hence maximal sensitivity, (4) product of transitive CA is transitive, (5) the set of periodic points of each period is finite if and only if all periodic points are shift-periodic, (6) recurrent points are residual for surjective CA, and (7) all surjective CA are semi-open.

2.1 The domain space of CA

Let A be a finite set with at least two points. Let us call the set A, the **alphabet**. For $n \in \mathbb{N}$, any $w = w_1 \cdots w_n \in A^n$ will be called a **word**. The length of a word w will be denoted by $|w|$. That is, $|w| = n$ if $w \in A^n$. Let $A^+ := \bigcup_{n=1}^{\infty} A^n$ be the collection of all non-empty words over the alphabet A. Next, consider the infinite product $A^{\mathbb{Z}}$ which is the set of all two-sided sequences with entries in A. For $x \in A^{\mathbb{Z}}$ and $j \in \mathbb{Z}$, x_j will denote the j^{th} coordinate of x. If $j \leq k$ are integers then $x_{[j,k]}$ means the word $x_j x_{j+1} \cdots x_k$. The space $A^{\mathbb{Z}}$ can be given a topological structure as well as a measure theoretical structure.

Topological structure on $A^{\mathbb{Z}}$: Consider A as a discrete space and provide $A^{\mathbb{Z}}$ with the product topology. Then, $A^{\mathbb{Z}}$ becomes a compact, totally disconnected space without isolated points, homeomorphic to the Cantor set. An admissible metric on $A^{\mathbb{Z}}$ is given by

$$d(x, y) = \sum_{j \in \mathbb{Z}} \frac{\rho(x_j, y_j)}{2^{|j|}}$$

where ρ is the discrete metric on A. There is a natural countable base of clopen sets, called **cylinders**, for the topology on $A^{\mathbb{Z}}$. A cylinder is a set of the form $\{x \in A^{\mathbb{Z}} : x_{[-k,k]} = w\}$ where $k \in \{0, 1, 2, \ldots\}$ and $w \in A^{2k+1}$. It will be denoted by U_w. In particular if $a \in A$, then $U_a = \{x \in A^{\mathbb{Z}} : x_0 = a\}$.

Measure structure on $A^{\mathbb{Z}}$: The cylinders mentioned above generate the Borel σ-algebra on $A^{\mathbb{Z}}$. There is a natural Borel probability measure μ on this Borel σ-algebra. For any cylinder U_w, we have $\mu[U_w] = |A|^{-|w|}$ and this determines uniquely the measure of other Borel sets. Note that μ is simply the product measure obtained after assigning the natural probability measure on subsets of A. The measure μ has *full support*, which is to say $\mu[U] > 0$ for every non-empty open set $U \subset A^{\mathbb{Z}}$.

Group structure on $A^{\mathbb{Z}}$: Since the alphabet set A is finite and discrete, it does not matter which finite set we choose. We may as well take $A = \{0, 1, \ldots, m-1\}$ for some integer $m \geq 2$, and then $A^{\mathbb{Z}}$ becomes an abelian group with coordinatewise addition modulo m. In fact, this group structure is friendly with the topological structure on $A^{\mathbb{Z}}$ so that $A^{\mathbb{Z}}$ becomes a topological group. Then, the Borel measure mentioned above turns out to be the normalized Haar measure on the Borel σ-algebra of $A^{\mathbb{Z}}$. The group structure on $A^{\mathbb{Z}}$ is relevant only when we consider a special class of CA known as *additive CA*.

2.2 The shift map

Usually denoted by the symbol σ, the **shift map** on $A^{\mathbb{Z}}$ is defined by $[\sigma(x)]_j = x_{j+1}$ for $x \in A^{\mathbb{Z}}$ and $j \in \mathbb{Z}$. That is, it shifts any sequence $x = \cdots x_{-2}x_{-1}x_0x_1x_2\cdots$ to the left by one position. This shift map is **measure-preserving** in the sense that $\mu[\sigma^{-1}(B)] = \mu[B]$ for every Borel set $B \subset A^{\mathbb{Z}}$. This is equivalent to saying that $\mu[\sigma^n(B)] = \mu[B]$ for every Borel set B and every $n \in \mathbb{Z}$, since σ is a homeomorphism. Also, for every $n \in \mathbb{Z} \setminus \{0\}$, σ^n is **ergodic**: if $Y \subset A^{\mathbb{Z}}$ is measurable and if $\sigma^n(Y) = Y$, then either $\mu[Y] = 0$ or $\mu[Y] = 1$.

If $A^{\mathbb{Z}}$ is given the group structure mentioned above, then σ is also a group isomorphism. For our possible need, some more well-known properties of the shift map are listed in the Theorem below; there may be some redundancy among the statements.

Theorem 2.2.1. *(1) σ^n is transitive for every $n \in \mathbb{Z} \setminus \{0\}$.*

(2) σ is mixing and hence weak mixing.

(3) $\{x \in A^{\mathbb{Z}} : O_\sigma(x)$ is dense in $A^{\mathbb{Z}}\}$ is a dense G_δ set with full measure.

(4) σ has a dense set of periodic points. The set of points satisfying $\sigma^n(x) = x$ has cardinality $|A|^n$.

2.3 Cellular automata

For us, CA always means one-dimensional CA, where d-dimensional CA is something whose phase space is $A^{\mathbb{Z}^d}$. There are two equivalent definitions for a CA. One definition is in terms of the shift map and the other in terms of the "local rule". That the two definitions are equivalent, is Hedlund's observation [35].

Definition 2.3.1. Any continuous map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ which commutes with the shift map (that is, $F \circ \sigma = \sigma \circ F$) is called a **cellular automata**.

Definition 2.3.2. A function $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is called a **cellular automata** if there exist $r \in \mathbb{N}$ and a map $f : A^{2r+1} \to A$ known as the **local rule** such that $[F(x)]_j = f(x_{j-r}, \ldots, x_0, \ldots, x_{j+r})$ for every $x \in A^{\mathbb{Z}}$ and $j \in \mathbb{Z}$.

In the above definition, r is referred to as the **radius** of the local rule of the CA. It is known that any CA is equivalent to (more precisely, topologically conjugate to) a CA (on possibly a different alphabet set) having a local rule of radius 1, see Alphabet Lemma in section 2.9. Thus, practically it may suffice to consider CA whose local rules are maps from A^3 to A.

Note that the shift map itself is a CA as it trivially satisfies the first definition. A local rule for σ is given by $f : A^3 \to A$, $f(a, b, c) = c$. As another example of a CA, consider the map $F : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ defined by $[F(x)]_j = x_j + x_{j+1} \pmod 2$. Observe that this CA is also a surjective group homomorphism of $\{0, 1\}^{\mathbb{Z}}$. On the other hand, the CA defined by $[F(x)]_j = 1 - x_j$ on $\{0, 1\}^{\mathbb{Z}}$ is not a group homomorphism. CA which are also group homomorphisms are called **additive CA** (also known as **linear CA** in some parts of the literature).

A map $F : \{0, 1, \ldots, m-1\}^{\mathbb{Z}} \to \{0, 1, \ldots, m-1\}^{\mathbb{Z}}$ is an additive CA if and only if F has the expression $[F(x)]_j = \sum_{i=-k}^{k} a_i x_{j+i} \pmod{m'}$ for fixed quantities $k \in \{0, 1, 2, \ldots\}$,

$m' \in \{0, 1, 2, \ldots, m\}$ and $a_i \in \mathbb{Z}$. The dynamics of additive CA is reasonably well-understood. Many of their properties could be characterized in terms of the coefficients a_i appearing in the linear expression [25], [55]. We will have occasion to deal with additive CA in more detail in the next chapter.

2.4 Minimal sets

It is possible to deduce many dynamical properties of a CA just from the first definition, where a CA is thought of as a continuous map commuting with the shift. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA throughout this section.

The notion of a **minimal set** is fundamental in the theory of dynamical systems. It is known that a continuous self map of a compact metric space has at least one minimal set. The idea of the proof is to apply *Zorn's lemma*, to find a minimal element of the collection of all non-empty closed invariant subsets of the phase space, ordered by inclusion. The minimal set can be as small as a periodic orbit or can be the entire space as in the case of any irrational rotation of the unit circle. Now, we ask: how "big" can be a minimal set of a CA? We show that it has to be very small both topologically as well as measure theoretically.

Theorem 2.4.1. *Any minimal set of F is a nowhere dense set of zero measure.*

Proof. Let M be a minimal set for F. Note that by definition M is closed. Since the measure has full support, it is enough to show that M is of zero measure. Assume that M has positive measure, and we will find a contradiction.

Since σ is measure-preserving homeomorphism, we have that $\mu[\sigma^n(M)] = \mu[M]$ for every $n \in \mathbb{Z}$. One cannot have infinitely many pairwise disjoint sets of equal positive measure inside a finite measure space (has our reader seen a proof of **Poincare's Recurrence Theorem**?). Therefore, $\sigma^n(M)$'s cannot be pairwise disjoint. Since σ is a homeomorphism, this implies that there exists $n \in \mathbb{N}$ such that $M \cap \sigma^n(M) \neq \emptyset$. Moreover, using the fact that $F \circ \sigma = \sigma \circ F$, it is easily verified that $\sigma^n(M)$ is also a minimal set for F. Any two minimal sets of F are either identical or disjoint - this is because, if their intersection is non-empty, then the intersection is also a minimal set. Hence we have that $\sigma^n(M) = M$.

By the ergodicity of σ^n, M must have full measure. But the measure μ has full support and M is a closed set. Therefore, $M = A^{\mathbb{Z}}$. Thus to complete the proof, it is enough to show that $A^{\mathbb{Z}}$ is not a minimal set for F. This is easy. The set of fixed points of σ is a non-empty, finite (hence closed), F-invariant subset properly contained in $A^{\mathbb{Z}}$. $\square$

The result may be the best possible since we cannot say that minimal sets should be small in the sense of cardinality. The shift map, which is a CA, has minimal sets which are uncountable; see topics like *Toeplitz shift* and *substitution shift* in the literature [23], which we do not discuss here.

Problem: Find out some non-trivial minimal sets for additive CA (other than the shift).

A stronger result regarding the topological part in the above Theorem is:

Theorem 2.4.2. *Any F-orbit is either dense or nowhere dense in $A^{\mathbb{Z}}$.*

It seems advantageous to have a more general version from which Theorem 2.4.2 could be deduced.

Theorem 2.4.3. *Let X be a compact metric space and let $f, g : X \to X$ be continuous maps such that $f \circ g = g \circ f$. Assume that g^n is transitive for every $n \in \mathbb{N}$. Then, any f-orbit is either dense or nowhere dense in X.*

Proof. Let $x \in X$ and let K be the closure of $\{x, f(x), f^2(x), \ldots\}$. Assume that K has non-empty interior. We will show that $K = X$. The hypothesis that g^n is transitive for every n implies that there are no isolated points in X (except when X is a singleton). Therefore, replacing x by some $f^n(x)$ if necessary, we may suppose that $x \in int[K]$. Since K has non-empty interior, and since g is transitive, there exists $n \in \mathbb{N}$ such that the interior of $K \cap g^{-n}(K)$ is non-empty. Then we can find $m \in \mathbb{N}$ such that $f^m(x) \in K \cap g^{-n}(K)$.

Now, K is f-invariant from the definition and $g^{-n}(K)$ is f-invariant because f commutes with g. Thus the closed set $K \cap g^{-n}(K)$ is f-invariant. Therefore, the closure of the f-orbit of the element $f^m(x)$ is contained in $K \cap g^{-n}(K)$. But, the closure of the f-orbit of $f^m(x)$ is K since $x \in int[K]$. Hence we deduce that $K \subset g^{-n}(K)$ which is to say $g^n(K) \subset K$. Since g^n is transitive and since K is assumed to have non-empty interior, we conclude that $K = X$. □

Now, to prove Theorem 2.4.2, take $X = A^{\mathbb{Z}}$, $f = F$ and $g = \sigma$ in Theorem 2.4.3.

