DYNAMICS OF CERTAIN TRANSCENDENTAL ENTIRE AND MEROMORPHIC FUNCTIONS

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CERTIFICATE

It is certified that the work contained in the thesis entitled "**Dynamics of certain Transcendental Entire and Meromorphic Functions**" by **Tarakanta Nayak** (Roll Number: 01612303) has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

September, 2006

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To the reminiscence of

my dear

JEJA

and all Grandfathers like him

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September, 2006 **Tarakanta Nayak**

Omm
Purnamadah Purnamidam
Purnat Purnamudachyate
Purnasya Purnamadaya
Purnamebabasishyate
Omm Shantih Shantih Shantih

— The Bruhadaranyak Upanishad

Abstract

Let $f : \mathbb{C} \to \widehat{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be a non-constant transcendental entire or meromorphic function. The function f^n, the n-times composition of f is called the n-th iterate of f. The **Fatou set** of the function f, denoted by $\mathcal{F}(f)$, is defined as $\{z \in \widehat{\mathbb{C}} : f^n(z)$ is defined for each $n = 0, 1, 2, \cdots$ and $\{f^n\}_{n=0}^{\infty}$ forms a normal family (in the sense of Montel) at $z\}$. The **Julia set** of f, denoted by $\mathcal{J}(f)$, is the complement of the Fatou set of f in the extended complex plane $\widehat{\mathbb{C}}$. Thus, the complex plane is divided into two disjoint subsets based on the long term behaviour of the sequences of iterates. For a given function f, the main problem in complex dynamics is to determine the Fatou set and the Julia set of f. In the present work, we have investigated the change of dynamics of transcendental entire and meromorphic functions mainly in one parameter families. The functions we consider are of the following kinds: (i) transcendental meromorphic functions with non-rational Schwarzian derivative and infinite order, (ii) transcendental entire functions of bounded type, (iii) transcendental meromorphic functions of bounded type, (iv) meromorphic functions which are not of bounded type and (v) real meromorphic functions. The present work is organized into six chapters. The preliminaries along with a brief survey are given in **Chapter 1**. The descriptions of other chapters of the thesis are as follows.

In **Chapter 2**, the dynamics of functions in the one parameter family $\mathcal{M} = \{f_\lambda(z) \equiv \lambda \tanh(e^z) : \lambda \in \mathbb{R} \setminus \{0\}\}$ is investigated. The function $\lambda \tanh(e^z)$ differs in many ways from its constituent functions e^z and $\lambda \tanh z$. The dynamics of $\lambda \tanh(e^z)$ for $z \in \mathbb{C}$ is explored and bifurcation in the dynamics of functions $f_\lambda \in \mathcal{M}$ at a critical parameter $\lambda^* \approx -3.2946$ is observed as follows. For $\lambda > \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is the basin of attraction of a real attracting fixed point of f_λ. The Fatou set $\mathcal{F}(f_{\lambda^*})$ is the parabolic basin corresponding to a real rationally indifferent fixed point of f_{λ^*} and the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the basin of attraction or the parabolic basin corresponding to an attracting or a rationally indifferent cycle of real 2-periodic points of f_λ for $\lambda < \lambda^*$. The topology

of the Fatou components is also determined as follows. For $\lambda > \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is infinitely connected whereas the Fatou set $\mathcal{F}(f_\lambda)$ for $\lambda \leq \lambda^*$, contains infinitely many strictly pre-periodic (pre-periodic but not periodic) components. Further, we have proved that, every component of the Fatou set of f_λ is simply connected for $\lambda \leq \lambda^*$. Finally, the Lebesgue measure of the Julia set of f_λ is found to be zero for each non-zero real λ.

Chapter 3 is devoted to the study of dynamics of a class of entire transcendental functions of bounded type. We define a class E of transcendental entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $a_n \geq 0$ for all n, $f(x) > 0$ for $x < 0$ and the set of all singular values of f is a bounded subset of $\mathbb{R}$. Letting $\mathrm{E}_0 \equiv \{f \in \mathrm{E} \ : \ f(0) = 0\}$ and $\mathrm{E}_1 \equiv \{f \in \mathrm{E} \ : \ f(0) \neq 0\}$, it is shown that, both the classes E_0 and E_1 are closed under composition. It is also shown that, for $P(z) = (z + a_1)^{m_1}(z + a_2)^{m_2} \cdots (z + a_n)^{m_n}$ where $a_1, a_2, \cdots, a_n$ are positive real numbers and $m_1, m_2, \cdots, m_n$ are non-negative integers, the functions $\Phi = P \circ f$ and $\Psi = h \circ P$ belong to E_1 when $f \in \mathrm{E}$ and $h \in \mathrm{E}_1$. It is shown for $f \in \mathrm{E}_1$ that there exists a positive real number λ^* (depending on f) such that the chaotic burst in the dynamics of functions in the one parameter family $\{f_\lambda \equiv \lambda f \ : \ \lambda > 0\}$ occurs at $\lambda = \lambda^*$. More precisely, if $f \in \mathrm{E}_1$ then, (i) for $0 < \lambda < \lambda^*$, the Fatou set of f_λ is the union of the basin of attraction of a real attracting fixed point and possibly wandering domains, (ii) for $\lambda = \lambda^*$, the Fatou set of f_λ is the union of the parabolic basin corresponding to a real rationally indifferent fixed point and possibly wandering domains and (iii) for $\lambda > \lambda^*$, the Fatou set of f_λ is empty or possibly contains wandering domains. For $f \in \mathrm{E}_0$, we show that the Fatou set of $f_\lambda \equiv \lambda f$ is the union of the basin of attraction of the superattracting fixed point 0 and possibly wandering domains for each $\lambda > 0$. A sufficient condition is provided for the Fatou set of f_λ to be connected. Lastly, the dynamics of the functions $\frac{\sinh^m z}{z^n}$, $m > n > 0$, both of m and n are either odd or even, and $I_{2n}(z)$ are discussed as examples in the class E_0 where $I_n(z)$ denotes the modified Bessel function of first kind and order n. It is found that the class E_1 contains interesting functions like $z^{-n}I_n(z)$ and $\frac{\sinh^n z}{z^n}$

for $n \in \mathbb{N}$. The dynamics of all these functions are discussed in detail and the pictures of the Julia sets are generated in some cases.

Chapter 4 deals with the chaotic burst in a class of meromorphic functions of bounded type. First, we define a class $\mathcal{E}$ of entire functions h such that (i) $h(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{C}$ where $a_n \geq 0$ for all $n > 0$, (ii) $a_0 = h(0) \geq 1$, (iii) $h(x) > 0$ for all $x < 0$ and (iv) the closure of all the singular values of h is a bounded subset of $\{x \in \mathbb{R} \ : \ x \neq 0\} \bigcup \{z \in \mathbb{C} \ : \ |z| = 1 \text{ and } z \neq \pm i\}$. Then, the class $\mathcal{M} \equiv \{f(z) = J^n(h(z)) \text{ for } z \in \mathbb{C} \ : \ n \in \mathbb{N} \text{ and } h \in \mathcal{E}\}$ is considered where J^n denotes the n-times composition of the Joukowski function $J(z) = z + \frac{1}{z}$. It is interesting to find that, for each natural number m, there is a function $f \in \mathcal{M}$ such that f has exactly m singular values. For $f \in \mathcal{M}$, the dynamics of functions in the one parameter family $\{f_\lambda = \lambda f \ : \ \lambda > 0\}$ is investigated and the chaotic burst at a critical parameter λ^* is observed. The Fatou set $\mathcal{F}(f_\lambda)$ for $0 < \lambda < \lambda^*$, is the union of the basin of attraction of a real attracting fixed point and possibly wandering domains whereas the Fatou set $\mathcal{F}(f_{\lambda^*})$ is the union of the parabolic basin corresponding to a real rationally indifferent fixed point and possibly wandering domains and the Fatou set $\mathcal{F}(f_\lambda)$ is empty or possibly contains wandering domains for $\lambda > \lambda^*$. The function $f(z) = e^z + 1 + \frac{1}{e^z+1}$ is shown to be in the class $\mathcal{M}$ and the dynamics of $f_\lambda(z) = \lambda(e^z + 1 + \frac{1}{e^z+1})$, $\lambda > 0$ is discussed in detail where a number of interesting results are proved. For $0 < \lambda < \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is shown to be connected and consequently, the Julia set $\mathcal{J}(f_\lambda)$ does not contain any continuum which disconnects $\widehat{\mathbb{C}}$. It is also proved that for $0 < \lambda < \lambda^*$, the Julia set of $\lambda J(e^z + 1)$ contains infinitely many bounded but not singleton components along with unbounded components. Further, each component of $\mathcal{J}(f_\lambda)$ containing a pole is bounded and no component of $\mathcal{J}(f_\lambda)$ contains more than one pole. We show that the Julia set of f_λ, $0 < \lambda < \lambda^*$ consists of two completely invariant subsets one of which is totally disconnected.

A class of meromorphic functions which are not of bounded type is investigated in

Chapter 5. Define $\mathcal{N} = \left\{f(z) = \frac{z^m}{\sinh^m z} \text{ for } z \in \mathbb{C} \ : \ m \in \mathbb{N}\right\}$. For each $f \in \mathcal{N}$, consider the one parameter family $\{f_\lambda(z) = \lambda f(z) \ : \ \lambda \in \mathbb{R} \setminus \{0\} \}$. It is proved that the functions in this family are not of bounded type and the Fatou sets of these functions do not contain any Baker domain or wandering domain. For each fixed m, there is a critical parameter λ^* depending upon m such that the dynamics of $f_\lambda(z) = \lambda \frac{z^m}{\sinh^m z}$ undergoes a sudden change when the parameter λ passes through $\lambda^* > 0$. For $0 < |\lambda| < \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is the basin of attraction of a real attracting fixed point of f_λ, the Fatou set $\mathcal{F}(f_{\lambda^*})$ is the parabolic basin corresponding to a real rationally indifferent fixed point of f_λ and, for $|\lambda| > \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is the basin of attraction or parabolic basin corresponding to a cycle of real 2-periodic points of f_λ. The topology of the Fatou components is also investigated and it is found that, the Fatou set $\mathcal{F}(f_\lambda),\ 0 < |\lambda| \leq \lambda^*$ is infinitely connected. For $|\lambda| > \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ contains infinitely many strictly pre-periodic components and each component of $\mathcal{F}(f_\lambda)$ is simply connected.

The meromorphic function that takes real values on the real line is known as real meromorphic. In **Chapter 6**, we consider the class $\mathcal{R}$ of real meromorphic functions f satisfying (i) $f(z) = \sum_{k=-\infty}^{\infty} A_k\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right)$, (ii) $A_k > 0$, $a_k \neq 0$ for $k \in \mathbb{Z}$ and (iii) $\sum_{k=-\infty}^{\infty} \frac{A_k}{a_k^2}$ converges. Then a subclass $\mathcal{R}^*$ of $\mathcal{R}$ is defined to contain those functions f for which (i) $f(z) = \sum_{k=1}^{\infty} \frac{A_k z}{a_k^2 - z^2}$, (ii) $A_k > 0,\ a_k \neq 0$ for $k \in \mathbb{N}$ and (iii) $\sum_{k=1}^{\infty} \frac{A_k}{a_k^2}$ converges. The class $\mathcal{R}^*$ contains the functions $\tan z = \sum_{k=1}^{\infty} \frac{z}{(\frac{(2k-1)\pi}{2})^2 - z^2}$, $\frac{3}{z} - \frac{z \sin z}{\sin z - z \cos z} = \sum_{k=1}^{\infty} \frac{2z}{a_k^2 - z^2}$ where a_k's are positive roots of $\tan z = z$ and $\frac{1}{2i} + \frac{1}{z} + \frac{1}{i(e^{iz} - 1)} = \sum_{k=1}^{\infty} \frac{2z}{4k^2\pi^2 - z^2}$. In the present chapter, the change in the nature of the Fatou set of functions in the family $\mathcal{S} \equiv \{h_{a,b,c}(z) \equiv a + bz - \frac{c}{z} + f(z) \ : \ a,\ b,\ c \in \mathbb{R},\ b,\ c \geq 0 \text{ and } f \in \mathcal{R}\}$ is investigated. The change in dynamics of $h_a(z) = a + f(z)$ for $f \in \mathcal{R}$ and $a \in \mathbb{R}$ is found as follows. Let $J = \{x \in \mathbb{R} \ : \ 0 < f'(x) < 1\}$, $J^* = \{x \in \mathbb{R} \ : \ f'(x) = 1\}$ and $\varphi(x) = x - f(x)$ for $x \in \mathbb{R}$. For $J = \emptyset$ and either $J^* = \emptyset$ or $a \notin \varphi(J^*)$, the Fatou set $\mathcal{F}(h_a)$ is either the union of upper and lower half-planes or, a completely invariant Baker domain. The

Fatou set $\mathcal{F}(h_a)$ is a parabolic domain corresponding to a real rationally indifferent fixed point for $J = \emptyset$, $J^* \neq \emptyset$ and $a \in \varphi(J^*)$. If the set J is non-empty then J can be written as a union of countably many intervals *i.e.,* $J = \bigcup_{n\in K} J_n$ where $K \subset \mathbb{Z}$ is some index set. Define $I_n = \varphi(J_n)$ and $I = \bigcup_{n\in K} I_n$. Let I_n^0 and ∂I_n denote the interior and the boundary of I_n respectively. For $a \in \bigcup_{n\in K} I_n^0$, the Fatou set $\mathcal{F}(h_a)$ is the attracting basin of a real attracting fixed point of $h_a(z)$ and the Fatou set is the parabolic basin corresponding to a real rationally indifferent fixed point of $h_a(z)$ when $a \in \bigcup_{n\in K} \partial I_n$. For $a \in \mathbb{R} \setminus \overline{\bigcup_{n\in K} I_n}$, the Fatou set $\mathcal{F}(h_a)$ is either the union of the lower and upper half-planes or a completely invariant Baker domain. Let $h_{b,c} \equiv h_{a,b,c} \in \mathcal{S}$ where $a = 0,\ b \geq 0,\ c > 0$ and $f \in \mathcal{R}^*$ be bounded on the imaginary axis. Then the Fatou set $\mathcal{F}(h_{b,c}),\ b \in [0,\ 1)$ is $H^+ \bigcup H^-$, where $H^+ = \{z \in \mathbb{C}\ :\ \Im(z) > 0\}$ and $H^- = \{z \in \mathbb{C}\ :\ \Im(z) < 0\}$ are the basins of attractions of a conjugate pair of attracting fixed points. For $b \geq 1$, the Fatou set $\mathcal{F}(h_{b,c})$ consists of Baker domains. Suppose $g_j, f_j \in \mathcal{R}^*$, $\lim_{y\to+\infty} \frac{g_j(iy)}{i} = l_j \neq 0$ and $\lim_{y\to+\infty} \frac{f_j(iy)}{i} = m_j \neq 0$ for $j = 1,\ 2, ...,\ n$. Let $h(z) = \sum_{j=1}^{n} \{\alpha_j g_j(z) - \frac{\beta_j}{f_j(z)}\}$ where $\alpha_j > 0$ and $\beta_j \geq 0$. Assume that at least one β_j is not zero. Then, it is shown that, the Fatou set of h is the union of two completely invariant attracting basins. For $h_b \equiv h_{a,b,c} \in \mathcal{S}$ where $a = c = 0,\ b \geq 0$ and $f \in \mathcal{R}^*$ is bounded on the imaginary axis, the Fatou set $\mathcal{F}(h_b)$ is shown to be the basin of attraction of 0 and $\mathcal{F}(h_b)$ is infinitely connected when $h_b'(0) < 1$. For $h_b'(0) = 1$, the Fatou set $\mathcal{F}(h_b)$ is the parabolic basin corresponding to the rationally indifferent fixed point 0 and is the disjoint union of simply connected petals, namely H^+ and H^-. The Fatou set $\mathcal{F}(h_b) = H^+ \bigcup H^-$ for $h_b'(0) > 1$. In this case, H^+ and H^- are invariant attracting basins if $0 \leq b < 1$ and are invariant Baker domains if $b \geq 1$. Finally, the dynamics of $T_a(z) = a + \tan z$ is studied for $a \in \mathbb{C} \setminus \mathbb{R}$. Since T_a and T_{-a} are conformally conjugate, we investigate the dynamics of T_a for $\Im(a) > 0$ and found that the Fatou set $\mathcal{F}(T_a)$ contains a completely invariant component U_a containing $a + i$ and H^+. For $a - i \in U_a$, we prove that $\mathcal{F}(T_a) = U_a$ which is the case for $a \in P_1 \bigcup P_2$ where

$P_1 = \{x + iy \in \mathbb{C} \ : \ x \in \mathbb{R} \text{ and } y > 1\} \bigcup \{x + i \in \mathbb{C} \ : \ x \neq \frac{2k+1}{2}\pi \text{ for any } k \in \mathbb{Z}\}$ and $P_2 = \{\pi k + iy \in \mathbb{C} \ : \ k \in \mathbb{Z} \text{ and } y > 0\}$. If $a - i \notin U_a$ then each Fatou component of T_a different from U_a is simply connected. It follows that T_a has no Herman rings for $a \in \mathbb{C}$.

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Chapter 1

Introduction

A dynamical system is one which evolves with time. Mathematically, it consists of the space of states of the system together with a rule for determining the state at a future point of time when present state is given. The basic goal of the mathematical theory of dynamical systems is to determine or characterize the long term behaviour of the system. A repeated application or iteration is the essence of the mathematical theory of discrete dynamical systems where the evolution takes place in a discrete manner. *Complex analytic dynamics* is the study of iteration of analytic functions defined on the complex plane.

Iteration theory of rational functions traces its origin in the works of two French mathematicians Pierre Fatou [65, 66] and Gaston Julia [74, 75] in early twentieth century. During the same time, Ritt [112] also studied rational iteration. After a long period of inactivity, in 1980, Mandelbrot used computer graphics successfully to explore complex dynamics [92, 93]. His discovery of Mandelbrot set which is a fractal set inspired many researchers to reinvestigate this field. Many beautiful pictures of similar kind are also due to Peitgen and Richter [104]. Afterwards, new mathematical tools are introduced into the field answering old questions and opening new vistas of investigation. For instance, Sullivan proved the non-existence of wandering domains for rational functions using quasi-conformal mappings [131, 132] and Douady and Hubbard investigated the dynamics of polynomials [33, 52, 53] from a larger perspective. Other important works on rational functions include Sullivan and McMullen [95, 133], Branner and Hubbard [34, 35], Lyubich [86–88] and Epstein [56].

Due to the presence of an essential singularity at infinity, the behaviour of transcendental entire and meromorphic functions are very much complicated in the neighbourhood of infinity. This fact very often calls for different techniques in the study of the iteration of transcendental functions. Though a number of basic results on the dynamics of rational functions get generalized to transcendental case, more subtle techniques are required for the proofs and new features arise in the later case. Unlike rational functions and transcendental entire functions, iteration of transcendental meromorphic functions does not lead to a dynamical system since forward orbit of any pre-image of ∞ terminates. Of course, all other points have well defined forward orbits. Fatou [67] generalized many of his results on the dynamics of rational functions to that of transcendental entire functions. In a series of papers [2, 3, 5–14], Baker *et al.* generalized many basic results on dynamics of rational functions to transcendental entire and meromorphic functions. Further, they found certain new features including Baker domains and wandering domains in the iteration of transcendental entire and meromorphic functions. Other important contributions in these lines are due to Siegel [117], Devaney *et al.* [27, 39–41, 43–48], Rippon *et al.* [108, 110, 111], Stallard [119–121, 123–129] , Eremenko and Lyubich [59, 61], Bergweiler *et al.* [20–26], Domínguez [49, 50], Fagella *et al.* [51, 62], Zheng *et al.* [137–144], Keen *et al.* [68, 78–82] and Herring [69, 70]. A good exposition of iteration of transcendental entire and meromorphic functions can be found in [18, 19, 29, 37, 58, 60, 96, 99, 130].

1.1 Basic Theory

A discrete dynamical system supposes that $(n+1)$-th state of the system, z_{n+1} is determined solely from a knowledge of the previous state z_n, that is $z_{n+1} = f(z_n)$ where f is a function. Let $z_0 = f^0(z_0)$. Set $z_n = f(z_{n-1}) = f^n(z_0)$ for $n = 1, 2, 3, \cdots$. Then, the sequence $\{f^n(z_0)\}_{n=0}^{\infty}$ is called the sequence of iterates or the (forward) orbit of the point z_0. In complex dynamics, the behaviour of the sequence of iterates of z_0 is mainly investigated for various initial points $z_0 \in \widehat{\mathbb{C}}$ where f is a complex function. A function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is called meromorphic in $\mathbb{C}$ if it is analytic everywhere in $\mathbb{C}$ except possibly at poles. A meromorphic function which is not rational is called transcendental.

1.1.1 The dynamical dichotomy: The Fatou and Julia set

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant transcendental meromorphic function and f^n denote n-times composition of f.

Definition 1.1.1. *The (forward) orbit of a point $z_0 \in \widehat{\mathbb{C}}$, denoted by $O^+(z_0)$ is defined as $\bigcup\limits_{n=0}^{\infty} f^n(z_0)$, the union being taken over all n for which $f^n(z_0)$ is defined.*

While studying the long term behaviour of the orbits of various points, it is observed that for certain initial points z_0, the orbits of all points z in some neighbourhood of z_0 exhibit similar behaviour. However, for other initial points z_0, the orbits of all points z in every neighbourhood of z_0 differ drastically. In other words, the forward orbits of some points remain stable under small perturbation which is not the case for other points. For a given transcendental meromorphic function f, the set of all points having stable forward orbits is known as the Fatou set or stable set of f and its complement in the extended complex plane is known as the Julia set or unstable set or chaotic set of f. Thus, the complex plane is divided into two disjoint subsets based on the behaviour of orbits of points. In order to assign a precise mathematical meaning to these ideas, we need to define the important concept of normality introduced by Montel [98] in 1927. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are in $\mathbb{C}$, then the Euclidean distance between z_1 and z_2 is defined as $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The chordal distance $\chi(z_1, z_2)$ between two points z_1 and z_2 in $\mathbb{C}$ is defined as $\frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$ and $\chi(z_1, \infty)$ is defined as $\frac{1}{\sqrt{1+|z_1|^2}}$.

Definition 1.1.2. *A sequence of functions $\{f_n\}$ converges spherically uniformly on compact subsets of a domain D to a function f if, for any compact subset $K \subseteq D$ and $\epsilon > 0$,* *there exists a number* $n_0 = n_0(K, \epsilon)$ *such that* $n \geq n_0$ *implies* $\chi(f_n(z), f(z)) < \epsilon$ *for all* $z \in K$.

If a sequence of functions converges uniformly with respect to the Euclidean metric on compact subsets of a domain, then it converges spherically uniformly on compact subsets of the domain. The converse is true when the limit function is bounded.

Definition 1.1.3. *A family* $\mathcal{F}$ *of functions meromorphic in a common domain* $D \subset \widehat{\mathbb{C}}$ *is said to be normal in* D *if every sequence* $\{f_n\}_{n>0} \subseteq \mathcal{F}$ *contains a subsequence which converges spherically uniformly on compact subsets of* D*. The limit function of* $\{f_n\}$ *is allowed to be* ∞ *in this case. The family* $\mathcal{F}$ *is said to be normal at a point* $z_0 \in D$ *if it is normal in some neighbourhood of* z_0*.*

The concept of equicontinuity can be used to characterize normality of a family of analytic functions [18, 114, 130]. The following result is due to Montel and is called the fundamental normality test that is used to check the normality of a family of meromorphic functions.

Theorem 1.1.1. *Let* $a_1, a_2, a_3 \in \mathbb{C}$ *be distinct and* $\mathcal{F}$ *be a family of meromorphic functions defined on a common domain* $\Omega \subset \mathbb{C}$ *such that* $f(z) \neq a_j$ *for all* $j \in \{1,\ 2,\ 3\}$*, all* $f \in \mathcal{F}$*, and all* $z \in \Omega$*. Then* $\mathcal{F}$ *is normal in* Ω*.*

The fundamental normality test assumes the following form for analytic functions.

Theorem 1.1.2. *Let* $a_1, a_2 \in \mathbb{C}$ *be distinct and* $\mathcal{F}$ *be a family of analytic functions defined on a common domain* $\Omega \subset \mathbb{C}$ *such that* $f(z) \neq a_j$ *for all* $j \in \{1,\ 2\}$*, all* $f \in \mathcal{F}$*, and all* $z \in \Omega$*. Then* $\mathcal{F}$ *is normal in* Ω*.*

Now, we define two basic objects of complex dynamics.

Definition 1.1.4. *The* ***Fatou set*** *of a meromorphic function f, denoted by $\mathcal{F}(f)$, is defined as*

$$\left\{ \begin{array}{lcl} z \in \widehat{\mathbb{C}} & : & f^n(z) \text{ is defined for each } n = 0,\ 1,\ 2,\ \cdots \text{ and } \{f^n\}_{n=0}^{\infty} \text{ forms} \\ & & \text{a normal family at the point } z \end{array} \right\}.$$

Definition 1.1.5. *The* ***Julia set*** *of f, denoted by $\mathcal{J}(f)$, is the complement of the Fatou set of f in the extended complex plane $\widehat{\mathbb{C}}$.*

If f is a transcendental function, $f(\infty)$ is undefined and the point at infinity is in the Julia set of f. By definition, the Fatou set is open and the Julia set is closed. Further, the Julia set is a nowhere dense subset of $\widehat{\mathbb{C}}$ unless it is equal to $\widehat{\mathbb{C}}$. Fatou conjectured that the Julia set may be equal to $\widehat{\mathbb{C}}$ for some function. Later, Baker proved this property for a function of the form $f(z) = kze^z$ [4]. Misiurewicz [97] also established that $\mathcal{J}(e^z) = \widehat{\mathbb{C}}$. The Julia set of the meromorphic function $i\pi \tan z$ is also known to be $\widehat{\mathbb{C}}$ [19]. Other fundamental properties of the Fatou and Julia sets of meromorphic functions are provided in the next few propositions.

Definition 1.1.6. *Given a function f, a set S is called forward invariant if, for all $z \in S$, $f(z) \in S$, unless $f(z)$ is undefined. A set S is called backward invariant if $w \in S$ implies that $z \in S$ for all z satisfying $f(z) = w$. A set S is said to be completely invariant if it is both forward and backward invariant.*

Proposition 1.1.1. *For a meromorphic function f, the Fatou set $\mathcal{F}(f)$ and the Julia set $\mathcal{J}(f)$ are completely invariant.*

Unlike the Julia set, the Fatou set can never be equal to the extended complex plane. Therefore, the Julia set is always non-empty.

Proposition 1.1.2. *The Julia set of a meromorphic function is a perfect set.*

It is an easy consequence of the above proposition that the Julia set of a meromorphic function is uncountable.

Definition 1.1.7. *The backward orbit of z_0, denoted by $O^-(z_0)$ is defined as* $\bigcup_{n=0}^{\infty} f^{-n}(z_0)$ *where* $f^{-n}(z_0) = \{z \in \mathbb{C} \ : \ f^n(z) = z_0\}$. *A point $z_0 \in \mathbb{C}$ is said to be an exceptional value for a function f if the set $O^-(z_0)$ is finite. If the set $O^-(z_0)$ is empty, the point z_0 is called an omitted value of f.*

The following proposition provides a characterization of the Julia set in terms of a non-exceptional value.

Proposition 1.1.3. *If $z_0 \in \mathfrak{J}(f)$ is not an exceptional value of f, then $\mathfrak{J}(f) = \overline{O^-(z_0)}$, the closure of the backward orbit of z_0.*

All transcendental meromorphic functions are classified into three classes by Bergweiler as follows.

- $E = \{f \ : \ f$ is transcendental entire$\}$.
- $P = \{f \ : \ f$ is transcendental meromorphic, has exactly one pole, and this pole is an omitted value$\}$.
- $M = \{f \ : \ f$ is transcendental meromorphic and has either at least two poles or exactly one pole which is not an omitted value$\}$.

There are major differences in the dynamics of functions belonging to the above three classes. The dynamics of functions in the class P were studied in [9, 22, 54, 63, 64, 78, 79, 83, 90, 91, 100]. The present work deals with the dynamics of functions belonging to E and M only.

For $f \in E$, the iterates $f^n(z)$ are defined for all $z \in \mathbb{C}$ and the point at ∞ is the only exceptional (omitted) value. The point at ∞ is not an exceptional value of f for $f \in M$. Therefore, the backward orbit $O^-(\infty)$ of ∞ is an infinite set where f^n fails to be defined for some n. According to Bergweiler [19], the function f in M is called a general meromorphic function. For general meromorphic functions, we have yet another characterization of the Julia set.

Proposition 1.1.4. *If $f \in M$, then $\mathfrak{J}(f) = \overline{O^{-}(\infty)}$.*

For a meromorphic function f, let $I(f) = \{z \in \mathbb{C} \ : \ f^n(z) \to \infty$ as $n \to \infty$ and $f^n(z) \neq \infty\}$. The points of the set $I(f)$ are known as escaping points of f. The following theorem gives a characterization of the Julia set in terms of escaping points.

Theorem 1.1.3. *If f is a transcendental entire or meromorphic function, then $\mathfrak{J}(f)$ is equal to the boundary of $I(f)$ and $I(f) \bigcap \mathfrak{J}(f) \neq \emptyset$.*

The above theorem is proved by Eremenko [57] for entire functions. Domínguez [50] proved it for meromorphic functions treating the cases of finitely many and infinitely many poles separately.

1.1.2 Periodic points

The forward orbits of certain points are finite sets and play a central role in the study of the dynamics of a function. The periodic points of a function are such points. The following is a brief review of the definitions and results concerning periodic points.

Definition 1.1.8. *A point $z_0 \in \mathbb{C}$ is said to be a* ***periodic point*** *of period p of the function $f(z)$ if p is a natural number such that $f^p(z_0) = z_0$. If p is the smallest natural number satisfying $f^p(z_0) = z_0$, then the point z_0 is called a periodic point of f of minimal or prime period p. The set $\{z_0,\ z_1 = f(z_0),\ z_2 = f^2(z_0),\ \ldots,\ z_{p-1} = f^{p-1}(z_0)\}$ is called a cycle of periodic points. The value $\lambda = (f^p)'(z_0)$ is called the* ***multiplier*** *or* ***eigenvalue*** *of the periodic point z_0 with minimal period p.*

For brevity, we write a periodic point of minimal period p as a p-periodic point. If for a point z_0, there is a natural number n_0 such that $f^{n_0}(z_0)$ is periodic, then z_0 is called a pre-periodic point. A periodic point of period one is called a fixed point. The periodic points are classified into three categories depending on the values of their multipliers.

- If $|\lambda| < 1$, then the periodic point z_0 is called attracting. An attracting periodic point is called superattracting if $\lambda = 0$.
- If $|\lambda| = 1$, then the periodic point z_0 is called indifferent or neutral. In this case, $\lambda = e^{2\pi i\alpha}$ for a real number α. The indifferent periodic point z_0 is called rationally indifferent if α is rational and is called irrationally indifferent otherwise. A rationally indifferent periodic point is also known as a parabolic periodic point.
- If $|\lambda| > 1$, then the periodic point z_0 is called repelling.

The local dynamics of the function $f(z)$ around a periodic point is completely dependent on whether the periodic point is attracting, indifferent or repelling and it also affects the global dynamics of the function f significantly. If $f^p(z_0) = z_0$ and $|\lambda| \neq 0,\ 1$, then there is a neighbourhood $N(z_0)$ of z_0 and an analytic homeomorphism $\phi : N(z_0) \to D$ such that $\phi(z_0) = 0$, $\phi'(z_0) = 1$ and $\phi(f^p(\phi^{-1}(z))) = \lambda z$ for all $z \in D = \{z \in \mathbb{C} : |z| < 1\}$. This important result, known as Kœnigs Linearization Theorem is proved by G. Kœnigs in 1884. If $\lambda = 0$, then the above result holds with $z \mapsto \lambda z$ replaced by $z \mapsto z^k$ for some $k \in \mathbb{N}$. If z_0 is an attracting p-periodic point of f, then all the points of $N(z_0)$ tend to z_0 under iteration of f^p. In other words, the nearby points of an attracting periodic point are attracted towards z_0 under iteration of f^p. If z_0 is a repelling p-periodic point, then there exists a neighbourhood $N(z_0)$ such that the following holds. For each $z \in N(z_0) \setminus \{z_0\}$, there is some $n \geq 1$ so that $f^{pn}(z) \notin N(z_0)$. However, the iterates of z may return to this neighbourhood later and may even land on z_0. When $|\lambda| = 1$ and z_0 is rationally indifferent, there are domains, all the points of which tend to z_0 under the iteration of f^p and z_0 lies on the boundary of the domains. These domains are called petals associated with z_0. It is worth noting that attracting periodic points lie in the Fatou set whereas rationally indifferent and repelling periodic points lie in the Julia set, and nothing can be said about irrationally indifferent periodic points in general. The behaviour of iterates of f in the neighbourhood of an irrationally indifferent periodic point is more intricate and the details can be found in [29, 96].

If f is a transcendental meromorphic function and $n \geq 2$, then f has infinitely many repelling periodic points of minimal period n [19]. A characterization of the Julia set of a transcendental meromorphic function in terms of repelling periodic points [12] is as follows.

Theorem 1.1.4. *Let f be a transcendental meromorphic function. Then $\mathfrak{J}(f)$ is the closure of the set of repelling periodic points of f.*

1.1.3 Singular values

Like periodic points, there are certain points in the plane that are equally decisive in the study of dynamics of a function. These points pertain to the mapping pattern of f^{-1} and they are known as singular values. The following is a brief review of the definitions concerning the singular values.

Definition 1.1.9. *A point z_c is said to be a critical point of the meromorphic function f if $f'(z_c) = 0$. The value $f(z_c)$ is called a critical value of f. A point a is called an asymptotic value of the function f if there exists a continuous curve $\gamma : [0, \infty) \to \widehat{\mathbb{C}}$ satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = a$. The curve γ is called an asymptotic path. All the critical and finite asymptotic values of a function are known as **singular values**. The set of all singular values of f is denoted by S_f.*

Definition 1.1.10. *A function f is called critically bounded or functions of bounded type if the set S_f is bounded. The class of all such functions is usually denoted by B. In particular, the function f is called critically finite or functions of finite type if the set S_f is finite. The class of all functions of finite type is denoted by S.*

A more general and often useful way of defining singular values of a function appears in [23] and is as follows.

Definition 1.1.11. *Let $a \in \widehat{\mathbb{C}}$ and $D_r(a)$ be a disc with center at a and radius r (in spherical metric). Let a component U_r of $f^{-1}(D_r(a))$ be chosen for $r > 0$ in such a way that $U_{r_1} \subset U_{r_2}$ for $r_1 < r_2$. Then, one of the following two possibilities occur.*

1. $\underline{\bigcap_{r>0} U_r = \{z\} \text{ for } z \in \mathbb{C}}$: *In this case $f(z) = a$. The point z is called an* ordinary point *if $a \in \mathbb{C}$ and $f'(z) \neq 0$, or if $a = \infty$ and z is a simple pole of f. If $a \in \mathbb{C}$ and $f'(z) = 0$, or if $a = \infty$ and z is a multiple pole of f, then z is called a* critical point *and the point a is called a* ***critical value***.

2. $\underline{\bigcap_{r>0} U_r = \emptyset}$: *In this case, it is said that the choice $r \to U_r$ defines a* ***(transcendental) singularity*** *of f^{-1}. It is also said that the singularity U lies over a and for every $r > 0$, the open set $U_r \subset \mathbb{C}$ is called a neighbourhood of the singularity of U. It can be seen that a point a is an asymptotic value of f if and only if there is a singularity lying over a.*

Note that a singularity may lie over a critical value which means that a point can be an asymptotic value and a critical value simultaneously. For example, the point $z = 1$ is a critical value as well as an asymptotic value for the function $\sin(e^z + \frac{\pi}{2})$. More generally, there can be many different singularities as well as critical and ordinary points lying over the same point a. For $\Omega_1,\ \Omega_2 \subseteq \widehat{\mathbb{C}}$, a meromorphic function $f\ :\ \Omega_1 \to \Omega_2$ is called a covering map if for every $w \in \Omega_2$, there is a neighbourhood N of w such that each component of $f^{-1}(N)$ is mapped homeomorphically onto N by f. If a disc $D \subseteq \widehat{\mathbb{C}}$ does not contain any critical value or asymptotic value of a meromorphic function f, then $f\ :\ f^{-1}(D) \to D$ is a covering. It is this fact that justifies the name *singularities of* f^{-1}. The singular values of a transcendental meromorphic function are precisely the points where some branch of f^{-1} fails to be defined. Iversen [72] classified the singularities as follows.

Definition 1.1.12. *Let $D_r(a)$ and U_r be as defined in Definition 1.1.11. A singularity U over a is called* ***direct*** *if there exists $r > 0$ such that $f(z) \neq a$ for $z \in U_r$. Then this is also true for all smaller r. We say that U is a logarithmic branch point or* ***logarithmic singularity*** *over a if $f : U_r \to D_r(a) \setminus \{a\}$ is a universal covering for some $r > 0$. The simplest kind of direct singularity is logarithmic branch point. A singularity U over a is called* ***indirect*** *if it is not direct.*

1.1.4 Components of the Fatou set

A maximally connected open subset of the Fatou set is called a component of the Fatou set or a Fatou component. Since the Fatou set is completely invariant, any Fatou component U is mapped into a Fatou component, of course not necessarily U.

Definition 1.1.13. *A component U of the Fatou set $\mathcal{F}(f)$ of a meromorphic function f is called p-periodic if p is the smallest natural number satisfying $f^p(U) \subseteq U$. The set $\{U, f(U), f^2(U), ..., f^{p-1}(U)\}$ is called a periodic cycle of Fatou components. The Fatou component U is said to be invariant if $p = 1$.*

Remark 1.1.1. *Let U be a backward invariant Fatou component of a meromorphic function f. If $w \in U$ and z is a point satisfying $f(z) = w$, then $z \in U$. This means that $w \in U \bigcap f(U)$. By continuity of f, $f(U)$ is connected and consequently, the Fatou component containing $f(U)$ and U is same. Therefore U is forward invariant. Thus, it is concluded that a Fatou component of a meromorphic function is backward invariant if and only if it is completely invariant.*

Definition 1.1.14. *A component U of the Fatou set $\mathcal{F}(f)$ is said to be pre-periodic if there exists a natural number k such that $f^k(U)$ is periodic. A pre-periodic Fatou component which is not periodic is called strictly pre-periodic.*

Note that the periodic Fatou components are also pre-periodic.

Definition 1.1.15. *If for some component W of the Fatou set $\mathcal{F}(f)$, $f^l(W)\bigcap f^m(W)=\emptyset$ for all $l, m\in\mathbb{N}$ with $l\neq m$, then W is called a wandering domain.*

Sullivan proved that wandering domains do not exist in the Fatou set of rational functions [132]. But this may occur in the Fatou set of transcendental meromorphic functions [11].

The classification of periodic Fatou components are done on the basis of the behaviour of the sequence $\{f^n\}$ on the components [19] as given below.