2.5 Transitivity

In 1996, the article "Transitive Cellular Automata are Sensitive" appeared in American Mathematical Monthly [26]. The proof of the title-result depends on analyzing the local rule. That is, the authors view CA through the second definition. Below (in Corollary 2.5.2) we deduce the same result in a more elegant way using the first definition of CA. In fact, our conclusion is much stronger. As we know, total transitivity, weak mixing and mixing are three of the stronger forms of transitivity. Mixing implies weak mixing and weak mixing implies total transitivity for all systems. Some or all of the reverse implications may hold for special classes of maps. For instance, all the three notions coincide for interval maps [19] and for subshifts of finite type (see Chapter 5). We do not know whether weak mixing implies mixing for CA. But what we are able to say is:

Proposition 2.5.1. *Any transitive F is weak mixing.*

Proof. Let $U_1, U_2, V_1, V_2 \subset A^{\mathbb{Z}}$ be non-empty open sets. We have to find $n \in \mathbb{N}$ such that $F^n(U_1) \cap V_1 \neq \emptyset$ and $F^n(U_2) \cap V_2 \neq \emptyset$. Since σ is weak mixing, there exists $k \in \mathbb{N}$ such that $\sigma^k(U_1) \cap U_2 \neq \emptyset$ and $\sigma^k(V_1) \cap V_2 \neq \emptyset$. Put $U = \sigma^k(U_1) \cap U_2$ and $V = \sigma^k(U_1) \cap U_2$. Then, U, V are non-empty open sets ($\because \sigma$ is a homeomorphism). By the transitivity of F, there exists $n \in \mathbb{N}$ such that $F^n(U) \cap V \neq \emptyset$. Since F commutes with σ, this gives what we wanted. □

This Proposition has some nice corollaries, the first of which is the promised one.

Corollary 2.5.2. *Transitive cellular automata are maximally sensitive.*

Proof. Weak mixing implies maximal sensitivity. □

Corollary 2.5.3. *Transitive cellular automata are Li-Yorke chaotic.*

Proof. Weak mixing implies Li-Yorke chaos [41]. □

Corollary 2.5.4. *The product of any two transitive cellular automata is again a transitive cellular automata.*

Proof. It is easy to see that the product of any two CA is again a CA. By definition, an **E-system** is a transitive map of a compact metric space with an invariant Borel measure having full support [39]. Transitivity on compact spaces implies surjectivity, and surjective CA are known to be measure-preserving (c.f. [49]). Thus, transitive CA are E-systems. It is known that any E-system is syndetically transitive and that the product of a syndetically transitive system with a weak mixing system is transitive (c.f. [39]). While considering the product of two transitive CA, think of one as an E-system and the other as a weak mixing system. □

2.6 Injectivity and surjectivity

The influence of the shift map on CA is evident in some more ways. Let $P_n(\sigma) = \{x \in A^{\mathbb{Z}} : \sigma^n(x) = x\}$. So $P_n(\sigma)$ is the set of all periodic points of σ with period dividing n. If F is a CA, then $P_n(\sigma)$ is F-invariant since F commutes with σ. Consequently any element of $P_n(\sigma)$ has a finite orbit under the action of F since $P_n(\sigma)$ is a finite set. Therefore, if F happens to be injective, then all elements of $P_n(\sigma)$ becomes periodic points for F. The set $P(\sigma)$ of periodic points of σ is the union of $P_n(\sigma)$'s. It follows that $P(\sigma) \subset P(F)$, whenever F is an injective CA. But $P(\sigma)$ is dense in $A^{\mathbb{Z}}$. This leads to two conclusions about injective CA (c.f. [49]).

The first conclusion is that any injective CA has a dense set of periodic points, which is clear from the above lines. The second conclusion is, any injective CA is surjective. This is argued as follows. The range of any CA, being the continuous image of a compact set, must be compact and hence closed. For an injective CA, the range contains a dense subset of $A^{\mathbb{Z}}$, namely the set of periodic points.

A surjective CA need not be injective (examples are easy). But there is a result which says that if a CA is surjective, then it is injective on a suitable dense subset of $A^{\mathbb{Z}}$ (on the so called "finite configurations"). This is the famous **Garden of Eden Theorem** (see [43]) into which we decide not to go. It is a long standing open question whether every surjective CA has a dense set of periodic points or not. This is settled in the affirmative for additive CA [30], and also for some other classes [18], [22]. The basic tools to understand surjective CA come from Hedlund's work. He connected surjectivity of a CA to the number of pre-images of elements.

Theorem 2.6.1. *[35] F is surjective if and only if $|F^{-1}(x)| < \infty$ for every $x \in A^{\mathbb{Z}}$.*

Actually, Hedlund's work gives a more accurate statement: F is surjective if and only if $|F^{-1}(x)| \leq |A|^{2r}$ for every $x \in A^{\mathbb{Z}}$, where r is the radius of the local rule of F. This is deducible from the more refined statement he proved, where a characterization of surjectivity for CA is given in terms of the local rule.

Assume that the local rule is $f : A^{2r+1} \to A$. If $n \in \mathbb{N}$, then we may suppose that f is also defined on words $w \in A^{n+2r+1}$ as follows. Let $w = w_1w_2\cdots w_{n+2r+1} \in A^{n+2r+1}$. We define $f(w) = f(w_1w_2\cdots w_{2r+1})f(w_2w_3\cdots w_{2r+2})\cdots f(w_{n+1}w_{n+2}\cdots w_{n+2r+1})$ so that $f(w) \in A^{n+1}$. In this way f gets defined on all words $w \in A^+$ having length $\geq 2r+1$. If $v \in A^+$, then $f^{-1}(v)$ is the set $\{w \in A^+ : f(w) = v\}$. Note that if $f(w) = v$, then $|w| = |v| + 2r$ so that $f^{-1}(v) = \{w \in A^{|v|+2r} : f(w) = v\}$.

Theorem 2.6.2 (Hedlund). *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automata given by a local rule $f : A^{2r+1} \to A$, $r \in \mathbb{N}$. Then, the following are equivalent:*
(i) F is not surjective.
(ii) There exist $u \in A^{2r}$, and $w, w' \in A^+$, $|w| = |w'|$, $w \neq w'$ such that $f(uwu) = f(uw'u)$.
(iii) For some $n \in \mathbb{N}$ ($n \geq 2r+1$), there exist more than $|A|^{2r}$ words of length n with the same f-image.

The Theorem does not give an explicit upper bound on the length of the words which are to be considered for the testing. We show that Hedlund's Theorem has the following variant which comes up with some upper bounds.

Theorem 2.6.3. *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automata given by a local rule $f : A^{2r+1} \to A$, $r \in \mathbb{N}$. Then, the following are equivalent:*

(i) F is not surjective.

(ii) There exist $u, v \in A^{2r}$, and $w, w' \in A^+$, $w \neq w'$ such that $|uwv| = |uw'v| \leq |A|^{4r}$ and $f(uwv) = f(uw'v)$.

(iii) For some $n \leq |A|^{4mr}$, there exist more than $|A|^{2r}$ words of length n with the same f-image, where m is the smallest integer greater than $2r \log_2 |A|$.

Proof. $(i) \Rightarrow (ii)$: Let $n \in \mathbb{N}$ be the smallest such that there exist $u, v \in A^{2r}$ and $w, w' \in A^+$, $w \neq w'$ such that $|uwv| = |uw'v| = n$ and $f(uwv) = f(uw'v)$. The existence of n is guaranteed by Theorem 2.6.2. Our job is to show that $n \leq |A|^{4r}$. For the convenience of arguing, write $uwv = b_1 b_2 \cdots b_n$ and $uw'v = c_1 c_2 \cdots c_n$ where $b_i, c_i \in A$. Of course, $b_1 \cdots b_{2r} = c_1 \cdots c_{2r} = u$ and $b_{n-2r+1} \cdots b_n = c_{n-2r+1} \cdots c_n = v$.

Claim-1: $b_{2r+1} \neq c_{2r+1}$ and $b_{n-2r} \neq c_{n-2r}$.
Proof: If $b_{2r+1} = c_{2r+1}$, the pair $(b_2 b_3 \cdots b_n,\ c_2 c_3 \cdots c_n)$ works as well in the place of the pair $(b_1 \cdots b_n,\ c_1 \cdots c_n)$, violating the minimality of n. Similarly, we have $b_{n-2r} \neq c_{n-2r}$.

Now, for $1 \leq i \leq n - 2r - 1$, let $\alpha_i = b_{i+1} \cdots b_{i+2r}$ and $\beta_i = c_{i+1} \cdots c_{i+2r}$.

Claim-2: $(\alpha_i, \beta_i) \neq (\alpha_j, \beta_j)$ for $i \neq j$.
Proof: Assume that $(\alpha_i, \beta_i) = (\alpha_j, \beta_j)$ for some $i < j$. Then, in the place of the pair $(b_1 b_2 \cdots b_n, c_1 c_2 \cdots c_n)$ we can consider the pair $(b_1 \cdots b_i b_{j+1} \cdots b_n, c_1 \cdots c_i c_{j+1} \cdots c_n)$, again contradicting the minimality of n (here, to say that the words are distinct, use Claim-1).

Claim-3: $\alpha_i \neq \beta_i$ for every i.
Proof: Suppose that $\alpha_i = \beta_i$ for some i. Since $b_{2r+1} \neq c_{2r+1}$, we have $i > 2r$. Then, we may consider the pair $(b_1 b_2 \cdots b_{i+2r},\ c_1 c_2 \cdots c_{i+2r})$ in the place of the pair $(b_1 \cdots b_n,\ c_1 \cdots c_n)$. A contradiction.

From Claim-2 and Claim-3, it follows that $n - 2r - 1 \leq |A|^{4r} - |A|^{2r}$. Since $|A|^{2r} \geq 2r + 1$, we get that $n \leq |A|^{4r}$. (we have actually produced an upper bound slightly better than $|A|^{4r}$, but the difference is negligible compared to $|A|^{4r}$).

$(ii) \Rightarrow (iii)$: Let u, v, w, w' be as given by (ii). Let $B = \{uwv, uw'v\}$. Then, for any $m \in \mathbb{N}$ and $y, z \in B^m$, $f(y) = f(z)$. So if $m > 2r \log_2 |A|$ then we get more than $|A|^{2r}$ elements in B^m with the same f-image. Any element of B^m has length

$|uwv|^m = |uw'v|^m \leq |A|^{4mr}$. (Here also, some slight improvements are possible on the upper bound).

$(iii) \Rightarrow (i)$ is included in Theorem 2.6.2. $\square$

Question: Is it possible to improve significantly the bounds given in the statements (ii) and (iii) of Theorem 2.6.3? In each case, what is the optimal bound?

Remark: The bounds given are too big to be of practical use. To test surjectivity (or injectivity) of a CA there are efficient algorithms which work in quadratic time [66].

2.7 Periodic points

For CA, if a dynamical phenomena is present locally, it is very likely to be present all over the domain space in a fairly uniform manner. As an example, we remark that a recent Theorem of Kurka [48] says that a CA either is sensitive or has a residual set of equicontinuity points. Again, this is a consequence of commuting with the shift, which we exploit further in this section to obtain some understanding on the nature of the set of periodic points of a CA. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA throughout. Note that the set $P(F)$ of periodic points of F is σ-invariant.

Proposition 2.7.1. *$P(F)$ is one of the following:*

(i) the whole space (in this case, $F^n = Identity$ for some $n \in \mathbb{N}$),

(ii) a dense set of first category and zero measure, or

(iii) a nowhere dense set of zero measure.

Proof. First we argue the topological part. Let $P_n(F) = \{x \in A^{\mathbb{Z}} : F^n(x) = x\}$. Each $P_n(F)$ is closed and σ-invariant. Since σ is transitive, an invariant closed set is either nowhere dense or the whole space. If $P_n(F)$ is the whole space for some n, then $F^n = Identity$ and $P(F) = A^{\mathbb{Z}}$. If $P_n(F)$ is nowhere dense for every n, then $P(F) = \bigcup_{n=1}^{\infty} P_n(F)$ is of first category. Again, by σ-invariance, $P(F)$ is dense or nowhere dense.

Now, we consider the measure part. If $\mu[P_n(F)] > 0$ for some $n \in \mathbb{N}$, then $P_n(F)$ has full measure since σ is ergodic. Then, $P_n(F) = A^{\mathbb{Z}}$ since $P_n(F)$ is closed. Therefore, if $P(F) \neq A^{\mathbb{Z}}$, then $P_n(F) \neq A^{\mathbb{Z}}$ for every n and hence $\mu[P(F)] \leq \sum_{n=1}^{\infty} \mu[P_n(F)] = 0$. □

Problem: Classify CA into the three classes mentioned in the above Proposition.

We have already observed that for the shift σ, $P_n(\sigma)$ is finite for every n. Dynamical systems where the set of periodic points of each period has finite cardinality, has been of interest to researchers. Such systems yield to techniques involving zeta functions, see for instance [23], p.60. We ask: can we characterize all CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ satisfying $|P_n(F)| < \infty$ for every $n \in \mathbb{N}$? Below, we provide a satisfactory answer. The necessary and sufficient condition is that $P(F) \subset P(\sigma)$. The proof uses both definitions of CA.

Theorem 2.7.2. *Let r be the radius of the local rule of F. Then, the following are equivalent:*

(i) $|P_n(F)| \leq |A|^{2nr}$ for every $n \in \mathbb{N}$.
(ii) $|P_n(F)| < \infty$ for every $n \in \mathbb{N}$.
(iii) $P(F) \subset P(\sigma)$.
If F is surjective, the following condition is equivalent to the above three:
(iv) $E(F) = P(\sigma)$.

Proof. $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (iii)$: Let $x \in P(F)$. Then, $F^n(x) = x$ for some $n \in \mathbb{N}$. It follows that $F^n(\sigma^k(x)) = \sigma^k(x)$ for any $k \in \mathbb{N}$. But $|P_n(F)| < \infty$. Therefore, $\sigma^{k+m}(x) = \sigma^k(x)$ for some $k, m \in \mathbb{N}$. Since σ is injective, we get $\sigma^m(x) = x$.

$(iii) \Rightarrow (i)$: Assume that $P(F) \subset P(\sigma)$ and let r be the radius of the local rule of F. Since $P_1(F^n) = P_n(F)$ and since the radius of the local rule of F^n is nr, we can replace F^n by F. Thus it is enough to show that $|P_1(F)| \leq |A|^{2r}$. Let if possible $|P_1(F)| > |A|^{2r}$. Then there are distinct $x, y \in P_1(F)$ with $x_{[1,\,2r]} = y_{[1,\,2r]}$. Define $z = (z_j)_{j\in\mathbb{Z}} \in A^{\mathbb{Z}}$ as $z_j = x_j$ for $j \leq 2r$, $z_j = y_j$ for $j \geq 1$. Since $x, y \in P_1(F) \subset P(\sigma)$ and $x \neq y$, we have that $z \notin P(\sigma)$. But $F(z) = z$, which is a contradiction.

Now, assume that F is surjective. $(iv) \Rightarrow (iii)$ is trivial.

$(ii) \Rightarrow (iv)$: Let $x \in E(F)$. Then, $F^{k+n}(x) = F^k(x)$ for some $k, n \in \mathbb{N}$. Let $y = F^k(x)$. It follows that $F^n(\sigma^i(y)) = \sigma^i(y)$ for any $i \in \mathbb{N}$. But $|P_n(F)| < \infty$. Therefore, as before $\sigma^i(y) = y$ for some $i \in \mathbb{N}$. Now, for any $m \in \mathbb{N}$,

$$F^k(\sigma^{mi}(x)) = \sigma^{mi}(F^k(x)) = \sigma^{mi}(y) = y.$$

Since F^k is surjective, $y \in A^{\mathbb{Z}}$ can have only finitely many pre-images under F^k [$\because$ Theorem 2.6.1]. We conclude that $\sigma^{mi}(x) = x$ for some $m \in \mathbb{N}$. Thus, $E(F) \subset P(\sigma)$. The reverse inclusion is true for all CA. $\square$

Note that $P_n(\sigma)$ is a finite F-invariant set and $P_m(\sigma) \cap P_n(\sigma) = P_k(\sigma)$ where $k = g.c.d.(m, n)$. Suppose that F is surjective. Then, applying Theorem 2.6.1 to points $x \in P_1(\sigma)$, we get that $P_n(\sigma) \setminus P_1(\sigma)$ is also F invariant if n is a large prime. Since any finite invariant set must contain a periodic point, F has a periodic point in $P_n(\sigma) \setminus P_1(\sigma)$. It follows that whenever F is surjective, $P(F)$ must be infinite. If we further assume $P(F) \subset P(\sigma)$, then F has periodic points of arbitrarily large periods, by Theorem 2.7.2.

There are non-trivial surjective CA having infinitely many fixed points. The surjective CA (Boyle et al, c.f. [49]) $F : \{0,1,2\}^{\mathbb{Z}} \to \{0,1,2\}^{\mathbb{Z}}$ given by

$$[F(x)]_i = \begin{cases} x_i + x_{i+1} \text{ (mod 2), if } x_i \neq 2 \\ 2, \text{ if } x_i = 2 \end{cases}$$

has uncountably many fixed points. Indeed, any $x \in \{0,2\}^{\mathbb{Z}}$ is fixed by F.

Problem: Let F be an arbitrary CA. Is it possible to give some kind of structural decomposition of F in terms of F_1 and F_2, where F_1 is a CA such that $|P_n(F_1)| < \infty$ for every n, and F_2 is a CA such that $F_2^n = Identity$ for some n?

2.8 Recurrent points

We have mentioned that it is not known whether every surjective CA has a dense set of periodic points. But we can say something definite about a bigger set, namely the set of recurrent points. First, in any dynamical system (X, f) where X is a metric space, with a metric say d, the set $R(f)$ of recurrent points is a G_δ set since $R(f) = \bigcap_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\{x \in X : d(x, f^m(x)) < \frac{1}{k}\}$. Next, since all surjective CA are measure preserving [35], by the topological version of Poincare Recurrence Theorem (c.f. [12]), one gets that the set of recurrent points is of full measure. Since the measure has full support, any set of full measure is a dense set. Hence we have the following, which seems to be missing in the existing literature on CA.

Proposition 2.8.1. *For any surjective CA, the set of recurrent points is a dense G_δ set.*

This Proposition can be derived in a slightly different way without resorting to Poincare's Theorem. In [49] it is shown that every point is non-wandering for surjective F: if U_w is a cylinder, then $U_w, F^{-1}(U_w), F^{-2}(U_w), \ldots$ are sets having equal positive measure; since the total measure is finite, two of the above sets, and hence U_w and some $F^{-n}(U_w)$, must intersect.

Now, to derive Proposition 2.8.1, apply the following general result:

Proposition 2.8.2. *Let (X, f) be a dynamical system where X is a complete metric space. If the set $\Omega(f)$ of non-wandering points is the whole of X, then $R(f)$ is a dense G_δ set.*

To prove Proposition 2.8.2, first we prove a simple lemma:

Lemma 2.8.3. Let (X, f) be a dynamical system and $U \subset X$ be non-empty, open. If all points of U are non-wandering, then for infinitely many $n \in \mathbb{N}$, $U \cap f^{-n}(U) \neq \emptyset$.

Proof. There is $n_1 \in \mathbb{N}$ with $V_1 := U \cap f^{-n_1}(U) \neq \emptyset$. Since $\Omega(f) \cap V_1 \neq \emptyset$, there is $n_2 \in \mathbb{N}$ such that $V_1 \cap f^{-n_2}(V_1) \neq \emptyset$. This implies $V_2 := U \cap f^{-(n_1+n_2)}(U) \neq \emptyset$. Next, find $n_3 \in \mathbb{N}$ such that $V_2 \cap f^{-n_3}(V_2) \neq \emptyset$. This implies $V_3 := U \cap f^{-(n_1+n_2+n_3)}(U) \neq \emptyset$, and so on. $\square$

Proof of Proposition 2.8.2. It is enough to establish that $R(f)$ is dense. Let $U \subset X$ be non-empty, open. We will construct a decreasing chain of closed balls in U in a particular way, with the diameters of the balls tending to 0. Completeness of the space ensures that the intersection of the balls is non-empty. We will show that the unique point in the intersection is a recurrent point.

Let B_1 be a (open) ball such that $\overline{B_1} \subset U$ and $diam[B_1] < 1$. Since all points in B_1 are non-wandering, there exists $n_1 \in \mathbb{N}$ such that $B_1 \cap f^{-n_1}(B_1) \neq \emptyset$. Choose a ball B_2 such that $\overline{B_2} \subset B_1 \cap f^{-n_1}(B_1)$ and $diam[B_2] < \frac{1}{2}$. Applying the Lemma to B_2, we can find $n_2 > n_1$ such that $B_2 \cap f^{-n_2}(B_2) \neq \emptyset$. Now, assume that we have chosen balls $B_1, \ldots, B_k$ within U and natural numbers $n_1 < n_2 < \cdots < n_k$ such that

(i) $\overline{B_{j+1}} \subset B_j \cap f^{-n_j}(B_j)$ for $1 \leq j < k$, and
(ii) $diam[B_j] < \frac{1}{j}$ for $i \leq j \leq k$.

At the $k+1^{th}$ step, choose a ball B_{k+1} such that $\overline{B_{k+1}} \subset B_k \cap f^{-n_k}(B_k)$ and $diam[B_{k+1}] < \frac{1}{k+1}$. Applying the Lemma to B_{k+1}, find $n_{k+1} > n_k$ such that $B_{k+1} \cap f^{-n_{k+1}}(B_{k+1}) \neq \emptyset$. Now, $\bigcap_{k=1}^{\infty} \overline{B_k} = \{x\}$ for some $x \in X$, since X is complete and since $diam[B_k] \to 0$. Note that for each $k \in \mathbb{N}$, $x \in \overline{B_{k+1}} \subset B_k \cap f^{-n_k}(B_k)$ and hence $f^{n_k}(x) \in B_k$. Since B_k's form a decreasing chain with intersection $\{x\}$, one has that $f^{n_k}(x) \to x$. □

2.9 Forward image of open sets

How do we decide whether there are any periodic points in specified regions of the phase space? For interval maps, one may use the Intermediate Value Theorem. For CA, at present we do not have results which can be used to check the existence of periodic points in specified regions. If we wish to make some progress in this direction, perhaps the first step is to achieve an understanding of the structure of the forward images of open sets for CA. The simplest question is, does the image of every non-empty open set has non-empty interior? For surjective CA, the answer is going to be affirmative. Note that if the CA is not surjective, then the forward image of the whole space itself is a nowhere dense set because of σ-invariance.

A known technical lemma (for instance, see [27]) allows us to reduce many questions concerning CA at the level of *words* to those at the level of the *alphabet*. We state it in a suitable form:

Lemma 2.9.1 (Alphabet Lemma). Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA with local rule $f : A^{2r+1} \to A$, $r \in \mathbb{N}$. Let $k \geq r$ and let $B = A^k$. Define $g : B^3 = A^{3k} \to B = A^k$ by $g(w = w_1 w_2 \cdots w_{3k}) = a_1 a_2 \cdots a_k$ where $a_j = f(w_{k+j-r} \cdots w_{k+j} \cdots w_{k+j+r})$, and the cellular automata $G : B^{\mathbb{Z}} \to B^{\mathbb{Z}}$ by $[G(y)]_i = g(y_{i-1}\, y_i\, y_{i+1})$, for $y \in B^{\mathbb{Z}}$, $i \in \mathbb{Z}$. Then, F is topologically conjugate to G. In fact, for any $p \in \mathbb{Z}$, the map $\phi_p : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ defined as $[\phi_p(x)]_i = x_{[ki+p,\, ki+p+k-1]}$ is a homeomorphism and satisfies $\phi_p \circ F = G \circ \phi_p$.

This lemma has interesting consequences which are folklore. For instance, one can easily deduce that any CA is topologically conjugate to a CA with local rule having radius ≤ 1, any sensitive CA is topologically conjugate to a 1-sensitive CA, etc.. Also, the Alphabet Lemma affords a simpler equivalent formulation of the open question as to whether periodic points are dense in surjective CA:

Question: Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be surjective and let $a \in A$. Does F always possess a periodic point in which a occurs?

Let us continue our discourse on the nature of forward images of open sets. The key observation is the following:

Lemma 2.9.2. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA. Then, for any set $D \subsetneq A$, we have $F(D^{\mathbb{Z}}) \neq A^{\mathbb{Z}}$.

Proof. Let $f : A^{2r+1} \to A$ be the local rule of F. Since $|D| < |A|$, for large n, $|D|^{n+2r} < |A|^n$, so that there exists $w \in A^n$ which cannot be the f-image of any word over D. $\square$

Now, a combination of Alphabet Lemma and Baire Category Theorem suffices to establish that every surjective CA is semi-open:

Theorem 2.9.3. *If F is surjective, then $int[F(U)] \neq \emptyset$ for any non-empty open set U.*

Proof. Since $\{U_w : w \in A^{2k+1},\ k \in \mathbb{N}\}$ (recall: $U_w = \{x \in A^{\mathbb{Z}} : x_{[-k,k]} = w\}$) is a base for the topology on $A^{\mathbb{Z}}$, it is enough to show that $int[F(U_w)] \neq \emptyset$ for any U_w. By the grace of Alphabet Lemma, it is enough to show that $int[F(U_a)] \neq \emptyset$ for every $a \in A$.