Theorem 1.1.5. *Let U be a p-periodic Fatou component of f. Then we have one of the following possibilities.*

1. *U is an **attracting domain**: In this case, U contains an attracting periodic point z_0 of minimal period p and $\lim\limits_{n\to\infty} f^{np}(z)=z_0$ for all $z\in U$. The component U is also known as immediate basin of attraction or immediate attracting basin. If z_0 is a superattracting periodic point, then U is called superattracting domain or Böttcher domain. The set $\{z\in\widehat{\mathbb{C}}\ :\ \lim_{n\to\infty} f^{np}(z)=z_i\}$ for some $z_i=f^i(z_0),\ i=0,\ 1,\ 2,\ ...,\ p-1$ is called the basin of attraction of the attracting periodic cycle $\{z_0,\ z_1,\ z_2,\ ...,\ z_{p-1}\}$.*

2. *U is a **parabolic domain**: In this case, the boundary ∂U of U contains a rationally indifferent periodic point z^* of minimal period p and $\lim\limits_{n\to\infty} f^{np}(z)=z^*$ for all $z\in U$. The component U is also called a Leau domain. The set $\{z\in\widehat{\mathbb{C}}\ :\ \lim_{n\to\infty} f^{np}(z)=z_i\}$ for some $z_i=f^i(z_0),\ i=0,\ 1,\ 2,\ ...,\ p-1$ is called the parabolic basin of the parabolic periodic cycle $\{z_0,\ z_1,\ z_2,\ ...,\ z_{p-1}\}$.*

3. *U is a **Siegel disc**: In this case, there exists an analytic homeomorphism $\varphi: U\to D$ where $D=\{z\ :|z|<1\}$, such that $\varphi(f^p(\varphi^{-1}(z)))=e^{i2\pi\alpha}z$ for some irrational number α.*

4. *U is a **Herman ring**: In this case, there exists an analytic homeomorphism $\varphi : U \to A$ where A is the annulus $\{z : 1 < |z| < r\}, r > 1$, such that $\varphi(f^p(\varphi^{-1}(z))) = e^{i2\pi\alpha}z$ for some irrational number α. Herman rings are also called Arnold rings.*

5. *U is a **Baker domain**: In this case, there exists a point z^* on the boundary ∂U of U such that $\lim\limits_{n\to\infty} f^{np}(z) = z^*$ for all $z \in U$ and $f^p(z^*)$ is not defined. Baker domains are also known as essentially parabolic domains or domains at infinity.*

Remark 1.1.2. 1. *If U is an attracting or parabolic domain, then all the limit functions of $\{f^n(z)\}_{n>0}$ for $z \in U$ are constants which are nothing but attracting or rationally indifferent periodic points. However, all the limit functions of $\{f^n(z)\}_{n>0}$ for z in a Siegel disc or a Herman ring are non-constants. In fact, Cremer [38] proved that if $f^n|_U$ has non-constant limit functions, then U is a Siegel disc or a Herman ring.*

2. *Let U be a p-periodic Siegel disc or a Herman ring. Then U is known as a rotational domain since the function $f^p|_U$ is conformally conjugate to a rotation on unit disc or an annulus. This implies that the map f^p is one-one on U. By Picard's theorem, each complex number except possibly two, has infinitely many pre-images under a transcendental meromorphic function. Therefore, a rotational domain U cannot be completely invariant. Further, there are infinitely many pre-periodic Fatou components that are mapped into U by some iterate of f.*

3. *Fatou discussed a Baker domain in [67]. Baker domains are so-called because Baker established fundamental properties about the growth of iterates in these domains. Since $z = \infty$ is the only point in $\widehat{\mathbb{C}}$ where a transcendental entire $f(z)$ is undefined, any limit function of $\{f^n\}_{n>0}$ on a Baker domain of a transcendental entire function f is ∞. For a Baker domain U of a function $f \in M$, a limit function of $\{f^n\}_{n>0}$ is either ∞ or belongs to the backward orbit $O^-(\infty)$ of ∞. Note that Baker domains do not exist for rational functions.*

The following theorem on Fatou components is proved by Herring [70].

Theorem 1.1.6. *Let f be a meromorphic function and $f : U \to V$ where U and V are two Fatou components of f. Then the set $V \setminus f(U)$ contains at most two points.*

It is interesting to note in the above theorem that, the set $V \setminus f(U)$ may not always contain the exceptional values of f.

The singular values are important due to their close connections with periodic and wandering Fatou components. One of the most useful theorems describing the relation between attracting domains, parabolic domains and rotational domains with the singular values of a meromorphic function is now presented [19].

Theorem 1.1.7. *Let f be a meromorphic function, and let $C = \{U_0,\ U_1, ...,\ U_{p-1}\}$ be a periodic cycle of components of $\mathcal{F}(f)$. Let $sing(f^{-1})$ denote the set of all critical values and finite asymptotic values and finite limit points of these values of f.*

1. *If C is a cycle of immediate attracting basins or Leau domains, then $U_j \bigcap sing(f^{-1}) \neq \emptyset$ for some $j \in \{0,\ 1, ...,\ p-1\}$. More precisely, there exists $j \in \{0,\ 1, ...,\ p-1\}$ such that $U_j \bigcap sing(f^{-1})$ contains a point which is not pre-periodic or such that U_j contains a periodic critical point (in which case C is called a cycle of superattracting domains).*

2. *If C is a cycle of rotational domains, then $\partial U \subset \overline{O^+(sing(f^{-1}))}$ for all $j \in \{0,\ 1, ...,\ p-1\}$.*

The relation between singular values and Baker domains are given in the next two theorems [19].

Theorem 1.1.8. *Let f be a meromorphic function, and let $C = \{U_0,\ U_1, ...,\ U_{p-1}\}$ be a periodic cycle of Baker domains of f. Let z_j denote the limit of $\{f^{np}(z)\}_{n>0}$ for $z \in U_j$*

and define $z_0 = z_p$. *Then* $z_j \in \bigcup_{n=0}^{p-1} f^{-n}(\infty)$ *for all* $j \in \{0,\ 1, ...,\ p-1\}$, *and* $z_j = \infty$ *for at least one* $j \in \{0,\ 1, ...,\ p-1\}$. *If* $z_j = \infty$, *then* z_{j+1} *is an asymptotic value of* f.

Theorem 1.1.9. *Let* f *be a meromorphic function, and let* $C = \{U_0,\ U_1, ...,\ U_{p-1}\}$ *be a periodic cycle of Baker domains of* f. *Then* ∞ *is in the derived set of* $\bigcup_{j=0}^{p-1} f^j(sing(f^{-1}))$ *where* $sing(f^{-1})$ *is as given in Theorem 1.1.7.*

Using *Logarithmic change of variables*, Eremenko and Lyubich [61] proved a remarkable result on non-existence of Baker domains for a certain class of entire functions.

Theorem 1.1.10. *If* $f \in B$ *is an entire function, then the Fatou set of* f *does not contain any Baker domain.*

Let $A_k(f) = \{z \in \mathbb{C} \ : \ f^k \text{ is not analytic at } z\}$ and define

$$S_n(f) = \bigcup_{k=0}^{n-1} f^k(S_f \setminus A_k(f)) \tag{1.1}$$

Let $P(f) = \{z \in \mathbb{C} \ : \ \text{for some } n \in \mathbb{N}, \text{some branch of } f^{-n} \text{ has a singularity at } z\}$. Then, Herring [69] showed that

$$P(f) = \bigcup_{n=1}^{\infty} S_n(f). \tag{1.2}$$

The set $P(f)$ is the forward orbits of all singular values of f as long as they are defined and note that $S_1(f) = S_f$. If f is entire, then the sets A_k are empty for each k and $P(f)$ is the forward orbit of all singular values. Theorem 1.1.10 has been extended to meromorphic functions by Rippon and Stallard [108].

Theorem 1.1.11. *If* f *is a transcendental meromorphic function for which* $S_n(f)$ *is bounded, then* f *has no Baker domains of period* n.

The wandering domains are also related to singular values. A number of classes of transcendental meromorphic functions not having wandering domains are known [19]. In particular, Baker *et al.* [14] proved non-existence of wandering domains for transcendental meromorphic functions in the class S. Bergweiler *et al.* [26] explored the connection between connectivity of wandering domains and weakly repelling (a fixed point z_0 of f is called weakly repelling if it is either repelling or parabolic with multiplier 1) fixed points.

Using hyperbolic metric, Zheng investigated the relations between forward orbits of singular values (wherever defined) and limit functions of iterates $\{f^n\}_{n>0}$ in a Fatou component. The following four theorems are due to Zheng [139].

Theorem 1.1.12. *Let f be a transcendental meromorphic function and let U be a wandering domain of f. Then all the limit points (including ∞ if it is a limit point) of $\{f^n|_U\}$ lie in the derived set of $P(f)$.*

Theorem 1.1.12 was proved by Bergweiler *et al.* [25] for entire functions.

Theorem 1.1.13. *Let f be a transcendental meromorphic function and let U be a component of $\mathcal{F}(f)$. If $f^{np}|_U \to q$ as $n \to \infty$, then either q lies in the derived set of $S_p(f)$ or is a periodic point of f of period $k \leq p$ and p is a multiple of k.*

As applications of above two theorems, Zheng proved non-existence of Baker domains and wandering domains for certain functions in a sub-class of B. The derived set of a set $A \subseteq \widehat{\mathbb{C}}$ is denoted by A'.

Theorem 1.1.14. *Let f be a transcendental meromorphic function and $f \in B$. If the set $\mathcal{J}(f) \bigcap P(f)'$ is finite and $P(f)' \bigcap O^-(\infty) \setminus \{\infty\} = \emptyset$, then f has no wandering domains.*

Theorem 1.1.15. *Let f be a transcendental meromorphic function and $f \in B$. If the set $S_p(f)' \bigcap J_p(\infty) \setminus \{\infty\}$ is empty where $J_p(\infty) = \bigcup_{n=0}^{p-1} f^{-n}(\infty)$, then f has no Baker domains of period $k \leq p$. Therefore, if $P(f)' \bigcap O^-(\infty) \setminus \{\infty\} = \emptyset$, then f has no Baker domains.*

1.1.5 Topology of the Fatou components

The mapping property of a function on a Fatou component is pivotal in determining the topology of the Fatou components and other important aspects of the dynamics. The topology of the Fatou components is discussed in this subsection.

Definition 1.1.16. *The connectivity of a domain* $\Omega \subseteq \widehat{\mathbb{C}}$ *is defined as the number of components of* $\widehat{\mathbb{C}} \setminus \Omega$. *Domains having connectivity one are known as* ***simply connected*** *whereas domains having connectivity more than one are called* ***multiply connected****. In particular, domains of connectivity* 2 *are called* ***doubly connected*** *and domains having connectivity* ∞ *are called* ***infinitely connected****.*

By definition, a Siegel disc is homeomorphic to the unit disc and hence is simply connected. Similarly, being homeomorphic to an annulus, Herman rings are doubly connected. Baker proved that a multiply connected Fatou component of an entire function is bounded and wandering [2]. This implies the following theorem [19].

Theorem 1.1.16. *If* $f \in E$, *then any pre-periodic Fatou component is simply connected. Consequently, the Fatou set of* f *does not contain Herman rings.*

The connectivity question of periodic Fatou components of meromorphic functions is settled by Bolsch [30, 31].

Theorem 1.1.17. *Let* f *be a meromorphic function, and let* U *be a periodic Fatou component. Then the connectivity of* U *is* 1, 2 *or* ∞.

Theorem 1.1.17 was earlier proved by Baker *et al.* [13] when U is an invariant Fatou component of a general meromorphic function. For completely invariant Fatou components, we have the following theorem [13].

Theorem 1.1.18. *Let* f *be a meromorphic function, and let* U *be a completely invariant Fatou component. Then the connectivity of* U *is* 1 *or* ∞.

Meromorphic functions having wandering domains of any connectivity including ∞ are constructed by Baker *et al.* in [11]. The same authors also proved that the connectivity of a pre-periodic Fatou component of a meromorphic function can be any natural number [13].

The number of completely invariant Fatou components of a rational function is at most two [18]. Baker proved that the Fatou set of a transcendental entire function contains at most one completely invariant domain [3]. Baker *et al.* also found that the number of completely invariant Fatou components of a transcendental meromorphic function belonging to the class S is at most two [13]. Two is best possible as shown for the function $\lambda \tan z$ for $\lambda \geq 1$ [42]. In this direction, Domínguez showed that, if $f(z)$ is a meromorphic function with at most finitely many poles, then there is at most one completely invariant Fatou component. The question of number of completely invariant Fatou components for a general meromorphic function, not necessarily belonging to the class S remains open. Further, Bergweiler put forward the following question: If the Fatou set $\mathcal{F}(f)$ of a meromorphic function f has two completely invariant components V_1 and V_2, is it true that $\mathcal{F}(f) = V_1 \bigcup V_2$? Cao and Wang [36] answered it as

Theorem 1.1.19. *Let f be a meromorphic function and $f \in S$. If $\mathcal{F}(f)$ contains two completely invariant components V_1 and V_2, then $\mathcal{F}(f) = V_1 \bigcup V_2$.*

Cao and Wang also answered the question for a special class of functions F defined by $F = \{f \ : \ f(z) = z + r(z)\exp(p(z)), \text{where } r \text{ is rational and } p \text{ is polynomial}\}$.

Theorem 1.1.20. *Let f be a function in $F \bigcap E$. If $\mathcal{F}(f)$ has a completely invariant component U, then $\mathcal{F}(f) = U$.*

Bergweiler and Eremenko investigated completely invariant Fatou components and proved the following [24].

Theorem 1.1.21. *Let f be a meromorphic function belonging to the class S, having two completely invariant domains D_j, $j = 1, 2$. Then*

1. *each* D_j *is the basin of attraction of an attracting or superattracting fixed point, or of a petal of a rationally indifferent fixed point with multiplier* 1,

2. $S_f \subset D_1 \bigcup D_2$,

3. *each* D_j *contains at most one asymptotic value, and if* a *is an asymptotic value and* $0 < \epsilon < dist(a, S_f \setminus \{a\})$, *then the set* $\{z \ : \ |f(z) - a| < \epsilon\}$ *has only one unbounded component,*

4. $\mathfrak{J}(f) \bigcup \{\infty\}$ *is a Jordan curve in* $\widehat{\mathbb{C}}$.

1.1.6 Structure and measure of the Julia sets

The Julia set of a transcendental meromorphic function is usually a very complicated set. In this subsection, some results relevant to our work on the structure and measure of the Julia set are presented.

Definition 1.1.17. *A maximally connected subset of* $\mathfrak{J}(f)$ *is said to be a component of the Julia set. A subset of* $\widehat{\mathbb{C}}$ *is said to be totally disconnected if each of its maximally connected subset is singleton.*

It is easy to see that the part of the Julia set of a transcendental meromorphic function lying in $\mathbb{C}$ is always unbounded. However, some or all components of $\mathfrak{J}(f) \bigcap \mathbb{C}$ can be bounded. The existence of bounded, in particular singleton components of the Julia set is investigated by Domínguez and the following four theorems are proved [50].

Theorem 1.1.22. *Let* $f(z)$ *be a transcendental meromorphic function. Suppose that* $\mathfrak{F}(f)$ *has a component* U *of connectivity at least three. Then singleton components of* $\mathfrak{J}(f)$ *are dense in* $\mathfrak{J}(f)$.

Theorem 1.1.23. *Let* $f(z)$ *be a transcendental meromorphic function. Suppose that* $\mathfrak{F}(f)$ *has three doubly connected components* $U_i,\ i = 1,\ 2,\ 3$ *such that either each component lies* *in the unbounded component of the complement of the other two or two of the components* $U_1,\ U_2$ *lie in the bounded component of* $\widehat{\mathbb{C}} \setminus U_3$ *but* U_1 *lies in the unbounded component of* $\widehat{\mathbb{C}} \setminus U_2$ *and* U_2 *lies in the unbounded component of* $\widehat{\mathbb{C}} \setminus U_1$. *Then singleton components of* $\mathfrak{J}(f)$ *are dense in* $\mathfrak{J}(f)$.

Theorem 1.1.24. *Let* $f(z)$ *be a transcendental meromorphic function. Suppose that* $\mathfrak{F}(f)$ *has multiply connected components* $A_i,\ i \in \mathbb{N}$, *all different, such that each* A_i *separates* 0 *and* ∞ *and* $f(A_i) \subset A_{i+1}$ *for* $i \in \mathbb{N}$. *Then* $\mathfrak{J}(f)$ *has a dense set of singleton components.*

The Julia set of a meromorphic function with infinitely many poles can be totally disconnected as shown for $\lambda \tan z$, $0 < |\lambda| < 1$. However, it is not true for meromorphic functions with finitely many poles.

Theorem 1.1.25. *If* $f(z)$ *is a transcendental meromorphic function with finitely many poles, then* $\mathfrak{J}(f)$ *cannot be totally disconnected.*

A Jordan arc γ in $\widehat{\mathbb{C}}$ is defined to be an homeomorphic image of the interval $[0,\ 1]$. Following Stallard [123], γ is said to be an analytic arc if the homeomorphism φ has a meromorphic extension in a neighbourhood of $[0,\ 1]$. If the extended φ is univalent at each point of $[0,\ 1]$, then γ is said to be a regular arc. If the interval $[0,\ 1]$ is replaced by the unit circle, then γ is called a Jordan curve.

Definition 1.1.18. *For a meromorphic function* f, γ *is said to be a free Jordan arc in* $\mathfrak{J}(f)$ *if there exists an homeomorphism* ψ *of the open unit disc onto a domain* D *in* $\widehat{\mathbb{C}}$ *such that* $\mathfrak{J}(f) \bigcap D$ *is the image of* $(-1, 1)$ *under* ψ *and* γ *is the image of some real interval* $[a,\ b]$ *where* $-1 < a < b < 1$.

The existence of free Jordan arcs in the Julia sets of meromorphic functions and related results are established by Stallard [123].

Theorem 1.1.26. *If f is a meromorphic function and $\mathfrak{J}(f)$ contains a free Jordan arc γ, then $\mathfrak{J}(f)$ is a Jordan arc or a Jordan curve. Further, if γ is analytic, then $\mathfrak{J}(f)$ is also an analytic curve.*

Theorem 1.1.27. *If f is a meromorphic function and $\mathfrak{J}(f)$ contains a free analytic Jordan arc, then $\mathfrak{J}(f)$ is a straight line, circle, segment of a straight line or an arc of a circle.*

Theorem 1.1.28. *There exist meromorphic functions g_0, g_1 and g_2 such that $\mathfrak{J}(g_0) = \mathbb{R}\bigcup\{\infty\}$, $\mathfrak{J}(g_1) = [0,\ \infty]$ and $\mathfrak{J}(g_2) = [-\infty,\ -1]\bigcup[1,\ \infty]$.*

Theorem 1.1.29. *Let $f(z)$ be a transcendental meromorphic function. Suppose that $\mathfrak{J}(f)$ is a Jordan arc or a Jordan curve. If $\mathfrak{J}(f)$ is not a straight line, circle, segment of a straight line or an arc of a circle, then $\mathfrak{J}(f)$ has no differentiable arc.*

Rippon and Stallard [109] proved that a transcendental meromorphic function with finitely many poles cannot have a free Jordan arc.

Even though the Julia set of a function has empty interior, it is not necessary that its Lebesgue measure is zero. McMullen [94] showed that the Julia set of $\sin(\alpha z + \delta)$ for $\alpha,\ \delta \neq 0$ has positive area. In a series of papers, Stallard [119–121, 124–129] investigated the Hausdorff dimension and measure of the Julia sets of entire and meromorphic functions. We briefly state the definition and the result related to the measure of the Julia set that are needed for our study.

Definition 1.1.19. *Let $m(A)$ denote the measure of $A \subset \widehat{\mathbb{C}}$ and $D_r(z)$ denote the disc of radius r with center at z. A subset E of $\widehat{\mathbb{C}}$ is said to be thin at ∞ if its density is bounded away from 1 in all sufficiently large discs, that is, if there exist positive R_0 and ϵ such that, for all complex z and every disc $D_r(z) = \{w : \ |w - z| < r\}$, $r > R_0$,*

$$density(E, D_r(z)) = \frac{m(E\bigcap D_r(z))}{m(D_r(z))} < 1 - \epsilon.$$

Recall that $P(f) = \{z \ : \text{ For some } n \in \mathbb{N}, \text{ some branch of } \ f^{-n} \ \text{ has a singularity at } z\}$ and set $P^*(f) = P(f) \setminus \{\infty\}$. Stallard [122] found a sufficient condition for the Julia sets to have zero Lebesgue measure.

Theorem 1.1.30. *Let f be a meromorphic function and $d(\overline{P^*(f)}, \mathfrak{J}(f)) > 0$ where $\overline{P^*(f)}$ is the closure of $P^*(f)$ in $\mathbb{C}$. If E is a measurable completely invariant subset of $\mathfrak{J}(f)$ such that E is thin at ∞, then $m(E) = 0$. In particular, the Julia set has measure zero if it is thin at ∞.*

1.1.7 Order and Schwarzian derivative

The order of an entire/meromorphic function plays an important role in determining the number of finite asymptotic values and hence in the study of its dynamics. Similarly, the Schwarzian derivative of a meromorphic function is very relevant in the study of its dynamics. The main property of the functions with polynomial Schwarzian derivatives is that these functions have finitely many asymptotic values and no critical values. In this subsection, the definitions and the results related to the order and Schwarzian derivative of a function are reviewed.

The Schwarzian derivative [55] $SD(f)$ of $f(z)$ is defined as

$$SD(f) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Meromorphic functions with polynomial or constant Schwarzian derivative are studied by Devaney and Keen [43, 44].

For a meromorphic function f, let $m(r, \infty) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$ where $\log^+ \alpha = \max(\log \alpha, 0)$ and $N(r, \infty) = \displaystyle\int_0^r \frac{n(t,\infty) - n(0,\infty)}{t} dt + n(0, \infty)\log r$ where $n(r, \infty)$ is the number of poles of $f(z)$ in the disc $\{z \ : \ |z| \leq r\}$, counted according to their multiplicities. Define $T(r) = m(r, \infty) + N(r, \infty)$. The function $T(r)$ is known as the *characteristic function* of $f(z)$. Other equivalent definitions of $T(r)$ can be found in [134].

Definition 1.1.20. *Let $T(r)$ denote the characteristic function of a meromorphic function $f(z)$. Then $f(z)$ is said to be of order ρ if $\rho = \limsup_{r\to\infty} \frac{\log T(r)}{\log r}$ so that $T(r) = O(r^{\rho+\epsilon})$ as $r \to \infty$ for every positive ϵ but not for $\epsilon < 0$.*

Above definition agrees with the order of entire functions as defined below [134].

Definition 1.1.21. *An entire function f is said to be of finite order if there is a positive number A such that, as $|z| = r \to \infty$, $|f(z)| < Ke^{r^A}$ for some constant K. The lower bound ρ of numbers A for which this is true is called the order of the function.*

The following three theorems [71] are useful to calculate the order of an entire function.

Theorem 1.1.31. *A necessary and sufficient condition that an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ should be of order ρ, is that $\liminf_{n\to\infty} \frac{\log \frac{1}{|a_n|}}{n \log n} = \frac{1}{\rho}$. If for certain n, $a_n = 0$, the terms in the above limit are to be taken as zero.*

Theorem 1.1.32. *The order of an entire function $f(z)$ and the order of $f'(z)$ are same.*

Theorem 1.1.33. *If f_1 and f_2 are two entire functions having orders ρ_1 and ρ_2 respectively, then orders of $\frac{f_1}{f_2}$ and $f_1 + f_2$ are at most max$\{\rho_1,\ \rho_2\}$.*

There is a close correlation between the order of an entire function and the number of finite asymptotic values of a function. In 1907, Denjoy conjectured that an entire function of order ρ can have at most 2ρ asymptotic values. The conjecture was proved in affirmative by Ahlfors [1]. Thus, we have the following Denjoy-Ahlfors theorem.

Theorem 1.1.34. *If an entire function is of finite order ρ, then it has at most 2ρ finite asymptotic values.*

Other results connecting the order of a function and the type of singular values are given in the next two theorems [23, 101].

Theorem 1.1.35. *Let f be an entire function with finite order ρ. If f has 2ρ finite asymptotic values, then none of them is a direct singularity of f^{-1}.*

Theorem 1.1.36. *If f is a meromorphic function of finite order and a is an asymptotic value of f, then a is a limit point of critical values $a_k \neq a$ or all singularities of f^{-1} over a are logarithmic.*

1.2 Motivation

A transcendental entire or meromorphic function has an essential singularity at ∞ and the pre-images of all the points in $\widehat{\mathbb{C}}$ except possibly two accumulate at ∞. For this reason, the transcendental entire or meromorphic functions cannot be defined at ∞ which ultimately results in a high degree of complexity in the study of their dynamics. A number of instruments for investigation of the dynamics of rational functions are no longer applicable for transcendental functions. Certain new features arise and more are expected in the dynamics of transcendental entire and meromorphic functions. The dynamical study of individual functions is very often useful and sometimes gives intuitions to prove more general results. The present work is an attempt to investigate the dynamics of certain kind of transcendental entire and meromorphic functions. If a small perturbation is applied to a function f in a family of functions, then the dynamics of the perturbed function may have essentially different features from that of f. The simplest kind of perturbation associated with a function f is the one parameter family $\{\lambda f \ : \ \lambda \text{ is a parameter }\}$. We are mainly interested in studying the changes in the dynamics of functions in the one parameter family $\{\lambda f \ : \ \lambda \text{ is a parameter }\}$ as the parameter λ changes.

A *bifurcation* is said to occur at a parameter value λ^* in the dynamics of a one parameter family of functions $\{f_\lambda \equiv \lambda f \ : \ \lambda \text{ is a real parameter }\}$, if there exists $\epsilon > 0$ such that whenever a and b satisfy $\lambda^* - \epsilon < a < \lambda^*$ and $\lambda^* < b < \lambda^* + \epsilon$, the dynamics of f_a is different from that of f_b. In other words, there is an abrupt change in the nature of the

Fatou set and the Julia set of the functions when the parameter value crosses through the point λ^*. Bifurcation in the dynamics of functions in a one parameter family is an intricate and important issue in complex dynamics, and has been observed in a number of families of transcendental entire and meromorphic functions. The Julia set of λe^z changes from a nowhere dense subset of $\widehat{\mathbb{C}}$ to the whole of extended complex plane when λ increases through the value $\frac{1}{e}$. This phenomena is referred to as *explosion* in the Julia sets or *chaotic burst* in the dynamics of functions in the one parameter family $\{\lambda e^z \ : \ \lambda > 0\}$. An extensive study of the dynamics of λe^z is carried out by Devaney *et al.* [27, 39–41, 45, 47, 48] and others [15–17, 28, 89, 102, 107, 115, 116, 135, 136]. The Julia set of $\lambda \tan z$ is a totally disconnected subset of $\mathbb{R} \bigcup \{\infty\}$ for $|\lambda| < 1$ whereas it is equal to $\mathbb{R} \bigcup \{\infty\}$ for $|\lambda| \geq 1$. This sudden change in dynamics in the tangent family is first reported by Keen and Kotus [42]. Later on, various aspects of dynamics of tangent are explored by Keen *et al.* [80–82], Jiang [73], Skorulski [118] and Oliveira [103]. In these lines, Kremer [84] and Peter *et al.* [105] studied the functions $\lambda z e^z$ and $\lambda + z + e^z$ respectively. Kapoor and Prasad [76, 77] and Prasad [106] have furthered it by studying certain entire functions belonging to the class $B \setminus S$. Sajid [113] studied certain transcendental meromorphic functions having non-rational Schwarzian derivatives including $\frac{z}{z+1}e^{-z}$, $\frac{\sinh z}{z^2}$ and $\frac{\sinh^2 z}{z^4}$.

In this work, we investigate the changes in the dynamics of functions in certain one parameter families of transcendental entire and meromorphic functions. The functions considered for the dynamical study are mainly (i) meromorphic functions with non-rational Schwarzian derivative and infinite order, (ii) entire and meromorphic functions of bounded type, (iii) meromorphic functions which are not of bounded type and (iv) real meromorphic functions.

1.3 Organization

The thesis consists of six chapters. Basic definitions and a brief literature survey of the dynamics of transcendental entire and meromorphic functions are provided in Chapter 1. The organization of the rest of the chapters is as follows.

In **Chapter** 2, dynamics of functions in the one parameter family $\{\lambda \tanh(e^z) \ : \ \lambda \in \mathbb{R}\backslash\{0\}\}$ is investigated. Certain dynamically relevant properties of $\lambda \tanh(e^z)$ are discussed in Section 2.1. Section 2.2 deals with the existence and nature of all the real periodic points of $\lambda \tanh(e^z)$. In Section 2.3, bifurcation in the dynamics of $\lambda \tanh(e^z)$ is proved to occur at a critical parameter $\lambda^* \approx -3.2946$. Section 2.4 describes the change in topology of the Fatou components as a result of above mentioned bifurcation. Finally, it is established in Section 2.5 that the Lebesgue measure of the Julia set of $\lambda \tanh(e^z)$ is zero for all nonzero λ.

Chapter 3 deals with the dynamics of functions in a class of entire transcendental functions which are of bounded type. We define a class E of transcendental entire functions $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ for which (i) $a_n \geq 0$ for all n, (ii) $f(x) > 0$ for $x < 0$ and (iii) the set of all singular values S_f is a bounded subset of $\mathbb{R}$. For each $f \in$ E, the dynamics of functions in the one parameter family $\{\lambda f \ : \ \lambda > 0\}$ is investigated. Let $\mathrm{E}_0 \equiv \{f \in \mathrm{E} \ : \ f(0) = 0\}$ and $\mathrm{E}_1 \equiv \{f \in \mathrm{E} \ : \ f(0) \neq 0\}$. We have shown in Section 3.1 that both the classes E_0 and E_1 are closed under composition and that the compositions of certain kind of polynomials with the functions in E yield functions belonging to E. The real dynamics of $\lambda f(z)$ is determined separately for $f \in \mathrm{E}_0$ and $f \in \mathrm{E}_1$ in Section 3.2. It is shown in Section 3.3 that, for $f \in \mathrm{E}_1$, there exists a positive real number λ^* (depending on f) such that a similar phenomena as chaotic burst occurs in the dynamics of functions in the one parameter family $\{\lambda f \ : \ \lambda > 0\}$ at $\lambda = \lambda^*$. For $f \in \mathrm{E}_0$, it is shown that the Fatou set of λf is always the union of the attracting basin of the superattracting fixed 0 and possibly wandering domains for $\lambda > 0$. It is proved that the Fatou set of λf is connected when it is an attracting basin and all the singular values of λf lie in the immediate attracting basin. Lastly, some examples of functions are discussed in Section 3.4.

In **Chapter 4**, the dynamics of functions in certain class of meromorphic functions of bounded type is investigated. Consider a class $\mathcal{E}$ of entire functions h such that (i) $h(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{C}$ where $a_n \geq 0$ for all $n > 0$, (ii) $a_0 = h(0) \geq 1$, (iii) $h(x) > 0$ for all $x < 0$ and (iv) the closure of all the singular values of h is a bounded subset of $\{x \in \mathbb{R} \; : \; x \neq 0\} \bigcup \{z \in \mathbb{C} \; : \; |z| = 1 \text{ and } z \neq \pm i\}$. Then, we consider the class $\mathcal{M} \equiv \{f(z) = J^n(h(z)) \text{ for } z \in \mathbb{C} \; : \; n \in \mathbb{N} \text{ and } h \in \mathcal{E}\}$ where J^n denotes the n-times composition of the Joukowski function $J(z) = z + \frac{1}{z}$. The change in dynamics of functions in the one parameter family $\mathcal{S} \equiv \{f_\lambda = \lambda f \; : \; \lambda > 0\}$ for $f \in \mathcal{M}$ is mainly explored in this chapter. General properties of f_λ are first proved in Section 4.1. For instance, it is proved that $S_p(f_\lambda)$ is bounded for each p and for each natural number m, there exists a function in $\mathcal{M}$ having exactly m singular values. The dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ is found in Section 4.2 and a similar phenomena as chaotic burst occurs in the dynamics of function in $\mathcal{S}$ is shown at some critical value λ^* (depending on f) in Section 4.3. An example $\lambda J(e^z + 1) \in \mathcal{S}$ is discussed in detail in Section 4.4. Besides chaotic burst, we establish that the Julia set of $\lambda J(e^z + 1)$ contains infinitely many bounded but not singleton components along with unbounded components whenever it is not equal to $\widehat{\mathbb{C}}$. Further, the Julia set can be expressed as a disjoint union of two completely invariant subsets one of which is totally disconnected.

Chapter 5 is devoted to the study of dynamics of certain meromorphic functions which are not of bounded type. Define $\mathcal{N} = \left\{ f(z) = \frac{z^m}{\sinh^m z} \text{ for } z \in \mathbb{C} : \; m \in \mathbb{N} \right\}$. The one parameter family $\mathcal{S} = \{f_\lambda(z) = \lambda f(z) \; : \; \lambda \in \mathbb{R} \setminus \{0\}\}$ for $f \in \mathcal{N}$ is considered. The functions $f_\lambda \in \mathcal{S}$ are proved to be not of bounded type in Section 5.1 along with other dynamically relevant properties of f_λ. Since f_λ and $f_{-\lambda}$ are conformally conjugate and hence have same dynamics, we only investigate the dynamics of $f_\lambda \in \mathcal{S}$ for $\lambda > 0$. The real dynamics of $f_\lambda,\ \lambda > 0$ is investigated in Section 5.2. The bifurcation in the dynamics of functions $f_\lambda \in \mathcal{S},\ \lambda > 0$ at a critical parameter λ^* is proved in Section 5.3. Finally, the effect of this bifurcation on the topology of Fatou components is investigated in Section 5.4.

The dynamics of certain real meromorphic functions is studied in **Chapter 6**. Let $\mathcal{R}$ be a class of real meromorphic functions f satisfying (i) $f(z) = \displaystyle\sum_{k=-\infty}^{\infty} A_k \left(\frac{1}{a_k - z} - \frac{1}{a_k}\right)$, (ii) $A_k > 0,\ a_k \neq 0$ for $k \in \mathbb{Z}$ and (iii) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{A_k}{a_k^2}$ converges. Then, a subclass $\mathcal{R}^*$ of $\mathcal{R}$ is considered that contains those functions f for which (i) $f(z) = \displaystyle\sum_{k=1}^{\infty} \frac{A_k z}{a_k^2 - z^2}$, (ii) $A_k > 0,\ a_k \neq 0$ for $k \in \mathbb{N}$ and (iii) $\displaystyle\sum_{k=1}^{\infty} \frac{A_k}{a_k^2}$ converges. The change in the nature of the Fatou set of functions in the family $\mathcal{S} \equiv \{h_{a,b,c}(z) \equiv a + bz - \frac{c}{z} + f(z)\ :\ a,\ b,\ c \in \mathbb{R},\ b,\ c \geq 0 \text{ and } f \in \mathcal{R}\}$ is investigated. Let $h_a(z) \equiv h_{a,0,0}(z) = a + f(z)$ where $f \in \mathcal{R}$. The dynamics of h_a is mainly investigated in Section 6.1. A sufficient condition for $f \in \mathcal{R}^*$ for being bounded on the imaginary axis is provided and the Fatou sets of $h_{b,c}(z) \equiv h_{0,b,c}(z) = bz - \frac{c}{z} + f(z)$ and $h_b(z) \equiv h_{0,b,0}(z) = bz + f(z)$ are explored in Section 6.2 when $f \in \mathcal{R}^*$ is bounded on the imaginary axis. Several examples are discussed in Section 6.3. In Section 6.4, the dynamics of $a + \tan z$ is explored for $a \in \mathbb{C} \setminus \mathbb{R}$ though these functions are not real meromorphic.

Chapter 2

Dynamics of $f_\lambda(z) = \lambda \tanh(e^z)$

The dynamics of transcendental meromorphic functions in the one parameter family

$$\mathcal{M} = \{f_\lambda(z) = \lambda\, f(z) \;:\; f(z) = \tanh(e^z) \text{ for } z \in \mathbb{C} \text{ and } \lambda \in \mathbb{R} \setminus \{0\}\ \}$$

is studied in the present chapter. The dynamics of the transcendental entire functions $\lambda e^z, (\lambda \in \mathbb{C} \setminus \{0\})$ have been extensively studied and a number of interesting properties of the Julia set of λe^z are proved [15–17, 27, 28, 39–41, 45, 47, 48, 89, 97, 102, 107, 115, 116, 135, 136]. Devaney and Keen [42–44] studied the dynamics of meromorphic functions with constant/polynomial Schwarzian derivatives and in particular, the dynamics of functions in the one parameter family $\{\lambda \tan z \;:\; \lambda \in \mathbb{R} \setminus \{0\}\ \}$. Jiang [73], Keen and Kotus [80] furthered the study of dynamics of the tangent family. It is worth to note that the dynamics of $\lambda \tanh z$ is essentially same as the dynamics of $\lambda \tan z$, since $\tanh z$ and $\tan z$ are conformally conjugate by the conjugating map $\psi(z) = iz$.

The real meromorphic function $\tan z$ maps the upper half-plane into itself and this simplifies the determination of the dynamics of $\lambda \tanh z$, $\lambda \in \mathbb{R} \setminus \{0\}$. But, the mapping properties of the real meromorphic function $\tanh(e^z)$ are comparatively more complicated. Further, the function $\tanh(e^z)$ is a meromorphic function having non-rational Schwarzian derivative and the order is infinite. However, the functions λe^z and $\lambda \tanh z$ have constant Schwarzian derivatives and finite orders. Even though the order of $f_\lambda(z) = \lambda \tanh(e^z)$ is infinite, it has only 3 finite asymptotic values, namely $\pm\lambda$ and 0. The asymptotic values of e^z and $\tanh z$ are logarithmic that are the simplest kind of direct singularities of the respective inverse functions. However, the asymptotic values $\pm\lambda$ of f_λ are logarithmic, and over the asymptotic value 0 there lie a direct singularity as well as ordinary points of the inverse function of f_λ. Thus, the properties of the function f_λ differ in many ways from those of λe^z and $\lambda\tanh z$ and this suggests that we explore the dynamics of f_λ.

The change in the dynamical behaviour of the functions $f_\lambda \in \mathcal{M}$ is also investigated in this chapter. In the dynamics of functions $f_\lambda \in \mathcal{M}$, we show that the bifurcation occurs only at one critical parameter $\lambda^* \approx -3.2946$. Then, certain topological properties of the Fatou sets of f_λ are proved. Finally, it is shown that the measure of the Julia set of f_λ is zero.

2.1 Properties of f_λ

In this section, we prove some basic results about the functions $f_\lambda \in \mathcal{M}$ that are relevant in the study of the dynamics of f_λ. The function $f_\lambda(z) = \lambda\tanh(e^z)$ is periodic of minimal period $2\pi i$ and maps the real line $\mathbb{R}$ onto $(0, \lambda)$ for $\lambda > 0$. The poles of $f_\lambda(z)$ are the zeros of $\cosh(e^z)$, and hence they satisfy $e^{2e^z} = -1 = e^{i\pi(2k+1)}$ for $k \in \mathbb{Z}$. Therefore, the set of poles of $f_\lambda(z)$ is $\{z = x+iy \in \mathbb{C} \ : \ x = \ln\left|\frac{\pi}{2}(2k+1)\right| \text{ and } y = \frac{\pi}{2}(2l+1) \text{ where } k \in \mathbb{Z} \text{ and } l \in \mathbb{Z}\}$. Further, all the poles are simple and lie in the right half-plane $\{z \in \mathbb{C} \ : \ \Re(z) \geq \ln(\frac{\pi}{2})\}$. Observe that $f_\lambda(z) = \lambda\tanh(e^z) = \lambda(1 + \frac{-2}{e^{2e^z}+1})$ and order of $\frac{-2}{e^{2e^z}+1}$ is infinity ([101], page-216) and therefore, the order of the function $f_\lambda(z) = \lambda\tanh(e^z)$ is infinite.