Let $D = A \setminus \{a\}$. Then, $F(D^{\mathbb{Z}})$ is a closed set, and by the previous Lemma, $F(D^{\mathbb{Z}}) \neq A^{\mathbb{Z}}$. Therefore, $int[F(A^{\mathbb{Z}} \setminus D^{\mathbb{Z}})] \neq \emptyset$ since F is surjective. But note that $A^{\mathbb{Z}} \setminus D^{\mathbb{Z}} = \bigcup_{n=-\infty}^{\infty} \sigma^n(U_a)$. That is,

$$\emptyset \neq int[F(\bigcup_{n=-\infty}^{\infty} \sigma^n(U_a))] = int[\bigcup_{n=-\infty}^{\infty} F(\sigma^n(U_a))].$$

Using Baire Category Theorem, we get that $int[F(\sigma^n(U_a))] \neq \emptyset$ for some $n \in \mathbb{Z}$. Since σ is a homeomorphism commuting with F, this implies that $int[F(U_a)] \neq \emptyset$. □

Corollary 2.9.4. *If F is surjective, $F(U_w) = \overline{int[F(U_w)]}$ for any cylinder U_w.*

Proof. Since F is semi-open, one can easily show that $F(U) \subset \overline{int[F(U)]}$ for any non-empty open set U. But $F(U_w)$ is closed as well. □

Problem: Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a surjective CA. Let Y be the collection of all $x \in A^{\mathbb{Z}}$ with the property that there exist a neighbourhood U of x and $n \in \mathbb{N}$ such that $F^n(x) \notin int[F^n(U)]$. From the semi-openness of F, it can be easily argued that Y is a set of first category. Is it possible to provide more details on the structure of Y?

Let (X, f) be a dynamical system, where X is a compact metric space. A point $x \in X$ is said to be **backward recurrent** if there exists a sequence (x_k) in X converging to x such that for each k, there is $n_k \in \mathbb{N}$ with $f^{n_k}(x_k) = x$. Recall from Topology that, a subset Y of X is said to be **residual** if its complement is a set of first category. Note that a dense G_δ set is residual.

Corollary 2.9.5. *If F is surjective, then the set of backward recurrent points is residual.*

Proof. Note that x is a backward recurrent point if and only if for any open set U containing x, there exists $n \in \mathbb{N}$ such that $x \in F^n(U)$. Therefore,

$$\{x \in A^{\mathbb{Z}} : x \text{ is not backward recurrent}\} = \bigcup_w \left(U_w \setminus \left[\bigcup_{n\in\mathbb{N}} F^n(U_w) \right] \right).$$

Thus, it is enough to show that for each w, $U_w \setminus \left[\bigcup_{n\in\mathbb{N}} F^n(U_w)\right]$ is nowhere dense. Since recurrent points are dense in $A^{\mathbb{Z}}$, $U_w \cap \left[\bigcup_{n\in\mathbb{N}} F^n(U_w)\right]$ is dense in U_w. Moreover,

$$\bigcup_{n\in\mathbb{N}} F^n(U_w) = \bigcup_{n\in\mathbb{N}} \overline{int[F^n(U_w)]} \subset \overline{\bigcup_{n\in\mathbb{N}} int[F^n(U_w)]}.$$

This implies that the open set $U_w \cap \left[\bigcup_{n\in\mathbb{N}} int[F^n(U_w)]\right]$ is dense in U_w. Hence $U_w \setminus \left[\bigcup_{n\in\mathbb{N}} F^n(U_w)\right]$ is nowhere dense. □

For a dynamical system (X, f) and $x \in X$, the **backward orbit** of x is $\{y \in X : f^n(y) = x \text{ for some } n \in \mathbb{N}\}$. Backward orbits are relevant, for instance, in the study of *Julia sets* [11]. Moreover, it is known that in the case of systems such as the tent map and the irrational rotation, every point of the phase space has a dense backward orbit (c.f. [42]).

Corollary 2.9.6. *If F is transitive, then the set of points having dense backward orbit, is residual.*

Proof. Since F is transitive, for any cylinder U_w, $\bigcup_{n\in\mathbb{N}} F^n(U_w)$ is dense and

$$\bigcup_{n\in\mathbb{N}} F^n(U_w) = \bigcup_{n\in\mathbb{N}} \overline{int[F^n(U_w)]} \subset \overline{\bigcup_{n\in\mathbb{N}} int[F^n(U_w)]}.$$

Therefore, $V_w := \bigcup_{n\in\mathbb{N}} int[F^n(U_w)]$ is open and dense. Hence $V = \bigcap_w V_w$ is a dense G_δ set. It can be seen that any point of V has a dense backward orbit. □

For a transitive CA, there may exist points whose backward orbit is not dense. For example consider the shift map and one of its fixed points. On the other hand, it is easy to verify that for $F : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ given by $[F(x)]_j = x_j + x_{j+1} \pmod 2$, every point has a dense backward orbit (essentially use: F is both *left permutive* and *right permutive*).

Problem: Characterize those transitive CA having the property that every point has a dense backward orbit.

2.10 Appendix: Cell maps

There is this *overlapping* phenomena when one applies the local rule of a CA which creates so much complication. Just for curiosity we ask: what if we rearrange matters so that there is no overlapping? Of course, this means we are no longer working with CA, but with a different class of continuous maps. For want of a better name, let us call the new class of maps, **cell maps**. Similar to CA, the cell map is also defined with respect to a local rule, but without the overlapping of arguments. A cell map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined in such a way that the value of x_i will affect the value of $[F(x)]_j$ for only one j. Hence its behavior is expectedly more easily understandable in terms of the local rule. For example, we can characterize sensitive maps and transitive maps among cell maps.

Definition 2.10.1. A map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a **cell map** if there is an integer $r \geq 0$ and a map $f : A^{2r+1} \to A$ such that $[F(x)]_i = f(x_{(2r+1)i-r}, \ldots, x_{(2r+1)i}, \ldots, x_{(2r+1)i+r})$ for $x \in A^{\mathbb{Z}}$, $i \in \mathbb{Z}$.

Example: Let $f : A^3 \to A$ be $f(abc) = c$. Then the corresponding cell map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by $F(x) = (x_{3i+1})_{i \in \mathbb{Z}}$.

Properties of cell maps: We will be very brief. Assume that $A = \{0, 1, \ldots, m-1\}$ and the cell map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by the local rule $f : A^{2r+1} \to A$. Here r is called the *radius* of the local rule.

1. A cell map F commutes with the shift map if and only if $r = 0$.

2. A cell map F is injective (surjective) if and only if f is injective (surjective).

3. If F is not injective, there exists $y \in A^{\mathbb{Z}}$ with $\{x \in A^{\mathbb{Z}} : F(x) = y\}$ uncountable.

4. If F is a cell map, define $A_0 \supset A_1 \supset A_2 \supset \cdots$ inductively as: $A_0 = A$ and $A_j = f(A_{j-1}^{2r+1})$ for $j \in \mathbb{N}$. Put $\tilde{A} = \bigcap_{j \geq 0} A_j$. Let $n, j \geq 0$. For $\alpha \in A^{(2r+1)^n}$, if k is given by the relation $2k + 1 = (2r+1)^n$, define $U_\alpha^j = \{x \in A^{\mathbb{Z}} : x_{[-k,k]} = \alpha$ and $x_i \in A_j$ if $|i| > k\}$ and $\tilde{U_\alpha} = \{x \in A^{\mathbb{Z}} : x_{[-k,k]} = \alpha$ and $x_i \in \tilde{A}$ if $|i| > k\}$. Write U_α^0 as U_α. Then:

   (i) $\{U_\alpha : \alpha \in A^{(2r+1)^n}, n \in \mathbb{N}\}$ is a clopen base for $A^{\mathbb{Z}}$.

   (ii) for $\alpha \in A^{(2r+1)^n}$, $F^j(U_\alpha) = U_\beta^j$ for some $\beta \in A^{(2r+1)^{n-j}}$, for each $j \in \{0, 1, \ldots, n\}$.

   (iii) for $j > 0$, $U_\beta^j = U_\beta$ if and only if F is surjective. If $U_\beta^j \neq U_\beta$, then U_β^j is a nowhere dense subset of U_β.

   (iv) $diam[U_\beta^j] = 0$ if $|A_j| = 1$, and $diam[U_\beta^j] = diam[U_\beta]$ if $|A_j| > 1$.

(v) If $|\tilde{A}| = 1$, then $|A_j| = 1$ and hence F^j is a constant map for some $j < m$.

(vi) If $r = 0$, then F satisfies $d(F(x), F(y)) \leq d(x, y)$ for every $x, y \in A^{\mathbb{Z}}$.

(vii) If $r > 0$, then, for any non-empty open set $U \subset A^{\mathbb{Z}}$, there exists $n \in \mathbb{N}$ such that $F^n(U) \supset \tilde{U}_i$ for some $i \in A = \{0, 1, \ldots, m-1\}$.

5. A cell map is surjective if and only if it is an open map [$\because$ (i), (ii), (iii) of (4)].

6. A cell map F is sensitive if and only if $r > 0$ and $|\tilde{A}| > 1$ [$\because$ (iv), (v), (vi), (vii) of (4)]. In particular, every surjective cell map with $r > 0$ is sensitive.

7. A cell map which is not sensitive is equicontinuous.

8. Every cell map has a dense set of eventually periodic points [$\because P_n(\sigma^{2r+1}) := \{x \in A^{\mathbb{Z}} : \sigma^{(2r+1)n}(x) = x\}$ is F-invariant].

9. Every transitive cell map is sensitive.

10. A surjective cell map need not have a dense set of periodic points. Let $A = \{0, 1\}$ and the local rule $f : A^3 \to A$ of the cell map F be $f(abc) = 0$ if and only if $abc = 000$. Then, F is onto but the open set $\{x \in A^{\mathbb{Z}} : x_{-1}x_0x_1 = 101\}$ contains no periodic points.

11. Recall from (4) that for $0 \leq i \leq m-1$, $U_i = \{x \in A^{\mathbb{Z}} : x_0 = i\}$. If F is a surjective cell map, then, with respect to $U_0, U_1, \ldots U_{m-1}$, F is a 'Markov map' in the sense that for any $i \in \{0, 1, \ldots, m-1\}$, $F(U_i)$ is a non-empty union of finitely many sets from $\{U_0, U_1, \ldots, U_{m-1}\}$. Hence we can associate a matrix M_F called a 'Markov matrix' to F. It is an $m \times m$ square matrix with entries 1 or 0. The $(i, j)^{th}$ entry is 1 if $F(U_i) \supset U_j$, and 0 otherwise. From the theory of Markov maps and (vii) of (4), we obtain: a surjective cell map F with $r > 0$ is transitive if and only if the corresponding Markov matrix M_F is irreducible (this means, for every (i, j), there exists n such that the $(i, j)^{th}$ entry of M_F^n is positive). F is totally transitive if and only if M_F^k is irreducible for every $k \in \mathbb{N}$.

12. There exist transitive cell maps which are not totally transitive and hence not weakly mixing (unlike CA).

Question: What is the necessary and sufficient condition for a cell map to have a dense set of periodic points?

Chapter 3

Set of periods of additive cellular automata

3.1 Introduction

Calculating the set of periods of dynamical systems has often been an interesting area of research, a classical result in this direction being Sarkovski's Theorem about interval maps. All possible sets of periods of continuous self-maps on zero-dimensional metric spaces, compact subsets of $\mathbb{R}$ and convex subsets of $\mathbb{R}^n$ are described in [61].

In this section, we determine completely the set of periods for a large class of one-dimensional additive CA. We do not deal with the topological aspects here. Our arguments are rather combinatorial in nature, and our proofs are built upon nothing heavier than some basic properties of primes.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the collection of all additive CA where the addition is done modulo n. Let p be any prime. The highlights are:

(i) For any $F \in \mathcal{F}_p$, the set of periods can be determined using some simple conditions on the coefficients in the linear expression of F.

(ii) For any $F \in \mathcal{F}_p$, the set of periods has only four possibilities: $\{1, m\}$ for some m where $1 \leq m < p$, $\mathbb{N} \setminus \{p^m : m \in \mathbb{N}\}$, $\mathbb{N} \setminus \{2p^m : m \in \mathbb{N} \cup \{0\}\}$ or the whole set $\mathbb{N}$.

(iii) If $F \in \mathcal{F}_p$, then our proof actually calculates the cardinality of the set $\{x : F^n(x) = x\}$, which is shown to be a power of p except when F is a root of identity.