Lemma 2.1.1. *Let g be a meromorphic function, h be a non-constant entire function and $g, h \in S$. Let $F(z) = g(h(z))$ be the composition function. If a is a finite asymptotic value of $F(z)$, then either a is an asymptotic value of g or there exists $b \in \mathbb{C}$ such that $g(b) = a$ and b is an asymptotic value of h. Consequently, the number of finite asymptotic values of the composite function F is at most the sum of the numbers of finite asymptotic values of* *the individual functions g and h.*

Proof. Let $\gamma : [0, \infty) \to \mathbb{C}$ be an asymptotic path corresponding to an asymptotic value a for the function $F(z)$. Let M denote the collection of all limit points of the set

$$\{h(\gamma(t_k)) \ : \ \{t_k\} \text{ is any sequence of positive real numbers which tends to } \infty \text{ as } k \to \infty\}.$$

Observe that $g(z) = a$ for every $z \in M$. Since g is a non-constant meromorphic function, the set M cannot have any limit point in $\mathbb{C}$. Therefore, $M \bigcap \mathbb{C}$ is a discrete subset of $\mathbb{C}$. Now we claim that M contains only one element in $\widehat{\mathbb{C}}$. If possible, the set M contains more than one element in $\widehat{\mathbb{C}}$. Suppose that m_1 and m_2 are in M with $m_1 \neq m_2$. Then, there exist open discs $B_1(m_1)$ and $B_2(m_2)$ such that $\overline{B_1(m_1)} \cap M = \{m_1\}$ and $\overline{B_2(m_2)} \cap M = \{m_2\}$. The curve $h(\gamma(t))$ intersects the discs $B_1(m_1)$ and $B_2(m_2)$ infinitely many times and also the boundaries $C_1 = \partial B_1(m_1)$ and $C_2 = \partial B_2(m_2)$ of these discs infinitely many times. Note that, if $\{h(\gamma(t)) \ : \ t \geq 0\} \cap C_i$ is a finite set S (say), then $S \bigcap M \neq \emptyset$ which is a contradiction to $\overline{B_i(m_i)} \cap M = \{m_i\}$ for $i = 1,\ 2$. Suppose that $\{h(\gamma(t)) \ : \ t \geq 0\} \cap C_i$ is an infinite set. Then, the intersecting points $\{h(\gamma(t)) \ : \ t \geq 0\} \cap C_i$ will have a limit point, l_i (say), since C_1 and C_2 are compact. This implies that $l_i \in M$ which is a contradiction to $\overline{B_i(m_i)} \cap M = \{m_i\}$ for $i = 1,\ 2$. Therefore, M is a singleton set in $\widehat{\mathbb{C}}$.

If $M = \{b\}$ where $b \in \mathbb{C}$, then $a = g(b)$ and b is an asymptotic value of $h(z)$. If $M = \{\infty\}$, then a is an asymptotic value of $g(z)$. Therefore, in both the cases, a corresponds either to an asymptotic value of h or to that of g. This completes the proof. ∎

The following proposition determines all the singular values of $f_\lambda \in \mathcal{M}$.

Proposition 2.1.1. *Let $f_\lambda \in \mathcal{M}$. Then $f_\lambda(z)$ has only three (finite) asymptotic values and no critical values.*

Proof. Since $f_\lambda'(z) = \lambda\, e^z\, \mathrm{sech}^2(e^z) \ \neq 0$ for any $z \in \mathbb{C}$, it follows that $f_\lambda(z)$ has no critical points and hence it has no critical values.

Turning to asymptotic values, by Lemma 2.1.1, it follows that $f_\lambda(z)$ will have at most 3 finite asymptotic values, since e^z has only one finite asymptotic value, namely, 0 and $\lambda\tanh(z)$ has two finite asymptotic values, namely, λ and $-\lambda$.

If $\gamma_1(t) = -t$ for $t \in [0,\ \infty)$, then $\lim\limits_{t\to\infty} f_\lambda(\gamma_1(t)) = 0$. If $\gamma_2(t) = t$ for $t \in [0,\ \infty)$, then $\lim\limits_{t\to\infty} f_\lambda(\gamma_2(t)) = \lambda$. When $\gamma_3(t) = t + i\pi$ for $t \in [0,\ \infty)$, $\lim\limits_{t\to\infty} f_\lambda(\gamma_3(t)) = -\lambda$. Therefore, 0 and $\pm\lambda$ are the three finite asymptotic values of $f_\lambda(z)$. $\square$

Two transcendental meromorphic functions $f,\ g : \mathbb{C} \to \widehat{\mathbb{C}}$ are called conformally conjugate if there is an analytic homeomorphism ψ on $\widehat{\mathbb{C}}$ such that $\psi(f(z)) = g(\psi(z))$ for all $z \in \widehat{\mathbb{C}}$. Since f is transcendental, the function $\psi(f(z))$ is undefined only at ∞ which means that $g(\psi(z))$ is undefined only at ∞. Therefore, any conformal conjugacy ψ existing between two meromorphic functions satisfies $\psi(\infty) = \infty$ and hence will be of the form $\psi(z) = az + b$ where a and b are complex constants with $a \neq 0$. In the following, we show that no two functions f_{λ_1} and f_{λ_2} in $\mathcal{M}$ are conformally conjugate. Suppose that there exists an analytic homeomorphism $\psi(z) = az + b$ for all $z \in \widehat{\mathbb{C}}$ between two functions f_{λ_1} and f_{λ_2} in $\mathcal{M}$ with $\lambda_1 \neq \lambda_2$. In the proof of Proposition 2.1.1, it is shown that the function f_{λ_i} has three finite asymptotic values, namely, 0, λ_i and $-\lambda_i$ for $i = 1,\ 2$. Note that $\pm\lambda_i$ are the exceptional values of f_{λ_i}. Now, the conjugacy map ψ is required to take the set $\{\lambda_1,\ -\lambda_1\}$ to $\{\lambda_2,\ -\lambda_2\}$. That is, either $\psi(\lambda_1) = \lambda_2$, $\psi(-\lambda_1) = -\lambda_2$ or $\psi(\lambda_1) = -\lambda_2$, $\psi(-\lambda_1) = \lambda_2$. This implies that $b = 0$ and consequently, $\psi(z) = az$. Therefore, $af_{\lambda_1}(z) = f_{\lambda_2}(az)$ for all $z \in \mathbb{C}$. It follows that $af'_{\lambda_1}(0) = af'_{\lambda_2}(0)$ and $\lambda_1 = \lambda_2$ which is not true.

Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(x) = \frac{x}{f(x)} + \frac{1}{f'(x)}$. Now, we discuss some properties of ϕ and thereby define the critical parameter λ^* that will be used later in this chapter. Note that, $\phi(x) = \frac{xf'(x)+f(x)}{f(x)f'(x)} = \frac{1}{\tanh(e^x)}\left(x + \frac{\tanh(e^x)}{e^x \operatorname{sech}^2(e^x)}\right) = \frac{1}{\tanh(e^x)}\left(x + \frac{e^{2e^x} - e^{-2e^x}}{4e^x}\right) = \frac{1}{4e^x\ \tanh(e^x)}$ $(4xe^x + e^{2e^x} - e^{-2e^x})$. Letting $\phi_1(x) = 4xe^x + e^{2e^x} - e^{-2e^x}$, we observe that $\phi_1'(x) = 2e^x(2x + 2 + e^{2e^x} + e^{-2e^x}) = 2e^x\phi_2(x)$, where $\phi_2(x) = 2x + 2 + e^{2e^x} + e^{-2e^x}$. The function $\phi_2'(x) = 2 + 2e^x(e^{2e^x} - e^{-2e^x}) > 0$ for $x < 0$. This implies that $\phi_2(x)$ is strictly increasing for

$x < 0$. Since $\phi_2(x) \to -\infty$ as $x \to -\infty$ and $\phi_2(x) \to 2+e^2+e^{-2} > 0$ as $x \to 0$, there exists a point $x_2 < 0$ such that $\phi_2(x) < 0$ for $x < x_2$, $\phi_2(x_2) = 0$ and $\phi_2(x) > 0$ for $x_2 < x < 0$ and consequently, $\phi_1'(x) < 0$ for $x < x_2$, $\phi_1'(x_2) = 0$ and $\phi_1'(x) > 0$ for $x_2 < x < 0$. Therefore, $\phi_1(x)$ is decreasing for $x < x_2$ and, is increasing for $x_2 < x < 0$. This shows that the function $\phi_1(x)$ attains the minimum value at the point x_2 and the minimum value $\phi_1(x_2)$ is negative, because $\phi_1(x) \to 0$ as $x \to -\infty$. Since $\phi_1(x) \to e^2 - e^{-2} > 0$ as $x \to 0$, there exists a unique point x^* with $x_2 < x^* < 0$ such that $\phi_1(x) < 0$ for $x < x^*$, $\phi_1(x) = 0$ for $x = x^*$ and $\phi_1(x) > 0$ for $x^* < x < 0$; and consequently,

$$\phi(x) \begin{cases} < 0 & \text{for } x < x^* < 0 \\ = 0 & \text{for } x = x^* \\ > 0 & \text{for } x^* < x < 0 \end{cases} \tag{2.1}$$

Observe that $\phi(x) > 0$ for $x \geq 0$. Define

$$\lambda^* = \frac{x^*}{f(x^*)} = \frac{-1}{f'(x^*)} \tag{2.2}$$

where x^* is the unique real root of the equation $\phi(x) = \frac{x}{f(x)} + \frac{1}{f'(x)} = 0$. Numerically, it is found that $x^* \approx -1.0789$ and $\lambda^* \approx -3.2946$. In this chapter, x^* and λ^* denote the numbers as defined by Equation (2.1) and Equation (2.2) respectively.

2.2 Real periodic points

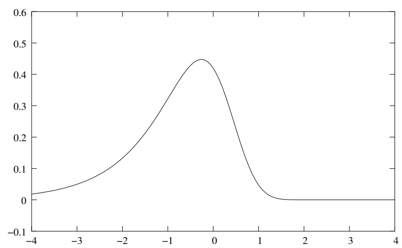
In this section, the real periodic points of $f_\lambda \in \mathcal{M}$ are investigated. The existence and nature of the real fixed points is proved in Theorem 2.2.1. In Theorem 2.2.2, it is proved that f_λ cannot have a real periodic point of prime period more than two. The existence and nature of the real periodic points of prime period 2 is analyzed in Theorem 2.2.3.

The function $f(x) = \tanh(e^x)$ is positive for all $x \in \mathbb{R}$. Since $f'(x) = e^x \frac{1}{\cosh^2 e^x} > 0$ for all $x \in \mathbb{R}$, the function $f(x)$ is strictly increasing on $\mathbb{R}$. It is easy to see that $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 1$ as $x \to +\infty$. Now, we find the nature of the function $f''(x) = e^x \frac{1}{\cosh^2 e^x}(1 - 2e^x \tanh(e^x))$ on $\mathbb{R}$. Observe that the function $\frac{d}{dx}(1 -$

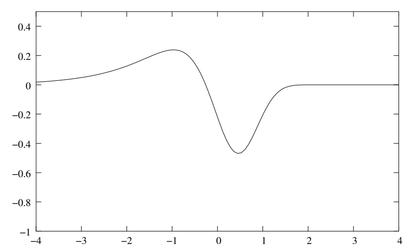
$2e^x\tanh(e^x)) = -2e^x(e^x\frac{1}{\cosh^2 e^x} + \tanh(e^x)) < 0$ for all $x \in \mathbb{R}$. Therefore, the function $\psi(x) = 1 - 2e^x\tanh(e^x)$ is a strictly decreasing on $\mathbb{R}$. Since $\lim_{x\to-\infty} 1 - 2e^x\tanh(e^x) = 1$ and $\lim_{x\to 0} 1 - 2e^x\tanh(e^x) = 1 - 2\,\dfrac{e^2-1}{e^2+1} = \dfrac{3-e^2}{e^2+1} < 0$, it follows that there exists a point $\hat{x} < 0$ such that $\psi(x) > 0$ for $x < \hat{x}$, $\psi(x) = 0$ for $x = \hat{x}$ and $\psi(x) < 0$ for $x > \hat{x}$. Consequently,

$$f''(x) = e^x\frac{1}{\cosh^2 e^x}(1 - 2e^x\tanh(e^x)) \begin{cases} > 0 & \text{for } x < \hat{x} \\ = 0 & \text{for } x = \hat{x} \\ < 0 & \text{for } x > \hat{x} \end{cases} \tag{2.3}$$

(See Figure 2.1(b)).



(a)



(b)

Figure 2.1: Graphs of (a) $f'(x)$ and (b) $f''(x)$.

This shows that the function $f'(x)$ increases in the interval $(-\infty,\ \hat{x})$, decreases in the interval $(\hat{x},\ \infty)$ and attains the maximum value at the point $\hat{x}$. Also $f'(x) \to 0$ as $|x| \to \infty$ (See Figure 2.1(a)). Define $\hat{\lambda}$ as $\frac{1}{f'(\hat{x})}$. It is numerically computed that $\hat{x} \approx -0.261$ and $\hat{\lambda} \approx 2.233$.

Theorem 2.2.1. *Let $f_\lambda \in \mathcal{M}$.*

1. *If $\lambda > \lambda^*$, then f_λ has a unique real fixed point a_λ (say) and that is attracting.*

2. *If $\lambda = \lambda^*$, then f_λ has a unique rationally neutral real fixed point at $x = x^*$, where x^* is the unique real root of $\phi(x) = \frac{x}{f(x)} + \frac{1}{f'(x)} = 0$.*

3. *If $\lambda < \lambda^*$, then f_λ has a unique real fixed point r_λ (say) and that is repelling.*

Proof. Set $h_\lambda(x) = f_\lambda(x) - x = \lambda f(x) - x$ where $f(x) = \tanh(e^x)$ for $x \in \mathbb{R}$ and λ is a nonzero real parameter. Then, $h'_\lambda(x) = \lambda f'(x) - 1$ and $h''_\lambda(x) = \lambda f''(x)$.

For all λ,

$$\lim_{x \to -\infty} h_\lambda(x) = +\infty \qquad \text{and} \qquad \lim_{x \to +\infty} h_\lambda(x) = -\infty \, .$$

Since $h_\lambda(x)$ is a continuous function on $\mathbb{R}$, it has a real zero. Consequently, the function f_λ has a real fixed point x_λ (say). Since $f(x) > 0$ for all $x \in \mathbb{R}$, the real fixed point of f_λ has the same sign as that of λ. If $\lambda > 0$, the function $h'_\lambda(x)$ is increasing from the value -1 to the value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1$ in the interval $(-\infty, \hat{x}]$ and it is decreasing from the value $h'_\lambda(\hat{x})$ to -1 in the interval $[\hat{x},\ \infty)$ where $\hat{x}$ satisfies $f''(\hat{x}) = 0$. If $\lambda < 0$, the function $h'_\lambda(x)$ is decreasing from the value -1 to the value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 < 0$ in the interval $(-\infty, \hat{x}]$ and it is increasing from the value $h'_\lambda(\hat{x})$ to -1 in the interval $[\hat{x},\ \infty)$. For $\lambda < 0$, it follows that the function $h_\lambda(x)$ is strictly decreasing and consequently, the real fixed point x_λ of f_λ is unique.

Case (1): $\lambda > \lambda^*$

Subcase (a): $\lambda \geq \hat{\lambda}$

In this case, the function $h'_\lambda(x)$ is increasing from the value -1 to the value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 \geq \hat{\lambda} f'(\hat{x}) - 1 = 0$ in the interval $(-\infty, \hat{x}]$ and it is decreasing from the value $h'_\lambda(\hat{x})$ to -1 in the interval $[\hat{x},\ \infty)$. Therefore, there exist two points $x_{1,\lambda}$ and $x_{2,\lambda}$ (say) with $x_{1,\lambda} \leq x_{2,\lambda}$ such that $h'_\lambda(x) = 0$ for $x = x_{1,\lambda}$ and $x = x_{2,\lambda}$. Further, $h'_\lambda(x) < 0$ for $x \in (-\infty,\ x_{1,\lambda}) \cup (x_{2,\lambda},\ \infty)$ and $h'_\lambda(x) > 0$ for $x \in (x_{1,\lambda},\ x_{2,\lambda})$. If $x_{2,\lambda} \leq 0$, then $-1 < h'_\lambda(x) < 0$ for all $x > 0$. Therefore, it follows that the real fixed point x_λ (which is positive as $\lambda > 0$ in this case) of f_λ is unique and attracting. When $x_{2,\lambda} > 0$, the function h_λ attains the maximum value at $x = x_{2,\lambda}$ in $(0,\ \infty)$. Since $0 < h_\lambda(0) < h_\lambda(x_{2,\lambda})$ and $h_\lambda(x)$ is decreasing in the interval $(x_{2,\lambda},\ \infty)$, it follows that $x_{2,\lambda} < x_\lambda$. Therefore, the real fixed point x_λ of f_λ is unique and attracting. Let us rename the fixed point x_λ as a_λ when $\lambda \geq \hat{\lambda}$.

Subcase (b): $0 < \lambda < \hat{\lambda}$

If $0 < \lambda < \hat{\lambda}$, the maximum value of $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 < \hat{\lambda} f'(\hat{x}) - 1 = 0$ for all $x \in \mathbb{R}$. It follows that $-1 < h'_\lambda(x) = f'_\lambda(x) - 1 < 0$ for all $x \in \mathbb{R}$. Therefore, the real fixed point x_λ of f_λ is unique and attracting. Rename the real fixed point x_λ as a_λ.

Subcase (c): $-\hat{\lambda} < \lambda < 0$

If $-\hat{\lambda} < \lambda < 0$, the minimum value of $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 > -\hat{\lambda} f'(\hat{x}) - 1 = -2$ for all $x \in \mathbb{R}$. It follows that $-2 < h'_\lambda(x) = f'_\lambda(x) - 1 < -1$ for all $x \in \mathbb{R}$. Therefore, the real fixed point x_λ of f_λ is attracting for f_λ. In this case, we rename x_λ as a_λ.

Subcase (d): $\lambda^* < \lambda \leq -\hat{\lambda}$

The function $h'_\lambda(x)$ is decreasing from the value -1 to the value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 \leq -\hat{\lambda} f'(\hat{x}) - 1 \leq -2$ in the interval $(-\infty, \hat{x}]$ and it is increasing from the value $h'_\lambda(\hat{x})$ to -1 in the interval $[\hat{x}, \infty)$. Since $h'_\lambda(\hat{x}) + 2 \leq 0$ for $\lambda^* < \lambda \leq -\hat{\lambda}$, there exist two points $y_{1,\lambda}$ and $y_{2,\lambda}$ (say) with $y_{1,\lambda} \leq y_{2,\lambda}$ such that $h'_\lambda(x) + 2 = 0$ for $x = y_{1,\lambda}$ and $x = y_{2,\lambda}$. Further, $h'_\lambda(x) + 2 > 0$ for $x \in (-\infty, y_{1,\lambda}) \cup (y_{2,\lambda}, \infty)$ and $h'_\lambda(x) + 2 < 0$ for $x \in (y_{1,\lambda}, y_{2,\lambda})$. Now, the parameter λ can be realized in two ways as $\lambda = \frac{-1}{f'(y_{1,\lambda})}$ and $\lambda = \frac{x_\lambda}{f(x_\lambda)}$ where $y_{1,\lambda}$ is the smaller root of $h'_\lambda(x) + 2 = 0$ and x_λ is the unique real fixed point of f_λ. It is noticed that $x^* < \hat{x} < 0$. Now we shall show that the points x_λ and $y_{1,\lambda}$ are in the interval $(x^*, \hat{x}]$ and $x_\lambda < y_{1,\lambda}$. Since $\lambda^* < \lambda \leq -\hat{\lambda}$, we have $\frac{-1}{f'(x^*)} < \frac{-1}{f'(y_{1,\lambda})} \leq \frac{-1}{f'(\hat{x})}$. Using the fact that $\frac{-1}{f'}$ is strictly increasing in $(-\infty, \hat{x})$, we get

$$x^* < y_{1,\lambda} \leq \hat{x} \ .$$

For all $x < 0$, $\frac{d}{dx}\left(\frac{x}{f(x)}\right) > 0$ implies that $\frac{x}{f(x)}$ is strictly increasing in $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$. The inequality $\lambda^* < \lambda \leq -\hat{\lambda}$ gives $\frac{x^*}{f(x^*)} < \frac{x_\lambda}{f(x_\lambda)} \leq \frac{-1}{f'(\hat{x})}$. Since $\hat{x} > x^*$, we have $\phi(\hat{x}) > 0$ and $\frac{\hat{x}}{f(\hat{x})} > \frac{-1}{f'(\hat{x})}$. Therefore, $\frac{x^*}{f(x^*)} < \frac{x_\lambda}{f(x_\lambda)} \leq \frac{-1}{f'(\hat{x})} < \frac{\hat{x}}{f(\hat{x})}$ which gives that

$$x^* < x_\lambda < \hat{x} \ .$$

Since $\phi(y_{1,\lambda}) > 0$, it follows that $\frac{y_{1,\lambda}}{f(y_{1,\lambda})} > \frac{-1}{f'(y_{1,\lambda})} = \frac{x_\lambda}{f(x_\lambda)}$. Since the function $\frac{x}{f(x)}$ is increasing for $x < 0$, we get $x_\lambda < y_{1,\lambda}$. Now, the function $h'_\lambda(x) + 2 > 0$ for $x < y_{1,\lambda}$. So, it follows that $-1 < f'_\lambda(x) < 0$ for $x < y_{1,\lambda}$ and in particular, $-1 < f'_\lambda(x_\lambda) < 0$. Therefore, the real fixed point x_λ is attracting and rename it as a_λ.

Case (2): $\lambda = \lambda^*$

By definition $\lambda^* = \frac{x^*}{f(x^*)} = \frac{-1}{f'(x^*)}$. Since the function $\frac{x}{f(x)}$ is one-to-one in the negative real axis, it follows that the real fixed point x_λ is equal to x^*. The real fixed point x^* is a rationally neutral fixed point, because $\lambda^* f'(x^*) = -1$.

Case (3): $\lambda < \lambda^*$

As in Subcase (d), the minimum value of $h'_\lambda(\hat{x}) < -2$. Therefore, there exist two points $y_{1,\lambda}$ and $y_{2,\lambda}$ (say) with $y_{1,\lambda} < y_{2,\lambda}$ such that $h'_\lambda(x) + 2 = 0$ for $x = y_{1,\lambda}$ and $x = y_{2,\lambda}$. Further, $h'_\lambda(x) + 2 > 0$ for $x \in (-\infty,\ y_{1,\lambda}) \cup (y_{2,\lambda},\ \infty)$ and $h'_\lambda(x) + 2 < 0$ for $x \in (y_{1,\lambda},\ y_{2,\lambda})$. Here our intention is to show that the fixed point x_λ lies in $(y_{1,\lambda},\ y_{2,\lambda})$ where $|f'_\lambda(x)| > 1$. From the deliberations made in Subcase (d) of Case (1), it is clear that $y_{1,\lambda} < \hat{x} < y_{2,\lambda}$. Now $\lambda < \lambda^*$ gives that $\frac{x_\lambda}{f(x_\lambda)} < \frac{x^*}{f(x^*)}$. Since $\frac{x}{f(x)}$ is an increasing function in $\mathbb{R}^-$ and $x_\lambda,\ x^*$ are in $\mathbb{R}^-$, we get $x_\lambda < x^*$. Therefore, $x_\lambda < x^* < \hat{x} < y_{2,\lambda}$. Observe that $\lambda < \lambda^*$ implies $\frac{-1}{f'(y_{1,\lambda})} < \frac{-1}{f'(x^*)}$. Since the function $\frac{-1}{f'(x)}$ is an increasing function in the interval $(-\infty,\ \hat{x})$ which contains $y_{1,\lambda}$ and x^*, it follows that $y_{1,\lambda} < x^*$. By Equation (2.1), $\phi(y_{1,\lambda}) < 0$ and that gives $\frac{y_{1,\lambda}}{f(y_{1,\lambda})} < \frac{-1}{f'(y_{1,\lambda})}$. But, $\lambda = \frac{-1}{f'(y_{1,\lambda})} = \frac{x_\lambda}{f(x_\lambda)}$. Therefore, $\frac{y_{1,\lambda}}{f(y_{1,\lambda})} < \frac{x_\lambda}{f(x_\lambda)}$ and consequently $y_{1,\lambda} < x_\lambda$. Therefore, the fixed point x_λ is repelling. Let us rename it as r_λ. □

Theorem 2.2.2. *Let $f_\lambda \in \mathcal{M}$. Then, f_λ has no real periodic point of prime period more than two.*

Proof. Let $g_\lambda(x) = f_\lambda(f_\lambda(x))$ for $x \in \mathbb{R}$. Then, the function $g_\lambda(x)$ is strictly increasing on $\mathbb{R}$, since $g'_\lambda(x) = \lambda f'(\lambda f(x))\lambda f'(x) > 0$ for all $x \in \mathbb{R}$. Therefore the function $g_\lambda(x)$ can have only fixed points on $\mathbb{R}$. If possible, let x_0 be a real periodic point of f_λ of prime period $p > 2$. Then, $g_\lambda(x_0) = f^2_\lambda(x_0) \neq x_0$ and $f^p_\lambda(x_0) = x_0$. If p is even, then $f^p_\lambda(x_0) = x_0 = g^{\frac{p}{2}}_\lambda(x_0)$. If p

is odd, then $f_\lambda^{2p}(x_0) = x_0 = g_\lambda^p(x_0)$. This shows that g_λ has a real periodic point of prime period greater than one which is a contradiction. Thus, we conclude that $f_\lambda(x)$ cannot have a real periodic point of prime period more than two. $\square$

The existence and the nature of the real periodic points of prime period 2 is explored in the following theorem.

Theorem 2.2.3. *Let $f_\lambda \in \mathcal{M}$.*

1. *If $\lambda > \lambda^*$, f_λ^2 has only one real fixed point a_λ which is an attracting fixed point of f_λ.*

2. *If $\lambda = \lambda^*$, f_λ^2 has only one real fixed point x^* which is a rationally neutral fixed point of f_λ.*

3. *If $\lambda < \lambda^*$, f_λ^2 has exactly three real fixed points. One of the fixed points of f_λ^2 is r_λ which is a repelling fixed point of f_λ. The other two fixed points of f_λ^2 are the periodic points of (prime) period 2 of f_λ and form an attracting or a parabolic 2-periodic cycle $\{a_{1\lambda}, a_{2\lambda}\}$ (say) with $a_{1\lambda} < r_\lambda < a_{2\lambda} < 0$.*

Proof. **Case 1:** $\lambda > \lambda^*$

If $\lambda > \lambda^*$, by Theorem 2.2.1(1), $f_\lambda(x)$ has a unique attracting fixed point a_λ on the real line. The fixed point a_λ of f_λ is also a fixed point of f_λ^2. Now, we show that f_λ^2 has no other real fixed points.

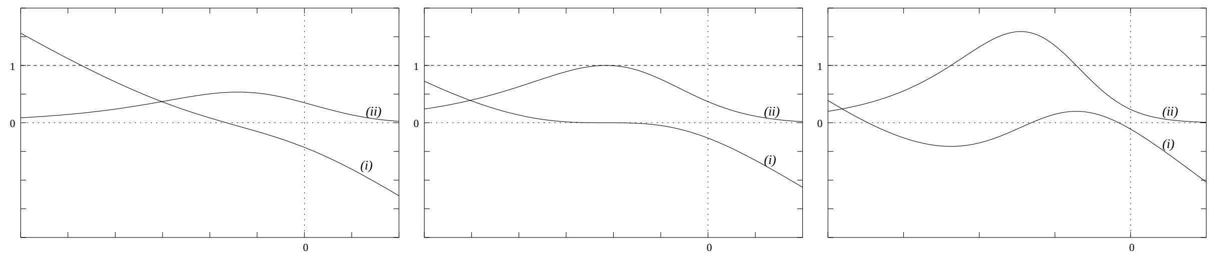
For $\lambda > 0$, f_λ is strictly increasing on $\mathbb{R}$. If $f_\lambda(x) \neq x$ for a point $x \in \mathbb{R}$, then $f_\lambda^n(x) \neq x$ for any integer $n > 1$. To see it, note that $f_\lambda(x) > x$ implies $f_\lambda^n(x) > f_\lambda^{n-1}(x)$ and $f_\lambda(x) < x$ implies $f_\lambda^n(x) < f_\lambda^{n-1}(x)$ for all $n \in \mathbb{N}$. Therefore, it follows that f_λ $(\lambda > 0)$ has no real periodic points of prime period $p = 2$.

Let $\lambda^* < \lambda < 0$. Suppose that there is a fixed point of f_λ^2 which is different from a_λ. As f_λ has only one real fixed point, any fixed point other than a_λ of f_λ^2 will be a 2-periodic cycle for f_λ. If f_λ has more than one 2-periodic cycles, then the outer most 2-periodic cycle is chosen for consideration. This is possible, because, if f_λ has two different 2-periodic cycles $\{a,\ b\}$ with $a < b$ and $\{c,\ d\}$ with $c < d$, then it follows from the fact f_λ is strictly decreasing for $\lambda < 0$ that the two different 2-periodic cycles satisfy $c < a < a_\lambda < b < d$ or $a < c < a_\lambda < d < b$. In the first case $\{c,\ d\}$ and in the second case $\{a,\ b\}$ is called the outer cycle.

Let $\{d_{1\lambda},\ d_{2\lambda}\}$ be the outermost 2-periodic cycle of f_λ such that $f_\lambda(d_{1\lambda}) = d_{2\lambda}$ and $f_\lambda(d_{2\lambda}) = d_{1\lambda}$ with $d_{1\lambda} < d_{2\lambda}$. Set $D_1 = (-\infty, d_{1\lambda})$ and $D_2 = (d_{2\lambda}, \infty)$. Since $f_\lambda^2(x) > x$ for each $x \in D_1$, the sequence $\{f_\lambda^{2n}(x)\}$ will be a monotonically increasing sequence and $d_{1\lambda} = \sup\{f_\lambda^{2n}(x)\ :\ x \in D_1 \text{ and } n \in \mathbb{N}\}$. Therefore, $f_\lambda^{2n}(x) \to d_{1\lambda}$ as $n \to \infty$. Similarly, $\{f_\lambda^{2n}(x)\}$ is a monotonically decreasing sequence converging to $d_{2\lambda}$ for each $x \in D_2$, since $f_\lambda^2(x) < x$ for $x \in D_2$ and $d_{2\lambda} = \inf\{f_\lambda^{2n}(x)\ :\ x \in D_2 \text{ and } n \in \mathbb{N}\}$. This shows that the cycle $\{d_{1\lambda},\ d_{2\lambda}\}$ can be either an attracting or a parabolic cycle. Note that $\lambda < d_{1\lambda} < a_\lambda < d_{2\lambda} < 0 < -\lambda$. This implies that $f_\lambda^{2n}(\lambda) \to d_{1\lambda}$, $f_\lambda^{2n}(0) \to d_{2\lambda}$ and $f_\lambda^{2n}(-\lambda) \to d_{2\lambda}$ as $n \to \infty$. Thus, all the singular values are attracted by the 2-periodic cycle $\{d_{1\lambda},\ d_{2\lambda}\}$. It is shown in Theorem 2.2.1 that a_λ is a real attracting fixed point of f_λ for $\lambda > \lambda^*$. So, the basin of attraction $A(a_\lambda)$ of the attracting fixed point a_λ must contain at least one singular value which is a contradiction to the fact that all three singular values tend either to $d_{1\lambda}$ or to $d_{2\lambda}$ under iterations of f_λ^2. Therefore, f_λ^2 cannot have any real fixed point other than a_λ if $\lambda^* < \lambda < 0$ (See Figure 2.2(a)).

Case 2: $\lambda = \lambda^*$:

If $\lambda = \lambda^*$, by Theorem 2.2.1(2), $f_\lambda(x)$ has a unique rationally neutral fixed point x^* on the real line. The fixed point x^* of f_{λ^*} is also a fixed point for $f_{\lambda^*}^2$. Further, it is rationally indifferent and the corresponding parabolic domain contains at least one singular value of f_λ. By similar arguments as in Case 1, one can show that $f_{\lambda^*}^2$ has no real periodic point of prime period 2 (See Figure 2.2(b)).



(a) (b) (c)

Figure 2.2: Graphs of (i) $f_\lambda^2(x) - x$ and (ii) $\left(f_\lambda^2\right)'(x)$ for (a) $\lambda > \lambda^*$, (b) $\lambda = \lambda^*$ and (c) $\lambda < \lambda^*$.

Case 3: $\lambda < \lambda^*$:

If $\lambda < \lambda^*$, by Theorem 2.2.1(3), $f_\lambda(x)$ has a unique repelling fixed point r_λ on the real line. The fixed point r_λ of f_λ is also a fixed point for f_λ^2. Now, we show that including r_λ, the function f_λ^2 has 3 fixed points on $\mathbb{R}$.

Let $x < r_\lambda$. Suppose that $f_\lambda^2(x) > x$. Since $f_\lambda^2(x)$ is strictly increasing on $\mathbb{R}$, it follows that $f_\lambda^{2n}(x) > f_\lambda^{2(n-1)}(x)$ for all $n \in \mathbb{N}$. But, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is bounded above by r_λ. Therefore, the sequence $\{f_\lambda^{2n}(x)\}$ converges to a point a (say). By the continuity of f_λ, it follows that the limit point a satisfies $f_\lambda^2(a) = a$. That means the point a is a non-repelling (attracting or rationally indifferent) periodic point of f_λ of prime period at most two. As f_λ does not have any real fixed point other than r_λ, the limit point a must be a periodic point of prime period 2. Similarly, the other possibility $f_\lambda^2(x) < x$ also leads to the same conclusion. Therefore, f_λ has a periodic point of prime period 2 on $\mathbb{R}$.

Now, we show that f_λ has a unique periodic point of prime period 2 on $\mathbb{R}$. Suppose that f_λ has more than one periodic point of prime period 2 on $\mathbb{R}$. Then, choose the outer most (in the sense defined earlier in **Case 1**) 2-periodic cycle of f_λ.

Let $\{o_{1\lambda},\ o_{2\lambda}\}$ be the outermost 2-periodic cycle of f_λ such that $f_\lambda(o_{1\lambda}) = o_{2\lambda}$ and

$f_\lambda(o_{2\lambda}) = o_{1\lambda}$ with $o_{1\lambda} < o_{2\lambda}$. As shown in case of $\lambda \in (\lambda^*,\ 0)$, it can be shown that the 2-periodic cycle $\{o_{1\lambda},\ o_{2\lambda}\}$ is either an attracting cycle or a parabolic cycle of f_λ and the singular values 0 and $\pm\lambda$ are attracted by this cycle. Now, let us consider the inner most 2-periodic cycle $\{i_{1\lambda},\ i_{2\lambda}\}$ (say) of f_λ with $f_\lambda(i_{1\lambda}) = i_{2\lambda}$ and $f_\lambda(i_{2\lambda}) = i_{1\lambda}$ with $i_{1\lambda} < i_{2\lambda}$. Observe that $f_\lambda(x) \in (r_\lambda,\ i_{2\lambda})$ for $x \in (i_{1\lambda},\ r_\lambda)$ and $f_\lambda(x) \in (i_{1\lambda},\ r_\lambda)$ for $x \in (r_\lambda,\ i_{2\lambda})$. This gives that the sequence $\{f_\lambda^{2n}(x)\}$ is bounded for $x \in (i_{1\lambda},\ i_{2\lambda})$. Since f_λ^2 is strictly increasing on $\mathbb{R}$ for $\lambda < \lambda^* < 0$, the sequence $\{f_\lambda^{2n}(x)\}$ is monotonic. Since r_λ is repelling, $\{f_\lambda^{2n}(x)\} \to i_{1\lambda}$ as $n \to \infty$ for $x \in (i_{1\lambda},\ r_\lambda)$ and $\{f_\lambda^{2n}(x)\} \to i_{2\lambda}$ as $n \to \infty$ for $x \in (r_\lambda,\ i_{2\lambda})$. This shows that the inner cycle $\{i_{1\lambda},\ i_{2\lambda}\}$ is also either attracting or parabolic(Here the cycle $\{i_{1\lambda},\ i_{2\lambda}\}$ cannot be irrationally indifferent as $(f_\lambda^2)'(i_{1\lambda}) = 1$). But, there is no singular value that can be attracted by the inner cycle $\{i_{1\lambda},\ i_{2\lambda}\}$, since all the singular values are already attracted by the outer most cycle $\{o_{1\lambda},\ o_{2\lambda}\}$. This rules out the existence of the inner most cycle $\{i_{1\lambda},\ i_{2\lambda}\}$. Therefore, the function f_λ has only one 2-periodic cycle $\{a_{1\lambda},\ a_{2\lambda}\}$ (say) that is either attracting or parabolic on $\mathbb{R}$ if $\lambda < \lambda^*$ (See Figure 2.2(c)). This completes the proof. ☐

2.3 Dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$

The dynamics of the function $f_\lambda(z) = \lambda\tanh(e^z)$ for $z \in \mathbb{C}$ is investigated in the present section. In Theorem 2.3.1, the dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ is determined. The dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ is studied in Theorem 2.3.2.

Proposition 2.3.1. *Let $f_\lambda \in \mathcal{M}$. Then, the Fatou set of f_λ does not contain wandering domain or Baker domain.*

Proof. For meromorphic functions of finite type, the non-existence of wandering domains is proved in [14] and the non-existence of Baker domains is proved in [19]. Since the function f_λ is of finite type by Proposition 2.1.1, it follows that the Fatou set of f_λ does not contain wandering domains or Baker domains. □

We determine the dynamics of f_λ on the real line in the following theorem.

Theorem 2.3.1. *Let $f_\lambda \in \mathcal{M}$.*

1. *If $\lambda > \lambda^*$, then $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for all $x \in \mathbb{R}$ where a_λ is the attracting real fixed point of f_λ.*

2. *If $\lambda = \lambda^*$, then $f_\lambda^n(x) \to x^*$ as $n \to \infty$ for all $x \in \mathbb{R}$ where x^* is the rationally neutral real fixed point of f_λ.*

3. *If $\lambda < \lambda^*$, then $f_\lambda^{2n}(x) \to a_{1\lambda}$ as $n \to \infty$ for $x < r_\lambda$ and $f_\lambda^{2n}(x) \to a_{2\lambda}$ as $n \to \infty$ for $x > r_\lambda$ where $\{a_{1\lambda},\ a_{2\lambda}\}$ is the attracting or parabolic real 2-periodic cycle and r_λ is the repelling real fixed point of f_λ.*

Proof. **Case 1:** $\lambda > \lambda^*$

By Theorem 2.2.1(1) and Theorem 2.2.3(1), the function $f_\lambda(z)$ has a unique real attracting fixed point a_λ and f_λ^2 has no fixed point other than a_λ on the real line. It is noted that f_λ^2 is strictly increasing and bounded on $\mathbb{R}$. Observe that $f_\lambda^2(x) > x$ for $x < a_\lambda$. This implies that $\{f_\lambda^{2n}(x)\}$ is a monotonically increasing bounded sequence and hence convergent. By continuity of f_λ^2, it follows that the limit point of $\{f_\lambda^{2n}(x)\}$ is a fixed point of f_λ^2 and therefore it equals to the only such point, namely, a_λ. Therefore, $f_\lambda^{2n}(x) \to a_\lambda$ as $n \to \infty$ for $x < a_\lambda$. Similarly, the same conclusion follows for $x > a_\lambda$ since $f_\lambda^2(x) < x$ for $x > a_\lambda$. Therefore, $\lim\limits_{n\to\infty} f_\lambda^{2n}(x) = a_\lambda$ for all $x \in \mathbb{R}$. Since a_λ is an attracting fixed point of the continuous function f_λ, it is concluded that $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for all $x \in \mathbb{R}$.

Case 2: $\lambda = \lambda^*$

By Theorem 2.2.1(2) and Theorem 2.2.3(2), the function $f_\lambda(z)$ has a unique rationally neutral real fixed point x^* and f_λ^2 has no fixed point other than x^* on the real line. Since

f_λ^2 is a strictly increasing, bounded function on $\mathbb{R}$ with $f_\lambda^2(x) > x$ for $x < x^*$ and $f_\lambda^2(x) < x$ for $x > x^*$, it follows by similar arguments as in the previous case that $f_\lambda^{2n}(x) \to x^*$ as $n \to \infty$ for $x \in \mathbb{R}$. Since f_λ is continuous and x^* is a fixed point of f_λ, it is concluded that $f_\lambda^n(x) \to x^*$ as $n \to \infty$ for all $x \in \mathbb{R}$.

Case 3: $\lambda < \lambda^*$

On the real line $\mathbb{R}$, the function f_λ has a unique repelling real fixed point r_λ by Theorem 2.2.1(3), and has a unique attracting or parabolic 2-periodic cycle $\{a_{1\lambda}, a_{2\lambda}\}$ with $a_{1\lambda} < r_\lambda < a_{2\lambda} < 0$ by Theorem 2.2.3(3). Observe that $f_\lambda^2(x) > x$ for $x < a_{1\lambda}$. Since f_λ^2 is strictly increasing on $\mathbb{R}$ and $f_\lambda^2(a_{1\lambda}) = a_{1\lambda}$, it follows that the sequence $\{f_\lambda^{2n}(x)\}$ is a monotonically increasing sequence and $\sup\{f_\lambda^{2n}(x) \ : \ n \in \mathbb{N}\} = a_{1\lambda}$ for $x \leq a_{1\lambda}$. This gives that $\lim\limits_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ for $x \leq a_{1\lambda}$. Since $f_\lambda^2(x) < x$ for $a_{1\lambda} < x < r_\lambda$, the sequence $\{f_\lambda^{2n}(x)\}$ is monotonically decreasing and bounded below by $a_{1\lambda}$. Therefore, $f_\lambda^{2n}(x) \to a_{1\lambda}$ as $n \to \infty$ for $a_{1\lambda} < x < r_\lambda$. For $x \in (r_\lambda, \ a_{2\lambda})$, the function f_λ^2 satisfies $f_\lambda^2(x) > x$. Consequently, the sequence $\{f_\lambda^{2n}(x)\}$ is monotonically increasing and converges to $a_{2\lambda}$. When $x \geq a_{2\lambda}$, the sequence $\{f_\lambda^{2n}(x)\}$ is decreasing and bounded below by $a_{2\lambda}$, since $f_\lambda^2(x) < x$ for $x > a_{2\lambda}$ and $f_\lambda^2(a_{2\lambda}) = a_{2\lambda}$. Therefore, $f_\lambda^{2n}(x) \to a_{2\lambda}$ as $n \to \infty$ for $x \geq a_{2\lambda}$ which completes the proof. $\square$

If a function preserves the real line, then the dynamics of the function on the real line can be indicated by phase portraits. *Phase portraits* are diagrams that represent possible beginning positions in a dynamical system and the change in these positions under the iteration of a function. Phase portraits of $f_\lambda(x) = \lambda \tanh(e^x)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ are given in Figure 2.3 for various values of λ.

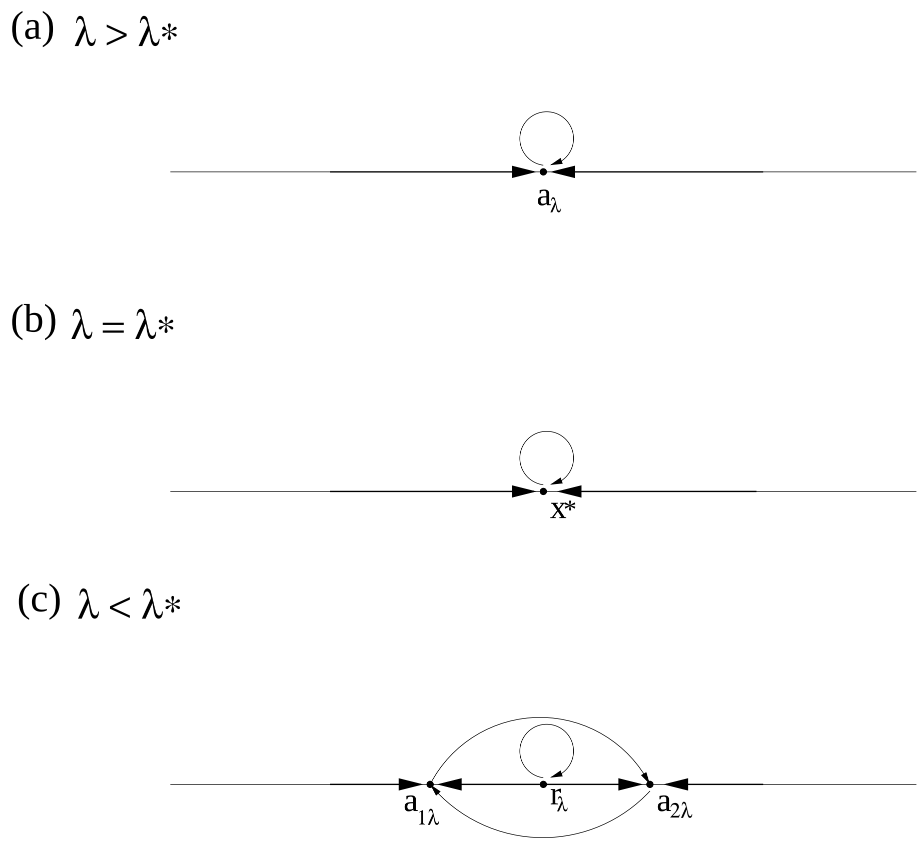


Figure 2.3: Phase portraits of $\lambda \tanh(e^x)$ for (a) $\lambda > \lambda^*$, (b) $\lambda = \lambda^*$ and (c) $\lambda < \lambda^*$.

In the dynamics of f_λ in the family $\mathcal{M}$, we show that the bifurcation occurs only at the critical parameter λ^*. We mainly prove the following result on the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$.

Theorem 2.3.2. *Let $f_\lambda \in \mathcal{M}$.*

1. *If $\lambda > \lambda^*$, then the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the basin of attraction $A(a_\lambda)$ where a_λ is the attracting real fixed point of f_λ.*

2. *If $\lambda = \lambda^*$, then the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the parabolic basin $P(x^*)$ where x^* is the rationally neutral real fixed point of f_λ.*

3. *If $\lambda < \lambda^*$, then the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the basin of attraction or the parabolic basin corresponding to the attracting or the parabolic real* 2*-periodic cycle* $\{a_{1\lambda},\ a_{2\lambda}\}$ *of* f_λ.

Proof. **Case 1:** $\lambda > \lambda^*$

By Theorem 2.2.1(1), the function $f_\lambda(z)$ has a unique real attracting fixed point a_λ on the real line. Let $A(a_\lambda) = \{z \in \widehat{\mathbb{C}}\ :\ f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}$ be the basin of attraction of the real attracting fixed point a_λ. By Theorem 2.3.1(1), the real line $\mathbb{R}$ is in the basin of attraction $A(a_\lambda)$ and in particular, all the singular values $\{\lambda,\ -\lambda,\ 0\}$ and their forward orbits are in $A(a_\lambda)$.

The Fatou set of $f_\lambda(z)$ has no basin of attraction other than $A(a_\lambda)$. To see this, assume, if possible, $A(z_\lambda)$ is a basin of attraction of an attracting periodic point $z_\lambda \neq a_\lambda$. Obviously, $A(z_\lambda)\bigcap A(a_\lambda) = \emptyset$. But, $A(z_\lambda)$ contains at least one singular value and its forward orbit. It contradicts the fact that all the singular values and their forward orbits are contained in $A(a_\lambda)$, since $A(z_\lambda)\bigcap A(a_\lambda) = \emptyset$ for $z_\lambda \neq a_\lambda$.

The Fatou set of $f_\lambda(z)$ cannot contain a parabolic domain. For, if the Fatou set of $f_\lambda(z)$ contains a parabolic domain U, then U must contain at least one singular value, which leads to a contradiction that all singular values are in $A(a_\lambda)$.

Again, the Fatou set of $f_\lambda(z)$ cannot contain a Siegel disc or a Herman ring. For, if possible, let the Fatou set of $f_\lambda(z)$ contain a Siegel disc or a Herman ring, then the boundary of Siegel disc / Herman ring is contained in the closure of the forward orbits of all singular values of $f_\lambda(z)$. But all the singular values and their forward orbits are contained in $A(a_\lambda)$, giving a contradiction.

By Proposition 2.3.1, the Fatou set of $f_\lambda(z)$ does not contain any Baker domain or wandering domain. Therefore, the Fatou set of $f_\lambda(z)$ is equal to the basin of attraction $A(a_\lambda)$ of the attracting real fixed point a_λ if $\lambda > \lambda^*$.

Case 2: $\lambda = \lambda^*$

The function $f_\lambda(z)$ has a unique rationally neutral real fixed point x^* on the real line by Theorem 2.2.1(2). Let $P(x^*) = \{z \in \widehat{\mathbb{C}} \ : \ f_\lambda^n(z) \to x^* \text{ as } n \to \infty\}$ be the parabolic basin corresponding to x^*. By Theorem 2.3.1(2), it follows that the real line $\mathbb{R}$ and in particular, all the singular values $\{\lambda, \ -\lambda, \ 0\}$ and their forward orbits are in the parabolic basin $P(x^*)$. Now, the Fatou set of $f_\lambda(z)$ for $\lambda = \lambda^*$ does not contain any other parabolic domain U other than $P(x^*)$. If the Fatou set of $f_{\lambda^*}(z)$ contains any other parabolic domain U $(\neq P(x^*))$, then U must contain at least one singular value which is not possible.

Since all singular values are in $P(x^*)$, the Fatou set of $f_{\lambda^*}(z)$ cannot contain a basin of attraction. The proofs of the fact that the Fatou set of $f_\lambda(z)$ for $\lambda = \lambda^*$ does not contain Siegel disc, Herman ring, Baker domain or wandering domain are similar to that of Case 1. Thus, all the possible stable domains other than the parabolic basin $P(x^*)$ are ruled out and hence the Fatou set of $f_{\lambda^*}(z)$ equals to the parabolic basin $P(x^*)$ corresponding to the rationally neutral real fixed point x^*.

Case 3: $\lambda < \lambda^*$

By Theorem 2.2.3(3), the function $f_\lambda(z)$ has an attracting or a parabolic real 2-periodic cycle $\{a_{1\lambda}, \ a_{2\lambda}\}$ with $a_{1\lambda} < r_\lambda < a_{2\lambda} < 0$ where r_λ is the unique repelling real fixed point. Let us denote the basin of attraction of the attracting real 2-periodic cycle or the parabolic basin corresponding to the parabolic real 2-periodic cycle by

$$\mathsf{A} = \{z \in \widehat{\mathbb{C}} \ : \ f_\lambda^{2n}(z) \to a_{1\lambda} \text{ or } f_\lambda^{2n}(z) \to a_{2\lambda} \text{ as } n \to \infty\} \ .$$

By Theorem 2.3.1(3), it follows that the real line $\mathbb{R}$ except the point r_λ and in particular, all the singular values $\{\lambda, \ -\lambda, \ 0\}$ and their forward orbits are in A. By proceeding in the same lines of arguments as in Case 1 or Case 2, we get that the Fatou set of $f_\lambda(z)$ does not contain any Herman ring, Siegel disc, Baker domain, wandering domain or any basin of attraction or parabolic basin other than A. Therefore, the Fatou set of $f_\lambda(z)$ is equal to

A for $\lambda < \lambda^*$. □

The above theorem gives the following characterization of the Julia set of $f_\lambda(z)$.

Corollary 2.3.1. *Let $f_\lambda \in \mathcal{M}$.*

1. *If $\lambda > \lambda^*$, then the Julia set $\mathfrak{J}(f_\lambda)$ is the complement of the basin of attraction $A(a_\lambda)$ where a_λ is the attracting real fixed point of f_λ.*

2. *If $\lambda = \lambda^*$, then the Julia set $\mathfrak{J}(f_\lambda)$ is the complement of the parabolic basin $P(x^*)$ where x^* is the rationally neutral real fixed point of f_λ.*

3. *If $\lambda < \lambda^*$, then the Julia set $\mathfrak{J}(f_\lambda)$ is the complement of the basin of attraction or the parabolic basin corresponding to the attracting or the parabolic real 2-periodic cycle $\{a_{1\lambda},\ a_{2\lambda}\}$.*

2.4 Topology of the Fatou components

In the present section, the topology of the Fatou components and existence of the pre-periodic Fatou components of f_λ are mainly investigated. In Proposition 2.4.1, it is shown that the Fatou set of f_λ contains the left half-plane $H_\lambda = \{z \in \mathbb{C} : \Re(z) < M_\lambda\}$, certain horizontal lines and horizontal half strips. Proposition 2.4.2 shows that the Julia set of f_λ ($\lambda > \lambda^*$) does not contain any unbounded component. We prove in Theorem 2.4.1 that the Fatou set of f_λ is connected for $\lambda > \lambda^*$ and the Fatou set contains infinitely many pre-periodic components for $\lambda \leq \lambda^*$. It is established in Theorem 2.4.2 that the Fatou set of f_λ for $\lambda > \lambda^*$ is infinitely connected. We prove that each component of the Fatou set of f_λ for $\lambda \leq \lambda^*$ is simply connected in Theorem 2.4.3.

Proposition 2.4.1. *Let $f_\lambda \in \mathcal{M}$. Then,*

1. *The Fatou set of f_λ contains the left half-plane $H_\lambda = \{z \in \mathbb{C} : \Re(z) < M_\lambda\}$ where M_λ is a real number depending on λ.*

2. *The Fatou set of f_λ contains the horizontal lines $L_{2k+1} = \{x + i(2k+1)\pi \ : \ x \in \mathbb{R}\}$ for every integer k. Further, there exists a real number $\delta \in (0, \ \frac{\pi}{2})$ depending upon λ such that the strip $S_{2k+1} \ = \ \{z \in \mathbb{C} \ : \ |\Im(z) - (2k+1)\pi| < \delta, \ \Re(z) \geq M_\lambda\}$ is contained in the Fatou set.*

Proof. 1. For every $f_\lambda \in \mathcal{M}$, the point $z = 0$ is always either in the immediate basin of attraction or in the parabolic domain by Theorem 2.3.1. Since $z = 0$ is in the Fatou set of f_λ, there exists a disc $D_r(0) = \{z \in \mathbb{C} \ : \ |z| < r\}$ for some $r > 0$ such that $D_r(0) \subset \mathcal{F}(f_\lambda)$.

Since e^z maps the left half-plane $\{z \in \mathbb{C} \ : \ \Re(z) < a\}$ where $a \in \mathbb{R}$ onto a punctured disc $D^*(0) = \{z \in \mathbb{C} \ : \ 0 < |z| < e^a\}$ and $\tanh(0) = 0$, we can find a real number M_λ depending on λ such that the left half-plane $H_\lambda = \{z \in \mathbb{C} \ : \ \Re(z) < M_\lambda\}$ is mapped into the open ball $D_r(0) \subset \mathcal{F}(f_\lambda)$ by the map $w = f_\lambda(z)$. Therefore, the Fatou set of f_λ contains the left half-plane $H_\lambda = \{z \in \mathbb{C} \ : \ \Re(z) < M_\lambda\}$.

2. The function e^z maps the horizontal lines $L_{2k+1} = \{x + i(2k+1)\pi \ : \ x \in \mathbb{R}\}$ where $k \in \mathbb{Z}$, onto the negative real axis $\{x \in \mathbb{R} \ : \ x < 0\}$. The function $\lambda\tanh(x)$ maps the negative real axis into a subset of the real axis. By Theorem 2.3.1, if $\lambda > \lambda^*$, the real line $\mathbb{R}$ is contained in the Fatou set of f_λ. Therefore, the horizontal lines $L_{2k+1} = \{x + i(2k+1)\pi \ : \ x \in \mathbb{R}\}$ where $k \in \mathbb{Z}$, are in the Fatou set of f_λ for $\lambda > \lambda^*$.

   If $\lambda \leq \lambda^* < 0$, the function $\lambda\tanh(x)$ maps the negative real axis into a subset of the positive real axis. By Theorem 2.3.1, if $\lambda \leq \lambda^*$, the positive real axis is contained in the Fatou set of f_λ. This gives that the horizontal lines $L_{2k+1} = \{x + i(2k+1)\pi \ : \ x \in \mathbb{R}\}$ where $k \in \mathbb{Z}$, are in the Fatou set of f_λ for $\lambda \leq \lambda^*$.

It is already shown that $-\lambda$ lies in the Fatou set of f_λ. So, there exists a disc $D_r(-\lambda)$ with center at $-\lambda$ and radius r such that $D_r(-\lambda)$ is a subset of the Fatou set. One can find a $\tilde{M}_\lambda < 0$ depending on λ so that $\lambda \tanh z$ maps the half-plane $\tilde{H} = \{z : \Re(z) < \tilde{M}_\lambda\}$ into $D_r(-\lambda)$. Now, we choose $\delta^* \in (0, \frac{\pi}{2})$ and $M^*_\lambda > 0$ depending on $\tilde{M}_\lambda$ such that the image of the strip $\{z \in \mathbb{C} : |\Im(z) - (2k+1)\pi| < \delta^*, \ \Re(z) > M^*_\lambda\}$ under e^z is an angular region $\{z \in \mathbb{C} : |\arg(z) - \pi| < \delta^*, \ |z| > e^{M^*_\lambda}\}$ lying in the left half-plane $\tilde{H}$ (See Figure 2.4). Therefore, $\{z \in \mathbb{C} : |\Im(z) - (2k+1)\pi| < \delta^*, \ \Re(z) > M^*_\lambda\}$ is in the Fatou set of f_λ. As the line segment $\{z \in L_{2k+1} : M_\lambda \leq \Re(z) \leq M^*_\lambda\}$ is in the Fatou set (here M_λ is as given earlier in this proposition), there exists a $\hat{\delta} \in (0, \frac{\pi}{2})$ such that the rectangular region $\{z \in \mathbb{C} : |\Im(z) - (2k+1)\pi| < \hat{\delta}, \ M_\lambda \leq \Re(z) \leq M^*_\lambda\}$ is in the Fatou set of f_λ. Choosing δ to be the minimum of δ^* and $\hat{\delta}$, it follows that $S_{2k+1} = \{z \in \mathbb{C} : |\Im(z) - (2k+1)\pi| < \delta, \ \Re(z) \geq M_\lambda\}$ is contained in the Fatou set.

$\square$

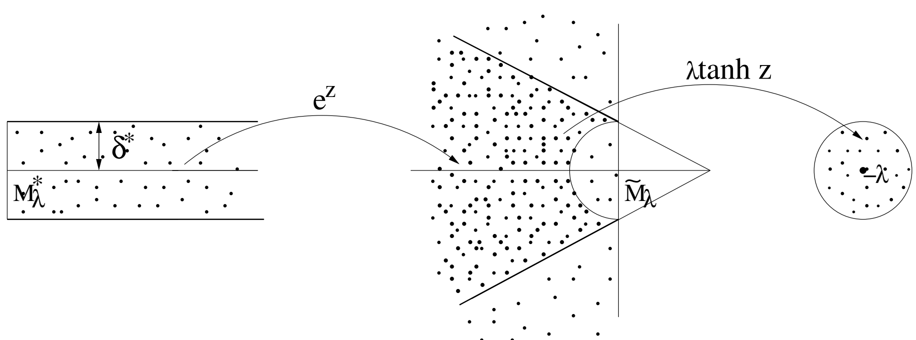


Figure 2.4: Mapping property of $\lambda \tanh(e^z)$.

Remark 2.4.1. *For $\lambda > \lambda^*$, it also follows that the Fatou set of f_λ contains the horizontal lines $L_{2k} = \{x + i\ 2k\pi : x \in \mathbb{R}\}$ for each $k \in \mathbb{Z}$ because $\mathbb{R} \subset \mathcal{F}(f_\lambda)$ and $f_\lambda(L_{2k}) \subset \mathbb{R}$.*

We prove in the following proposition that the Julia set of f_λ for $\lambda > \lambda^*$ cannot contain an unbounded component in $\mathbb{C}$.

Proposition 2.4.2. *Let $f_\lambda \in \mathcal{M}$. If $\lambda > \lambda^*$, then every component of $\mathcal{J}(f_\lambda)\bigcap\mathbb{C}$ is bounded.*

Proof. Let, on the contrary, γ be an unbounded component of $\mathcal{J}(f_\lambda)\bigcap\mathbb{C}$. Then a sequence t_n can be found in γ such that $\lim_{n\to\infty} t_n = \infty$. It follows from Proposition 2.4.1 and Remark 2.4.1 that γ lies in a horizontal strip whose boundary is $L_k \bigcup L_{k+1} \bigcup\{\infty\}$ for some $k \in \mathbb{Z}$ and the set $\{\Re(t_n) \ : \ n \in \mathbb{N}\}\bigcap\{x \in \mathbb{R} \ : \ x > 0\}$ is unbounded. Now, observe that the image $\gamma_1 = e^\gamma$ of γ under the mapping $w = e^z$ is unbounded .

If γ_1 intersects L_k for some $k \in \mathbb{Z}$, then the map $\lambda\tanh(z)$ takes each such intersecting point to a real number which is in the Fatou set. In this way, there is a point common to γ and the Fatou set of f_λ which is not possible. Therefore, γ_1 lies in some horizontal strip bounded by two consecutive L_k's. Since γ_1 is unbounded, there exists a sequence s_n on γ_1 such that $\lim_{n\to\infty} \Re(s_n) = \infty$ or $\lim_{n\to\infty} \Re(s_n) = -\infty$. But, in both the cases, $\lambda\tanh(s_n)$ tends to an asymptotic value of f_λ as $n \to \infty$. Since all the three asymptotic values lie in the Fatou set, there is a sequence $\{z_n\}_{n>0}$ on γ such that $e^{z_n} = s_n$ and $f_\lambda(z_n)$ is a subset of the Fatou set of f_λ for sufficiently large n. By the complete invariance of the Fatou set, there are points z_n on γ which are in the Fatou set. It gives a contradiction. Therefore, any component of $\mathcal{J}(f_\lambda)\bigcap\mathbb{C}$ is bounded. $\square$

The following theorem shows the existence of pre-periodic components in the Fatou set of f_λ for certain range of values of λ.

Theorem 2.4.1. *Let $f_\lambda \in \mathcal{M}$.*

1. *For $\lambda > \lambda^*$, the Fatou set of f_λ is connected.*
2. *For $\lambda \le \lambda^*$, the Fatou set of f_λ contains infinitely many strictly pre-periodic (pre-periodic but not periodic) components.*

Proof. 1. Let V be a component of the Fatou set of f_λ different from the immediate basin of attraction $IM(a_\lambda)$ of the attracting fixed point a_λ. Then, there exists a natural number k such that $f_\lambda^k(V) \subseteq IM(a_\lambda)$. Let W be the Fatou component containing $f_\lambda^{k-1}(V)$. If U_1 and U_2 are two Fatou components of a meromorphic function f such that $f : U_1 \to U_2$, then $U_2 \setminus f(U_1)$ contains at most two points [70]. The two exceptional values λ and $-\lambda$ of f_λ are in $IM(a_\lambda)$. Therefore, it follows that $f_\lambda(W) \subseteq IM(a_\lambda) \setminus \{\lambda, \ -\lambda\}$. Let $D_r(\lambda)$ be a disc of radius $r > 0$ with center λ such that $D_r(\lambda)$ is contained in $IM(a_\lambda)$. Let $U(r)$ be a component of $f_\lambda^{-1}(D_r(\lambda))$ in W. If $r_2 < r_1 < r$, then there are components $U(r_2)$ of $f_\lambda^{-1}(D_{r_2}(\lambda))$ and $U(r_1)$ of $f_\lambda^{-1}(D_{r_1}(\lambda))$ in $U(r) \subset W$ such that $U(r_2) \subset U(r_1)$. Since there is only one singularity of f_λ^{-1} lying over λ and that is logarithmic, $\bigcap_{r>0} U(r) = \emptyset$ and $U(r)$ is unbounded [23]. Thus the component W is unbounded. Also, observe that $\mathbb{R} \subset IM(a_\lambda)$ and $IM(a_\lambda)$ is unbounded. Thus, there are at least two unbounded components, namely, W and $IM(a_\lambda)$ of the Fatou set. Consequently, there is an unbounded component of $\mathcal{J}(f_\lambda) \bigcap \mathbb{C}$ contained in the boundary of W or $IM(a_\lambda)$. But, it is not possible by Proposition 2.4.2. Therefore, the Fatou set of f_λ for $\lambda > \lambda^*$ contains only one component and hence the Fatou set is connected.

2. By Theorem 2.3.2, it follows that the Fatou set $\mathcal{F}(f_\lambda)$ for $\lambda < \lambda^*$ is equal to the basin of attraction or the parabolic basin corresponding to the attracting or the parabolic real 2-periodic cycle $\{a_{1\lambda}, \ a_{2\lambda}\}$ of f_λ. Let $\{a_{1\lambda}, a_{2\lambda}\}$ be an attracting cycle. Let $IM(a_{1\lambda})$ be the component of the Fatou set containing the point $a_{1\lambda}$ and $IM(a_{2\lambda})$ be the component of the Fatou set containing the point $a_{2\lambda}$. Then, $(-\infty, \ r_\lambda) \subset IM(a_{1\lambda})$ and $(r_\lambda, \ \infty) \subset IM(a_{2\lambda})$. Let $L_{2k} = \{x + i \ 2k\pi \ : \ x \in \mathbb{R}\}$ where $k \in \mathbb{Z}$ and $k \neq 0$. Then, $f_\lambda : L_{2k} \to (\lambda, 0)$ is a bijection and it maps $L_{2k}^- = \{x + i \ 2k\pi \ : \ -\infty < x < r_\lambda = f_\lambda^{-1}(r_\lambda)\}$ and $L_{2k}^+ = \{x + i \ 2k\pi \ : \ r_\lambda = f_\lambda^{-1}(r_\lambda) < x < \infty\}$ to $(r_\lambda, \ 0)$ and $(\lambda, \ r_\lambda)$ respectively. This gives that L_{2k}^+ and

L_{2k}^- lie in two different components of the Fatou set. It is clear that some left half-plane H_λ, all horizontal lines $L_{2k+1} = \{x + i\ (2k+1)\pi\ :\ x \in \mathbb{R}\}$ for $k \in \mathbb{Z}$ and $L_{2k}^- = \{x + i\ 2k\pi\ :\ -\infty < x < r_\lambda\}$ are in $IM(a_{1\lambda})$. Further, L_{2k}^+ lies in a component, W_k (say) of the Fatou set which is different from $IM(a_{1\lambda})$ and $IM(a_{2\lambda})$. For each nonzero integer k, we can find such component W_k which contains the line L_{2k}^+ and $W_k \bigcap W_l = \emptyset$ for $k \neq l$. These components W_k's are pre-periodic but not periodic.

If $\{a_{1\lambda}, a_{2\lambda}\}$ is a parabolic cycle, then there will be two different components of the Fatou set containing $(-\infty, a_{1\lambda})$ and $(a_{2\lambda}, \infty)$. Considering them as $IM(a_{1\lambda})$ and $IM(a_{2\lambda})$, it can be observed that $IM(a_{1\lambda})$ contains some half-plane H_λ and horizontal lines L_{2k+1}. Similar arguments as in previous paragraph gives the existence of infinitely many Fatou components W_k containing $L_{2k}^+ = \{x + i\ 2k\pi\ : f_\lambda^{-1}(a_{1\lambda}) < x < \infty\}$ for nonzero integer k which are different from $IM(a_{1\lambda})$ and $IM(a_{2\lambda})$. These components are pre-periodic but not periodic.

For $\lambda = \lambda^*$, there are two petals P_1 and P_2 containing $(-\infty,\ x^*)$ and $(x^*,\ +\infty)$ respectively. The Fatou component P_1 contains a left half-plane and horizontal lines L_{2k+1}. The rest of the proof is similar to the case $\lambda < \lambda^*$ and it is concluded that the Fatou set of f_{λ^*} contains infinitely many strictly pre-periodic Fatou components.

$\square$

Remark 2.4.2. *For $\lambda \leq \lambda^*$, all the singular values of f_λ are in the immediate basin of attraction or in the petals corresponding to the parabolic fixed point which are not completely invariant.*

Given a domain $E \subset \widehat{\mathbb{C}}$, the connectivity $c(E)$ of E is the number of components of $\widehat{\mathbb{C}} \setminus E$. The topology of the Fatou set of f_λ for $\lambda > \lambda^*$ is determined in the following theorem.

Theorem 2.4.2. *Let $f_\lambda \in \mathcal{M}$. If $\lambda > \lambda^*$, then the Fatou set of f_λ is infinitely connected.*

Proof. It is shown in [13] that the connectivity of an invariant Fatou component is either 1, 2 or ∞, 2 being the case for Herman rings. For $\lambda > \lambda^*$, the Fatou set of f_λ is equal to the basin of attraction of the attracting fixed point a_λ and the connectivity of the Fatou set is either 1 or ∞. If the connectivity of the Fatou set is 1, then the Julia set is connected. Since ∞ is in the Julia set, there is an unbounded component of $\mathcal{J}(f_\lambda) \bigcap \mathbb{C}$. But this is impossible by Proposition 2.4.2. Therefore, the Fatou set of f_λ for $\lambda > \lambda^*$ is infinitely connected. $\square$

As a consequence of the infinite connectivity of $\mathcal{F}(f_\lambda)$ for $\lambda > \lambda^*$, we make the following remark on the Julia set of f_λ for $\lambda > \lambda^*$.

Remark 2.4.3. *Let w be a pre-pole of f_λ. If it is not a singleton component of the Julia set, then there will be a component γ in the Julia set that contains w and $f_\lambda^k(\gamma) \subset \mathcal{J}(f_\lambda)$ is a component containing the point $z = \infty$ for some natural number k. But, it is not possible for $\lambda > \lambda^*$ by Proposition 2.4.2. Thus, every pre-pole is a singleton component of the Julia set for $\lambda > \lambda^*$. Since pre-poles are dense in $\mathcal{J}(f_\lambda)$, we conclude that the singleton components of the Julia set are dense in the Julia set of f_λ for $\lambda > \lambda^*$. This can also be concluded from the previous theorem and using Theorem 1.1.22.*

Let I_1 be the component of the Fatou set containing the interval $(-\infty, a_{1\lambda})$ when $\lambda < \lambda^*$. When $\lambda = \lambda^*$, let I_1 denote the Fatou component containing the interval $(-\infty, x^*)$. Let I_2 denote the Fatou component containing $f_\lambda(I_1)$.

Lemma 2.4.1. *Let $f_\lambda \in \mathcal{M}$ with $\lambda \leq \lambda^*$. Let V be a component of the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ. If γ is a Jordan curve in V and the bounded component B of $\widehat{\mathbb{C}} \setminus \gamma$ intersects the Julia set, then B does not contain any pole of f_λ.*

Proof. Since V is a Fatou component, $f_\lambda(V)$ is contained in a Fatou component, say V_1. Let γ be a Jordan curve in V and the bounded component B of $\widehat{\mathbb{C}} \setminus \gamma$ intersects the Julia set.

Suppose that V_1 is different from I_1. Let us assume that B contains a pole. Then, $B_1 = f_\lambda(B)$ contains $\{z : |z| > M\}$ for some $M > 0$. Since I_1 is unbounded, B_1 intersects I_1. This means that there are points in B whose f_λ-images belong to I_1. Consequently, there is a Fatou component, say W in B such that $f_\lambda(W) \subseteq I_1$. The assumption that V_1 is different from I_1 guarantees $W \bigcap V = \emptyset$. In [70], it has been proved that for any meromorphic function $f : A_1 \to A_2$, the cardinality of the set $A_2 \setminus f(A_1)$ is at most two where A_1 and A_2 are two Fatou components of f. Using this result, it follows that $E = I_1 \setminus f_\lambda(W)$ contains at most two points. Since $\lambda \in I_1$, there exists a neighbourhood N_λ of the point λ which is completely contained in I_1 and $N_\lambda \bigcap E = \{\lambda\}$. Therefore, there is a component of $f_\lambda^{-1}(N_\lambda)$ in W. As there is only one singularity lying over $-\lambda$ and it is logarithmic, every component of $f_\lambda^{-1}(N_\lambda)$ is unbounded [23]. Consequently, W is unbounded and B is also unbounded which is not true. Therefore, it follows that B contains no pole of f_λ.

Suppose that V is a Fatou component such that $f_\lambda(V) \subseteq I_1$. Since I_2 is unbounded and, there is only one singularity lying over $-\lambda$ and it is logarithmic, the same arguments given in the previous paragraph with I_1 replaced by I_2 are applied to conclude that B contains no pole of f_λ. The proof is complete. □

The following theorem gives information about the topology of the Fatou components of f_λ for $\lambda \leq \lambda^*$.

Theorem 2.4.3. *Let $f_\lambda \in \mathcal{M}$. If $\lambda \leq \lambda^*$, then every component of the Fatou set of f_λ is simply connected.*

Proof. Let V be any component of the Fatou set of f_λ for $\lambda \leq \lambda^*$. Suppose that V is not simply connected. Let γ be a Jordan curve in V for which the bounded component U of $\widehat{\mathbb{C}} \setminus \gamma$ contains at least one component of $\widehat{\mathbb{C}} \setminus V$. Set $U_n = f_\lambda^n(U)$ for $n = 0, 1, 2, \cdots$. By Lemma 2.4.1, it follows that U does not contain any pole. Since the boundary of U also does not contain any pole, the component $U_1 = f_\lambda(U)$ is a bounded domain. Also, the boundary of U_1 is a subset of $f_\lambda(\partial U)$. Since the boundary ∂U of U is the Jordan curve γ which is in the Fatou set, the image $f_\lambda(\partial U)$ is in a Fatou component, and hence ∂U_1 is in a Fatou component. If U_1 does not contain a pole, the boundary of U_2 lies in a Fatou component by repeating the above arguments. As $U \bigcap \mathcal{J}(f_\lambda) \neq \emptyset$, after finite number of steps, we can find a natural number n_0 for which U_{n_0} contains a pole which gives a contradiction to Lemma 2.4.1. Therefore, it is concluded that the component V of the Fatou set of f_λ for $\lambda \leq \lambda^*$ is simply connected. $\square$

2.5 Measure of the Julia set

In this section, the (Lebesgue) measure of the Julia set of $f_\lambda \in \mathcal{M}$ is determined.

Theorem 2.5.1. *Let $f_\lambda \in \mathcal{M}$. Then, the Julia set of f_λ has measure zero.*

Proof. It is already shown that each singular value of f_λ is in an attracting basin or a parabolic basin. This implies that $d(\overline{P_{f_\lambda}},\ \mathcal{J}(f_\lambda)) > 0$. In view of Theorem 1.1.30, it is enough to show that $\mathcal{J}(f_\lambda)$ is thin at ∞ in order to show that the measure of $\mathcal{J}(f_\lambda)$ is zero.

Let $M \equiv M(\lambda)$ and $\delta \equiv \delta(\lambda)$ be two real numbers such that $H_\lambda = \{z \in \mathbb{C} : \Re(z) < M\}$ and $S_{2k+1} = \{z \in \mathbb{C} :\ |\Im(z) - (2k+1)\pi| < \delta,\ \Re(z) \geq M\}$ are in the Fatou set of f_λ which is possible by Proposition 2.4.1.

Now, consider the square $S(z,r) = \{w :\ |\Re(w) - \Re(z)| < \frac{r\sqrt{2}}{2}, |\Im(w) - \Im(z)| < \frac{r\sqrt{2}}{2}\}$ having its sides parallel to the co-ordinate axes and it is contained in the disc $D(z,r)$ with center at z and radius r. For a rectangle R having its sides parallel to co-ordinate axes with vertical side length 2π and horizontal side length h, $R \bigcap \mathcal{F}(f_\lambda) \supset R \bigcap (\bigcup_{k\in\mathbb{Z}} S_{2k+1})$. This implies that $m(R \bigcap \mathcal{F}(f_\lambda)) > m(R \bigcap (\bigcup_{k\in\mathbb{Z}} S_{2k+1})) > 2\delta h$. If $j = [\frac{r\sqrt{2}}{2\pi}]$ is the greatest integer not exceeding $\frac{r\sqrt{2}}{2\pi}$, then $S(z,r)$ will contain j different rectangles each having its sides parallel to co-ordinate axes with vertical side length 2π and horizontal side length

$r\sqrt{2}$. This gives that $m(\mathfrak{F}(f_\lambda)\bigcap S(z,r)) > j2\delta r\sqrt{2} \geq (\frac{r\sqrt{2}}{2\pi} - 1)(2\delta r\sqrt{2}) = \frac{2\delta r^2}{\pi} - 2\delta r\sqrt{2}$. Consequently, $m(\mathfrak{F}(f_\lambda)\bigcap D_r(z)) > \frac{2\delta r^2}{\pi} - 2\delta r\sqrt{2} = 2\delta(\frac{r^2}{\pi} - r\sqrt{2})$ and

$$density(\mathfrak{F}(f_\lambda), D_r(z)) = \frac{m(\mathfrak{F}(f_\lambda)\bigcap D_r(z))}{m(D_r(z))} > \frac{2\delta}{\pi r^2}\left(\frac{r^2}{\pi} - r\sqrt{2}\right).$$

Now, $density(\mathfrak{F}(f_\lambda),\ D_r(z)) > \frac{2\delta}{\pi}(\frac{1}{\pi} - \frac{\sqrt{2}}{r}) > \frac{\delta}{\pi^2}$ for $r > 2\sqrt{2}\pi$.

Letting $\epsilon = \frac{\delta}{\pi^2}$ and $R_0 = 2\sqrt{2}\pi$, it is concluded that $density(\mathfrak{F}(f_\lambda),\ D_r(z)) > \epsilon$ for all $z \in \mathbb{C}$ and all $r > R_0$. Since $density(\mathfrak{F}(f_\lambda),\ D_r(z)) + density(\mathfrak{J}(f_\lambda),\ D_r(z)) = 1$, it follows that $density(\mathfrak{J}(f_\lambda),\ D_r(z)) < 1 - \epsilon$ for all $z \in \mathbb{C}$ and all $r > R_0$. Therefore, the Julia set of f_λ is thin at ∞ which completes the proof. $\square$

A comparison between the dynamics of $\lambda\tanh(e^z)$, $\lambda\tanh z$ and λe^z is given in the Table 2.1.