(iv) Using our results, the set of periods of any additive CA, where the addition is done modulo some square-free positive integer, is easily obtained.

3.2 Basic Lemma

Let $f : X \to X$ be a map (X can be just a set; the topological structure on X is not needed). Let $Per(f) = \{n \in \mathbb{N} :$ there is a periodic point x for f whose period is $n\}$. Let $P_n(f) := \{x \in X : f^n(x) = x\} = \{x \in X :$ the period of x is a divisor of $n\}$. We have the following simple result, which will be used many a time.

Lemma 3.2.1. Let $f : X \to X$ be a map and let $|P_n(f)|$ denote the cardinality of $P_n(f)$. If $\sum_{d|n \text{ and } d<n} |P_d(f)| < |P_n(f)|$, then $n \in Per(f)$. On the other hand, if $|P_d(f)| = |P_n(f)| < \infty$ for some proper divisor d of n, then $n \notin Per(f)$.

Let $m \geq 2$ be an integer and let $A = \{0, 1, \ldots, m-1\}$. Recall that an **additive CA** is a map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ which has the form $F(x)_i = \sum_{j=-k}^{k} a_j x_{i+j} \pmod{m'}$ for some fixed natural number $m' \leq m$, fixed $k \geq 1$ and fixed integers a_j. Since we are interested only in the periodic points of F, and since these periodic points lie in the range of F, without loss of generality we may assume $m' = m$.

3.3 Using the binomial coefficients

We require a technical result about the divisibility of the integer coefficients of a certain polynomial by a prime. The proof of this technical result uses an elementary property of binomial coefficients, which is given below. The binomial coefficient $\dfrac{n!}{j!(n-j)!}$ is denoted by ${}^nC_j$.

Lemma 3.3.1. Let p be a prime, $n \in \mathbb{N}$ and let m be the largest integer such that p^m divides n. Then, the smallest $j \geq 1$ such that ${}^nC_j$ is non-zero modulo p, is $j = p^m$.

The following simple technical result, which has a somewhat complicated appearance, might be known. A proof is provided for the sake of completeness.

Lemma 3.3.2. Let p be a prime, let $k \in \mathbb{N}$, and let $a_0, a_1, \dots, a_k$ be integers such that a_0 and a_k are non-zero modulo p. Also, let $l \geq 1$ be the smallest integer such that a_l is non-zero modulo p. Fix $n \in \mathbb{N}$ and write $n = p^m r$, where $m \geq 0$ and $p \nmid r$. Let β_t be the coefficient of x^t in the polynomial $(a_0 + a_1 x + \cdots + a_k x^k)^n$. Then, the smallest integer $t \geq 1$ such that β_t is non-zero modulo p, is $t = lp^m$.

Proof. For the convenience of writing, put $q = p^m$ so that $n = qr$. Now,

$$\begin{aligned}(a_0 + a_1 x + \cdots + a_k x^k)^n &= \left[(a_0 + a_1 x + \cdots + a_k x^k)^q\right]^r \\ &\equiv [a_0^q + a_1^q x^q + \cdots + a_k^q (x^q)^k]^r \pmod p\end{aligned}$$

since $z \mapsto z^p$ is a morphism modulo p. By the choice of l in the hypothesis, it is clear that the smallest non-zero power of x occurring in the above expression is $(x^q)^l$, whose coefficient is $r a_0^{q(r-1)} a_l^q$. Hence the required t is $t = ql = p^m l$. □

For later use, observe that that $\beta_t = \sum a_{r_1} a_{r_2} \cdots a_{r_n}$, where the sum is taken over all n-tuples $(r_1, r_2, \dots, r_n)$ of non-negative integers such that $r_1 + \cdots + r_n = t$.

3.4 $Per(F)$ for $F \in \mathcal{F}_p$, where p is prime

Convention for this entire section: *p is a prime, $A = \{0, 1, \dots, p-1\}$, $k \in \mathbb{N}$, $a_j \in \mathbb{Z}$ for $-k \leq j \leq k$ and $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is an additive CA given by $F(x)_i = \sum_{j=-k}^{k} a_j x_{i+j}$ (mod p). It is allowed that $a_j \equiv 0$ (mod p).*

To determine $Per(F)$, first we compute $|P_n(F)|$. The idea behind the computation of $|P_n(F)|$ can be explained with a few words. Observe that

$$F^n(x)_i = \sum_{t=-kn}^{kn} \left(\sum_{r_1 + r_2 \cdots + r_n = t} a_{r_1} a_{r_2} \cdots a_{r_n} \right) x_{i+t} \pmod p, \quad \text{where } r_j \in \{-k, \dots, 0, \dots, k\}.$$

Note the similarity between the coefficient of x_{i+t} in the above expression, and the coefficient β_t mentioned in Lemma 3.3.2, which we will exploit soon. From the above expression, it is clear that $[F^n(x) - x]_i$ has a linear expression involving x_{i+t}'s for $-kn \leq t \leq kn$, where some coefficients may vanish modulo p. Assume for the moment that there is at least one non-vanishing coefficient. Let t_0, t_1 be respectively the smallest and greatest t such that the coefficient of x_{i+t} is non-zero modulo p in the linear expression for $[F^n(x) - x]_i$. Note that t_0 and t_1 are independent of i, but they depend on n and a_j's (and of course on p). We wish to conclude that $|P_n(F)| = p^{t_1 - t_0}$.

This is argued as follows. When we look for an element x satisfying $F^n(x) = x$, or equivalently $[F^n(x)-x]_i = 0$ for every i, we note that any $t_1 - t_0$ consecutive positions of x can take arbitrarily values of A, and any fixed collection of values for one set of $t_1 - t_0$ consecutive positions of x determines uniquely the values for all other positions of x in a recursive manner using the conditions $[F^n(x) - x]_i = 0$. This is a simple consequence of the fact that if $q \in A \setminus \{0\} = \{1, 2, \ldots, p-1\}$, then $\{qr \pmod p) : r \in A\} = A$. Thus, we have that $|P_n(F)| = |A|^{t_1 - t_0} = p^{t_1 - t_0}$.

For the convenience of writing let us make the following definitions. We say F is of

(i) **type-1** if $a_j \equiv 0 \pmod p$ for all $j \neq 0$.

(ii) **type-2** if $\exists l \in \{1, \ldots, k\}$ such that a_{-k}, a_l are non-zero modulo p and $a_j \equiv 0 \pmod p$ for $l < j \leq k$.

(iii) **type-3** if $\exists l \in \{1, \ldots, k\}$ such that a_l, a_k are non-zero modulo p and $a_j \equiv 0 \pmod p$ for $-k \leq j < l$.

(iv) **type-4** if $\exists l \in \{1, \ldots, k\}$ such that a_0, a_l, a_k are non-zero modulo p and $a_j \equiv 0 \pmod p$ for $-k \leq j < 0$ and $0 < j < l$.

Note that for any F, either F or its mirror image (obtained by interchanging a_j and a_{-j}) has to belong to one of the above types. Therefore, it is enough to calculate $Per(F)$ for the four types of F mentioned above. If F is of type-1, then F is either identically zero or F is a root of the identity. Therefore, in this case $Per(F)$ can be determined directly.

Theorem 3.4.1. *If F is of type-1, that is if F has the form $F(x)_i = a_0 x_i \pmod p$, then $Per(F) = \{1, m\}$ for some $m \in \{1, \ldots, p-1\}$.*

Proof. For any additive CA, $1 \in Per(F)$ since the element $\cdots 000 \cdots$ is fixed by F. If $a_0 \equiv 0 \pmod p$, then $Per(F) = \{1\}$. If $a_0 \not\equiv 0 \pmod p$, then $Per(F) = \{1, m\}$, where $m \in \{1, \ldots, p-1\}$ is the smallest such that $a_0^m \equiv 1 \pmod p$. □

For the other types, we have to calculate $|P_n(F)|$ first. If F is of type-2 or type-3, then this calculation is easy.

Proposition 3.4.2. *If F is of type-2, then $|P_n(F)| = p^{(k+l)n}$ for every n. If F is of type-3, then $|P_n(F)| = p^{kn}$ for every n.*

Proof. If F is of type-2, then the smallest (greatest) t such that the coefficient of x_{i+t} in the linear expression for $[F^n(x) - x]_i$ is non-zero modulo p, is same as the smallest (greatest) t such that the coefficient of x_{i+t} in the linear expression for $[F^n(x)]_i$ is non-zero modulo p. But the linear expression for $F^n(x)_i$ starts with the term $a_{-k}^n x_{i-kn}$ and ends with $a_l^n x_{i+ln}$ and therefore $|P_n(F)| = p^{(k+l)n}$.

If F is of type-3, then the linear expression for $[F^n(x) - x]_i$ starts with $-x_i \equiv (p-1)x_i \pmod p$ and ends with $a_k^n x_{i+kn}$ so that $|P_n(F)| = p^{kn}$. □

This gives:

Theorem 3.4.3. *If F is of type-2 or type-3, then $Per(F) = \mathbb{N}$.*

Proof. In both the cases, $|P_n(F)|$ is of the form p^{rn} for some constant $r \geq 1$. We have already noted that $1 \in Per(F)$ always. So let $n \geq 2$. Then, $\sum_{d|n \text{ and } d<n} |P_d(F)| = \sum_{d|n \text{ and } d<n} p^{rd} \leq \sum_{j=0}^{r\lfloor n/2 \rfloor} p^j < p^{rn/2+1} \leq p^{rn} = |P_n(F)|$. Hence by Lemma 3.2.1, $n \in Per(F)$. □

In computing $|P_n(F)|$, there is some difficulty when F is of type-4. Here, the linear expression for $[F^n(x)]_i$ starts with the term $a_0^n x_i$. So if n is such that $a_0^n \equiv 1 \pmod p$ then, the coefficient of x_i in the linear expression for $[F^n(x) - x]_i$ vanishes modulo p, and hence to determine the first non-vanishing coefficient, we have to resort to more refined techniques. For this purpose we will use Lemma 3.3.2 proved in the previous section.

Proposition 3.4.4. *If F is of type-4, then $|P_n(F)| = p^{\alpha(n)}$, where*

$$\alpha(n) = \begin{cases} kn, \ \text{if } a_0^n \not\equiv 1 \ (\text{mod } p) \\ kn - lp^m, \ \text{if } a_0^n \equiv 1 \ (\text{mod } p) \text{ and if } n = p^m r \text{ where } p \nmid r. \end{cases}$$

Proof. Note that the coefficient of x_{i+t} in $[F^n(x)]_i$ is precisely β_t mentioned in Lemma 3.3.2. □

Thus we can have:

Theorem 3.4.5. *Suppose that F is of type-4. Then,*
(i) if $l = k$ and $a_0 \equiv 1$ (mod p), then $Per(F) = \mathbb{N} \setminus \{p^m : m \in \mathbb{N}\}$.
(ii) if $l = k$, $a_0 \not\equiv 1$ (mod p) and $a_0^2 \equiv 1$ (mod p), then $Per(F) = \mathbb{N} \setminus \{2p^m : m \in \mathbb{N} \cup \{0\}\}$.
(iii) in all other cases, $Per(F) = \mathbb{N}$.

Proof. Again note that $1 \in Per(F)$ always.

Proof of (i): From Proposition 4.1.2, we have $|P_n(F)| = p^{\alpha(n)}$, $\alpha(n) = k(n - p^m)$ for every n, where $n = p^m r$, $p \nmid r$. If $r = 1$, then $n = p^m$ so that $|P_n(F)| = p^{k(n-p^m)} = p^0 = 1$. Hence $p^m \notin Per(F)$ for $m \geq 1$ by Lemma 3.2.1. If $r \geq 2$, then without much difficulty we can see that for proper divisors d of n, we have that $\alpha(d)$'s are distinct and $\alpha(d) \leq k(\lfloor n/2 \rfloor - 1)$. Hence $\sum_{d|n \text{ and } d<n} |P_d(F)| \leq \sum_{j=0}^{k(\lfloor n/2 \rfloor - 1)} p^j < p^{kn/2} \leq |P_n(F)|$, and therefore $n \in Per(F)$.