Dynamics of $f_\lambda(z) = \lambda\tanh(e^z),\ \lambda \neq 0$	**Dynamics of** $E_\lambda(z) = \lambda e^z,\ \lambda \neq 0$	**Dynamics of** $T_\lambda(z) = \lambda\tanh z,\ \lambda \neq 0$
The order of f_λ is ∞.	The order of E_λ is 1.	The order of T_λ is 1.
The Schwarzian derivative of f_λ is a transcendental function.	The Schwarzian derivative of E_λ is constant.	The Schwarzian derivative of T_λ is constant.
f_λ has no critical values.	E_λ has no critical values.	T_λ has no critical values.
f_λ has three asymptotic values 0, λ and $-\lambda$.The point 0 is an indirect and, each of $\{\lambda,\ -\lambda\}$ is a direct singularity of f_λ^{-1}.	E_λ has one asymptotic value 0. The point 0 is a direct singularity of E_λ^{-1}.	T_λ has two asymptotic values $-\lambda$ and λ. Each of $-\lambda$ and λ is a direct singularity of T_λ^{-1}.
f_λ is periodic with period $2\pi i$.	E_λ is periodic with period $2\pi i$.	T_λ is periodic with period πi.
f_λ is neither even nor odd.	E_λ is neither even nor odd.	T_λ is even.
Bifurcation in the dynamics of f_λ occurs at one critical parameter $\lambda^* \approx -3.2946$.	Bifurcation in the dynamics of E_λ occurs at two critical parameters $-e$ and $\frac{1}{e}$.	Bifurcation in the dynamics of T_λ occurs at two critical parameters -1 and 1.
The Fatou set of f_λ has infinitely many components and each component is simply connected for $\lambda < \lambda^*$.	The Fatou set of E_λ has only one component and it is simply connected for $-e < \lambda < \frac{1}{e}$.	The Fatou set of T_λ has only one component and it is infinitely connected for $-1 < \lambda < 1$.
The Julia set of f_λ is connected for $\lambda < \lambda^*$.	The Julia set of E_λ is connected for $-e < \lambda < \frac{1}{e}$.	The Julia set of T_λ is totally disconnected for $-1 < \lambda < 1$.
The Fatou set of f_λ is infinitely connected for $\lambda > \lambda^*$.	The Fatou set of E_λ is empty for $\lambda > \frac{1}{e}$.	The Fatou set of T_λ has two components and each is simply connected for $\lambda \leq -1$ and $\lambda \geq 1$.
The Julia set $\mathfrak{J}(f_\lambda)$ has a totally disconnected subset for $\lambda > \lambda^*$.	The Julia set $\mathfrak{J}(E_\lambda)$ is $\widehat{\mathbb{C}}$ and hence connected for $\lambda > \frac{1}{e}$.	The Julia set $\mathfrak{J}(T_\lambda)$ is $i\mathbb{R}\bigcup\{\infty\}$ for $\lambda \leq -1$ and $\lambda \geq 1$.

Table 2.1: Comparison between the dynamics of $\lambda\tanh(e^z)$, λe^z and $\lambda\tanh z$.

Chapter 3

Dynamics of certain entire functions of bounded type

In the present chapter, we define a class of entire transcendental functions and investigate the occurrence of bifurcation and chaotic burst in the dynamics of functions in the one parameter family $\{\lambda f \ : \ \lambda > 0\}$ for each f belonging to the class.

Define

$$\mathrm{E} \equiv \left\{ f \ : \ \begin{array}{l} \text{(i) } f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n \text{ for } z \in \mathbb{C} \text{ where } a_n \geq 0 \ \text{ for all } n \geq 0 \\ \text{(ii) } f(x) > 0 \ \text{ for all } x < 0 \\ \text{(iii) The set } S_f \ \text{ is a bounded subset of } \mathbb{R} \end{array} \right\}.$$

Let

$$\mathrm{E}_0 \equiv \{f \in \mathrm{E} \ : \ f(0) = 0\} \qquad \text{and} \qquad \mathrm{E}_1 \equiv \{f \in \mathrm{E} \ : \ f(0) \neq 0\}\ .$$

For each $f \in \mathrm{E}$, consider the one parameter family $\mathcal{S} = \{f_\lambda \equiv \lambda f \ : \ \lambda > 0\}$. It is worth noting that the class E contains the interesting functions such as $I_{2n}(z)$ and $z^{-n} I_n(z)$ for $n \in \mathbb{N}$ where $I_n(z)$ is the modified Bessel function of first kind and order n. The class E_1 includes the functions $\dfrac{\sinh z}{z}$, $I_0(z)$ and e^z whose dynamics have already been studied. The change in the dynamics of functions in the one parameter family $\mathcal{S}$ is the main subject of investigation of this chapter. For each $f \in \mathrm{E}_0$, it is shown that the Julia set of f_λ is the complement of the basin of attraction of the superattracting fixed point 0 for each $\lambda > 0$. When $f \in \mathrm{E}_1$, the Julia set of f_λ is shown to change from a nowhere dense subset of the complex plane to the whole plane as the parameter λ crosses a critical parameter λ^* (its value depending on f). We find a necessary condition for the Fatou set of f_λ to be connected for $f \in \mathrm{E}$ and $\lambda > 0$. A number of interesting examples are discussed at the end of the chapter.

3.1 Properties of E

It is easy to observe that, if $\lambda > 0$, then $\lambda + f \in \mathrm{E}_1$ whenever $f \in \mathrm{E}$ and $f_\lambda \equiv \lambda f \in \mathrm{E}_\mathrm{j}$ for $f \in \mathrm{E}_\mathrm{j}$, $j = 0,\ 1$. Besides this, certain compositions of functions also yield functions in the class E as is shown in Proposition 3.1.1.

Remark 3.1.1. *Let g and h be two entire functions and $g \circ h$ be their composition. Let S_g and S_h denote the set of singular values of g and h respectively. From the arguments used in Lemma 2.1.1, it follows that $S_{g\circ h} \subseteq S_g \bigcup \{g(z) :\ z \in S_h\}$. If S_g and S_h are bounded subsets of $\mathbb{R}$ and g is an entire function preserving the real axis, then $g(S_h) = \{g(z)\ :\ z \in S_h\}$ is a bounded subset of $\mathbb{R}$. Therefore, $S_{g\circ h}$ is a bounded subset of $\mathbb{R}$.*

Proposition 3.1.1. *Let $f \in \mathrm{E}$, $g \in \mathrm{E}_0$ and $h \in \mathrm{E}_1$. Let $P(z) = (z + a_1)^{m_1}(z + a_2)^{m_2}...(z + a_n)^{m_n}$ be a non-constant polynomial where $a_1,\ a_2,\ \cdots,\ a_n$ are positive real numbers and $m_1,\ m_2,\ \cdots,\ m_n$ are non-negative integers. Then,*

1. *$\phi = h \circ f \in \mathrm{E}_1$ and $\psi = g \circ h \in \mathrm{E}_1$. In particular, the class E_1 is closed under composition.*

2. *The class E_0 is closed under composition.*

3. *$\Phi = P \circ f \in \mathrm{E}_1$ and $\Psi = h \circ P \in \mathrm{E}_1$.*

Proof. 1. Let $\phi(z) = h(f(z))$ for $z \in \mathbb{C}$ where $h \in \mathrm{E}_1$ and $f \in \mathrm{E}$. If $h(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{C}$, then $h(f(z)) = \sum_{n=0}^{\infty} a_n (f(z))^n = \sum_{n=0}^{\infty} b_n z^n$ for $z \in \mathbb{C}$ (say). All the coefficients in the Taylor series of $(f(z))^n$ about the origin are non-negative, so all b_n's are non-negative. It is obvious that $\phi(x) = h(f(x)) > 0$ for $x < 0$ and $\phi(0) = h(f(0)) > 0$. As f and h are in E, S_f and S_h are bounded subsets of $\mathbb{R}$ and h is an entire function that preserves the real axis. The set $S_{h\circ f}$ is a bounded subset of $\mathbb{R}$ by Remark 3.1.1. Thus, $\phi = h \circ f \in \mathrm{E}_1$ for $h \in \mathrm{E}_1$ and $f \in \mathrm{E}$. Taking f in E_1, it is seen that the class E_1 is closed under composition.

It can be shown similarly that all the coefficients of the Taylor series of $\psi = g \circ h$ about the origin are non-negative and $S_{g\circ h}$ is a bounded subset of $\mathbb{R}$ for all $g \in \mathrm{E}_0$. Since $g(h(x)) > 0$ for all $x \leq 0$, it follows that $\psi = g \circ h \in \mathrm{E}_1$.

2. Let g and $\widetilde{g}$ be in E_0. It follows by similar arguments used in the first paragraph of this proposition that, all the coefficients of the Taylor series of $g \circ \widetilde{g}$ about the origin are non-negative and $S_{g\circ\widetilde{g}}$ is a bounded subset of $\mathbb{R}$. Clearly, $g(\widetilde{g}(0)) = 0$. Since $\widetilde{g}(x) > 0$ for $x < 0$, $g(\widetilde{g}(x)) > 0$ for all $x < 0$. Therefore, $g \circ \widetilde{g}$ belongs to E_0.

3. Observe that all the coefficients in the Taylor series of $\Phi = P \circ f$ and $\Psi = h \circ P$ about the origin are non-negative. Since all zeros of $P(z)$ are real, the zeros of $P'(z)$ are real by Lucas Theorem. Further, $P(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ which gives that the critical values of P are real. As P has no finite asymptotic value, S_P is a finite subset of $\mathbb{R}$. For any function f in E, the set of all singular values S_f is a bounded subset of $\mathbb{R}$ and, P and f preserve the real axis. So S_Φ and S_Ψ are bounded subsets of $\mathbb{R}$ by Remark 3.1.1. Clearly, $P(f(x)) > 0$ for $x \leq 0$ and $h(P(x)) > 0$ for $x \leq 0$. Thus, $\Phi = P \circ f$ and $\Psi = h \circ P$ belong to E_1 for all $f \in \mathrm{E}$ and $h \in \mathrm{E}_1$.

$\square$

3.2 Dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$

In this section, the dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ where $f_\lambda \in \mathcal{S}$ is studied. For the functions f in the class E_0, the dynamics of $f_\lambda \equiv \lambda f$ on $\mathbb{R}^+ = \{x \in \mathbb{R} \ : \ x > 0\}$ is described in Theorem 3.2.1. For the functions f in the class E_1, the dynamics of $f_\lambda \equiv \lambda f$ on $\mathbb{R}^+$ is described in Theorem 3.2.2.

Theorem 3.2.1. *Let $f_\lambda \equiv \lambda f$ where $f \in \mathrm{E}_0$ and $\lambda > 0$. Then, f_λ has only two real periodic points 0 and r_λ with $0 < r_\lambda$ where 0 is a superattracting fixed point and r_λ is a repelling fixed point. Further, $\lim_{n\to\infty} f_\lambda^n(x) = 0$ for $0 \le x < r_\lambda$ and $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for $x > r_\lambda$.*

Proof. Let $f_\lambda(x) = \lambda \sum\limits_{n=0}^{\infty} a_n x^n$ for $x \in \mathbb{R}$ where $a_n \ge 0$ for all $n \ge 0$. Observe that $f_\lambda(x) > 0$ for $x \in \mathbb{R}$ with $x \ne 0$ and $f'_\lambda(x) > 0$ for $x > 0$. Therefore, any nonzero real periodic point of f_λ lies only in $\mathbb{R}^+$ and it must be a fixed point. Since $f \in \mathrm{E}_0$, $f_\lambda(0) = \lambda f(0) = 0$. Therefore, the point $x = 0$ is a fixed point of f_λ. Since $f_\lambda(x) > 0$ for $x < 0$, $f_\lambda(0) = 0$ and $f_\lambda(x) > 0$ for $x > 0$, it follows that $f'_\lambda(0) = 0$ and hence the point $x = 0$ is a superattracting fixed point of f_λ. Thus, $f_\lambda(x) = \lambda \sum\limits_{n=2}^{\infty} a_n x^n$ for $x \in \mathbb{R}$ and $a_n \ge 0$ for all $n \ge 2$.

Let $g_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$. Then, $g'_\lambda(x) = f'_\lambda(x) - 1 = \lambda\left(\sum_{n=2}^{\infty} n\, a_n\, x^{n-1}\right) - 1$ and $g''_\lambda(x) = f''_\lambda(x) = \lambda \sum\limits_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}$. It shows that $g''_\lambda(x) > 0$ for $x > 0$ and $g'_\lambda(x) \to +\infty$ as $x \to +\infty$. Since $g'_\lambda(0) = -1$ and g'_λ is increasing in $\mathbb{R}^+$, there is a unique $x_\lambda > 0$ such that $g'_\lambda(x) < 0$ for $x \in (0,\ x_\lambda)$, $g'_\lambda(x_\lambda) = 0$ and $g'_\lambda(x) > 0$ for $x > x_\lambda$. It shows that g_λ decreases in $(0,\ x_\lambda)$ and increases thereafter. Since $g_\lambda(0) = 0$ and $\lim_{x\to+\infty} g_\lambda(x) = +\infty$, there exists a unique point r_λ in $(x_\lambda,\ \infty)$ such that $g_\lambda(x) < 0$ for $x \in (0,\ r_\lambda)$, $g_\lambda(r_\lambda) = 0$ and $g_\lambda(x) > 0$ for $x \in (r_\lambda,\ \infty)$. Therefore, the point r_λ is a unique real fixed point of f_λ and is repelling as $f'_\lambda(x) > 1$ in $(x_\lambda,\ \infty)$.

For $0 < x < r_\lambda$, $f_\lambda(x) < x$. This gives that the sequence $\{f_\lambda^n(x)\}_{n>0}$ is decreasing and bounded below by 0. Therefore, $\lim_{n\to\infty} f_\lambda^n(x) = 0$ for $0 < x < r_\lambda$. The sequence

$\{f_\lambda^n(x)\}_{n>0}$ is increasing and unbounded for $x > r_\lambda$. So $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ when $x > r_\lambda$. $\square$

Remark 3.2.1. *Let $A = \{x \in \mathbb{R} \ : \ x < 0$ and $f_\lambda(x) \in [0,\ r_\lambda)\}$ and $B = \{x \in \mathbb{R} \ : \ x < 0$ and $f_\lambda(x) \in (r_\lambda,\ \infty)\}$ where $f \in \mathrm{E}_0$ and $\lambda > 0$. It follows by Theorem 3.2.1 that $f_\lambda^n(x) \to 0$ as $n \to \infty$ for $x \in A$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in B$.*

Theorem 3.2.2. *Let $f_\lambda \equiv \lambda f$ where $f \in \mathrm{E}_1$ and $\lambda > 0$. Then, there exists a unique positive real number λ^* such that*

1. *For $0 < \lambda < \lambda^*$, f_λ has only two real fixed points a_λ and r_λ with $a_\lambda < r_\lambda$ where a_λ is attracting and r_λ is repelling. Further, $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for $0 \le x < r_\lambda$ and $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for $x > r_\lambda$.*

2. *For $\lambda = \lambda^*$, f_λ has only one real fixed point x^* where x^* is the unique real solution of $f(x) - xf'(x) = 0$ and x^* is rationally indifferent. Further, $\lim_{n\to\infty} f_\lambda^n(x) = x^*$ for $0 \le x < x^*$ and $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for $x > x^*$.*

3. *For $\lambda > \lambda^*$, f_λ has no real fixed point. Further, $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for all $x \ge 0$.*

Proof. As $f \in \mathrm{E}_1$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in \mathbb{R}$ where $a_n \ge 0$ for all $n \ge 0$. It is easy to see that $f(x) > 0$ for all $x \in \mathbb{R}$. Observe that $f'(x) > 0$ and $f''(x) > 0$ for $x > 0$. Therefore, $f'(x)$ is increasing and tends to $+\infty$ as $x \to +\infty$. Further, we conclude that $\lim_{x\to+\infty} f(x) - x = +\infty$ and $\lim_{x\to+\infty} f(x) - xf'(x) = -\infty$.

For $\lambda > 0$, $f_\lambda(x)$ is positive for any real number x and $f_\lambda'(x)$ is positive for $x > 0$. Therefore, any real periodic point of f_λ must be a fixed point.

Let $g_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$. Observe that $g_\lambda''(x) = f_\lambda''(x) > 0$ for $x > 0$. Therefore, $g_\lambda'(x)$ is increasing in $\mathbb{R}^+$ and $g_\lambda'(x) \to +\infty$ as $x \to +\infty$. Suppose that $f'(0) \neq 0$. Then, $f'(0) > 0$ since $f'(x) > 0$ for $x > 0$. If $\lambda \ge \frac{1}{f'(0)}$, then $g_\lambda'(0) = f_\lambda'(0) - 1 \ge 0$. This gives that

$g'_\lambda(x) > g'_\lambda(0) \geq 0$ for all $x > 0$. Therefore, g_λ is strictly increasing in $\mathbb{R}^+$. As $g_\lambda(0) > 0$, $g_\lambda(x)$ has no zero in $\mathbb{R}^+$. In other words, $f_\lambda(x)$ has no fixed point in $\mathbb{R}$ when $\lambda \geq \frac{1}{f'(0)}$.

If $0 < \lambda < \frac{1}{f'(0)}$, then $g'_\lambda(0) < 0$. Also, if $f'(0) = 0$, then $g'_\lambda(0) < 0$ for all λ. As $g'_\lambda(x)$ is increasing and tends to $+\infty$ in $\mathbb{R}^+$, there exists a real number $x_\lambda > 0$ such that $g'_\lambda(x) < 0$ for $x \in (0,\ x_\lambda)$, $g'_\lambda(x) = 0$ for $x = x_\lambda$ and $g'_\lambda(x) > 0$ for $x > x_\lambda$. It shows that $g_\lambda(x)$ decreases in $(0,\ x_\lambda)$ and attains the minimum value at $x = x_\lambda$, and then increases to $+\infty$ in $(x_\lambda,\ \infty)$. This gives that $\lambda = \dfrac{1}{f'(x_\lambda)}$.

Consider $\phi(x) = f(x) - xf'(x)$ for $x \in \mathbb{R}^+$. Since $\phi'(x) = -xf''(x) < 0$ for all $x > 0$, the function $\phi(x)$ is decreasing in $(0,\ \infty)$. Using the facts that $\phi(0) = f(0) > 0$ and $\lim_{x\to+\infty} \phi(x) = -\infty$, by the continuity of ϕ, it follows that there exists unique $x^* > 0$ such that

$$\phi(x) \begin{cases} > 0 & \text{for} \quad x < x^* \\ = 0 & \text{for} \quad x = x^* \\ < 0 & \text{for} \quad x > x^* \end{cases} \tag{3.1}$$

In the sequel, let

$$\lambda^* = \frac{1}{f'(x^*)} \tag{3.2}$$

where x^* is the positive real root of $\phi(x) = f(x) - xf'(x)$ for $f \in \mathrm{E}_1$.

Since $\frac{1}{f'(x)}$ is decreasing in $\mathbb{R}^+$ and $x^* > 0$, it follows that $\lambda^* < \frac{1}{f'(0)}$ if $f'(0) \neq 0$.

1. If $0 < \lambda < \lambda^*$, then $\frac{1}{f'(x_\lambda)} < \frac{1}{f'(x^*)}$ and $x_\lambda > x^*$. It follows from Equation (3.1) that $\phi(x_\lambda) < 0$. Consequently, $g_\lambda(x_\lambda) = f_\lambda(x_\lambda) - x_\lambda < 0$. Therefore, there exists two real numbers a_λ and r_λ (say) with $0 < a_\lambda < x_\lambda < r_\lambda$ such that $g_\lambda(a_\lambda) = 0 = g_\lambda(r_\lambda)$. Hence, f_λ has exactly two real fixed points a_λ and r_λ. Observe that $0 < f'_\lambda(a_\lambda) < f'_\lambda(x_\lambda) = 1 < f'_\lambda(r_\lambda)$, since $0 < a_\lambda < x_\lambda < r_\lambda$. Therefore, a_λ is attracting and r_λ is repelling fixed points of f_λ. Note that $f_\lambda(x) > x$ for $0 < x < a_\lambda$ and $f_\lambda(x) < x$ for $a_\lambda < x < r_\lambda$. Since $f_\lambda(x)$ is increasing in $\mathbb{R}^+$, the sequence $\{f_\lambda^n(x)\}_{n>0}$ is increasing and bounded above by a_λ for $0 \leq x < a_\lambda$. Similarly, the sequence $\{f_\lambda^n(x)\}_{n>0}$ is decreasing and bounded below by a_λ for $a_\lambda < x < r_\lambda$. Hence, $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for

$0 \leq x < r_\lambda$ by monotone convergence theorem. Now, $f_\lambda(x) > x$ and $f'_\lambda(x) > 1$ for $x > r_\lambda$. This implies that the sequence $\{f_\lambda^n(x)\}_{n\geq 0}$ is increasing and not bounded above. Consequently, $\lim_{n\to\infty} f_\lambda^n(x) = \infty$.

2. When $\lambda = \lambda^*$, it can be shown that $g_\lambda(x_\lambda) = 0$ and $x_\lambda = x^*$ by similar arguments as used in the case $0 < \lambda < \lambda^*$. As $g_{\lambda^*}(x^*)$ is the minimum value of $g_{\lambda^*}(x)$, the point x^* is the only zero of $g_{\lambda^*}(x)$. Hence, $f_{\lambda^*}(x)$ has only one fixed point x^* and it is rationally indifferent. The sequence $\{f_{\lambda^*}^n(x)\}_{n>0}$ is increasing and bounded above by x^* for $0 \leq x < x^*$. By monotone convergence theorem, it follows that $\lim_{n\to\infty} f_{\lambda^*}^n(x) = x^*$ for $0 \leq x < x^*$. For $x > x^*$, the sequence $\{f_{\lambda^*}^n(x)\}_{n>0}$ is increasing and not bounded above. Therefore, $f_{\lambda^*}^n(x)$ tends to $+\infty$ as $n \to \infty$.

3. Let $\lambda > \lambda^*$. If $f'(0) = 0$, then there exists a x_λ such that $f'_\lambda(x_\lambda) = 1$ and $\frac{1}{f'(x_\lambda)} > \frac{1}{f'(x^*)}$. Similarly, if $f'(0) \neq 0$ and $\lambda^* < \lambda < \frac{1}{f'(0)}$, then there exists a x_λ such that $f'_\lambda(x_\lambda) = 1$ and $\frac{1}{f'(x_\lambda)} > \frac{1}{f'(x^*)}$. This implies that $x_\lambda < x^*$ and $\phi(x_\lambda) > 0$. Therefore, $g_\lambda(x) > g_\lambda(x_\lambda) > 0$ for all $x > 0$ showing that $f_\lambda(x)$ has no fixed point for $\lambda > \lambda^*$. If $f'(0) \neq 0$ and $\lambda \geq \frac{1}{f'(0)}$, then it is already shown in the beginning that f_λ has no fixed point. In all the cases, observe that $f_\lambda(x) > x$ for all $x \geq 0$. Therefore, the sequence $\{f_\lambda^n(x)\}_{n\geq 0}$ is strictly increasing and not bounded above for $x \geq 0$ and hence $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for all $x \geq 0$, if $\lambda > \lambda^*$.

$\square$

Remark 3.2.2. *1. Let $A = \{x \in \mathbb{R} : x < 0$ and $f_\lambda(x) \in [0, r_\lambda)\}$ and $B = \{x \in \mathbb{R} : x < 0$ and $f_\lambda(x) \in (r_\lambda, \infty)\}$ where $f \in \mathrm{E}_1$ and $0 < \lambda < \lambda^*$. Then, it follows by Theorem 3.2.2(1) that $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in A$ and $f_\lambda^n(x) \to +\infty$ as $n \to \infty$ for $x \in B$.*

2. Let $A = \{x \in \mathbb{R} : x < 0$ and $f_{\lambda^*}(x) \in [0, x^*)\}$ and $B = \{x \in \mathbb{R} : x < 0$ and $f_{\lambda^*}(x) \in (x^*, \infty)\}$ where $f \in \mathrm{E}_1$. Then, it follows by Theorem 3.2.2(2) that

$f^n_{\lambda^*}(x) \to x^*$ as $n \to \infty$ for $x \in A$ and $f^n_{\lambda^*}(x) \to +\infty$ as $n \to \infty$ for $x \in B$.

3. *If $f \in \mathrm{E}_1$ and $\lambda > \lambda^*$, then $f_\lambda(x) > 0$ for $x < 0$. Since $f^n_\lambda(x) \to +\infty$ as $n \to \infty$ for $x \geq 0$ by Theorem 3.2.2(3), it also happens that $f^n_\lambda(x) \to +\infty$ as $n \to \infty$ for $x < 0$. Therefore, $f^n_\lambda(x) \to +\infty$ as $n \to \infty$ for all $x \in \mathbb{R}$, if $f \in \mathrm{E}_1$ and $\lambda > \lambda^*$.*

From Theorem 3.2.2 and Remark 3.2.2, it follows that the dynamics of f_λ for $\lambda \in (0,\ \lambda^*)$ is different from the dynamics of f_μ for $\mu \in (\lambda^*, \infty)$. That is, the dynamics of the function f_λ in $\mathcal{S} = \{f_\lambda \equiv \lambda f\ :\ \lambda > 0\}$ where $f \in \mathrm{E}_1$ changes suddenly when the parameter λ crosses the value λ^*. Thus, a bifurcation occurs in the dynamics of functions in the one parameter family $\mathcal{S}$ where $f \in \mathrm{E}_1$ at the parameter value λ^*.

3.3 Dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$

In this section, it is shown that rotational domains, wandering domains and Baker domains are non-existent in the Fatou set of functions in E. Then an investigation of dynamics of functions f_λ is made separately for $f \in \mathrm{E}_0$ and $f \in \mathrm{E}_1$.

It is well known that the Fatou set of an entire transcendental function does not contain *Herman ring* [19]. If $f \in \mathrm{E}$, we prove that the Siegel discs, Baker domains and wandering domains also do not exist in the Fatou set of f in Theorem 3.3.1 and Theorem 3.3.2.

Theorem 3.3.1. *Let $f \in \mathrm{E}$. Then, $\mathcal{F}(f)$ does not contain a Siegel disc.*

Proof. Suppose that the Fatou set of f contains a Siegel disc D (say). For $f \in \mathrm{E}$, it follows from the facts $f(x) \geq 0$ for $x \in \mathbb{R}$ and S_f is a bounded subset of $\mathbb{R}$ that the closure of the forward orbits of all singular values of f is properly contained in $\mathbb{R}$. It is known that the boundary ∂D of the Siegel disc D lies in the closure of the set of forward orbits of all singular values of f. Therefore, $\partial D \subsetneq \mathbb{R}$ and the set $(\partial D)^c$ (=the complement of ∂D) is path connected and $D \subseteq (\partial D)^c$. Now, we claim that the complement of the closure of D, denoted by $(\overline{D})^c$, is empty. If possible, let there be a point z^* in $(\overline{D})^c = (D \bigcup \partial D)^c = D^c \bigcap (\partial D)^c$. Then, $\{z^*\} \bigcup D$ is a subset of $(\partial D)^c$ and a path γ can be found in $(\partial D)^c$ joining the point z^* and a point of D. Since $z^* \in D^c$, the path γ must intersect ∂D which is not possible. Therefore, $(\overline{D})^c$ is an empty set. As any component of the Fatou set other than D must be in $(\overline{D})^c$, the Fatou set of f cannot contain any component other than D. Since the Fatou set of f is completely invariant, it follows that D is completely invariant. By Picard's theorem, all points of D, except at most one have infinitely many pre-images. Since D is completely invariant, f is not one-to-one on D which is a contradiction to the definition of Siegel disc. Therefore, the Fatou set of f does not contain a Siegel disc. □

Theorem 3.3.2. *Let $f \in \mathrm{E}$. Then, the Fatou set of f does not contain Baker domain.*

Proof. Since $f \in B$, $f^k \in B$ for each $k \in \mathbb{N}$ where B denotes the class of functions for which the set of all singular values is bounded. It is shown in [60] that the sequence $\{f^{kn}(z)\}_{n>0}$ does not converge to ∞ for any $k \in \mathbb{N}$ and $z \in \mathcal{F}(f)$. Therefore, the Fatou set of f cannot contain a Baker domain.

□

The dynamics of $f_\lambda(z) = \lambda f(z)$ for $z \in \mathbb{C}$ where $f \in \mathrm{E}_0$ and $\lambda > 0$ is described in the following theorem.

Theorem 3.3.3. *Let $f_\lambda(z) = \lambda f(z)$ for $z \in \mathbb{C}$ where $f \in \mathrm{E}_0$ and $\lambda > 0$. Then, the Fatou set of f_λ is the union of a basin of attraction of the superattracting fixed point 0 of f_λ and possibly wandering domains.*

Proof. If $f \in \mathrm{E}_0$ and $\lambda > 0$, then $f_\lambda \equiv \lambda f \in \mathrm{E}_0$. By Theorems 3.3.1 and 3.3.2, it follows that the Fatou set of f_λ does not contain rotational domains and Baker domains. Therefore, the Fatou set of f_λ contains only the basins of attractions, parabolic basins and possibly wandering domains. Suppose that U is an immediate basin of attraction (or a parabolic domain) associated with a non-real attracting (or rationally indifferent) periodic point c_λ of period p of f_λ. Then, there is at least one singular value $\tilde{w}$ (say) of f_λ such that $f_\lambda^{np}(\tilde{w})$ converges to c_λ as $n \to \infty$. It shows that there exists a natural number n_0 such that $f_\lambda^{np}(\tilde{w})$ are non-real for all $n > n_0$ which is a contradiction to the fact that the forward orbits of all singular values are subset of the real line. Therefore, it is concluded that the immediate basins of attractions, parabolic domains (and their pre-images) corresponding to the real periodic points and possibly wandering domains are the only possible components of the Fatou set of f_λ. It is proved in Theorem 3.2.1 that the superattracting fixed point $x = 0$ is the only such real periodic point of f_λ for $\lambda > 0$. Therefore, the Fatou set $\mathcal{F}(f_\lambda)$ is the union of the basin of attraction of the superattracting fixed point 0 of f_λ and possibly wandering domains. □

The following computationally useful characterization of the Julia set of f_λ for $f \in \mathrm{E}_0$ is an immediate consequence of Theorem 3.3.3.

Corollary 3.3.1. *Let $f_\lambda(z) = \lambda f(z)$ for $z \in \mathbb{C}$ where $f \in \mathrm{E}_0$ and $\lambda > 0$. Then, the Julia set of f_λ is the complement of the union of the basin of attraction of the superattracting fixed point 0 of f_λ and possibly wandering domains.*

Remark 3.3.1. *From Theorem 3.3.3, it is easy to see that there is no occurrence of bifurcation in the dynamics of the one parameter family $\mathcal{S}$ where $f \in \mathrm{E}_0$.*

The following theorem describes the dynamics of $f_\lambda(z) = \lambda f(z)$ for $z \in \mathbb{C}$ where $f \in \mathrm{E}_1$ and $\lambda > 0$.

Theorem 3.3.4. *Let $f_\lambda(z) = \lambda f(z)$ for $z \in \mathbb{C}$ where $f \in \mathrm{E}_1$ and $\lambda > 0$.*

1. *For $0 < \lambda < \lambda^*$, the Fatou set of f_λ is the union of the basin of attraction of the real attracting fixed point a_λ and possibly wandering domains.*

2. *For $\lambda = \lambda^*$, the Fatou set of f_λ is the union of the parabolic basin corresponding to the real rationally indifferent fixed point x^* and possibly wandering domains.*

3. *For $\lambda > \lambda^*$, the Fatou set is empty or possibly contains wandering domains.*

Proof. If $f \in \mathrm{E}_1$ and $\lambda > 0$, then $f_\lambda \equiv \lambda f \in \mathrm{E}_1$. The Fatou set of f_λ has no rotational domains, Baker domains by Theorems 3.3.1 and 3.3.2. Suppose that U is an immediate basin of attraction or a parabolic domain of f_λ associated with a non-real attracting or rationally indifferent periodic point of period p. Then, there is at least one singular value $\tilde{w}$ of f_λ and a natural number n_0 such that $f_\lambda^{np}(\tilde{w})$ is non-real for all $n > n_0$ which is not possible since $O^+(S_{f_\lambda}) \subseteq [0,\ \infty)$. Therefore, if U is an immediate basin of attraction or a parabolic domain of f_λ, then the periodic point of f_λ associated with U is real. Let λ^* be given in Equation (3.2).

1. For $0 < \lambda < \lambda^*$, it follows from Theorem 3.2.2(1) that f_λ has only one real attracting fixed point a_λ. Therefore, the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the union of the basin of attraction of a_λ and possibly wandering domains for $0 < \lambda < \lambda^*$.

2. For $\lambda = \lambda^*$, f_λ has only one real fixed point x^* where x^* is the unique real solution of $f(x) - xf'(x) = 0$ and is the real rationally indifferent fixed point by Theorem 3.2.2(2). Therefore, it follows that the Fatou set of f_{λ^*} is the union of the parabolic basin corresponding to the real rationally indifferent fixed point x^* and possibly wandering domains.

3. For $\lambda > \lambda^*$, the function f_λ has no real fixed point by Theorem 3.2.2(3) and $f_\lambda^n(x) \to \infty$ for all $x \in \mathbb{R}$ as $n \to \infty$ by Remark 3.2.2(3). Therefore, it is concluded that the Fatou set $\mathcal{F}(f_\lambda)$ is empty set or possibly contains wandering domains for $\lambda > \lambda^*$.

$\square$

Corollary 3.3.2. *Let $f_\lambda(z) = \lambda f(z)$ for $z \in \mathbb{C}$ where $f \in \mathrm{E}_1$ and $\lambda > 0$. Then,*

1. *For $0 < \lambda < \lambda^*$, the Julia set of f_λ is the complement of the union of the basin of attraction of the real attracting fixed point a_λ and possibly wandering domains.*

2. *For $\lambda = \lambda^*$, the Julia set of f_λ is the complement of the union of the parabolic basin corresponding to the real rationally indifferent fixed point x^* and possibly wandering domains.*

3. *For $\lambda > \lambda^*$, the Julia set of f_λ is equal to $\widehat{\mathbb{C}}$ or complement of wandering domains.*

Remark 3.3.2. *For each $f \in \mathrm{E}_1$, it follows from Corollary 3.3.2 that if f_λ does not contain any wandering domain then the Julia set of f_λ is a nowhere dense subset of $\widehat{\mathbb{C}}$ for $0 < \lambda \leq \lambda^*$. If the parameter λ exceeds the value λ^*, then the Julia set of f_λ explodes and becomes equal to the extended complex plane. Thus, the chaotic burst occurs in the dynamics of the functions in the one parameter family $\mathcal{S}$ for $f \in \mathrm{E}_1$ at the parameter value $\lambda = \lambda^*$ whenever f_λ does not have any wandering domain for $\lambda > 0$.*

Remark 3.3.3. *Observe that, if f is an even function (i.e., $f(-z) = f(z)$ for all z) in the class E, then $f_\lambda \equiv \lambda f$ and $f_{-\lambda} \equiv -\lambda f$ are conformally conjugate for every nonzero real parameter λ with the conjugating map $\psi(z) = -z$; and the dynamics of f_λ and $f_{-\lambda}$ are same. Therefore, if f is an even function in E, then dynamics of functions in $\{f_\lambda \,:\, \lambda < 0\}$ follows from Theorems 3.3.3 and 3.3.4.*

3.3.1 Connected Fatou set

If f is an entire transcendental function, then each pre-periodic component of the Fatou set of f is simply connected. For the functions f in E, we show in the following that the Fatou set of f is (simply) connected in certain cases.

Lemma 3.3.1. *Let f be a transcendental entire function and the set of all singular values of f be contained in a bounded Jordan domain D containing 0 and with smooth boundary. Then each component of $f^{-1}(D^c)$ is a simply connected domain whose boundary is a single non-closed analytic curve in $\mathbb{C}$ both ends of which tend to ∞.*

Proof. It is known that, if $D_{\frac{r}{2}}(0) = \{z : |z| < \frac{r}{2}\}$ contains all the singular values of a transcendental entire function f, then every component of $f^{-1}(\mathbb{C} \setminus D_r(0))$ is a simply connected domain bounded by a single non-closed analytic curve in $\mathbb{C}$ both ends of which tend to ∞ [61]. Since D is homeomorphic to $D_r(0)$, the same proof works. □

Theorem 3.3.5. *Let $f_\lambda \equiv \lambda f$ where $f \in \mathrm{E}$ and $\lambda > 0$. Suppose that the Fatou set of f_λ is a basin of attraction of an attracting fixed point a_λ. If all the singular values of f_λ are in the immediate basin of attraction of a_λ, then the Fatou set of f_λ is connected and each maximally connected subset of $\mathcal{J}(f_\lambda) \setminus \{\infty\}$ is unbounded.*

Proof. Let $I(a_\lambda)$ be the immediate basin of attraction of the attracting fixed point a_λ and $D \subset I(a_\lambda)$ be a Jordan domain with smooth boundary containing all the singular values of f_λ and 0 such that $f_\lambda(D) \subset D$. Then, $f_\lambda^{-1}(D^c)$ does not intersect D and $f_\lambda^{-1}(D) = \mathbb{C} \setminus \overline{f_\lambda^{-1}(D^c)}$. Therefore, $D \subset f_\lambda^{-1}(D)$. By Lemma 3.3.1, each component of $f_\lambda^{-1}(D^c)$ is a simply connected domain whose boundary is a single non-closed analytic curve in $\mathbb{C}$ both ends of which tend to ∞. In other words, $\overline{f_\lambda^{-1}(D^c)}$ is connected and its boundary has no self intersections in $\mathbb{C}$ which means that $f_\lambda^{-1}(D)$ is (simply) connected. Since each component of $f_\lambda^{-1}(I(a_\lambda))$ intersects $f_\lambda^{-1}(D)$, $f_\lambda^{-1}(I(a_\lambda))$ is connected. Further, $a_\lambda \in f_\lambda^{-1}(I(a_\lambda)) \bigcap I(a_\lambda)$ implies that $I(a_\lambda)$ is backward invariant. Therefore, the Fatou set $\mathcal{F}(f_\lambda)$ is equal to $I(a_\lambda)$ and is connected. Since the Fatou set of f_λ is simply connected, each maximally connected subset of $\mathcal{J}(f_\lambda) \setminus \{\infty\}$ is unbounded. □

Remark 3.3.4. *Theorem 3.3.5 does not guarantee the existence of a curve in the Julia set.*

3.4 Examples

In the present section, the dynamics of some interesting functions f in the class E are described.

First, we prove a proposition on the number of (finite) asymptotic values of f that is needed for our discussion.

Proposition 3.4.1. *Let f be an entire transcendental function of order (growth) one and $\tilde{w}$ be a finite asymptotic value of f. If all the critical values of f have only one limit point in $\mathbb{C}$ and that is equal to $\tilde{w}$, then $\tilde{w}$ is the only finite asymptotic value of f.*

Proof. Suppose that w^* is a finite asymptotic value of f with $w^* \neq \tilde{w}$. It is well known that an entire function of finite order (growth) ρ has at most 2ρ finite asymptotic values. Since f has order (growth) one, the function f has exactly two finite asymptotic values. If an entire function of finite order (growth) ρ has 2ρ finite asymptotic values, then none is direct (Page 307, [101]). Therefore, the asymptotic values $\tilde{w}$ and w^* are indirect singularities of f^{-1}. Now, by Theorem 1 in [23], it follows that any indirect singularity of f^{-1} must be a limit point of critical values of f. By hypothesis, all the critical values of f has only one finite limit point $\tilde{w}$ and hence $\tilde{w} = w^*$ which is a contradiction. Hence, it is concluded that $\tilde{w}$ is the only finite asymptotic value of f. $\square$

For $n = 0, 1, 2, \cdots$, define

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{2k+n} \qquad \text{for } z \in \mathbb{C}.$$

The entire transcendental function $J_n(z)$ is known as the Bessel function of first kind and order n. Let

$$I_n(z) = \frac{J_n(iz)}{i^n} \qquad \text{for } z \in \mathbb{C}\ .$$

Then, the function $I_n(z)$ is called the modified Bessel function of first kind and order n.

The orders (growth) of $J_n(z)$, $I_n(z)$ and $z^{-n}I_n(z)$ are computed for each non-negative integer n in the following proposition. Let $\rho(f)$ denote the order of the entire function f.

Proposition 3.4.2. *For each non-negative integer n, the orders (growth) of $J_n(z)$, $I_n(z)$ and $z^{-n}I_n(z)$ are equal to one.*

Proof. The Bessel functions J_n satisfies the recurrence relation $J_{n+1}(z) = \frac{n}{z}J_n(z) - J_n'(z)$ for $n = 0,\ 1,\ 2,\ \ldots$ (page-93, [32]). Since for any two entire functions g and h, $\rho(g \pm h) = \rho(\frac{g}{h}) = max\{\rho(g), \rho(h)\}$ provided $g \neq h$ (c.f. Theorem 1.1.33) and the order (growth) of g is equal to the order (growth) of derivative of g (c.f. Theorem 1.1.32), the order (growth) of J_{n+1} is equal to the order (growth) of J_n for $n = 0,\ 1,\ 2, \ldots.$ Further, it is observed that the order (growth) of $J_n(z),\ I_n(z)$ and $z^{-n}I_n(z)$ are equal for $n = 0,\ 1,\ 2, \ldots.$ Therefore, it is enough to show that the order of J_0 is 1.

The Taylor series of $J_0(z)$ about the point $z = 0$ is given by $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,k!}\left(\frac{z}{2}\right)^{2k}$ for $z \in \mathbb{C}$. Using the characterization of order of entire functions in terms of coefficients of the Taylor series [71] and Stirling formula $k! = k^k e^{-k}\sqrt{2\pi k}$ for large enough k, it is easy to see that the order of J_0 is 1. $\square$

3.4.1 Example I: $B_n(z) = z^{-n}I_n(z)$, $n \geq 0$

For $n = 0,\ 1,\ 2,\ 3,\ \cdots$, let

$$z^{-n}I_n(z) = \frac{z^{-n}J_n(iz)}{i^n} = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k+n}\ k!\,(k+n)!} \tag{3.3}$$

for $z \in \mathbb{C}$. The following proposition locates all the singular values of $z^{-n}I_n(z)$.

Proposition 3.4.3. *For any non-negative integer n, the function $B_n(z) = z^{-n}I_n(z)$ has infinitely many singular values and all of them lie in $[\frac{-1}{2^n n!},\ \frac{1}{2^n n!}]$.*

Proof. Observe that $B_n'(z) = \frac{d}{dz}\{\frac{J_n(iz)}{(iz)^n}\} = \frac{-iJ_{n+1}(iz)}{(iz)^n}$. Since the set of all zeros of $J_{n+1}(z)$ is an unbounded subset of $\mathbb{R}$, the set of all critical points of $B_n(z)$ (that are the zeros of

$J_{n+1}(iz))$ is an unbounded subset of the imaginary axis. Let all the critical points whose imaginary part is positive be arranged in an increasing sequence, say $\{z_k = ix_k\}_{k>0}$. Then $B_n(z_k) = B_n(ix_k) = \frac{J_n(-x_k)}{(-x_k)^n} \neq 0$ for each k because the roots of J_n and J_{n+1} interlace and $J_{n+1}(-x_k) = \pm J_{n+1}(x_k) = 0$ for each k. As $B_n(iy) = \frac{J_n(-y)}{(-y)^n} \to 0$ as $y \to +\infty$, $\lim_{k\to\infty} B_n(z_k) = 0$. Since B_n is an even function, $-z_k$ is also a critical point of B_n for each k and $\lim_{k\to\infty} B_n(-z_k) = 0$. It shows that $B_n(z_k)$ are distinct for infinitely many values of k. Therefore $B_n(z)$ has infinitely many critical values. Since $B_n(iy) \to 0$ as $|y| \to \infty$, the point $z = 0$ is an asymptotic value of $B_n(z)$. It also follows that 0 is the only limit point of all the critical values of $B_n(z)$. By Proposition 3.4.1, the point $z = 0$ is the only finite asymptotic value of B_n.

If $z_k = ix_k$ is a critical point of $B_n(z)$ then

$$\frac{n}{x_k^n} J_n(x_k) = \frac{J_{n+1}(x_k) + J_n'(x_k)}{x_k^{n-1}} = \frac{J_n'(x_k)}{x_k^{n-1}}$$

by using the recurrence relation $\frac{n}{x} J_n(x) = J_{n+1}(x) + J_n'(x)$ (Page 93, [32]) and the fact that $J_{n+1}(x_k) = 0$. Again by recurrence relation $\frac{n}{x} J_n(x) = J_{n-1}(x) - J_n'(x)$, we get

$$\frac{n}{x_k^n} J_n(x_k) = \frac{J_{n-1}(x_k) - \frac{n}{x_k} J_n(x_k)}{x_k^{n-1}} = \frac{J_{n-1}(x_k)}{x_k^{n-1}} - \frac{n}{x_k^n} J_n(x_k).$$

which gives that

$$\frac{J_n(x_k)}{x_k^n} = \frac{1}{2n} \frac{J_{n-1}(x_k)}{x_k^{n-1}}$$

for each $n = 1, 2, 3, ...$, and hence

$$\frac{J_n(x_k)}{x_k^n} = \frac{1}{2n\ 2(n-1)...2} \frac{J_0(x_k)}{x_k^0} = \frac{1}{2^n\ n!} J_0(x_k).$$

Since $|J_0(x)| \leq 1$ for $x \in \mathbb{R}$, $|\frac{J_n(x_k)}{x_k^n}| \leq \frac{1}{2^n\ n!}$. Thus, all the critical values lies in $[\frac{-1}{2^n n!},\ \frac{1}{2^n n!}]$.

Therefore all the singular values of $B_n(z)$ are in $[\frac{-1}{2^n n!},\ \frac{1}{2^n n!}]$. $\square$

From Equation (3.3), it is obvious that all the coefficients in the Taylor series of $B_n(z) = z^{-n} I_n(z)$ about origin are positive, $B_n(x) > 0$ for all $x < 0$ and $B_n(0) = \frac{1}{2^n n!}$. This fact along with Proposition 3.4.3 shows that the function $B_n(z) = z^{-n}I_n(z)$ is in E_1 for each non-negative integer n.

For a fixed non-negative integer n, the dynamics of functions in the one parameter family $\{\lambda B_n(z) = \lambda z^{-n} I_n(z) \ : \ \lambda > 0\}$ follows from Theorem 3.3.4. Thus, there is a critical parameter λ_n^* for this family such that $\mathcal{F}(\lambda B_n)$ is the union of the basin of attraction of a real attracting fixed point and possibly wandering domains for $0 < \lambda < \lambda_n^*$ and is the union of the parabolic basin corresponding to a real rationally indifferent fixed point and possibly wandering domains for $\lambda = \lambda_n^*$. The Fatou set of λB_n is empty or contains possibly wandering domains when $\lambda > \lambda_n^*$. If a_λ denotes the attracting fixed point of λB_n for $0 < \lambda < \lambda_n^*$, then $[0,\ a_\lambda]$ is in the immediate basin of attraction of a_λ by Theorem 3.2.2. As $B_n(z)$ is an even function, it follows that $[-a_\lambda, 0]$ is in the immediate basin of attraction of a_λ. Since λB_n is increasing in $\mathbb{R}^+$ and $0 < a_\lambda$, it follows that $B_n(0) = \frac{1}{2^n n!} < a_\lambda$. By Proposition 3.4.3, all the singular values of λB_n are in $[\frac{-1}{2^n n!},\ \frac{1}{2^n n!}]$ and hence in the immediate basin of attraction of a_λ. By Theorem 3.3.5, $\mathcal{F}(\lambda B_n)$ is connected for $0 < \lambda < \lambda_n^*$ and hence the Fatou set is the basin of attraction. In other words, $\mathcal{J}(\lambda B_n)$ is a nowhere dense subset of the complex plane for $0 < \lambda < \lambda_n^*$ and is equal to the entire complex plane or possibly the complement of wandering domains for $\lambda > \lambda_n^*$. Here a similar phenomena as *chaotic burst* occurs at the parameter value $\lambda = \lambda_n^*$ in the dynamics of $\{\lambda B_n(z) = z^{-n} I_n(z) \ : \ \lambda > 0\}$.

Since B_n is an even function, dynamics of functions in $\{\lambda B_n \ : \ \lambda < 0\}$ follows from Remark 3.3.3.

For $n = 0$, the dynamics of functions in the family $\{\lambda I_0(z) \ : \ \lambda > 0\}$ is studied in [77].

For $n = 1$, it is numerically found that the critical parameter λ_1^* for the family $\{\lambda z^{-1} I_1(z) \ : \ \lambda > 0\}$ is approximately equal to 2.529. The pictures of the Julia sets of $2.5z^{-1}I_1(z)$ and $2.5291z^{-1}I_1(z)$ are computationally generated using an algorithm (Refer [77]) based on Corollary 3.3.2 and are given in Figures 3.1 and 3.2.

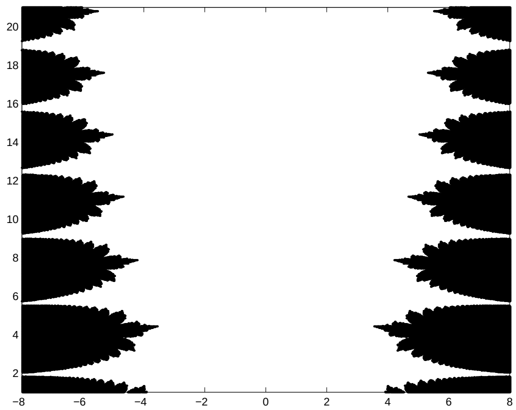


Figure 3.1: Julia set of $2.5\frac{I_1(z)}{z}$

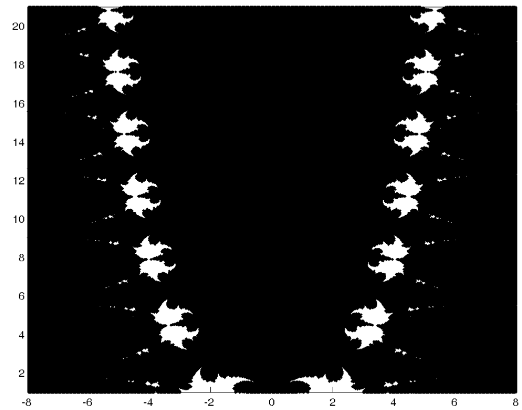


Figure 3.2: Julia set of $2.5291\frac{I_1(z)}{z}$

3.4.2 Example II: $I_{2n}(z)$, $n > 0$

For $n = 1, 2, 3, \cdots$, $I_{2n}(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,(k+2n)!}\left(\frac{z}{2}\right)^{2k+2n}$ for $z \in \mathbb{C}$. All the coefficients in the Taylor series of I_{2n} about origin are positive. Further, $I_{2n}(x) > 0$ for all $x < 0$ and, $I_{2n}(0) = 0$. Note that $I'_{2n}(z) = i^{-2n+1}J'_{2n}(iz)$. Observe that the order of $J_{2n}(z)$ is one. The function $J_{2n}(z)$ is real for real z and it has only real zeros. By Laguerre Theorem, all the roots of $J'_{2n}(z)$ are real and separated from each other by zeros of $J_{2n}(z)$. Thus, all the critical points of $I_{2n}(z)$ are purely imaginary and are separated by the zeros of $I_{2n}(z)$. Observe that, the number of critical points is infinite and they form an unbounded subset of the imaginary axis. Further, 0 is not a critical value and $I_{2n}(z) \to 0$ when $z \to \infty$ along positive (or negative) imaginary axis. Therefore, 0 is an asymptotic value. By the same argument used in Proposition 3.4.3, it follows that 0 is the only limit point of all critical values of $I_{2n}(z)$. By Proposition 3.4.1, the point $z = 0$ becomes the only finite asymptotic value of I_{2n}. Thus, $I_{2n} \in \mathrm{E}_0$ for $n > 0$. The Fatou set $\mathcal{F}(\lambda I_{2n}(z))$ is the union of the basin of attraction of the superattracting fixed point 0 and possibly wandering domains for all $\lambda > 0$. It is easy to see that, if λ is sufficiently large then some critical values of $\lambda I_{2n}(z)$, $n > 0$ are not in the immediate basin of attraction of the superattracting fixed point 0. The pictures of the Julia sets of $I_2(z)$ and $5I_2(z)$ are computationally generated and are given in Figures 3.3 and 3.4.

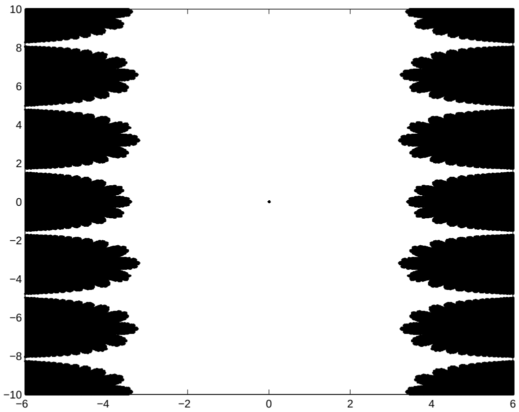


Figure 3.3: Julia set of $I_2(z)$

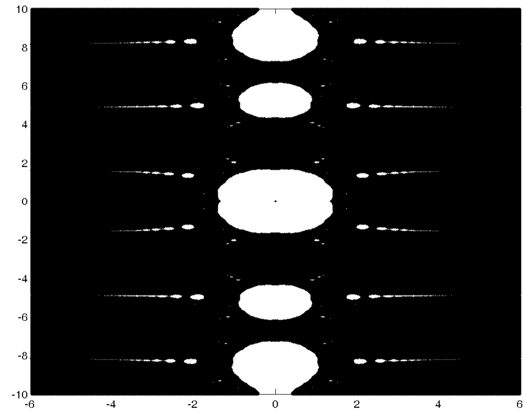


Figure 3.4: Julia set of $5I_2(z)$

3.4.3 Example III: $S_{m,n}(z) = \dfrac{\sinh^m z}{z^n}$, $m \geq n > 0$

The Taylor series of sinh z about origin is given by $\displaystyle\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$. For $m, n \in \mathbb{N}$ and $m \geq n$, all the coefficients of the Taylor series of $S_{m,n}(z) = \dfrac{\sinh^m z}{z^n}$ are positive. Let both m and n be even or both of them be odd. Then $S_{m,n}(x) > 0$ for $x < 0$.

Note that $S'_{m,n}(z) = \dfrac{\sinh^{m-1} z(mz \cosh z - n \sinh z)}{z^{n+1}}$. Since all the solutions of $\alpha \tan z = z$ are real for $0 < \alpha \leq 1$, it follows that $\dfrac{n}{m} \tanh z = z$ has only purely imaginary solutions when $\frac{n}{m} \leq 1$ and these solutions form an unbounded set. Further, the solutions of $\frac{\sinh^{m-1} z}{z^{n+1}} = 0$ are purely imaginary. Therefore, all the critical points of $S_{m,n}(z)$ are purely imaginary. Note that, $S_{m,n}$ takes the imaginary axis to a bounded interval in the real axis. So all the critical values of $S_{m,n}$ are in a bounded interval of the real axis. The function $S_{m,n}(z)$ tends to 0 as $z \to \infty$ along either positive or negative imaginary axis. So the point $z = 0$ is an asymptotic value of $S_{m,n}$ and, same argument used in Proposition 3.4.3 gives that it is the only limit point of all the critical values of $S_{m,n}$. By Proposition 3.4.1, the point $z = 0$ is the only finite asymptotic value of $S_{m,n}$. Note that $S_{m,n}(0) = 1$ for $n = m$ and $S_{m,n}(0) = 0$ for $m > n$. Thus, $\dfrac{\sinh^m z}{z^n} \in \mathrm{E}_0$ for $m > n$ and $\dfrac{\sinh^m z}{z^n} \in \mathrm{E}_1$ for

$m = n$ when $m,\ n \in \mathbb{N},\ m \geq n$ and both m and n are even or both of them are odd. The dynamics of the functions in $\{\lambda S_{m,n} \ : \ \lambda > 0\}$ follows from Theorem 3.3.3 when $m > n$ and from Theorem 3.3.4 when $m = n$. Thus, the Fatou set of $\lambda S_{m,n}$ is the union of a basin of attraction of the superattracting fixed point 0 and possibly wandering domains when $m > n$ and $\lambda > 0$. However, for $m = n$, the functions in the one parameter family $\{\lambda S_{m,m} \ : \ \lambda > 0\}$ exhibit a phenomena similar to *chaotic burst* as follows. There is a critical parameter, say λ_m^* such that $\mathcal{F}(S_{m,m})$ is the union of the basin of attraction of a real attracting fixed point and possibly wandering domains for $0 < \lambda < \lambda_m^*$ and is the union of the parabolic basin corresponding to a real rationally indifferent fixed point and possibly wandering domains for $\lambda = \lambda_m^*$. The Fatou set of $\lambda S_{m,m}$ is empty or possibly contains wandering domains for $\lambda > \lambda_m^*$.

The case $m = n = 1$ is studied in [106].

For $m = n = 2$, the critical parameter λ_2^* is found to be approximately equal to 0.7618. The computationally generated pictures of the Julia sets of $0.75\frac{\sinh^2 z}{z^2}$ and $0.7619\frac{\sinh^2 z}{z^2}$ are given in Figures 3.5 and 3.6.

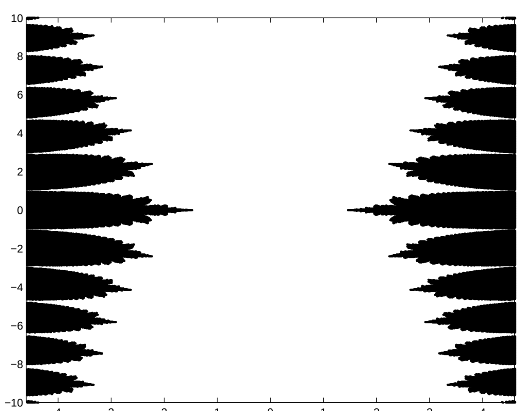


Figure 3.5: Julia set of $0.75\frac{\sinh^2 z}{z^2}$

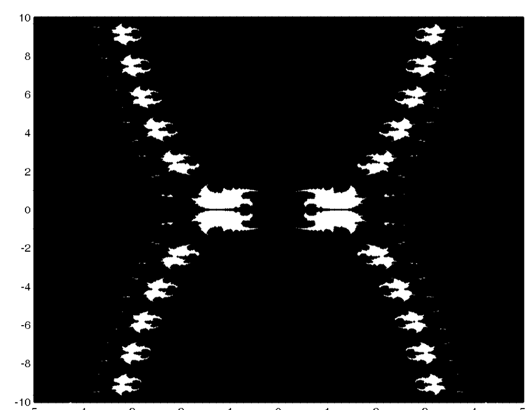


Figure 3.6: Julia set of $0.7619\frac{\sinh^2 z}{z^2}$

3.4.4 Example IV: $P(f)$ where $f \in \mathrm{E}$

Let $P(z) = (z + a_1)^{m_1}(z + a_2)^{m_2}...(z + a_n)^{m_n}$ be a non-constant polynomial where a_1, a_2, $\cdots$, a_n are positive real numbers and m_1, m_2, $\cdots$, m_n are non-negative integers. Then, it follows from Proposition 3.1.1 (3) that the functions $\Phi = P \circ f \in \mathrm{E}_1$ and $\Psi = h \circ P \in \mathrm{E}_1$ for $f \in \mathrm{E}$ and $h \in \mathrm{E}_1$.

Let $P(z) = (z+1)(z+2)$ and $f(z) = I_0(z)$. Set $\Phi(z) = P(f(z)) = (I_0(z)+1)(I_0(z)+2)$ for $z \in \mathbb{C}$. Then, there exists a critical parameter $\lambda^* \approx 0.155$ such that the Fatou set $\mathcal{F}(\lambda\Phi)$ is the union of the attracting basin of a real fixed point and possibly wandering domains for $0 < \lambda < \lambda^*$, is the union of the parabolic basin corresponding to a real rationally indifferent fixed point and possibly wandering domains for $\lambda = \lambda^*$ and is an empty set or possibly contains wandering domains for $\lambda > \lambda^*$. Thus, a sudden change occurs in the one parameter family $\{\lambda\Phi \ : \ \lambda > 0\}$ where $\Phi(z) = P(f(z)) = (I_0(z) + 1)(I_0(z) + 2)$ at the parameter value $\lambda = \lambda^* \approx 0.155$. The computationally generated pictures of the Julia sets of $0.15\Phi(z)$ and $0.1556\Phi(z)$ are given in Figures 3.7 and 3.8.

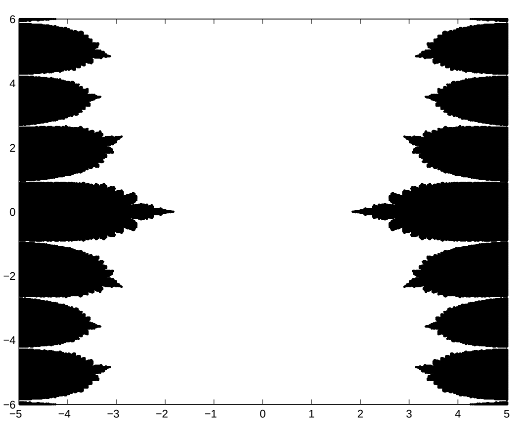


Figure 3.7: Julia set of $0.15(I_0(z) + 1)(I_0(z) + 2)$

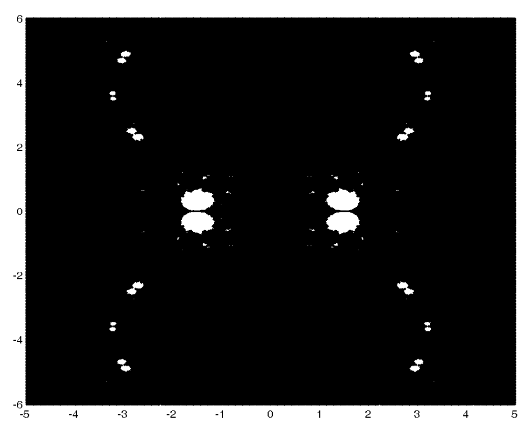


Figure 3.8: Julia set of $0.1556(I_0(z) + 1)(I_0(z) + 2)$

3.4.5 Example V: e^{bz+ce^z}, $b \in \mathbb{N}$ and $c > 0$

Let $f(z) = e^{bz+ce^z}$ for $z \in \mathbb{C}$ where $b \in \mathbb{N}$ and $c > 0$. All the coefficients of the Taylor series of the functions e^z and $bz + ce^z$ are non-negative which in turn gives that all the coefficients of the Taylor series of $f(z)$ are non-negative. Clearly, $f(x) > 0$ for all $x \in \mathbb{R}$. The function f has only one finite asymptotic value 0 and the only critical value is $e^{b \ln|\frac{-b}{c}|-c}$ if b is even and $-e^{b \ln|\frac{-b}{c}|-c}$ if b is odd. Therefore, $f \in \mathrm{E}_1$ and chaotic burst in the dynamics of functions in the one parameter family $\{\lambda f \ : \ \lambda > 0\}$ occurs at a critical parameter λ^* (depending on f) by Theorem 3.3.4.

Chapter 4

Dynamics of certain meromorphic functions of bounded type

In this chapter, we define a class of transcendental meromorphic functions and prove chaotic burst in the one parameter family $\{\lambda f \ : \ \lambda > 0\}$ for each f in the class.

Let

$$\mathcal{E} \equiv \left\{ h(z) \ : \ \begin{array}{l} (i) \ h(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n \text{ for } z \in \mathbb{C} \text{ where } a_n \geq 0 \text{ for all } n > 0 \\ \text{(ii) } a_0 = h(0) \geq 1 \\ \text{(iii) } h(x) > 0 \text{ for all } x < 0 \\ \text{(iv) The set } \overline{S_h} \text{ is a bounded subset of } \mathcal{C}^* \bigcup \mathbb{R}^* \end{array} \right\}$$

where $\overline{S_h}$ is the closure of the set of singular values of h, $\mathcal{C}^* = \{z \in \mathbb{C} \ : \ |z| = 1 \text{ and } z \neq \pm i\}$ and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. It is important to note that the class E defined in Chapter 3 and the class $\mathcal{E}$ defined above have non-empty intersection. For instance, $\frac{\sinh z}{z} + 1$ and $e^z + 1$ are in $\mathcal{E} \bigcap \mathrm{E}$. Define

$$\mathcal{M} \equiv \{f(z) = J^n(h(z)) \text{ for } z \in \mathbb{C} \ : \ \ n \in \mathbb{N} \text{ and } h \in \mathcal{E}\}$$

where J^n denotes the n-times composition of the Joukowski function $J(z) = z + \frac{1}{z}$. For $f \in \mathcal{M}$, the dynamics of functions in the one parameter family

$$\mathcal{S} \equiv \{f_\lambda(z) = \lambda f(z) \ : \ \lambda > 0\}$$

is investigated in this chapter.

The chaotic burst in the one parameter family $\mathcal{S}$ at certain critical parameter (depending on f) is proved. The function $J(e^z+1)$ is an example from the class $\mathcal{M}$ and, the dynamics of functions in the one parameter family $\{\lambda(e^z+1+\frac{1}{e^z+1})\ :\ \lambda>0\}$ is discussed in detail and some additional properties of the Julia set are proved. For instance, it is established that, whenever the Julia set of $\lambda(e^z+1+\frac{1}{e^z+1})$ is not equal to $\widehat{\mathbb{C}}$, it contains singleton components, bounded (but not singleton) components and unbounded components. Further, it can be written as union of two completely invariant sets one of which is totally disconnected.

4.1 Properties of f_λ

Some basic properties of $f_\lambda \in \mathcal{S}$ are proved in this section which are required for determining the dynamics of f_λ. Before that, we make some useful observations on the function $z+\frac{1}{z}$.

The Joukowski function $J(z) = z+\frac{1}{z}$ is an odd meromorphic function having only one pole at $z=0$ and two critical values 2 and -2 in $\mathbb{C}$. On $\mathbb{R}^+ = \{x \in \mathbb{R}\ :\ x>0\}$, the map $J(x)$ decreases in $(0,1)$, attains its minimum at $x=1$, increases in $(1,\ \infty)$ and $J(x) \to +\infty$ as $x \to +\infty$. For each $x \in \mathbb{R}^+$, $\{J^n(x)\}_{n>0}$ is an increasing sequence tending to ∞ as $n \to \infty$. Observe that $J(z)$ has only one fixed point ∞ in $\widehat{\mathbb{C}}$ which is rationally indifferent and, the two singular (critical) values are in the parabolic domain corresponding to ∞. Therefore, the Fatou set of $J(z)$ is the parabolic basin corresponding to ∞. A detailed treatment of dynamics of $J(z)$ appears in [18].

For $f \in \mathcal{M}$, there is a natural number n such that $f(z) = J^n(h(z))$ for $z \in \mathbb{C}$ where $h \in \mathcal{E}$. Note that $J^{-1}(\infty) = \{0,\ \infty\}$, $J^{-2}(\infty) = \{0,\ \infty,\ i,\ -i\}$ and $\Re(J(z)) = x(1+\frac{1}{x^2+y^2}) = 0$ if and only if $\Re(z) = x = 0$ for each nonzero $z = x+iy$. Set $\mathcal{P}_n = \{z \in \widehat{\mathbb{C}}\ :\ J^n(z) = \infty\}$ for $n \in \mathbb{N}$. Then, $\infty \in \mathcal{P}_n$ for all $n \in \mathbb{N}$ because ∞ is a fixed point of $J(z)$ and all the finite elements of $\mathcal{P}_n$ are on the imaginary axis. For $n=1$, $\mathcal{P}_1 = \{0,\ \infty\}$ and the poles of $\lambda J(h(z))$ are precisely the zeros of $h(z)$ as h is entire. The function $\lambda J(h(z))$ is entire if 0 is an exceptional value of $h(z)$. For $n>1$, the set $\mathcal{P}_n$ contains more than one point in $\mathbb{C}$. The function $h(z)$ is entire for which it takes each finite complex value infinitely often except possibly one. Therefore, the function $f_\lambda(z) = \lambda J^n(h(z))$ is meromorphic with infinitely many poles for each $n > 1$. Further, all the poles of $f_\lambda(z)$ are in $\{z \in \mathbb{C} \ : \ \Re(h(z)) = 0\}$ for $n > 1$. In Proposition 4.1.1, it is shown that the set $S_p(f_\lambda)$ is bounded for each $f_\lambda \in \mathcal{S}$ and $p \in \mathbb{N}$.

Proposition 4.1.1. *Let $f_\lambda \in \mathcal{S}$. Then, the set $S_p(f_\lambda)$ is bounded for each natural number p. In particular, $S_1(f_\lambda) = S_{f_\lambda}$ is a bounded subset of $\mathbb{R}^*$.*

Proof. A critical point of $f_\lambda(z)$ is a critical point of $h(z)$ or a solution of $J^j(h(z)) = \pm 1$ for $j = 0,\ 1,\ 2,\ \cdots,\ n-1$ since $f'_\lambda(z) = \lambda h'(z) \prod_{j=0}^{n-1} \left\{1 - \frac{1}{(J^j(h(z)))^2}\right\}$. Let $\tilde{z}$ be a critical point of $h(z)$. Then $h(\tilde{z})$ is a critical value of $h(z)$ and $\lambda J^n(h(\tilde{z}))$ is a critical value of $f_\lambda(z)$. Thus $\lambda J^n(h(\tilde{z})) \in \lambda J^n(S_h)$ where S_h is the set of all singular values of $h(z)$. Now, let $\hat{z}$ be a solution of $J^j(h(z)) = \pm 1$ for j, $0 \leq j \leq n-1$. Then $\lambda J^{n-j}(J^j(h(\hat{z}))) = \lambda J^{n-j}(\pm 1)$ is a critical value of $f_\lambda(z)$. In this way, $\lambda J^{n-j}(\pm 1)$ is a critical value of f_λ for each $j \in \{0,\ 1,\ ...,\ n-1\}$. Since $\lambda J^n(z)$ has no finite asymptotic value, any finite asymptotic value a of $f_\lambda(z)$ is of the form $\lambda J^n(\tilde{w})$ where $\tilde{w}$ is a finite asymptotic value of $h(z)$. Therefore, $a \in \lambda J^n(S_h)$ and $S_{f_\lambda} \subset \lambda J^n(S_h) \bigcup \{\lambda J^{n-j}(\pm 1) \ : \ 0 \leq j \leq n-1\}$. Since $\overline{S_h}$ is a bounded subset of $\{z : \ |z| = 1 \text{ and } z \neq \pm i\} \bigcup \mathbb{R}^*$, $\lambda J^n(S_h)$ is a bounded subset of $\mathbb{R}^*$ and hence S_{f_λ} is a bounded subset of $\mathbb{R}^*$. For any $x \in \mathbb{R}$, $h(x) > 0$ and $\lambda J^n(h(x)) \in \mathbb{R}^*$. In this way, $\{z \in \mathbb{C} \ : \ f_\lambda^n(z) = \infty \text{ for some } n \in \mathbb{N}\} \bigcap \mathbb{R} = \emptyset$ and we conclude that $S_p(f_\lambda) = \bigcup_{k=0}^{p-1} f_\lambda^k(S_{f_\lambda} \setminus A_k(f_\lambda)) = \bigcup_{k=0}^{p-1} f_\lambda^k(S_{f_\lambda})$ (c.f. Equation (1.1)) is bounded for each p. $\square$

Since the singular values of λf and f are multiples of each other, it is clear from Proposition 4.1.1 that the function $f \in \mathcal{M}$ is of bounded type. But, the number of singular values of f may be finite or infinite. For example, $f(z) = J(\frac{\sinh z}{z} + 1)$ is in $\mathcal{M}$ and the set of singular values of f is infinite. At the same time, the class $\mathcal{M}$ contains plenty of functions having finite number of singular values as suggested by the next proposition before which we prove a lemma.

Lemma 4.1.1. *Let $m \in \mathbb{N}$ and a be a positive real number. Then, the function $J^m(e^z+a)$ has $2m+1$ singular values for $a \neq 1$ and the function $J^m(e^z+a)$ has $2m$ singular values for $a=1$.*

Proof. Any finite asymptotic value of f is either a finite asymptotic value of $J^m(z)$ or a J^m-image of a finite asymptotic value of e^z+a. The function $J^m(z)$ has no finite asymptotic value and a is the only finite asymptotic value of e^z+a. Hence, $J^m(a)$ is the only finite asymptotic value of $f(z)=J^m(e^z+a)$.

Set $J^0(z)=z$. Note that $f'(z)=\prod_{j=0}^{m-1} J'(J^j(e^z+a))e^z=\prod_{j=0}^{m-1}\left\{1-\frac{1}{(J^j(e^z+a))^2}\right\}e^z$ and the critical points of $f(z)$ are precisely the solutions of $J^j(e^z+a)=1$ or $J^j(e^z+a)=-1$ for $j \in \{0,\ 1,\ 2,\ \cdots,\ m-1\}$. There are two distinct roots of $J(z)=1$ namely, $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$ which gives by inductive argument that, $J^j(z)=J(J^{j-1}(z))=1$ has at least two distinct solutions for each $j>0$. The entire function $h(z)=e^z+a$ cannot omit two values in $\mathbb{C}$. This implies that $J^j(e^z+a)=1$ has at least one solution for each j, $j=1,\ 2,\ 3,\cdots,\ m-1$. Let j be fixed and z_j be a solution of $J^j(e^z+a)=1$. Then $f(z_j)=J^m(e^{z_j}+a)=J^{m-j}(J^j(e^{z_j}+a))=J^{m-j}(1)$ is a critical value of $f(z)=J^m(e^z+a)$. Similarly, it follows that $J^j(e^z+a)=-1$ has at least one solution for each j and $J^{m-j}(-1)$ is a critical value of $f(z)$ for each j, $j=1,\ 2,\ 3,\cdots,\ m-1$. For $j=0$, there are two cases. If $a\neq 1$, then each of the equations $J^0(e^z+a)=e^z+a=-1$ and $J^0(e^z+a)=e^z+a=1$ has solutions and the resulting critical values are $J^m(-1)$ and $J^m(1)$. If $a=1$, then the equation $J^0(e^z+a)=e^z+a=-1$ has a solution where as $J^0(e^z+a)=e^z+a=1$ has no solution and the only resulting critical value is $J^m(-1)$. Therefore, the set of all critical values of $J^m(e^z+a), a\neq 1$ is $\{J^{m-j}(1)\ :\ j\in\mathbb{N} \text{ and } 0\le j\le m-1\}\bigcup\{J^{m-j}(-1)\ :\ j\in \mathbb{N} \text{ and } 0\le j\le m-1\}$ and that of $J^m(e^z+1)$ is $\{J^{m-j}(1)\ :\ j\in\mathbb{N}$ and $1\le j\le$

$m-1\} \bigcup \{J^{m-j}(-1) \;:\; j \in \mathbb{N}$ and $0 \leq j \leq m-1\}$. As $J^j(1) \neq J^k(1)$ for $j \neq k$ and $J^m(-1) = -J^m(1)$, it is concluded that the number of critical values of $J^m(e^z + a)$ is $2m$ and $2m-1$ for $a \neq 1$ and $a = 1$ respectively. Therefore, the function $J^m(e^z + a)$ has $2m+1$ singular values for $a \neq 1$ and the function $J^m(e^z + a)$ has $2m$ singular values for $a = 1$. □

Proposition 4.1.2. *For each natural number n, there is a function f in $\mathcal{M}$ such that f has exactly n singular values.*

Proof. Let $h(z) = e^z + a$ for $z \in \mathbb{C}$ where $a > 0$. Observe that all the coefficients in the Taylor series of $h(z)$ about origin are non-negative, $h(0) = 1 + a > 0$, $h(x) > 0$ for $x < 0$ and the set of singular values of h is $\{a\}$. Therefore, $h \in \mathcal{E}$ and $J^n(h) \in \mathcal{M}$ for all $n \in \mathbb{N}$. We assert that the function $J^n(e^z + a)$ has exactly n singular values for suitable values of n and a. Two cases arise.

Case I: n is odd

If n is odd, then consider the function $J^m(e^z + a)$ in $\mathcal{M}$ where $m = \frac{n-1}{2}$ and $a \neq 1$. By Lemma 4.1.1, the function $J^m(e^z + a)$ has $2m + 1 = n$ singular values.

Case II: n is even

If n is even, then consider the function $J^m(e^z + 1)$ in $\mathcal{M}$ where $m = \frac{n}{2}$. By Lemma 4.1.1, the function $J^m(e^z + 1)$ has $2m = n$ singular values. □

Remark 4.1.1. *The set of positive critical values of $J^n(e^z + a)$, $a \neq 1$ is $\{J^j(1) \;:\; j \in \mathbb{N}, 1 \leq j \leq n\}$. Note that $J^k(1) > J^{k-1}(1)$ for all natural number k. Therefore, $J^n(1)$ is the largest positive critical value of $J^n(e^z + a)$. Similarly, $J^n(-1)$ is the smallest negative critical value of $J^n(e^z + 1)$ because $J(-z) = -J(z)$ for all $z \in \mathbb{C}$. Since $J^n(a)$ is the only finite asymptotic value of $J^n(e^z+a)$ it is concluded that, all the singular values of $J^n(e^z+1)$ lie in $[-J^n(1), J^n(1)] \bigcup \{J^n(a)\}$. For $a = 1$, the asymptotic value of $J^n(e^z + 1)$ is $J^n(1)$. Similarly, it follows that, all the singular values of $J^n(e^z+a)$, $a = 1$ lies in $[-J^n(1), J^n(1)]$.*

Now, we make some preliminary observations on the behaviour of f_λ on $\mathbb{R}$ which will be used later in determining the real dynamics of f_λ.

Proposition 4.1.3. *Let $f \in \mathcal{M}$. Then, $f(x)$ for $x \in \mathbb{R}$ satisfies the following.*

1. $f(x) > 0$ *for all* $x \in \mathbb{R}$.

2. $f'(x)$ *and* $f''(x) > 0$ *for all* $x > 0$.

3. $\lim_{x \to +\infty} f(x) - x = \lim_{x \to +\infty} f'(x) = +\infty$.

4. $\lim_{x \to +\infty} f(x) - xf'(x) = -\infty$.

Proof. Let n be the natural number such that $f(x) = J^n(h(x))$ where $h \in \mathcal{E}$.

1. Since $h(x) > 0$ for all $x \in \mathbb{R}$ and $J^n(x) > 0$ for $x > 0$, it follows that $f(x) = J^n(h(x)) > 0$ for $x \in \mathbb{R}$.