Proof of (ii): Since $a_0 > 1$ (mod p), p must be an odd prime. We have $|P_n(F)| = p^{\alpha(n)}$, where $\alpha(n)$ as given in Proposition 4.1.2. Let $n = p^m r$, where $p \nmid r$. We note that $\alpha(n) = kn$ if r is odd and hence $n \in Per(F)$ in this case by Lemma 3.2.1, since $\sum_{d|n \text{ and } d<n} |P_d(F)| \leq \sum_{j=0}^{k\lfloor n/2 \rfloor} p^j < p^{kn/2+1} \leq p^{kn} = |P_n(F)|$. If $r \geq 4$ is even, then $(n - p^m) \geq 3n/4$ so that $\sum_{d|n \text{ and } d<n} |P_d(F)| < p^{kn/2+1} \leq p^{3kn/4} \leq |P_n(F)|$, and hence $n \in Per(F)$. In the remaining case, $r = 2$ so that $n = 2p^m$. Then, $|P_n(F)| = p^{k(n-p^m)} = p^{kn/2} = |P_{n/2}(F)|$ and hence $2p^m \notin Per(F)$ for $m = 0, 1, 2, \ldots$, by Lemma 3.2.1.

Proof of (iii): Here, we have three subcases. For the arguments below, let $n = p^m r$, $p \nmid r$.

Subcase-1: Suppose that $l = k$, $a_0 \not\equiv 1$ (mod p) and $a_0^2 \not\equiv 1$ (mod p). In particular p must be an odd prime. We have that $\alpha(n) = kn$ if $r = 1$ or 2, and hence $n \in Per(F)$ in these cases by the typical argument since $\alpha(d) \leq kd$ for every divisor d of n. If $r \geq 3$ and $n \geq 6$, then, $\sum_{d|n \text{ and } d<n} |P_d(F)| < p^{kn/2+1} \leq p^{2kn/3} \leq |P_n(F)|$, and hence $n \in Per(F)$. The possibly remaining cases are $n = 2, 3, 4, 5$. We can show that $2, 3, 4, 5 \in Per(F)$ from the data $|P_1(F)| = p^k$, $|P_2(F)| = p^{2k}$, $|P_3(F)| \geq p^{2k}$, $|P_4(F)| \geq p^{3k}$, $|P_5(F)| \geq p^{4k}$, using Lemma 3.2.1.

Subcase-2: Suppose $l < k$ and $a_0 \equiv 1 \pmod p$. If $r = 1$, then $\alpha(n) = (k-l)p^m > (k-l)p^{m-1} = \alpha(n/p)$. Hence $p^m \in Per(F)$ for every m. If $r \geq 2$, then $kr/2 > l$ so that $\sum_{d|n \text{ and } d<n} |P_d(F)| < p^{kn/2+1} \leq p^{p^m(kr/2+1)} \leq p^{p^m(kr-l)} \leq |P_n(F)|$. Therefore, $n \in Per(F)$.

Subcase-3: Suppose $l < k$ and $a_0 \not\equiv 1 \pmod p$. If $r = 1$, then $\alpha(n) = kp^m > kp^{m-1} = \alpha(n/p)$. Hence $p^m \in Per(F)$ for every m. If $r \geq 2$, then as above, $\sum_{d|n \text{ and } d<n} |P_d(F)| < p^{kn/2+1} \leq p^{p^m(kr-l)} \leq |P_n(F)|$. Therefore, $n \in Per(F)$. □

We have two corollaries combining some of the results we obtained in this section:

Corollary 3.4.6. *Let F be an additive CA, where the addition is done modulo a prime p. If F is not a root of identity, then $|P_n(F)|$ is a non-negative integral power of p for every n.*

Proof. Proposition 3.4.2 and Proposition 4.1.2. □

Corollary 3.4.7. *Let F be an additive CA, where the addition is done modulo a prime p. Then, $Per(F)$ has only four possibilities: $\{1, m\}$ for some m where $1 \leq m < p$, $\mathbb{N} \setminus \{p^m : m \in \mathbb{N}\}$, $\mathbb{N} \setminus \{2p^m : m \in \mathbb{N} \cup \{0\}\}$ or the whole set $\mathbb{N}$.*

Proof. Theorem 3.4.1, Theorem 3.4.3 and Theorem 3.4.5. □

3.5 $Per(F)$ for $F \in \mathcal{F}_n$, where n is square-free

For this section, let $A_n = \{0, 1, \ldots, n-1\}$ for $n \in \mathbb{N}$. Note that A_n is a group with addition modulo n. If n is square-free, $n = p_1 \cdots p_t$ for distinct primes p_j. Basic group theory tells that the map $m \mapsto (m_1, \ldots, m_t)$, where m_j is $m \pmod{p_j}$, is a group isomorphism from A_n onto $A_{p_1} \times \cdots \times A_{p_t}$. This group isomorphism induces a bijection (in fact, a group isomorphism) $\Phi : A_n^{\mathbb{Z}} \to A_{p_1}^{\mathbb{Z}} \times \cdots \times A_{p_t}^{\mathbb{Z}}$.

For $F \in \mathcal{F}_n$, if we associate $F_j \in \mathcal{F}_{p_j}$ by declaring $F_j(x)_i$ to be $F(x)_i \pmod{p_j}$, then it is not difficult to verify that $\Phi \circ F = (F_1 \times \cdots \times F_t) \circ \Phi$. From the previous section we know $Per(F_j)$ for every j. Therefore, we feel that it should be possible to determine $Per(F)$ also. The following two simple results, whose proofs are omitted, help us to say that our guess is correct.

Lemma 3.5.1. Let $f : X \to X$ and $g : Y \to Y$ be maps. If $h : X \to Y$ is a bijection such that $h \circ f = g \circ h$, then $Per(f) = Per(g)$.

Lemma 3.5.2. For self-maps of spaces, if $f = f_1 \times \cdots \times f_t$, then we have $Per(f) = \{l.c.m.(r_1, \ldots, r_t) : r_j \in Per(f_j)\}$, where 'l.c.m.' stands for 'least common multiple'.

Therefore, we have the following tool to determine $Per(F)$ for $F \in \mathcal{F}_n$, when n is square-free.

Theorem 3.5.3. *Let $n = p_1 \cdots p_t$ be a product of distinct primes and let $F \in \mathcal{F}_n$. For $1 \leq j \leq t$, let $F_j \in \mathcal{F}_{p_j}$ be as given above. Then, $Per(F) = \{l.c.m(r_1, \ldots, r_t) : r_j \in Per(F_j)\}$.*

We illustrate the use of this Theorem with an example.

Example 3.5.4. Let $F : A_6^{\mathbb{Z}} \to A_6^{\mathbb{Z}}$ be given by $F(x)_i = 5x_i + x_{i+1} \pmod 6$. Since $6 = 2.3$, define $F_1 : A_2^{\mathbb{Z}} \to A_2^{\mathbb{Z}}$ and $F_2 : A_3^{\mathbb{Z}} \to A_3^{\mathbb{Z}}$ by declaring $F_1(x)_i$ to be $F(x)_i \pmod 2$ and $F_2(x)_i$ to be $F(x)_i \pmod 3$. Then, $F_1(x)_i = x_i + x_{i+1} \pmod 2$ and $F_2(x)_i = 2x_i + x_{i+1} \pmod 3$. By Theorem 3.4.5(i), $Per(F_1) = \mathbb{N} \setminus \{2^m : m \in \mathbb{N}\}$, and by Theorem 3.4.5(ii), $Per(F_2) = \mathbb{N} \setminus \{2.3^m : m \in \mathbb{N} \cup \{0\}\}$. By Theorem 3.5.3, $Per(F) = \{l.c.m.(r_1, r_2) : r_1 \in Per(F_1) \text{ and } r_2 \in Per(F_2)\}$. Hence, $Per(F) = \mathbb{N} \setminus \{2\}$.

Remark: Let p be a prime. Determining $Per(F)$ for $F \in \mathcal{F}_{p^m}$, where $m > 1$, seems to be difficult. The essential reason is the following. While considering $F \in \mathcal{F}_p$, to compute $|P_n(F)|$ we used the fact that if $A = \{0, 1, \ldots, p-1\}$ and if $q \in A \setminus \{0\}$, then $\{qr \pmod p : r \in A\} = A$. The corresponding result is no longer true of we replace p with p^m, $m > 1$.

3.6 On $P_n(F)$

Fine, we have determined the possibilities for the set of periods. But, what about the periodic points themselves? Can we say anything about the periodic points of an additive CA? At present we do not have much to say.

Problem: Is it possible to determine completely the sets $P_n(F) = \{x : F^n(x) = x\}$ when F is an additive CA?

Some elementary hints are given below. Let $A = \{0, 1, \ldots, m-1\}$ and let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an additive CA.

1. Since $P_n(F)$ is the kernel of the group homomorphism $Identity - F^n$, it follows that $P_n(F)$ is a subgroup of $A^{\mathbb{Z}}$, for each n.

2. We can write F as $F = \sum_{j=-k}^{k} a_j \sigma^j$ where σ is the shift map. Then it quickly follows that any two additive CA on the same phase space commute with each other. Therefore, if G is any additive CA having the same phase space as that of F, then $P_n(F)$ must be G-invariant for every n. This considerably reduces the choices for the the subgroup $P_n(F)$.

3. It is possible that if $|P_n(F)| < \infty$, then this cardinality has some relation to $m = |A|$. For instance, we ask: is it true that if $|P_n(F)| < \infty$, then any prime dividing $|P_n(F)|$ must divide m? A non-trivial example where $P_n(F)$ is infinite for an additive F is the following: let $F : \mathbb{Z}_4^{\mathbb{Z}} \to \mathbb{Z}_4^{\mathbb{Z}}$ be $F(x)_n = x_n + 2x_{n+1} \pmod 4$. Then, every element of $\{0, 2\}^{\mathbb{Z}}$ is fixed by F and hence $P_1(F)$ is uncountable.

3.7 Set of periods of a general CA

Once we drop the additivity requirement from a CA, the calculation of the set of periods seemingly becomes almost impossible. We can consider a weaker question.

Problem: Let $\mathcal{S} = \{M \subset \mathbb{N} : M = Per(F)$ for some cellular automata $F\}$. Is it possible to describe $\mathcal{S}$ completely? That is, which subsets of $\mathbb{N}$ can arise as the set of periods of some CA?

Some partial answers are given below:

1. $\mathcal{S}$ is countable as there are only countably many CA.

2. $\mathbb{N} = Per(\sigma) \in \mathcal{S}$ and for every $k \in \mathbb{N}$, $\{k\} = Per(\Gamma_k) \in \mathcal{S}$, where Γ_k is the CA whose local rule is a k-cycle on an alphabet of k symbols. Also, $k\mathbb{N} = Per(\sigma \times \Gamma_k) \in \mathcal{S}$.

3. If $M_1, M_2 \in \mathcal{S}$, then $M_1 \cup M_2 \in \mathcal{S}$.

This is proved as follows. For $i = 1, 2$, let F_i be a CA with local rule $f_i : A_i^3 \to A_i$ such that $Per(F_i) = M_i$. Let $k \in M_1$ and let $A_3 = \{0, 1, \ldots, k-1\}$. We may assume that A_1, A_2, A_3 are pairwise disjoint. Let $B = \bigcup_{i=1}^3 A_i$. We define $G : B^{\mathbb{Z}} \to B^{\mathbb{Z}}$ in such a way that $G|_{A_i^{\mathbb{Z}}} = F_i$ for $i = 1, 2$, $G|_{A_3^{\mathbb{Z}}} = \Gamma_k$ and G has no periodic points in $B^{\mathbb{Z}} \setminus \left[\bigcup_{i=1}^3 A_i^{\mathbb{Z}}\right]$. Then, it will follow that $Per(G) = M_1 \cup M_2$. To achieve our objective, the local rule of G, $g : B^3 \to B$, is defined as below:

$$g(abc) = \begin{cases} f_i(abc), \text{ if } abc \in A_i^3 \text{ for } i \in \{1,2\}, \\ b + 1 (\text{mod } k), \text{ if } abc \in A_3^3, \\ 0, \text{ otherwise.} \end{cases}$$

To see that G has no periodic points in $B^{\mathbb{Z}} \setminus \left[\bigcup_{i=1}^3 A_i^{\mathbb{Z}}\right]$, we argue as follows. If $x \in B^{\mathbb{Z}} \setminus \left[\bigcup_{i=1}^3 A_i^{\mathbb{Z}}\right]$, then there exist $m \in \mathbb{Z}$ and $i, j \in \{1, 2, 3\}$, $i \neq j$ such that $x_m \in A_i$ and $x_{m+1} \in A_j$. Then, we have $[G^n(x)]_{[m,m+1]} \in A_3^2$ for every $n \in \mathbb{N}$. But $x_{[m,m+1]} \notin A_3^2$ by the choice of m. Hence $G^n(x) \neq x$.

4. Every (non-empty) finite subset of $\mathbb{N}$ belongs to $\mathcal{S}$ [$\because$ 2 and 3].

5. $\mathbb{N} \setminus \{1, 2, \ldots, k\} \in \mathcal{S}$ for every $k \in \mathbb{N}$.

The proof is as follows: Let $A = \{0, 1, \ldots, k\}$ and B be a finite set of $k+1$ elements, disjoint with A. Put $C = A \cup B$. We define $F : C^{\mathbb{Z}} \to C^{\mathbb{Z}}$ in such a way that $F|_{A^{\mathbb{Z}}}$ is Γ_{k+1} and $F|_{B^{\mathbb{Z}}}$ is almost like σ except that there will not be any periodic point of period $\leq k$. Also, as done before, we take care that F has no periodic points in $C^{\mathbb{Z}} \setminus [A^{\mathbb{Z}} \cup B^{\mathbb{Z}}]$. The local rule $f : C^{2k+1} \to C$ of F is given by:

$$f(c_{-k} \cdots c_0 \cdots c_k) = \begin{cases} c_0 + 1 (\text{mod } k+1), \text{ if } c_{-k} \cdots c_0 \cdots c_k \in A^{2k+1}, \\ c_1, \text{ if } \{c_i : -k \leq i \leq k\} = B, \\ 0, \text{ otherwise.} \end{cases}$$

Then, it is not difficult to see that $Per(F) = \mathbb{N} \setminus \{1, 2, \ldots, k\}$.

6. Every cofinite subset of $\mathbb{N}$ belongs to $\mathcal{S}$ [$\because$ 3, 4 and 5].

7. If $M \in \mathcal{S}$ and $k \in \mathbb{N}$, then $kM \in \mathcal{S}$.

We proceed as follows. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be such that $Per(F) = M$. Also, let $f : A^3 \to A$ be the local rule of F. Note that $Per(F \times \Gamma_k)$ may not be equal to kM. So we have to make some slight modification. Since the idea is simple, we do not insist on notational rigor. Let $n \in M$. Let $A_0, \ldots, A_{k-1}$ be k-copies of A and let $A_k = \{0, 1, \ldots, n-1\}$. Let B be the disjoint union $\bigcup_{i=0}^{k} A_i$. To obtain $G : B^{\mathbb{Z}} \to B^{\mathbb{Z}}$ with $Per(G) = kM$, we define its local rule $g : B^3 \to B$ as:

$$g(abc) = \begin{cases} b \in A_{i+1}, \text{ if } abc \in A_i^3 \text{ for some } i \in \{0, 1, \ldots, k-2\} \\ f(abc) \in A_0, \text{ if } abc \in A_{k-1}^3 \\ b + 1 (\text{mod } n), \text{ if } abc \in A_k \\ 0 \in A_k, \text{ otherwise.} \end{cases}$$

8. For any $k \in \mathbb{N}$, every cofinite subset of $k\mathbb{N}$ belongs to $\mathcal{S}$ [$\because$ 6 and 7].

Some "vague feelings" are presented as questions:

Question-1: Is it true that if $M \in \mathcal{S}$ is infinite, then M contains a multiple of every natural number (and hence for example, the set $\{1, 3, 5, 7, \ldots\}$ cannot belong to $\mathcal{S}$)?

Question-2: Is it true that every infinite $M \in \mathcal{S}$ has positive *upper density*?

Chapter 4

Added flavor on Topological Dynamics

4.1 Weak mixing and mixing

Any closed invariant subsystem of the shift dynamical system is called a subshift. **Subshifts of finite type** and **sofic shifts** are two major classes of subshifts [51]. They appear as useful tools in many branches of Mathematics such as the study of hyperbolic flows and coding theory. In this section we show that weak mixing implies mixing for all subshifts of finite type and many sofic shifts.

For comparison, one may recall that (1) transitivity implies weak mixing for CA and (2) total transitivity implies mixing for interval maps.

Let A, B be finite sets with$|B| \leq |A|$, let $\phi : A \to B$ be a surjection and let G be a directed graph with A as the vertex set. E_G will stand for the edge set and we will write $ij \in E_G$ to mean that there is an edge from vertex i to vertex j in the directed graph G, where $i, j \in A$. Let $X_G = \{x \in A^{\mathbb{Z}} : x_m x_{m+1} \in E_G \text{ for every } m\}$ and $Y_G^{\phi} = \{y \in B^{\mathbb{Z}} : \text{ there exists } x \in X_G \text{ such that } \phi(x_m) = y_m \text{ for every } m\}$. The dynamical system obtained by restricting the shift map σ to X_G, or simply (X_G, σ), is called a **subshift of finite type** and similarly the pair (Y_G^{ϕ}, σ) is called a **sofic shift**. [Strictly speaking, what we mean is that any subshift of finite type or sofic shift can be realized in the above fashion. For original definitions and equivalence to our description, see [23], [51].]

Note that the map $\Phi : X_G \to Y_G^\phi$ given by $[\Phi(x)]_m = \phi(x_m)$ is a continuous surjection and that $\Phi \circ (\text{shift on } X_G) = (\text{shift on } Y_G^\phi) \circ \Phi$. That is, (Y_G^ϕ, σ) is a factor of (X_G, σ) via the factor map Φ.

Many of the dynamical properties of (X_G, σ) and (Y_G^ϕ, σ) can be expressed in terms of the directed graph G. We are mainly interested in the formulation of three properties: transitivity, weak mixing and mixing. It is known that

Proposition 4.1.1. *[9] A compact dynamical system which is totally transitive and has a dense set of periodic points is weak mixing.*

For $i, j \in A$ we define $N(i, j)$ to be the collection of all $n \in \mathbb{N}$ such that there is a path of length (= number of edges) n from vertex i to vertex j in the directed graph G. Then without much effort one can conclude the following facts (see also [23], [51]). Recall that by a **thick** subset of $\mathbb{N}$ we mean a subset containing arbitrarily large blocks of consecutive numbers.

Proposition 4.1.2. *Let (X_G, σ), (Y_G^ϕ, σ) and $N(i, j)$ be as described above. Then,*

(a) (X_G, σ) is transitive if and only if $N(i, j) \neq \emptyset$ for every $i, j \in A$.

(b) (X_G, σ) is weak mixing if and only if $N(i, j)$ is thick for every $i, j \in A$.

(c) (X_G, σ) is mixing if and only if $N(i, j)$ is cofinite for every $i, j \in A$.

(d) If (X_G, σ) is transitive, then the set of periodic points of σ is dense in X_G.

(e) If (X_G, σ) is transitive, totally transitive, weak mixing, or mixing, or has a dense set of periodic points, then so is the case with (Y_G^ϕ, σ).

Now, consider $i, j, k \in A$ and assume that in the directed graph G, there is a path of length n from vertex i to vertex j and a path of length m from vertex j to vertex k. Then by juxtaposition, we get a path of length $n + m$ from i to k. This yields:

Lemma 4.1.3. For every $i, j, k \in A$, $N(i, j) + N(j, k) \subset N(i, k)$. In particular, if $n \in N(i, i)$, then $n\mathbb{N} \subset N(i, i)$.

Theorem 4.1.4. *If a subshift (X_G, σ) of finite type is weak mixing, then it is mixing.*

Proof. Consider $i, j \in A$. By hypothesis and Lemma 4.1.3 we have that $n\mathbb{N} \subset N(i,i)$ for some $n \in \mathbb{N}$, $N(i,j)$ is thick and $N(i,i)+N(i,j) \subset N(i,j)$. Hence $N(i,j)$ is cofinite. Thus, by Proposition 4.1.2(c), (X_G, σ) is mixing. □

Corollary 4.1.5. *For a subshift of finite type, "total transitivity = weak mixing = mixing".*

Proof. Proposition 4.1.1, Proposition 4.1.2(d) and Theorem 4.1.4. □

Transitive subshifts of finite type which are not totally transitive can be easily constructed. For example, take $A = \{a, b, c, d, e, f\}$ and $E_G = \{ab, bc, cd, de, ef, fa, ad\}$.

As we know, the shift map is mixing, and we have seen that a transitive subshift of finite type need not be mixing. But we have the following compensation.

Theorem 4.1.6. *Let (X_G, σ) be a transitive subshift of finite type, which is not a periodic orbit. Then, for some $n \in \mathbb{N}$, the full shift on two symbols is a factor of (X_G, σ^n).*

Proof. Since (X_G, σ) is transitive and X_G is infinite, there exists a cyclic path C in the directed graph G which omits at least one vertex. Also, transitivity tells us that there must be a path from some vertex of C to the omitted vertex. Using this, we can find vertices $a \neq b$ in G such that for some $k \in \mathbb{N}$, there are paths P_{aa} and P_{ab} of length k from a to a and a to b, respectively. Again, by transitivity, there must be a path P_{ba} from b to a. Let l be its length. We show that for $n = 2k + l$, there are paths of length n from a to a, a to b, b to a and b to b.

Using the symbol "+" for the juxtaposition of paths, the required paths are given below:

a to a: $P_{ab} + P_{ba} + P_{aa}$.
a to b: $P_{ab} + P_{ba} + P_{ab}$.
b to a: $P_{ba} + P_{aa} + P_{aa}$.
b to b: $P_{ba} + P_{aa} + P_{ab}$.

This gives that the full shift $(\{a,b\}^{\mathbb{Z}},\sigma)$ is a factor of (X_G,σ^n), where a factor map $\Psi : X_G \to \{a,b\}^{\mathbb{Z}}$ can be defined as follows:

$$[\Psi(x)]_m = \begin{cases} x_{nm}, \text{ if } x_{nm} \in \{a,b\}, \\ a, \text{ otherwise.} \end{cases} \qquad \square$$

A sufficient condition for a transitive subshift of finite type to be totally transitive (= mixing) is given below:

Proposition 4.1.7. *Let (X_G,σ) be a transitive subshift of finite type. If $N(i,i)$ contains finitely many (≥ 2) integers with no common factor (in particular, if $1 \in N(i,i)$) for some $i \in A$, then (X_G,σ) is mixing.*

Proof. Consider $j,k \in A$. By transitivity, $N(j,i)$ and $N(i,k)$ are non-empty. Since $N(i,i)$ is closed under addition by Lemma 4.1.3, if $N(i,i)$ contains finitely many ($\geq$ 2) integers with no common factor, then $N(i,i)$ is cofinite. Again by Lemma 4.1.3, $N(j,i) + N(i,i) + N(i,k) \subset N(j,k)$. Hence $N(j,k)$ is cofinite and thus (X_G,σ) is mixing. $\square$

Now, we turn our attention to sofic shifts. We will make use of the factor map $\Phi : X_G \to Y_G^{\phi}$ which intertwines the dynamics in (X_G,σ) and that in (Y_G^{ϕ},σ). The trick is to translate the problem into (X_G,σ) and then to work with $N(i,j)$'s.

Theorem 4.1.8. *Let (Y_G^{ϕ},σ) be a Sofic shift. Assume that $N(i,i) \neq \emptyset$ for every $i \in A$(= vertex set of G) and that (Y_G^{ϕ},σ) is weak mixing. Then, (Y_G^{ϕ},σ) is mixing.*

Proof. Recall that the continuous surjection $\Phi : X_G \to Y_G^{\phi}$ defined by $[\Phi(x)]_m = \phi(x_m)$ satisfies $\Phi \circ (\text{shift on } X_G) = (\text{shift on } Y_G^{\phi}) \circ \Phi$, where $\phi : A \to B$ is a surjection.

Let $s \in B^{2l+1}$, $t \in B^{2k+1}$ be such that s,t appear in some $y,y' \in Y_G^{\phi}$ respectively. Since the topology on Y_G^{ϕ} is the subspace topology inherited from $B^{\mathbb{Z}}$, to show that (Y_G^{ϕ},σ) is mixing, it is enough to show that the following set is cofinite: $N_B(s,t) = \{n \in \mathbb{N} :$ there is $y \in Y_G^{\phi}$ such that $y_{[-l,l]} = s$ and $[\sigma^n(y)]_{[-k,k]} = t\}$.