2. Observe that
$$f'(x) = h'(x) \prod_{i=0}^{n-1} J'(J^i(h(x))) = h'(x) \prod_{i=0}^{n-1} \left\{1 - \frac{1}{(J^i(h(x)))^2}\right\}$$
and $f''(x) =$
$$h''(x) \prod_{i=0}^{n-1} \left\{1 - \frac{1}{(J^i(h(x)))^2}\right\} + h'(x) \sum_{j=0}^{n-1} 2\frac{\frac{d}{dx} J^j(h(x))}{(J^j(h(x)))^3} \left\{ \prod_{i=0, i\neq j}^{n-1} 1 - \frac{1}{(J^i(h(x)))^2} \right\}.$$
Since $h(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{C}$ where $a_n \geq 0$ for all n, the functions $h(x)$, $h'(x)$ and $h''(x)$ are positive for $x > 0$. The function $h(x)$ is increasing in $(0,\ \infty)$ and $h(0) \geq 1$. This gives that $h(x) > 1$ for $x > 0$ and consequently, $J^i(h(x)) > 1$ for $i \geq 0$ and $x > 0$. Since $J'(x) > 0$ for $x > 1$, it follows that $\frac{d}{dx} J^i(h(x)) = \prod_{k=1}^{i} J'(J^{i-k}(h(x)))h'(x) > 0$ for $x > 0$ and each i. Therefore, $f'(x)$ and $f''(x)$ are positive for $x > 0$.

3. For each $i \in \mathbb{N}$, $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} J^n(h(x)) = \lim_{x\to+\infty} \left(J^{n-1}(h(x)) + \frac{1}{J^{n-1}(h(x))}\right)$ and $\lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty} h'(x) \prod_{i=0}^{n-1} \left\{1 - \frac{1}{(J^i(h(x)))^2}\right\}$. Now $\lim_{x\to+\infty} J^k(h(x)) = +\infty$ for $k = 0, 1, 2, \cdots, n$ which gives that $\lim_{x\to+\infty} f(x) - x = \lim_{x\to+\infty} h(x) - x = +\infty$ and $\lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty} h'(x) = +\infty$.

4. Let $h(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in \mathbb{R}$. Then $h'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $h(x) - xh'(x) = a_0 + (a_1 - a_1)x + (a_2 - 2a_2)x^2 + ...$ and $\lim_{x\to+\infty} f(x) - xf'(x) = \lim_{x\to+\infty} h(x) - xh'(x) = -\infty$.

$\square$

4.2 Dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$

The image of all the singular values of f_λ lie in the real line and $f_\lambda(\mathbb{R}) \subset \mathbb{R}$. Therefore, the dynamics of f_λ on the real line is important for determining the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$. The dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ is studied in this section.

Consider $\phi(x) = f(x) - xf'(x)$ for $x \geq 0$. As $\phi'(x) = -xf''(x) < 0$ for $x > 0$, $\phi(x)$ is decreasing in $\mathbb{R}^+ = \{x \in \mathbb{R} \ : \ x > 0\}$. Observe that $\phi(0) = f(0) > 0$ and from Proposition 4.1.3 that $\lim_{x\to+\infty} \phi(x) = -\infty$. By the continuity of ϕ, there exists unique $x^* > 0$ such that

$$\phi(x) \begin{cases} > 0 & \text{for } 0 < x < x^* \\ = 0 & \text{for } x = x^* \\ < 0 & \text{for } x > x^* \end{cases} \tag{4.1}$$

Define $\lambda^* = \frac{1}{f'(x^*)}$ where x^* is the positive solution of $\phi(x) = 0$. If $f'(0) \neq 0$, it is noted that $\lambda^* < \frac{1}{f'(0)}$ since $\frac{1}{f'(x)}$ is decreasing in $\mathbb{R}^+$ and $x^* > 0$. The value of λ^* depends on n and h where $f(z) = J^n(h(z))$ and $h \in \mathcal{E}$.

Theorem 4.2.1. *Let $f_\lambda \in \mathcal{S}$. Then, f_λ has no real periodic points of period greater than one and*

1. *For* $0 < \lambda < \lambda^*$, f_λ *has only two real fixed points* a_λ *and* r_λ *with* $a_\lambda < r_\lambda$ *such that* a_λ *is attracting and* r_λ *is repelling. Further,* $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ *for* $0 \leq x < r_\lambda$ *and* $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ *for* $x > r_\lambda$.

2. *For* $\lambda = \lambda^*$, f_λ *has a real rationally indifferent fixed point* x^* *where* x^* *is the unique positive solution of* $f(x) - xf'(x) = 0$. *Further,* $\lim_{n\to\infty} f_\lambda^n(x) = x^*$ *for* $0 \leq x < x^*$ *and* $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ *for* $x > x^*$.

3. *For* $\lambda > \lambda^*$, f_λ *has no real fixed points. Further,* $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ *for all* $x \in \mathbb{R}$.

Proof. For $\lambda > 0$, the function $f_\lambda(x)$ is positive for all $x \in \mathbb{R}$. Therefore, any real periodic point of $f_\lambda(x)$ lies in $\mathbb{R}^+$. Since $f_\lambda(x)$ is increasing in $\mathbb{R}^+$, any real periodic point of $f_\lambda(x)$ cannot have prime period greater than one.

Define $g_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$. Suppose that $f'(0) \neq 0$. Since $f'(x)$ is positive for $x > 0$ and is continuous on $\mathbb{R}$, $f'(0) > 0$. Note that $g'_\lambda(0) = f'_\lambda(0) - 1 \geq 0$ for $\lambda \geq \frac{1}{f'(0)}$. Since $g''_\lambda(x) = f''_\lambda(x) > 0$ for $x > 0$, $g'_\lambda(x)$ is increasing in $\mathbb{R}^+$ and $g'_\lambda(x) > g'_\lambda(0) \geq 0$ for all $x > 0$ if $\lambda \geq \frac{1}{f'(0)}$. This gives that the function $g_\lambda(x)$ is strictly increasing on $\mathbb{R}^+$. As $g_\lambda(0) > 0$, $g_\lambda(x)$ has no zeros in $\mathbb{R}^+$. In other words, $f_\lambda(x)$ has no fixed points in $\mathbb{R}$ when $\lambda \geq \frac{1}{f'(0)}$. For $0 < \lambda < \frac{1}{f'(0)}$, $g'_\lambda(0) = f'_\lambda(0) - 1 < 0$. The function $g'_\lambda(x) = f'_\lambda(x) - 1$ is increasing in $\mathbb{R}^+$ and tends to $+\infty$ as x tends to $+\infty$. Therefore, there is a unique $x_\lambda > 0$ such that $g'_\lambda(x) = f'_\lambda(x) - 1 < 0$ for $0 < x < x_\lambda$, $g'_\lambda(x_\lambda) = f'_\lambda(x_\lambda) - 1 = 0$ and $g'_\lambda(x) = f'_\lambda(x) - 1 > 0$ for $x > x_\lambda$.

Now, let us suppose that $f'(0) = 0$. Then $g'_\lambda(0) = f'_\lambda(0) - 1 < 0$ for all $\lambda > 0$. Since the function $g'_\lambda(x) = f'_\lambda(x) - 1$ is increasing in $\mathbb{R}^+$ and tends to $+\infty$ as x tends to $+\infty$, there exist a unique $x_\lambda > 0$ such that $g'_\lambda(x) = f'_\lambda(x) - 1 < 0$ for $0 < x < x_\lambda$, $g'_\lambda(x_\lambda) = f'_\lambda(x_\lambda) - 1 = 0$ and $g'_\lambda(x) = f'_\lambda(x) - 1 > 0$ for $x > x_\lambda$.

When $f'(0) \neq 0$, for each λ with $0 < \lambda < \frac{1}{f'(0)}$ and when $f'(0) = 0$, for each $\lambda > 0$, there exists a positive real number x_λ such that g_λ decreases in $(0,\ x_\lambda)$, attains its minimum at

x_λ and then increases in $(x_\lambda,\ \infty)$. Further, $\lambda = \frac{1}{f'(x_\lambda)}$.

1. If $0 < \lambda < \lambda^*$, then $\frac{1}{f'(x_\lambda)} < \frac{1}{f'(x^*)}$. Since $\frac{1}{f'(x)}$ is strictly decreasing in $\mathbb{R}^+$, $x_\lambda > x^*$. It follows from Equation (4.1) that $\phi(x_\lambda) < 0$ which gives that $g_\lambda(x_\lambda) = f_\lambda(x_\lambda) - x_\lambda < 0$. As $\lim_{x\to+\infty} g_\lambda(x) = +\infty$ and $g_\lambda(0) > 0$, there are exactly two points, say a_λ and r_λ with $0 < a_\lambda < x_\lambda < r_\lambda$ such that $g_\lambda(a_\lambda) = g_\lambda(r_\lambda) = 0$. The points a_λ and r_λ are the real fixed points of $f_\lambda(x)$. Obviously, $0 < f_\lambda'(a_\lambda) < f_\lambda'(x_\lambda) = 1$ and $f_\lambda'(r_\lambda) > f_\lambda'(x_\lambda) = 1$. Therefore, a_λ is attracting and r_λ is repelling. Note that $f_\lambda(x) > x$ for $0 \le x < a_\lambda$ and $f_\lambda(x) < x$ for $a_\lambda < x < r_\lambda$. Since $f_\lambda(x)$ is increasing in $\mathbb{R}^+$, the sequence $\{f_\lambda^n(x)\}_{n>0}$ is increasing and bounded above by a_λ for $0 \le x < a_\lambda$ and, is decreasing and bounded below by a_λ for $a_\lambda < x < r_\lambda$. Hence, $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for $0 \le x < r_\lambda$ by monotone convergence theorem. Since $f_\lambda(x)$ is increasing in $\mathbb{R}^+$ and $f_\lambda(x) > x$ for all $x > r_\lambda$, it follows that $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for $x > r_\lambda$.

2. Let $\lambda = \lambda^*$. Proceeding on the similar lines of the arguments given for $0 < \lambda < \lambda^*$, it is arrived that $g_\lambda(x_\lambda) = 0$ for $\lambda = \lambda^*$ and $x_\lambda = x^*$. As $g_{\lambda^*}(x_\lambda)$ is the minimum value of $g_{\lambda^*}(x)$ in $\mathbb{R}^+$, the point $x_\lambda = x^*$ is the only zero of $g_{\lambda^*}(x)$ and hence it is the only real fixed point of $f_{\lambda^*}(x)$. Clearly, x^* is rationally indifferent. Further, the sequence $\{f_\lambda^n(x)\}_{n>0}$ is increasing and bounded above by x^* for $0 \le x < x^*$ which gives that $\lim_{n\to\infty} f_{\lambda^*}^n(x) = x^*$. For $x > x^*$, the sequence $\{f_{\lambda^*}^n(x)\}_{n>0}$ is increasing and unbounded above. Therefore, $\lim_{n\to\infty} f_{\lambda^*}^n(x) = \infty$ for $x > x^*$.

3. **Case I:** $f'(0) = 0$ and $\lambda > \lambda^*$

   If $f'(0) = 0$, then there exists a x_λ such that $\frac{1}{f'(x_\lambda)} > \frac{1}{f'(x^*)}$. It follows that $x_\lambda < x^*$ and by Equation (4.1), $\phi(x_\lambda) > 0$. Therefore, $g_\lambda(x) > g_\lambda(x_\lambda) = 0$ for all $x > 0$ showing that f_λ has no fixed points on $\mathbb{R}$.

   Case II: $f'(0) \neq 0$ and $\lambda^* < \lambda < \frac{1}{f'(0)}$

   There exists x_λ such that $g_\lambda(x)$ attains the minimum value at $x = x_\lambda$. Again, it follows from Equation (4.1) that $\phi(x_\lambda) > 0$ and consequently $g(x_\lambda) > 0$. Therefore, $g_\lambda(x) > 0$ for all $x \in \mathbb{R}$ and hence f_λ has no fixed points on $\mathbb{R}$ in this case.

Case III: $f'(0) \neq 0$ and $\lambda \geq \frac{1}{f'(0)}$

It is already shown in the beginning of the proof that f_λ has no fixed points on $\mathbb{R}$ when $\lambda \geq \frac{1}{f'(0)}$.

Since $f_\lambda(x) > 0$ for all $x \in \mathbb{R}$, $f_\lambda(x) > x$ for all $x \in \mathbb{R}^+$ and f_λ has no fixed points on $\mathbb{R}$, it is concluded that $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for all $x \in \mathbb{R}$ if $\lambda > \lambda^*$.

$\square$

4.3 Dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$

In the present section, the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ where $f_\lambda \in \mathcal{S}$ is studied. Non-existence of rotational domains, Baker domains and wandering domains in the Fatou set of f_λ are established in Proposition 4.3.1 and Proposition 4.3.2. In Theorem 4.3.1, the dynamics of functions in the one parameter family $\mathcal{S}$ is described.

Proposition 4.3.1. *Let $f_\lambda \in \mathcal{S}$. Then the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ does not contain any rotational domain.*

Proof. Let D be a rotational domain in the Fatou set of f_λ and ∂D denote the boundary of D. Since all the singular values of $f_\lambda(z)$ are in a bounded subset of $\mathbb{R}^*$ and $f_\lambda(x) > 0$ for all $x \in \mathbb{R}$, there are negative real numbers which are not in $\overline{P(f_\lambda)}$, the closure of the set of forward orbits of all singular values of f_λ whenever these are defined (c.f. Equation (1.2)). In other words, the set $\overline{P(f_\lambda)}$ is a proper subset of $\mathbb{R}$. It is known that the boundary ∂D of D is contained in the closure $\overline{P(f_\lambda)}$ of $P(f_\lambda)$ which consequently gives that ∂D is properly contained in $\mathbb{R}$. Note that $(\partial D)^c = \widehat{\mathbb{C}} \setminus \partial D$ is path connected and $D \subseteq (\partial D)^c$. Now, we claim that $(\overline{D})^c$ is an empty set. If possible, let there be a point z^* in $(\overline{D})^c = (D \bigcup \partial D)^c = D^c \bigcap (\partial D)^c$. Then $\{z^*\} \bigcup D$ is a subset of $(\partial D)^c$ and a path γ can be found in $(\partial D)^c$

joining z^* and a point of D. Since $z^* \in D^c$, the path γ must intersect ∂D which is not possible. Therefore, $(\overline{D})^c$ is an empty set. As any component of the Fatou set other than D must be in $(\overline{D})^c$, it is not possible for a component of the Fatou set of f_λ other than D to exist. Since $\mathcal{F}(f_\lambda)$ is completely invariant, it follows that D is completely invariant. All the points of D, except at most two have infinitely many pre-images by Picard's theorem. It shows that f_λ is not one-one on D which leads to a contradiction to the definition of rotational domains (c.f. Remark 1.1.2(2)). Therefore, it is concluded that the Fatou set of f_λ does not contain any rotational domain.

□

Proposition 4.3.2. *Let $f_\lambda \in \mathcal{S}$. Then the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ does not contain Baker domain.*

Proof. Since $S_p(f_\lambda)$ is bounded for each $p \in \mathbb{N}$, there is no component in the Fatou set of f_λ on which $f_\lambda^{np}(z) \to \infty$ as $n \to \infty$ [108]. Therefore there is no Baker domain in $\mathcal{F}(f_\lambda)$. □

Theorem 4.3.1. *Let $f_\lambda \in \mathcal{S}$. Then, the dynamics of f_λ is as follows.*

1. *If $0 < \lambda < \lambda^*$, then the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the union of the attracting basin of a real fixed point and possibly wandering domains.*

2. *If $\lambda = \lambda^*$, then the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the union of the parabolic basin corresponding to a real rationally indifferent fixed point and possibly wandering domains.*

3. *If $\lambda > \lambda^*$, then the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is an empty set or possibly contains wandering domains.*

Proof. The Fatou set of f_λ does not contain any rotational domain or Baker domain by Propositions 4.3.1 and 4.3.2.

Since all the singular values of f_λ are in $\mathbb{R}$ and $f_\lambda(\mathbb{R}) \subseteq \mathbb{R}^+$, by similar arguments used in Theorem 3.3.4 of Chapter 3, it follows that any periodic Fatou component is associated to a real non-repelling periodic point. By Theorem 4.2.1, the function f_λ has no real periodic point of prime period greater than one.

1. If $0 < \lambda < \lambda^*$, then f_λ has only one attracting fixed point a_λ. Therefore, $\mathcal{F}(f_\lambda)$ is the union of $A(a_\lambda)$ and possibly wandering domains for $0 < \lambda < \lambda^*$ where $A(a_\lambda)$ is the basin of attraction of a_λ.

2. If $\lambda = \lambda^*$, then f_λ has only one fixed point x^* and that is rationally indifferent. Consequently, $\mathcal{F}(f_{\lambda^*})$ is the union of $P(x^*)$ and possibly wandering domains where $P(x^*)$ is the parabolic basin corresponding to x^*.

3. If $\lambda > \lambda^*$, the function f_λ has no fixed points. Therefore, $\mathcal{F}(f_\lambda) = \emptyset$ or possibly contains wandering domains for $\lambda > \lambda^*$.

□

4.4 An example: $f(z) = J(e^z + 1)$

In this section, the dynamics of functions in the one parameter family $\{f_\lambda \equiv \lambda f \ : \ \lambda > 0\}$ where $f(z) = J(h(z))$, $h(z) = e^z + 1$ and $J(z) = z + \frac{1}{z}$ is discussed in detail. Observe that all the coefficients in the Taylor series of the function $h(z) = e^z + 1$ about origin are non-negative, $h(x) > 0$ for all $x < 0$ and $h(0) = 1$. The point $z = 1$ is the only singular value of $e^z + 1$. Therefore, $h \in \mathcal{E}$ and $f = J(h) \in \mathcal{M}$. In fact, $J^n(h(z)) \in \mathcal{M}$ for each $n \in \mathbb{N}$. The set of poles of the function $J(e^z + 1)$ is $\{i\pi(2k+1) \ : \ k \in \mathbb{Z}\}$.

Let $\lambda^* = \frac{1}{f'(x^*)}$ where x^* is the positive solution of $f(x) - xf'(x) = 0$. Numerically, it is found that $x^* \approx 1.36415$ and $\lambda^* \approx 0.2666$. The graphs of $f_\lambda(x)$ are given in Figure 4.1.

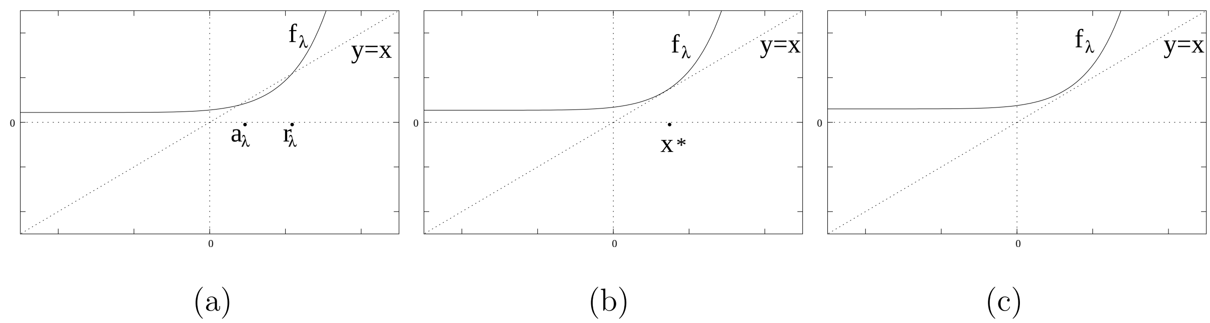


Figure 4.1: Graphs of $f_\lambda(x) = \lambda J(e^x + 1)$ for (a) $\lambda < \lambda^*$ (b) $\lambda = \lambda^*$ and (c) $\lambda > \lambda^*$.

For $0 < \lambda < \lambda^*$, f_λ has an attracting fixed point a_λ and a repelling fixed point r_λ and, $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for $x \in [0,\ r_\lambda)$ and $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for all $x > r_\lambda$. Since $f_\lambda(x)$ is increasing on $\mathbb{R}$, $f_\lambda(x) \in (0,\ a_\lambda)$ for all $x < 0$. Therefore, $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for all $x \in (-\infty,\ r_\lambda)$ when $0 < \lambda < \lambda^*$. The function f_{λ^*} has only one rationally indifferent fixed point x^*. Further, $\lim_{n\to\infty} f_{\lambda^*}^n(x) = x^*$ for $x \in [0,\ x^*)$ and $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for all $x > x^*$. Since $f_{\lambda^*}(x) \in (0,\ x^*)$ for all $x < 0$, $\lim_{n\to\infty} f_{\lambda^*}^n(x) = x^*$ for all $x \in (-\infty,\ x^*)$. If $\lambda > \lambda^*$, then $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for all $x \in \mathbb{R}$ since $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for all $x > 0$ and $f_\lambda(\mathbb{R}^-) \subseteq \mathbb{R}^+$. Thus, the dynamics of $f_\lambda(x) = \lambda J(e^x+1)$ for $x \in \mathbb{R}$ changes suddenly when the parameter λ passes through λ^*. The dynamics of $f_\lambda(x)$ on the real line is explained by the phase portraits in Figure 4.2.

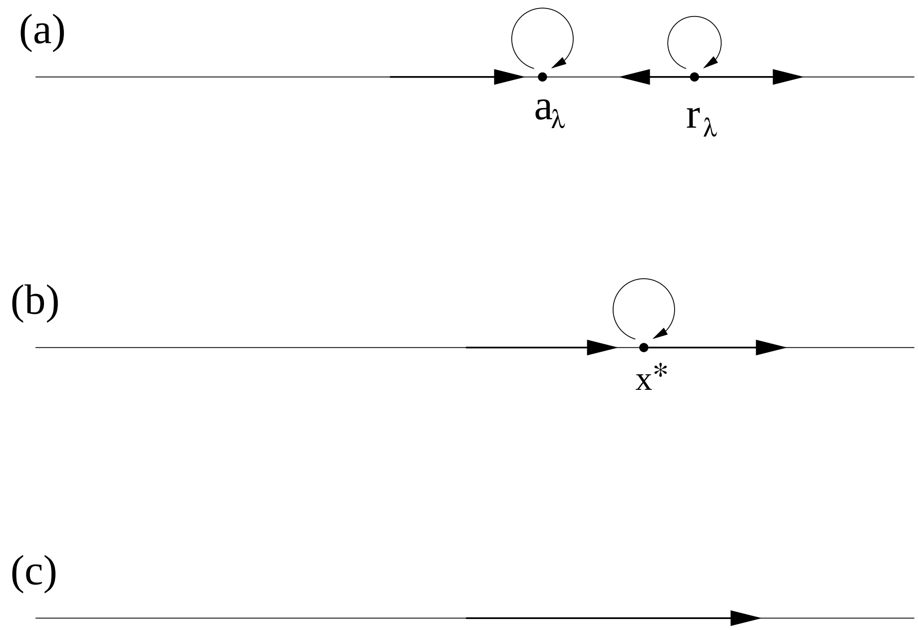


Figure 4.2: Phase portraits of $f_\lambda(x) = \lambda J(e^x + 1)$ for (a) $0 < \lambda < \lambda^*$, (b) $\lambda = \lambda^*$ and (c) $\lambda > \lambda^*$.

The dynamics of functions in the one parameter family $\{f_\lambda(z) = \lambda J(e^z + 1) \;:\; \lambda > 0\}$ follows from Theorem 4.3.1. The Fatou set $\mathcal{F}(f_\lambda)$ of $f_\lambda(z)$ does not contain any wandering domain since f_λ is critically finite. It is the attracting basin of a real attracting fixed point of $f_\lambda(z)$ for $0 < \lambda < \lambda^*$. The Fatou set $\mathcal{F}(f_{\lambda^*})$ is the parabolic basin corresponding to a real rationally indifferent fixed point x^*. If $\lambda > \lambda^*$, then $\mathcal{F}(f_\lambda)$ is an empty set. We now prove some additional properties of the Fatou set and the Julia set of $\lambda J(e^z + 1)$.

Theorem 4.4.1. *Let* $\mathcal{S} = \{f_\lambda(z) = \lambda J(h(z)) \;:\; \lambda > 0\}$ *where* $h(z) = e^z + 1$ *and* $J(z) = z + \frac{1}{z}$. *For* $0 < \lambda < \lambda^*$, *the Fatou set* $\mathcal{F}(f_\lambda)$ *is connected and consequently, the Julia set* $\mathcal{J}(f_\lambda)$ *does not contain any continuum that disconnects* $\widehat{\mathbb{C}}$.

Proof. When $0 < \lambda < \lambda^*$, the function f_λ has only two real fixed points a_λ and r_λ with $0 < a_\lambda < r_\lambda$ such that a_λ is attracting and r_λ is repelling. The function $f_\lambda(x)$ is increasing on $\mathbb{R}$ since $f'_\lambda(x) = \lambda e^x(1 - \frac{1}{(e^x+1)^2}) > 0$ for all $x \in \mathbb{R}$. Note that $\lim_{x\to-\infty} f_\lambda(x) = \lambda J(1) =$

2λ. Thus, the function f_λ maps $(-\infty, 0)$ onto $(2\lambda, f_\lambda(0))$ and $0 < 2\lambda < f_\lambda(0) < a_\lambda = f_\lambda(a_\lambda) < r_\lambda = f_\lambda(r_\lambda)$. The interval $[0, r_\lambda)$ is in the immediate basin of attraction $I(a_\lambda)$ of a_λ by Theorem 4.2.1. Therefore, $(-\infty, r_\lambda) \subset I(a_\lambda)$ and in particular, $2\lambda \in I(a_\lambda)$.

Let $D_r(2\lambda)$ be a disc of radius r with center at 2λ such that $D_r(2\lambda) \subset I(a_\lambda)$. Note that 2λ is a critical value of $\lambda J(z)$ corresponding to the critical point 1. The point $z = 1$ is the only pre-image of 2λ and each $z \in D_r(2\lambda) \setminus \{2\lambda\}$ has exactly two pre-images under the map $w = \lambda J(z)$. Therefore, $(\lambda J)^{-1}(D_r(2\lambda))$ is an open connected set, say $N(1)$ containing 1 and $\lambda J : N(1) \setminus \{1\} \to D_r(2\lambda) \setminus \{2\lambda\}$ is a two fold surjective map. Since 0 is a pole of $\lambda J(z)$, $0 \notin N(1)$. Now, let $D_\epsilon(1)$ be a disc around 1 and of radius ϵ such that $D_\epsilon(1) \subset N(1)$. If $E(z) = e^z + 1$, then $E^{-1}(D_\epsilon(1))$ is equal to the left half-plane $H_{\ln \epsilon} = \{z \in \mathbb{C} : \Re(z) < \ln \epsilon\}$. Thus, there is an open set $\tilde{D} = \lambda J(D_\epsilon(1))$ contained in $D_r(2\lambda)$ such that $f_\lambda^{-1}(\tilde{D})$ is equal to $H_{\ln \epsilon}$ (See Figure 4.3).

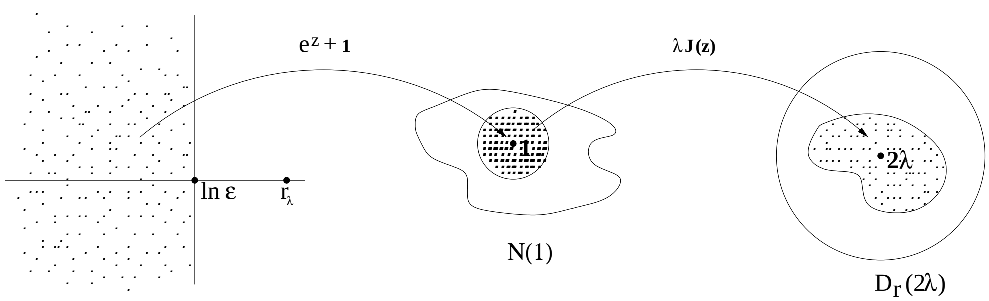


Figure 4.3: Mapping property of $f_\lambda(z) = \lambda J(e^z + 1)$ for $0 < \lambda < \lambda^*$.

This means that each component of $f_\lambda^{-1}(I(a_\lambda))$ intersects $H_{\ln \epsilon}$ (c.f. Theorem 1.1.6). Observe that $(-\infty,\ \ln \epsilon) \subset H_{\ln \epsilon} \subset \mathcal{F}(f_\lambda)$ and $[r_\lambda,\ \infty) \subset \mathcal{J}(f_\lambda)$ as a result of which it follows that $\ln \epsilon < r_\lambda$ and $H_{\ln \epsilon} \bigcap I(a_\lambda)$ is a non-empty set containing $(-\infty,\ \ln \epsilon)$. Thus each component of $f_\lambda^{-1}(I(a_\lambda))$ intersects $I(a_\lambda)$ since $H_{\ln \epsilon}$ is connected. In other words, the set $I(a_\lambda)$ is backward invariant. By definition, $I(a_\lambda)$ is connected and forward invariant. Hence, $\mathcal{F}(f_\lambda) = I(a_\lambda)$ and is connected.

If possible, let the Julia set $\mathcal{J}(f_\lambda)$ contain a continuum σ that disconnects $\widehat{\mathbb{C}}$. Then

$\sigma^c = \widehat{\mathbb{C}} \setminus \sigma$ has at least two non-empty components. Since the Fatou set $\mathcal{F}(f_\lambda)$ is non-empty and $\mathcal{J}(f_\lambda)$ has empty interior, the Fatou set intersects at least two of the components of σ^c. This contradicts the connectedness of $\mathcal{F}(f_\lambda)$. Therefore, $\mathcal{J}(f_\lambda)$ does not contain any continuum that disconnects $\widehat{\mathbb{C}}$. $\square$

Remark 4.4.1. *From the proof of Theorem 4.4.1, it is clear that there is a real number M_λ such that $\{z \in \mathbb{C} : \Re(z) < M_\lambda\}$ is in the Fatou set of f_λ for $0 < \lambda < \lambda^*$.*

By Theorem 4.3.1, the Julia set of f_λ contains $[r_\lambda,\ \infty)$ and the Fatou set of f_λ contains $(-\infty,\ r_\lambda)$ whenever $0 < \lambda < \lambda^*$. Let $L_k^+ = \{x + 2\pi ki\ :\ x \in [r_\lambda,\ \infty)\}$ and $L_k^- = \{x + 2\pi ki\ :\ x \in (-\infty,\ r_\lambda)\}$ for $k \in \mathbb{Z}$. Then $L_k^+ \subset \mathcal{J}(f_\lambda)$ and $L_k^- \subset \mathcal{F}(f_\lambda)$ for each $k \in \mathbb{Z}$ as $f_\lambda(z)$ is a $2\pi i$ periodic function. Thus, there are infinitely many unbounded curves in $\mathcal{J}(f_\lambda) \setminus \{\infty\}$. We prove in the following lemma that any unbounded component in the Julia set is contained in a horizontal half-strip of width less than 4π which is bounded to the left and unbounded to the right.

Lemma 4.4.1. *Let $0 < \lambda < \lambda^*$ and τ be an unbounded component (maximally connected subset) of $\mathcal{J}(f_\lambda)$. Then $\tau \subset \{z \in \widehat{\mathbb{C}}\ :\ |\Im(z) - \Im(z_0)| < 4\pi$ and $\Re(z) > R_0\}$ for some $z_0 \in \mathbb{C}$ and $R_0 \in \mathbb{R}$. Consequently, each component of $\mathcal{J}(f_\lambda)$ contains at most one pole.*

Proof. If possible, let the component τ intersects two half-lines $L_{n_1}^+$ and $L_{n_2}^+$ for some integers n_1 and n_2 with $n_1 \neq n_2$. Then $L_{n_1}^+ \bigcup \tau \bigcup L_{n_2}^+ \bigcup \{\infty\}$ is an unbounded continuum disconnecting $\widehat{\mathbb{C}}$ which is not possible by Theorem 4.4.1. Therefore, τ intersects L_n^+ for at most one value of n which means that, $|\Im(z) - \Im(z_0)| < 4\pi$ for each $z \in \tau$ and some $z_0 \in \mathbb{C}$. Further, there is a real number M_λ such that $\{z : \Re(z) < M_\lambda\}$ is in the Fatou set of f_λ by Remark 4.4.1. This implies $\Re(z) > R_0$ for each $z \in \tau$ and some $R_0 \in \mathbb{R}$. Hence $\tau \subset \{z \in \widehat{\mathbb{C}}\ :\ |\Im(z) - \Im(z_0)| < 4\pi$ and $\Re(z) > R_0\}$ for some $z_0 \in \mathbb{C}$ and $R_0 \in \mathbb{R}$.

The set of poles of $f_\lambda(z)$ is $\{i\pi(2k+1)\ :\ k \in \mathbb{Z}\}$. If a component ρ of the Julia set $\mathcal{J}(f_\lambda)$ contains two poles, then by $2\pi i$ periodicity of f_λ, ρ contains infinitely many poles and hence $\{\Im(z) : z \in \rho\}$ will become unbounded which is not possible. Therefore, each component of $\mathcal{J}(f_\lambda)$ contains at most one pole. $\square$

The existence of non-singleton bounded components in the Julia set of f_λ for $0 < \lambda < \lambda^*$ is proved in the following theorem.

Theorem 4.4.2. *Let $\mathcal{S} = \{f_\lambda(z) = \lambda J(h(z)) \ : \ \lambda > 0\}$ where $h(z) = e^z + 1$ and $J(z) = z + \frac{1}{z}$. For $0 < \lambda < \lambda^*$, the Julia set $\mathcal{J}(f_\lambda)$ of f_λ has infinitely many bounded components which are not singletons. In particular, each component of $\mathcal{J}(f_\lambda)$ containing a pole is bounded and non-singleton.*

Proof. Let w_0 be a pole of f_λ and ρ be the component of $\mathcal{J}(f_\lambda) \setminus \{\infty\}$ containing w_0. We claim that ρ is bounded and is not a singleton set. Since all the poles of $f_\lambda(z)$ are simple, the function $f_\lambda(z)$ is one-one in a sufficiently small disc $D_\epsilon(w_0)$ with center w_0 and radius ϵ. The set $f_\lambda(D_\epsilon(w_0))$ contains a neighbourhood of ∞ and it is already shown that $[r_\lambda, \infty)$ is in the Julia set. Therefore, $(x_0,\ \infty)\bigcup\{\infty\} \subset f_\lambda(D_\epsilon(w_0))\bigcap\mathcal{J}(f_\lambda)$ for some $x_0 \geq r_\lambda$ and consequently, $D_\epsilon(w_0)$ intersects a component ζ of $f_\lambda^{-1}((x_0,\ \infty)\bigcup\{\infty\})$. Now, $w_0 \in \zeta$ and $\zeta \subset \mathcal{J}(f_\lambda)$ because $\infty \in \mathcal{J}(f_\lambda)$ and the Julia set is completely invariant. Clearly $\zeta \subseteq \rho$ and ζ is not singleton. Therefore, ρ is not a singleton component of the Julia set.

Now, we assert that the component ρ of the Julia set $\mathcal{J}(f_\lambda)$ is bounded. If possible, let ρ be unbounded. Then $\rho^* = \rho \setminus \{w_0\}$ has an unbounded component. Note that, if ρ^* contains two unbounded components, then ρ contains a continuum that disconnects $\widehat{\mathbb{C}}$ which is not possible by Theorem 4.4.1. Also, if ρ^* contains a pole of f_λ, then ρ contains two poles of f_λ which is not possible by Lemma 4.4.1. Therefore, ρ^* contains exactly one unbounded component, say $\tilde{\rho}$, and ρ^* contains no pole. Let $\tilde{\rho}_1$ be a connected subset of $\tilde{\rho}\bigcup\{w_0\}$ contained in $D_\epsilon(w_0)$ and observe that $\tilde{\rho} \setminus \tilde{\rho}_1$ has an unbounded and connected subset(See Figure 4.4). Let it be $\tilde{\rho}_2$.

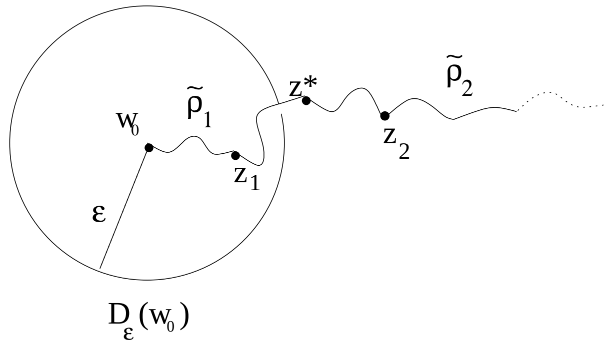


Figure 4.4: Pictures of $\tilde{\rho}_1$ and $\tilde{\rho}_2$.

By Lemma 4.4.1, $\tilde{\rho}_2$ is in $\{z \in \widehat{\mathbb{C}} \ : \ |\Im(z) - \Im(z_0)| < 4\pi \text{ and } \Re(z) > R_0\}$ for some $z_0 \in \mathbb{C}$ and $R_0 \in \mathbb{R}$ which gives that $f_\lambda(\tilde{\rho}_2) = \lambda J(e^{\tilde{\rho}_2} + 1)$ is an unbounded component. The set $f_\lambda(\tilde{\rho}_1)$ is also unbounded since $\tilde{\rho}_1$ contains the pole w_0. If the two sets $f_\lambda(\tilde{\rho}_1)$ and $f_\lambda(\tilde{\rho}_2)$ do not intersect, then $f_\lambda(\tilde{\rho}) \bigcup \{\infty\}$ forms a continuum in the Julia set disconnecting $\widehat{\mathbb{C}}$ which is not possible by Theorem 4.4.1. Hence $f_\lambda(\tilde{\rho}_1) \bigcap f_\lambda(\tilde{\rho}_2)$ is non-empty. Let $\tilde{z}_1 \in \tilde{\rho}_1$ and $\tilde{z}_2 \in \tilde{\rho}_2$ such that $f_\lambda(\tilde{z}_1) = f_\lambda(\tilde{z}_2)$. Let ρ' be the connected subset of $\tilde{\rho}$ containing $\tilde{z}_1$ and $\tilde{z}_2$. Then ρ' is bounded. Since the two singular values of f_λ are in the Fatou set which is connected and unbounded, we can find a simply connected domain in $\mathbb{C} \setminus S_{f_\lambda}$ containing $f_\lambda(\rho')$. Let $z^* \in \rho'$ and g be the branch of f_λ^{-1} that satisfies $g(f_\lambda(z^*)) = z^*$. Then g is single valued on $f_\lambda(\rho')$ by the Monodromy Theorem. So f_λ is one-one on ρ'. It does not agree with the fact that $f_\lambda(\tilde{z}_1) = f_\lambda(\tilde{z}_2)$. Therefore, it is concluded that the component ρ of the Julia set $\mathcal{J}(f_\lambda)$ is bounded.

Since ρ contains exactly one pole of f_λ and f_λ has infinitely many poles, there are infinitely many bounded components of $\mathcal{J}(f_\lambda)$ which are not singletons. It is obvious from the previous paragraph that each component of $\mathcal{J}(f_\lambda)$ containing a pole is bounded. $\square$

Corollary 4.4.1. *Let $f_\lambda(z) = \lambda J(h(z))$ where $J(z) = z + \frac{1}{z}$ and $h(z) = e^z + 1$. Then, the singleton components are dense in $\mathcal{J}(f_\lambda)$ for $0 < \lambda < \lambda^*$.*

Proof. Since the Julia set of f_λ for $0 < \lambda < \lambda^*$ contains infinitely many bounded components by Theorem 4.4.2, the connectivity of the Fatou set $\mathcal{F}(f_\lambda)$ is infinity. The corollary follows from the fact that, if the Fatou set of a transcendental meromorphic function has a component of connectivity at least three, then singleton components are dense in the Julia set of the function [50]. □

A point $w_0 \in \mathbb{C}$ is called a pre-pole of a meromorphic function $f(z)$ if there is a natural number k such that $f^k(w_0) = \infty$.