For the map $\phi : A \to B$ given above and for any $v = v_1v_2\cdots v_n \in A^n$, we put $\phi(v) = \phi(v_1)\phi(v_2)\cdots\phi(v_n)$. Now, let $J = J(s,t)$ be the (finite) collection of all ordered pairs

(v, w) such that $v \in A^{2l+1}$, $w \in A^{2k+1}$, $\phi(v) = s$, $\phi(w) = t$ and $N_A(v, w) \neq \emptyset$, where $N_A(v, w) = \{n \in \mathbb{N} :$ there is $x \in X_G$ such that $x_{[-l,\, l]} = v$ and $[\sigma^n(x)]_{[-k,\, k]} = w\}$. Then, $N_B(s, t) = \bigcup_{(v,w)\in J} N_A(v, w)$, which is thick since (Y_G^{ϕ}, σ) is assumed to be weak mixing. For $(v, w) \in J$, let $i_v \in A$ be the rightmost letter of v and let $j_w \in A$ be the leftmost letter of w. Then, $N_A(v, w) \setminus [N(i_v, j_w) + l + k]$ is finite. We conclude that $\bigcup_{(v,w)\in J} [N(i_v, j_w) + l + k] = \left[\bigcup_{(v,w)\in J} N(i_v, j_w)\right] + l + k$ is thick.

Hence, $Z := \bigcup_{(v,w)\in J} N(i_v, j_w)$ is also thick. If we show that Z is cofinite, then the steps can be retraced to establish the cofiniteness of $N_B(s, t)$, completing the proof.

By hypothesis on (X_G, σ), $N(i_v, i_v) \neq \emptyset$. Therefore, by Lemma 4.1.3 there exists $n_v \in \mathbb{N}$ such that $n_v\mathbb{N} \subset N(i_v, i_v)$. Let $n = \prod_{(v,w)\in J} n_v$. Then, $n\mathbb{N} \subset N(i_v, i_v)$ for every v such that $(v, w) \in J$. Also, $N(i_v, i_v) + N(i_v, j_w) \subset N(i_v, j_w)$ by Lemma 4.1.3. Thus, we have $n\mathbb{N} + Z \subset Z$. Since Z is thick, we get that Z is cofinite. □

Corollary 4.1.9. *Let (Y_G^{ϕ}, σ) be a sofic shift, which is a factor of the subshift of finite type (X_G, σ) and assume that (X_G, σ) has a dense set of periodic points. Then, for (Y_G^{ϕ}, σ), "total transitivity = weak mixing = mixing".*

Proof. Since the set of periodic points is dense in X_G, $N(i, i) \neq \emptyset$ for every $i \in A$. Now, use Proposition 4.1.1, Proposition 4.1.2(e) and Theorem 4.1.8. □

Remarks: (i) Note that we have not assumed the transitivity of (X_G, σ) in the hypothesis of Theorem 4.1.8. (Y_G^{ϕ}, σ) can be transitive even if (X_G, σ) is not. (ii) We do not know whether the assumption "$N(i, i) \neq \emptyset$ for every $i \in A$" can be dropped from the hypothesis of Theorem 4.1.8. More generally, we do not know whether there is a Sofic shift which is totally transitive but not mixing.

4.2 Continuous maps in the enveloping semigroup

The idea of the *enveloping semigroup* is due to Ellis (see [29], [47], [58]). With the help of the enveloping semigroup, properties regarding the asymptotic behavior of dynamical systems can be expressed as neat algebraic statements. For instance, a surjective dynamical system is equicontinuous if and only if the corresponding enveloping semigroup is a group of homeomorphisms [2].

Let (X, f) be a dynamical system, where X is a compact metric space. Consider X^X, the collection of all maps (need not be continuous) from X to X. Equip X^X with product topology, which is same as the topology of pointwise convergence. Now, $\{f^n : n \in \mathbb{N}\}$ is a subspace of X^X. The closure of $\{f^n : n \in \mathbb{N}\}$ in X^X is called the **enveloping semigroup** of (X, f), and it will be denoted by $Env(X, f)$ or $Env(f)$. It is indeed a semigroup under the composition of maps.

In this section we investigate the continuity property of maps which are the limit points of the enveloping semigroup of certain dynamical systems. The enveloping semigroup is in general not metrizable, and hence for arguments involving limit points one has to use *nets* instead of sequences.

Lemma 4.2.1. Let X be a compact metric space, $f : X \to X$ be continuous and $g \in Env(f)$. Then, $P(f) \subset P(g)$.

Proof. First note that any $g \in Env(f)$ commutes with f. Let $\{n_\alpha\}$ be a net such that $f^{n_\alpha} \to g$, and let $x \in P(f)$. Now, $\{f^{n_\alpha}(x) : n_\alpha\}$ is finite and $f^{n_\alpha} \to g$. Therefore, $g(x) = f^j(x)$ for some j. If $f^n(x) = x$, then since g commutes with f, $g^n(x) = f^{jn}(x) = x$. Thus, $x \in P(g)$. □

Let (X, f) be a compact dynamical system with an admissible metric d. A point $y \in X$ is said to be an **asymptotically periodic point** for f if there is a periodic point $x \in X$ such that $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$.

Lemma 4.2.2. Let X be a compact metric space, $f : X \to X$ be continuous and $g \in Env(f)$ be a limit point. If $x \in X$ is an asymptotically periodic point of f, then, $g(x) \in P(f)$.

Proof. Similar to the previous proof. □

Lemma 4.2.3. [25] Let $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the shift map. If $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a surjective CA, then $F^{-1}(P(\sigma)) \subset P(\sigma)$.

Proposition 4.2.4. *Let σ be the shift map. If $F \in Env(\sigma)$ is continuous, then $F = \sigma^n$ for some $n \in \mathbb{N}$.*

Proof. Let $F \in Env(\sigma)$ be continuous. Then, F is a CA. Since σ is surjective, it is easy to see that F is also surjective. Now, σ has asymptotically periodic points which are not periodic. Therefore, by the last two Lemmas, F cannot be a limit point of $Env(\sigma)$. □

Lemma 4.2.5. Let X, Y be compact metric spaces, and let $f : X \to X$, $g : Y \to Y$, $h : X \to Y$ be maps such that $h \circ f = g \circ h$. If f, h are continuous and if h is onto, then g is continuous.

Proof. We show that pre-image of closed sets are closed. Let $K \subset Y$ be closed. Then, $L := (g \circ h)^{-1}(K) = (h \circ f)^{-1}(K)$ is closed in X since f, h are continuous. Since h is onto, $g^{-1}(K) = h(L)$. Since X, Y are compact and h is continuous, $h(L)$ is closed in Y. □

Proposition 4.2.6. *Let $f : [0, 1] \to [0, 1]$ be a continuous map having a periodic point of period different from a power of 2. If $g \in Env(f)$ is continuous, then $g = f^n$ for some $n \in \mathbb{N}$.*

Proof. By the hypothesis on f, there exist $k \in \mathbb{N}$ and closed, f^k-invariant set $X \subset [0, 1]$ such that the shift map σ (on two symbols) is a factor of (X, f^k) (c.f. [60]). Let h be a factor map connecting $f^k|_X$ to σ.

Since $g \in Env(f)$, $f^{n\alpha} \to g$ for some net $\{n_\alpha\}$. We may assume that there is $j \in \{0, 1, \ldots, k-1\}$ such that $n_\alpha = m_\alpha k - j$ for every α. Also, by the compactness of $Env(\sigma)$, we may assume that $\{\sigma^{m_\alpha}\}$ converges to some F. Then, we have $h \circ [g \circ f^j] = F \circ h$. Therefore, by Lemma 4.2.5, F must be continuous, and hence by the previous Proposition, $F = \sigma^n$ for some n. Thus, m_α must be eventually n since σ^n is not a limit point of $Env(\sigma)$. It follows that n_α is eventually $nk - j$. Hence, $g = \lim f^{n_\alpha} = f^{nk-j}$. Thus g is not a limit point. □

For a periodic orbit P of (X, f), let $Asy[P]$ be the collection of all $y \in X$ such that there is $x \in P$ with $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$.

Proposition 4.2.7. *Let X be a compact metric space and let $f : X \to X$ be continuous. If f has distinct periodic orbits P, Q such that $Asy[P]$, $Asy[Q]$ are both dense in X, then any $g \in Env(f)$ which is a limit point, is nowhere continuous on X.*

Proof. Let $g \in Env(f)$ be a limit point, and let $x \in X$. Then, $g(x) \notin P$ or $g(x) \notin Q$. Assume that $g(x) \notin P$. Find a net $\{x_\alpha\}$ in $Asy[P]$ converging to x. Then, for each α, $g(x_\alpha) \in P$ by Lemma 4.2.2, and hence $\lim_\alpha g(x_\alpha) \neq g(x)$. □

Corollary 4.2.8. *Let X be a compact metric space and let $f : X \to X$ be transitive. If f has two distinct periodic orbits, then any $g \in Env(f)$ which is a limit point, is nowhere continuous on X.*

Proof. It is not difficult to verify (using the abundance of points with dense orbits) that the hypothesis of the previous proposition is satisfied. □

Corollary 4.2.9. *Let (X, f) be one of the following:*
(i) a transitive system on [0,1],
(ii) a transitive CA, or
(iii) a transitive subshift of finite type.
Then any $g \in Env(f)$ which is a limit point, is nowhere continuous on X.

Proof. In each case, verify the hypothesis of the previous corollary. □

4.3 ω-limit sets of the shift map

Let (X, f) be a dynamical system with X a compact metric space. For $x \in X$, the **ω-limit set** of x is the set of limit points of the f-orbit of x. It is denoted by $\omega(f, x)$. Note that $y \in \omega(f, x)$ if and only if there is an increasing sequence (n_k) of natural numbers such that $f^{n_k}(x) \to y$. The set $\omega(f, x)$ is closed and f-invariant. See for instance [1] to know more about ω-limit sets. In this section we characterize the ω-limit sets of the shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ in terms of words over A.

If $Y \subset A^{\mathbb{Z}}$, let $W_Y = \{w \in A^+ : w \text{ appears in some } y \in Y\}$. For $x \in A^{\mathbb{Z}}$, let us call $x_0x_1x_2\ldots$ the *right part* of x. It is clear that

Lemma 4.3.1. If $Y = \omega(\sigma, x)$, then $w \in W_Y$ if and only if w appears in the right part of x infinitely often.

Now, the characterization runs as follows.

Theorem 4.3.2. *Let $Y \subset A^{\mathbb{Z}}$ be non-empty and closed. Then the following are equivalent.*
(i) $Y = \omega(\sigma, x)$ for some $x \in A^{\mathbb{Z}}$.
(ii) For every $u, w \in W_Y$ and every $n \in \mathbb{N}$, there exists $v \in A^+$ such that any subword of uvw of length $\leq n$ belongs to W_Y.

Proof. $(i) \Rightarrow (ii)$: Let $u, w \in W_Y$ and $n \in \mathbb{N}$ be given. Then, the right part of x can be written as $v_1 u v_2 w v_3 u v_4 w \cdots$, where $v_j \in A^+$. We claim that for some even j, any subword of $u v_j w$ of length $\leq n$ belongs to W_F. Suppose not. Then, for every even j, there is $s_j \in A^n \setminus W_Y$ such that s_j is a subword of $u v_j w$. Since s_j varies over a finite set, some s_j must repeat infinitely often, call it s. Then, s appears in the right part of x infinitely many times, but by choice $s \notin W_F$, which is a contradiction.

$(ii) \Rightarrow (i)$: Write $W_Y = \{w_1, w_2, w_3, \ldots\}$ so that $|w_n| \leq |w_{n+1}|$ for every n. For each n, find $s_n \in A^+$ such that any subword of $w_n s_n w_{n+1}$ of length $\leq |w_n|$ belongs to W_Y. Then, one has that for each n, any word of length $\leq |w_n|$ appearing in the sequence $w_n s_n w_{n+1} s_{n+1} w_{n+2} s_{n+2} \cdots$ belongs to W_Y - call this observation $(*)$.

Let $x \in A^{\mathbb{Z}}$ be such that the right part of x is $w_1 s_1 w_2 s_2 w_3 s_3 \cdots$. We claim that $Y = \omega(\sigma, x)$.

Let $y \in Y$ and let $k \in \mathbb{N}$. Then, $y_{[-k,k]} \in W_Y$ so that $y_{[-k,k]} = w_n$ for some n. Hence by $(*)$, the word $y_{[-k,k]}$ appears in the right part of x infinitely often. Since k is arbitrary, we get $y \in \omega(\sigma, x)$. Thus, $Y \subset \omega(\sigma, x)$.

To establish the reverse inclusion, let $y \notin Y$. Since Y is closed, there is $k \in \mathbb{N}$ such that $y_{[-k,k]} \notin W_Y$. Because of $(*)$, the word $y_{[-k,k]}$ can appear only finitely many times in the right part of x. Therefore, $y \notin \omega(\sigma, x)$. □

Problem: Characterize the ω-limit set of additive CA in terms of words.

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