Corollary 4.4.2. *Let $f_\lambda(z) = \lambda J(h(z))$ where $J(z) = z + \frac{1}{z}$ and $h(z) = e^z + 1$. Also, let $0 < \lambda < \lambda^*$. If ζ is a component of $\mathcal{J}(f_\lambda)$ containing a pre-pole, then ζ is bounded.*

Proof. Let on the contrary, ζ be an unbounded component of the Julia set $\mathcal{J}(f_\lambda)$ containing a pre-pole $\hat{w}$. Then, for some $\hat{n}$, $f_\lambda^{\hat{n}}(\hat{w}) = \infty$. Observe that $\zeta \subset \{z \in \widehat{\mathbb{C}} \,:\, |\Im(z) - \Im(z_0)| < 4\pi$ and $\Re(z) > R_0\}$ for some $z_0 \in \mathbb{C}$ and $R_0 \in \mathbb{R}$ by Lemma 4.4.1. By the property of f_λ, the set $f_\lambda(\zeta)$ is unbounded. Further, $f_\lambda(\zeta)$ is contained in $\{z \in \widehat{\mathbb{C}} \,:\, |\Im(z) - \Im(z_0')| < 4\pi$ and $\Re(z) > R_0'\}$ for some $z_0' \in \mathbb{C}$ and $R_0' \in \mathbb{R}$ by Lemma 4.4.1. Repeating this argument, it is seen that the set $f_\lambda^n(\zeta)$ is unbounded for each n. Since $f_\lambda(z)$ is meromorphic, $f_\lambda^n(\zeta)$ is connected for all $n \in \mathbb{N}$. Therefore, $f_\lambda^{\hat{n}-1}(\zeta)$ is a component of the Julia set $\mathcal{J}(f_\lambda)$ containing a pole $f_\lambda^{\hat{n}-1}(\hat{w})$ of f_λ. But in Theorem 4.4.2, it is shown that each component of the Julia set of f_λ containing a pole is bounded leading to a contradiction. Therefore, it is concluded that each component ζ of $\mathcal{J}(f_\lambda)$ containing a pre-pole is bounded when $0 < \lambda < \lambda^*$. □

The following remark determines the number of pre-poles that can lie in a bounded component of the Julia set of f_λ for $0 < \lambda < \lambda^*$.

Remark 4.4.2. *Let $0 < \lambda < \lambda^*$ and σ be a bounded component of $\mathcal{J}(f_\lambda)$. Let $z_1,\ z_2 \in \sigma$ be two pre-poles of f_λ. Then, there are two natural numbers n_1 and n_2 such that $f_\lambda^{n_i}(z_i) = \infty$ for $i = 1,\ 2$. Since f_λ is meromorphic, $f_\lambda^n(\sigma)$ is connected for all $n \in \mathbb{N}$. If $n_1 =$*

n_2, then the component $f_\lambda^{n_1-1}(\sigma)$ of the Julia set contains two poles which is not possible by Lemma 4.4.1. So $n_1 \neq n_2$ and let $n_1 > n_2$. Then $f_\lambda^{n_1-1}(\sigma)$ contains a pole of f_λ. Since $f_\lambda^{n_2}(\sigma)$ is unbounded, $f_\lambda^{n}(\sigma)\}$ is bounded for all $n \geq n_2$ by similar arguments used in Corollary 4.4.2. Therefore, $f_\lambda^{n_1-1}(\sigma)$ is unbounded which is not possible by Theorem 4.4.2. Thus any bounded component of $\mathfrak{J}(f_\lambda)$ contains at most one pre-pole when $0 < \lambda < \lambda^*$.

A bounded component σ of $\mathfrak{J}(f_\lambda)$ contains at most one pre-pole. The following remark answers the question whether σ can contain (pre) periodic points also when a pre-pole lies on σ.

Remark 4.4.3. *Suppose that a bounded component σ of the Julia set of f_λ contains a pre-pole of f_λ. Then, there is a natural number n_0 such that $f_\lambda^{n_0}(\sigma)$ is unbounded. By using the arguments used in Corollary 4.4.2, it follows that $f_\lambda^{n}(\sigma)$ is unbounded for all $n > n_0$. Since the function f_λ is meromorphic, $f_\lambda^{n}(\sigma)$ is connected for all $n \in \mathbb{N}$. If σ contains a (pre) periodic point z_0 of f_λ, then a natural number $n^* > n_0$ can be found such that $f_\lambda^{n^*}(z_0) = z_0$ and consequently, $f_\lambda^{n^*}(\sigma) = \sigma$. Now $f_\lambda^{n^*}(\sigma)$ is unbounded whereas σ is bounded leading to a contradiction. Therefore, any bounded component of $\mathfrak{J}(f_\lambda)$ containing a pre-pole cannot contain a (pre) periodic point of f_λ where $0 < \lambda < \lambda^*$.*

It is well known that, the Julia set of a meromorphic function is completely invariant. In the following theorem, we show that the Julia set of f_λ, $0 < \lambda < \lambda^*$ consists of two completely invariant subsets one of which is totally disconnected.

Theorem 4.4.3. *Let $\mathcal{S} = \{f_\lambda(z) = \lambda J(h(z)) \ : \ \lambda > 0\}$ where $h(z) = e^z + 1$ and $J(z) = z + \frac{1}{z}$. Then, for $0 < \lambda < \lambda^*$, the Julia set of f_λ can be expressed as a union of two completely invariant subsets one of which is totally disconnected.*

Proof. By Corollary 4.4.1, there are singleton components of the Julia set of f_λ. Let $\mathfrak{J}_1(f_\lambda)$ consist of all the singleton components of $\mathfrak{J}(f_\lambda)$ and $\mathfrak{J}_2(f_\lambda) = \mathfrak{J}(f_\lambda) \setminus \mathfrak{J}_1(f_\lambda)$. Then $\mathfrak{J}_1(f_\lambda)$ is totally disconnected by definition. Since there are curves in the Julia set $\mathcal{J}(f_\lambda)$ containing ∞ and the function f_λ is one-one in sufficiently small neighbourhood of each pole, the components of $\mathcal{J}(f_\lambda)$ containing a pole is not singleton. Therefore, $\mathcal{J}_1(f_\lambda)$ does not contain any pole of f_λ. Since $\mathcal{J}(f_\lambda)$ is completely invariant, $\mathcal{J}_1(f_\lambda)$ is completely invariant if and only if $\mathcal{J}_2(f_\lambda)$ is completely invariant.

Let $z_1 \in \mathcal{J}_1(f_\lambda)$. Since the set S_{f_λ} of all the singular values of $f_\lambda(z)$ is a subset of the Fatou set $\mathcal{F}(f_\lambda)$ for $0 < \lambda < \lambda^*$ and $f_\lambda : \widehat{\mathbb{C}} \setminus f_\lambda^{-1}(S_{f_\lambda}) \to \widehat{\mathbb{C}} \setminus S_{f_\lambda}$ is a covering map, the function $f_\lambda(z)$ is locally one-one on $\mathcal{J}(f_\lambda) \setminus \{\infty\}$. Therefore, the image and pre-images of z_1 are singleton components of $\mathcal{J}(f_\lambda)$. Hence, $\mathcal{J}_1(f_\lambda)$ is completely invariant. $\square$

Remark 4.4.4. *Domínguez [50] proved that the Julia set of a meromorphic function with finitely many poles cannot be totally disconnected. However, the Julia set of a meromorphic function with infinitely many poles may not be totally disconnected, for example* $\mathcal{J}(\tan z) = \mathbb{R} \bigcup \{\infty\}$. *The function* $f_\lambda(z) = \lambda J(e^z + 1)$ *has infinitely many poles and the Julia set of* f_λ $0 < \lambda < \lambda^*$ *is not totally disconnected. But, the Julia set of* f_λ *has a completely invariant and totally disconnected proper subset* $\mathcal{J}_1(f_\lambda)$ *for* $0 < \lambda < \lambda^*$.

A comparison between the dynamics of $\lambda(e^z + 1 + \frac{1}{e^z+1})$, λe^z and $\lambda \tan z$ is given in the Table 4.1.

Dynamics of $f_\lambda(z) = \lambda J(e^z+1),\ \lambda>0$	**Dynamics of** $E_\lambda(z) = \lambda e^z,\ \lambda>0$	**Dynamics of** $T_\lambda(z) = \lambda \tan z,\ \lambda>0$
f_λ has one critical value -2λ.	E_λ has no critical values.	T_λ has no critical values.
f_λ has one asymptotic value 2λ.	E_λ has one asymptotic value 0.	T_λ has two asymptotic values $-i\lambda$ and $i\lambda$.
f_λ is periodic with period $2\pi i$.	E_λ is periodic with period $2\pi i$.	T_λ is periodic with period π.
f_λ is neither even nor odd.	E_λ is neither even nor odd.	T_λ is even.
Bifurcation in the dynamics of f_λ occurs at one critical parameter $\lambda^* \approx 0.27$.	Bifurcation in the dynamics of E_λ occurs at one critical parameter $\frac{1}{e}$.	Bifurcation in the dynamics of T_λ occurs at one critical parameter 1.
The Fatou set of f_λ is infinitely connected for $0 < \lambda < \lambda^*$.	The Fatou set of E_λ is simply connected for $0 < \lambda < \frac{1}{e}$.	The Fatou set of T_λ is infinitely connected for $0 < \lambda < 1$.
For $0 < \lambda < \lambda^*$, the Julia set of f_λ (i) has infinitely many singleton components, (ii) has infinitely many non-singleton bounded components (the Julia set is not totally disconnected), (iii) has infinitely many unbounded components.	For $0 < \lambda < \frac{1}{e}$, the Julia set of E_λ (i) has no singleton components, (ii) has no bounded components (the Julia set is connected), (iii) has only one unbounded component.	For $0 < \lambda < 1$, the Julia set of T_λ (i) has infinitely many singleton components, (ii) has no non-singleton bounded components (the Julia set is totally disconnected), (iii) has no unbounded components.
The Julia set of f_λ is $\widehat{\mathbb{C}}$ for $\lambda > \lambda^*$.	The Julia set of E_λ is $\widehat{\mathbb{C}}$ for $\lambda > \frac{1}{e}$.	The Julia set of T_λ is $\mathbb{R}\bigcup\{\infty\}$ for $\lambda \geq 1$.

Table 4.1: Comparison between the dynamics of $\lambda J(e^z+1)$, λe^z and $\lambda \tan z$.

Chapter 5

Dynamics of $f_\lambda(z) = \lambda \frac{z^m}{\sinh^m z}$

The dynamics of entire functions $\lambda \frac{\sinh^m z}{z^n}$, $\lambda > 0,\ m \geq n$ are investigated in Chapter 3 where both of m and n are either even or odd natural numbers. In the present chapter, the dynamics of the meromorphic functions $\lambda \frac{z^m}{\sinh^m z}$ which are not of bounded type is studied. Define

$$\mathcal{N} = \left\{ f(z) = \frac{z^m}{\sinh^m z} \text{ for } z \in \mathbb{C} :\ m \in \mathbb{N} \right\}.$$

For $f \in \mathcal{N}$, consider the one parameter family

$$\mathcal{S} = \{ f_\lambda(z) = \lambda f(z)\ :\ \lambda \in \mathbb{R} \setminus \{0\} \}.$$

The two functions f_λ and $f_{-\lambda}$ are conformally conjugate by the conjugating map $\psi(z) = -z$ and consequently, the dynamics of f_λ and $f_{-\lambda}$ are essentially same. For this reason, we investigate the dynamics of the functions $f_\lambda \in \mathcal{S}$ for $\lambda > 0$. A Fatou component of a transcendental entire/meromorphic function can be a Baker domain or a wandering domain. However a number of classes of transcendental entire/meromorphic functions of bounded type not having these domains in their Fatou sets are known (c.f. Theorem 1.1.11 and Theorem 1.1.14). In this chapter, non-existence of Baker domains and wandering domains for f_λ is established in spite of the fact that f_λ is not in the class B. Using this, the occurrence of bifurcation in the dynamics of functions in the one parameter family

$\{f_\lambda \in \mathcal{S} \ : \ \lambda > 0\}$ at a parameter value is proved. The topology of the Fatou components of f_λ, $\lambda > 0$ is also explored.

5.1 Properties of f_λ

Some dynamically relevant properties of functions $f_\lambda \in \mathcal{S}$ are proved in this section. The function $f_\lambda(z) = \lambda \frac{z^m}{\sinh^m z}$ is a meromorphic function with poles at $\{i\pi k \ : \ k \in \mathbb{Z} \setminus \{0\}\}$. All the poles are multiple if $m > 1$ and simple if $m = 1$. The function $f_\lambda(z)$ is one-one around each of its poles only when $m = 1$. The point $z = 0$ is an omitted value of f_λ and hence an asymptotic value of $f_\lambda(z)$. Further, the function $f_\lambda(z)$ is even and not periodic. In Proposition 5.1.1, we prove that the Julia set of f_λ is symmetric with respect to both real and imaginary axes. More importantly, the function $f_\lambda(z)$ is shown to be not in the class B in Proposition 5.1.2.

Proposition 5.1.1. *Let $f_\lambda \in \mathcal{S}$. If $z \in \mathcal{J}(f_\lambda)$, then $-z \in \mathcal{J}(f_\lambda)$ and $\overline{z} \in \mathcal{J}(f_\lambda)$.*

Proof. Let $z \in \mathcal{J}(f_\lambda)$. Since $f_\lambda(-z) = f_\lambda(z)$ for all $z \in \mathbb{C}$ and $\mathcal{J}(f_\lambda)$ is completely invariant, $-z \in \mathcal{J}(f_\lambda)$. Observe that $f_\lambda(\overline{z}) = \overline{f_\lambda(z)}$ and consequently, $f_\lambda^n(\overline{z}) = \overline{f_\lambda^n(z)}$ for all $z \in \mathbb{C}$ and $n \in \mathbb{N}$. For $z \in \mathcal{J}(f_\lambda)$, the sequence $\{f_\lambda^n\}_{n>0}$ is not normal at z. This gives that $\{\overline{f_\lambda^n}\}_{n>0}$ is also not normal at z. Therefore, $\{f_\lambda^n\}_{n>0}$ is not normal at $\bar{z}$ and $\overline{z} \in \mathcal{J}(f_\lambda)$. $\square$

Proposition 5.1.2. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$. Then, the set of all the critical values of f_λ is an unbounded subset of $\mathbb{R} \setminus (-\lambda, \ \lambda)$ and 0 is the only finite asymptotic value of f_λ.*

Proof. Observe that $f_\lambda'(z) = \lambda \frac{m z^{m-1}}{\sinh^{m-1} z} \{\frac{\sinh z - z \cosh z}{\sinh^2 z}\} = 0$ and $\frac{m z^{m-1}}{\sinh^{m-1} z} \neq 0$ for $z \in \mathbb{C}$. Further, the point $z = 0$ is the only common zero of $\sinh z - z \cosh z$ and $\sinh^2 z$ and is a zero of $\frac{\sinh z - z \cosh z}{\sinh^2 z}$. Therefore, the solutions of $f_\lambda'(z) = 0$ are precisely the solutions of $\sinh z - z \cosh z = 0$ *i.e.*, the solutions of $\tanh z = z$. It is easy to see that the set of all the solutions of $\tanh z = z$ is an unbounded subset of the imaginary axis. If $\tanh(iy) = iy$ for some $y \in \mathbb{R}$, then $\tanh(-iy) = -\tanh(iy) = -iy$. Therefore, the set of all the critical points of $f_\lambda(z)$ is symmetric with respect to origin and is an unbounded subset of the imaginary axis. Let $\{iy_k\}_{k>0}$ be the sequence of critical points in the positive imaginary axis arranged in increasing order of their modulli. Then $-iy_k$ is also a critical point of $f_\lambda(z)$ for each k. Since $f_\lambda(z)$ is an even function, $\lim_{k\to\infty} f_\lambda(iy_k) = \lim_{k\to\infty} f_\lambda(-iy_k) = \lim_{k\to\infty} \lambda\frac{i^m y_k^m}{i^m \sin^m y_k} = \infty$. Therefore, the set of all the critical values of f_λ is unbounded. Every critical point iy_k of $f_\lambda(z)$ satisfies $\tanh(iy_k) = iy_k$ and consequently, $\frac{iy_k}{\sinh(iy_k)} = \frac{1}{\cosh(iy_k)}$. The critical value $f_\lambda(iy_k) = \lambda(\frac{iy_k}{\sinh(iy_k)})^m = \lambda(\frac{1}{\cosh(iy_k)})^m = \lambda(\frac{1}{\cos y_k})^m$ is real. Since $|\cos y| < 1$ for all $y \in \mathbb{R}$, it follows that $|f_\lambda(iy_k)| > \lambda$. Therefore, the set of all the critical values of $f_\lambda(z)$ is an unbounded subset of $\mathbb{R} \setminus (-\lambda,\ \lambda)$.

All the critical points of $\frac{\sinh z}{z}$, *i.e.*, the roots of $\frac{z\cosh z - \sinh z}{z^2}$ are purely imaginary and form an unbounded set. Since $\lim_{|y|\to\infty} \frac{\sinh iy}{iy} = \lim_{|y|\to\infty} \frac{\sin y}{y} = 0$, 0 is an asymptotic value of $\frac{\sinh z}{z}$ and is the only limit point of all the critical values of $\frac{\sinh z}{z}$. Since the order of $\frac{\sinh z}{z}$ is one, it can have at most two finite asymptotic values (c.f. Theorem 1.1.34). Further, if there are exactly two finite asymptotic values of $\frac{\sinh z}{z}$, then both the asymptotic values are indirect singularities of the inverse function of $\frac{\sinh z}{z}$ (c.f. Theorem 1.1.35). If f is a meromorphic function of finite order and a is an asymptotic value of f, then a is a limit point of critical values $a_k \neq a$ or all singularities of f^{-1} are logarithmic (a special case of direct singularity) [23]. Therefore, if there is a finite asymptotic value $\hat{w}$ of $\frac{\sinh z}{z}$ other than 0, then both 0 and $\hat{w}$ are indirect singularities of inverse function of $\frac{\sinh z}{z}$ and the limit points of critical values of $\frac{\sinh z}{z}$. Since the critical values of $\frac{\sinh z}{z}$ accumulate only at 0, $\hat{w}$ cannot be an asymptotic value of $\frac{\sinh z}{z}$. Thus, 0 is the only finite asymptotic value of $\frac{\sinh z}{z}$. Since $\frac{\sinh z}{z}$ is an entire function, ∞ is also an asymptotic value. This implies that the function $\frac{z}{\sinh z}$ has only one finite asymptotic value, namely 0. Hence, 0 is the only finite asymptotic value of $f_\lambda(z) = \lambda\frac{z^m}{\sinh^m z}$ for $m \in \mathbb{N}$. $\square$

Remark 5.1.1. *For $z = x + iy \neq 0$,*

$$\left|\frac{z^m}{\sinh^m z}\right| = \left\{\left(\frac{x^2+y^2}{\sinh^2 x + \sin^2 y}\right)^{\frac{1}{2}}\right\}^m.$$

If $\{\Im(z) \;:\; z \in \gamma\}$ is bounded and $\lim_{t\to\infty} |\Re(\gamma(t))| = \infty$ for a path $\gamma \;:\; [0,\ \infty) \to \mathbb{C}$, then $\lim_{t\to\infty} f_\lambda(\gamma(t)) = 0$. If $\{\Re(z) \;:\; z \in \gamma\}$ is bounded and $\lim_{t\to\infty} |\Im(\gamma(t))| = \infty$ for a path $\gamma \;:\; [0,\ \infty) \to \mathbb{C}$, then $\lim_{t\to\infty} f_\lambda(\gamma(t)) = \infty$.

5.2 Dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$

In the present section, the dynamics of f_λ ($\lambda > 0$) on the real line is studied. In Theorem 5.2.1, the existence and nature of real fixed points of f_λ are investigated. The change in the nature and existence of real periodic points leads to a bifurcation in the dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ at a critical parameter value and that is proved in Theorem 5.2.2.

Consider the function $\phi(x) = xf'(x) + f(x) = x\frac{mx^{m-1}}{\sinh^{m+1} x}(\sinh x - x\cosh x) + \frac{x^m}{\sinh^m(x)} = \frac{x^m}{\sinh^{m+1}(x)}((m+1)\sinh x - mx\cosh x)$ for $x \geq 0$. If $p(x) = (m+1)\sinh x - mx\cosh x$, then $p'(x) = \cosh x - mx\sinh x$ and $p''(x) = (1-m)\sinh x - mx\cosh x$. Observe that $p''(x) < 0$ for $x \in \mathbb{R}^+ = \{x \in \mathbb{R} \;:\; x > 0\}$, since $m \geq 1$. Therefore, the function $p'(x)$ is decreasing on $\mathbb{R}^+$. Since $p'(0) = 1$ and $\lim_{x\to+\infty} p'(x) = -\infty$, by continuity of $p'(x)$, it follows that, there is a unique $\hat{x} > 0$ such that $p'(x) > 0$ for $0 \leq x < \hat{x}$, $p'(\hat{x}) = 0$ and $p'(x) < 0$ for $x > \hat{x}$. Therefore, $p(x)$ increases in $[0,\ \hat{x})$, attains its maximum at $\hat{x}$ and decreases thereafter. It follows from the facts $p(0) = 0$ and $\lim_{x\to+\infty} p(x) = -\infty$ that, there is a unique positive $x^* > \hat{x}$ such that $p(x) > 0$ for $0 < x < x^*$, $p(x^*) = 0$ and $p(x) < 0$ for $x > x^*$. Since $\frac{x^m}{\sinh^{m+1} x} > 0$ for all $x > 0$, it follows that

$$\phi(x) = \frac{x^m}{\sinh^{m+1} x} p(x) \begin{cases} > 0, & \text{for } 0 < x < x^* \\ = 0, & \text{for } x = x^* \\ < 0, & \text{for } x > x^* \end{cases} \tag{5.1}$$

Define

$$\lambda^* = \frac{x^*}{f(x^*)} \tag{5.2}$$

where x^* is the unique positive real root of the equation $\phi(x) = xf'(x) + f(x) = 0$.

Remark 5.2.1. *Let $x^*(m)$ is the root of $\phi(x) = xf'(x) + f(x)$ for $f(x) = \frac{x^m}{\sinh^m x}$ and $\lambda^*(m) = \frac{x^*(m)}{f(x^*(m))}$ be the corresponding critical parameter. For $m = 1,\ 2,\ 3$, the graphs of $\phi(x)$ are given in Figure 5.1.*

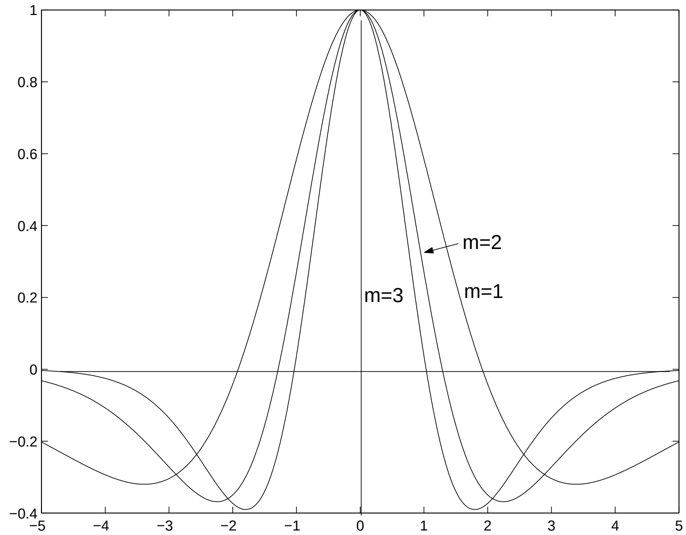


Figure 5.1: Graph of $\phi(x)$ for $m = 1$, $m = 2$ and $m = 3$.

Numerically, it is computed that $x^*(1) \approx 1.915$, $x^*(2) \approx 1.2878$, $x^*(3) \approx 1.03402$ and $\lambda^*(1) \approx 3.3198$, $\lambda^*(2) \approx 2.1772$, $\lambda^*(3) \approx 1.7926$.

The following theorem shows that $f_\lambda(z)$ has a unique real fixed point for each $\lambda > 0$. However, the nature of the fixed point changes when the parameter λ passes through the critical parameter λ^*.

Theorem 5.2.1. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$. Then, the function $f_\lambda(z)$ has a unique real fixed point x_λ. Further,*

1. *The fixed point x_λ is attracting for $0 < \lambda < \lambda^*$.*

2. *The fixed point x_λ is rationally indifferent for $\lambda = \lambda^*$.*

3. *The fixed point x_λ is repelling for $\lambda > \lambda^*$.*

Proof. Since $f_\lambda(x) > 0$ for all $x \in \mathbb{R}$, each real periodic point of f_λ is positive. The function $f'_\lambda(x) = \lambda \frac{mx^{m-1}}{\sinh^{m+1} x}(\sinh x - x\cosh x) < 0$ for $x > 0$ and hence $f_\lambda(x)$ is decreasing on $\mathbb{R}^+$. Let $g_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$. Since $f'_\lambda(x) < 0$ for $x > 0$, $g'_\lambda(x) = f'_\lambda(x) - 1 < 0$ and consequently, $g_\lambda(x)$ is decreasing on $\mathbb{R}^+$. Now, $g_\lambda(0) = \lambda > 0$, $\lim_{x\to+\infty} g_\lambda(x) = -\infty$ and $g_\lambda(x)$ is continuous on $\mathbb{R}^+$. By the intermediate value theorem, there exists a unique positive x_λ such that $g_\lambda(x_\lambda) = 0$. In other words, $f_\lambda(x)$ has a unique positive fixed point x_λ and $\lambda = \frac{x_\lambda}{f(x_\lambda)}$. Note that the function $\frac{x}{f(x)}$ is increasing on $\mathbb{R}^+$, since $\frac{d}{dx}\left(\frac{x}{f(x)}\right) = \frac{f(x)-xf'(x)}{(f(x))^2} > 0$ for $x > 0$.

1. For $0 < \lambda < \lambda^*$, $\frac{x_\lambda}{f(x_\lambda)} < \frac{x^*}{f(x^*)}$ which gives that $x_\lambda < x^*$. By Equation (5.1), $\phi(x_\lambda) > 0$. This implies that $\frac{\phi(x_\lambda)}{f(x_\lambda)} = \frac{xf'(x_\lambda)+f(x_\lambda)}{f(x_\lambda)} = f'_\lambda(x_\lambda)+1 > 0$. Since $f'_\lambda(x)$ is negative on $\mathbb{R}^+$, it follows that $-1 < f'_\lambda(x_\lambda) < 0$ and the fixed point x_λ is attracting for $0 < \lambda < \lambda^*$.

2. For $\lambda = \lambda^*$, $x_\lambda = x^*$ and $\phi(x_\lambda) = 0$ by similar arguments used in case (1). Now, by Equation (5.1), it follows that $\frac{\phi(x_\lambda)}{f(x_\lambda)} = 0$ implying $f'_{\lambda^*}(x_\lambda) = -1$. Therefore, the fixed point $x_\lambda = x^*$ is rationally indifferent if $\lambda = \lambda^*$.

3. For $\lambda > \lambda^*$, it follows $x_\lambda > x^*$ by similar arguments used in case (1). Again by Equation (5.1) and by the fact $x_\lambda > x^*$, we have $\phi(x_\lambda) < 0$. It shows that $\frac{\phi(x_\lambda)}{f(x_\lambda)} = f'_\lambda(x_\lambda) + 1 < 0$ and hence $f'_\lambda(x_\lambda) < -1$. Therefore, x_λ is a repelling fixed point of f_λ for $\lambda > \lambda^*$.

$\square$

Now, we investigate the possibility of the real periodic points of f_λ with prime period greater than one. The function $f_\lambda(x)$ is decreasing on $\mathbb{R}^+$, $f_\lambda(\mathbb{R}) = (0,\ \lambda]$ and f_λ has a unique real fixed point x_λ by Theorem 5.2.1. It is easy to see that $f_\lambda(0) = \lambda > f_\lambda(x) > x_\lambda$

for $0 < x < x_\lambda$ and $f_\lambda(x) < x_\lambda < f_\lambda(0) = \lambda$ for $x > x_\lambda > 0$. In other words, $f_\lambda((0,\ x_\lambda)) = (x_\lambda,\ \lambda)$ and $f_\lambda(x_\lambda,\ \infty) = (0,\ x_\lambda)$. This gives that $f_\lambda^n(x) \neq x$ for $x \in \mathbb{R}^+ \setminus \{x_\lambda\}$ and odd n. Therefore, $f_\lambda(x)$ does not have any real periodic point of odd period other than x_λ. Observe that $f_\lambda(x) > 0$ and $f'_\lambda(x) < 0$ for $x > 0$ and $\lambda > 0$. So $(f_\lambda^2)'(x) = f'_\lambda(f_\lambda(x))f'_\lambda(x) > 0$ and $f_\lambda^2(x)$ is increasing on $\mathbb{R}^+$. Consequently, if $f_\lambda^2(x) > x$ (or $f_\lambda^2(x) < x$) for some $x \in \mathbb{R}^+$, then $f_\lambda^{2n}(x) > f_\lambda^{2(n-1)}(x)$ (or $f_\lambda^{2n}(x) < f_\lambda^{2(n-1)}(x)$) for all n. It shows that the function $f_\lambda^2(x)$ does not have any real periodic point of period greater than 1 and hence $f_\lambda(x)$ has no real periodic point of even period greater than 2. Therefore, a real periodic point of f_λ other than x_λ is of prime period exactly equal to 2, if it exists. Also, each cycle $\{x_{1\lambda},\ x_{2\lambda}\}$ of real 2 periodic points satisfies $x_{1\lambda} < x_\lambda < x_{2\lambda}$. Let us assume that f_λ has two different 2 periodic real cycles $\{a,\ b\}$ with $0 < a < b$ and $\{c,\ d\}$ with $0 < c < d$. Since $f_\lambda(x)$ is strictly decreasing on $\mathbb{R}^+$ for $\lambda > 0$, it follows that $c < a < a_\lambda < b < d$ or $a < c < x_\lambda < d < b$. In the first case $\{c,\ d\}$ and in the second case $\{a,\ b\}$ is called the outer cycle. In the first case $\{a,\ b\}$ and in the second case $\{c,\ d\}$ is called the inner cycle. The following proposition shows that whenever such a 2 periodic cycle exists, it is attracting or rationally indifferent and all the singular values of $f_\lambda(z)$ goes to this cycle under iteration of f_λ^2.

Proposition 5.2.1. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$. If f_λ has a real 2-periodic cycle, then $\lim_{n\to\infty} f_\lambda^{2n}(x) = y_{1\lambda}$ or $y_{2\lambda}$ for all $x \in [0,\ y_{1\lambda}]\bigcup[y_{2\lambda},\ +\infty)$ where $\{y_{1\lambda},\ y_{2\lambda}\}$ is the outermost 2 periodic cycle. In particular, the cycle $\{y_{1\lambda},\ y_{2\lambda}\}$ is either attracting or rationally indifferent and all the singular values of f_λ tend to $\{y_{1\lambda},\ y_{2\lambda}\}$ under iteration of f_λ^2.*

Proof. It is observed earlier that any periodic point of the function f_λ is of prime period one or two and each 2-periodic cycle $\{a,\ b\}$ satisfies $a < x_\lambda < b$ where x_λ is the fixed point of f_λ. Since $\{y_{1\lambda},\ y_{2\lambda}\}$ is the outermost 2-periodic cycle, $f_\lambda(x) \neq x$ for all $x > y_{2\lambda}$. If possible, let $f_\lambda^2(x) > x$ for some $x > y_{2\lambda}$. Then, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is increasing and bounded above by λ, and hence $f_\lambda^{2n}(x)$ converges to l, say. Obviously, $l > y_{2\lambda}$. By the continuity of f_λ^2 it follows that the point l must be a periodic point of f_λ of period at most two. It contradicts the fact that $\{y_{1\lambda},\ y_{2\lambda}\}$ is the outermost 2 periodic cycle. Therefore, we conclude that $f_\lambda^2(x) < x$ for all $x > y_{2\lambda}$. Since $f_\lambda^2(x)$ is increasing, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is decreasing and bounded below by $y_{2\lambda}$ and consequently, $\lim_{n\to\infty} f_\lambda^{2n}(x) = y_{2\lambda}$ for $x > y_{2\lambda}$. Similarly, it can be proved that $f_\lambda^2(x) > x$ and $\lim_{n\to\infty} f_\lambda^{2n}(x) = y_{1\lambda}$ for all $0 < x < y_{1\lambda}$. Therefore, $\lim_{n\to\infty} f_\lambda^{2n}(x) = y_{1\lambda}$ or $y_{2\lambda}$ for all $x \in [0,\ y_{1\lambda}] \bigcup [y_{2\lambda},\ +\infty)$.

Each interval containing $y_{1\lambda}$ contains points tending to $y_{1\lambda}$ under iteration of f_λ^2. Therefore, $y_{1\lambda}$ cannot be a repelling periodic point of f_λ^2 and is either attracting or rationally indifferent. Thus, $\{y_{1\lambda},\ y_{2\lambda}\}$ is either attracting or rationally indifferent. As $(-y_{2\lambda},\ y_{2\lambda}) \subset (-\lambda,\ \lambda)$ and f_λ is an even function, $\lim_{n\to\infty} f_\lambda^{2n}(x) = y_{1\lambda}$ or $y_{2\lambda}$ for all $x \in \mathbb{R} \setminus (-\lambda,\ \lambda)$. Since all the singular values of f_λ are in $\mathbb{R} \setminus (-\lambda,\ \lambda)$, it is concluded that all the singular values of f_λ tend to $\{y_{1\lambda},\ y_{2\lambda}\}$ under iteration of f_λ^2. $\square$

The dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ is determined in the following theorem.

Theorem 5.2.2. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$.*

1. *If $\lambda < \lambda^*$, then $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for all $x \in \mathbb{R}$ where a_λ is the unique real attracting fixed point of f_λ.*

2. *If $\lambda = \lambda^*$, then $\lim_{n\to\infty} f_\lambda^n(x) = x^*$ for all $x \in \mathbb{R}$ where $x^*$is the unique real rationally indifferent fixed point of f_λ.*

3. *If $\lambda > \lambda^*$, then $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ or $a_{2\lambda}$ for all $x \in \mathbb{R} \setminus \{r_\lambda,\ -r_\lambda\}$ where r_λ is the unique real repelling fixed point of f_λ and $\{a_{1\lambda},\ a_{2\lambda}\}$ is the real attracting or rationally indifferent 2-periodic cycle.*

Proof. All the singular values of $f_\lambda(z)$ are on $\mathbb{R} \setminus (-\lambda,\ \lambda)$ by Proposition 5.1.2. If there is a 2-periodic cycle, then the cycle is in $(0,\ \lambda)$ and by Proposition 5.2.1, all the singular values tend to the outermost 2-cycle under iteration of f_λ^2.

1. Let $f_\lambda^2(x) > x$ (or $f_\lambda^2(x) < x$) for some $x > 0$. Since $f_\lambda^2(x)$ is increasing on $\mathbb{R}^+$, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is increasing and bounded above by λ (or decreasing and bounded below by 0). Therefore, $f_\lambda^{2n}(x)$ converges to $\hat{x}$, say. Now, by continuity of f_λ, the point $\hat{x}$ is a periodic point of $f_\lambda(x)$ of period one or two. If possible, let $\hat{x}$ be a periodic point of f_λ with prime period 2. Then, there is an outermost 2-cycle of f_λ and all the singular values of f_λ tends to the outermost 2-periodic cycle under iteration of f_λ^2 which is a contradiction to the fact that the basin of attraction of a_λ must contain at least one singular value of f_λ. Therefore, $\hat{x}$ is not a 2-periodic point and is a fixed point. Since f_λ has only one real fixed point a_λ for $0 < \lambda < \lambda^*$, $\hat{x} = a_\lambda$ and $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_\lambda$ for all $x \in \mathbb{R}^+$. By continuity of f_λ, it follows that $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for all $x \in \mathbb{R}^+$. Since $f_\lambda(\mathbb{R}^- \bigcup \{0\}) \subset \mathbb{R}^+$, $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for all $x \in \mathbb{R}$.

2. Let $f_\lambda^2(x) > x$ (or $f_\lambda^2(x) < x$). Since $f_\lambda^2(x)$ is increasing on $\mathbb{R}$, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is increasing and bounded above by λ (or decreasing and bounded below by 0). Proceeding as in Case 1, it is easy to see that the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ converges to x^* for all $x \in \mathbb{R}^+$. By continuity of f_λ, it follows that $\lim_{n\to\infty} f_\lambda^n(x) = x^*$ for all $x \in \mathbb{R}^+$. Since $f_\lambda(\mathbb{R}^- \bigcup \{0\}) \subset \mathbb{R}^+$, $\lim_{n\to\infty} f_\lambda^n(x) = x^*$ for all $x \in \mathbb{R}$.

3. If $\lambda > \lambda^*$, then the unique real fixed point of f_λ is repelling. Therefore, we can find a real number x sufficiently close to the fixed point r_λ such that $f_\lambda^2(x) > x$. Since $f_\lambda^2(x)$ is increasing on $\mathbb{R}^+$, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is increasing and bounded above by λ. Therefore, $\{f_\lambda^{2n}(x)\}$ converges to $\hat{x}$, say. By continuity of f_λ^2, it follows that $\hat{x}$ is a 2-periodic point of f_λ. If possible, let there are more than one 2-periodic cycles of periodic points. If $\{i_{1\lambda},\ i_{2\lambda}\}$ is the innermost real cycle of 2-periodic point of f_λ, then $i_{1\lambda} < r_\lambda < i_{2\lambda}$ and, $f_\lambda(x) \in (r_\lambda,\ i_{2\lambda})$ for all $x \in (i_{1\lambda},\ r_\lambda)$ and $f_\lambda(x) \in (i_{1\lambda},\ r_\lambda)$ for all $x \in (r_\lambda,\ i_{2\lambda})$. Further, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ converges either to $i_{1\lambda}$ or to

$i_{1\lambda}$ for $x \in (i_{1\lambda},\ i_{2\lambda}) \setminus r_\lambda$ by same arguments used in the previous cases. Therefore, $\{i_{1\lambda},\ i_{2\lambda}\}$ is either an attracting or a rationally indifferent cycle and at least one singular value of f_λ tends to this cycle under iteration of f_λ^2. All the singular values of f_λ tend to the outermost 2-cycle under iteration of f_λ by Proposition 5.2.1 leading to a contradiction. Hence, f_λ has exactly one 2-periodic cycle. Let it be $\{a_{1\lambda},\ a_{2\lambda}\}$. By Proposition 5.2.1, $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ or $a_{2\lambda}$ for all $x \in [0,\ a_{1\lambda}] \bigcup [a_{2\lambda},\ +\infty)$. If $x \in (r_\lambda,\ a_{2\lambda}]$, then $f_\lambda^2(x) > x$ and $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{2\lambda}$. Similarly, it is easily seen that $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ for all $x \in [a_{1\lambda},\ r_\lambda)$. Since $f_\lambda(z)$ is an even function, it follows that $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ or $a_{2\lambda}$ for all $x \in \mathbb{R}^- \setminus \{-r_\lambda\}$. Therefore, if $\lambda > \lambda^*$ it is concluded that $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ or $a_{2\lambda}$ for all $x \in \mathbb{R} \setminus \{r_\lambda,\ -r_\lambda\}$ where r_λ is the repelling fixed point of f_λ and $\{a_{1\lambda},\ a_{2\lambda}\}$ is the attracting or rationally indifferent 2-periodic cycle.

$\square$

Remark 5.2.2. *All the singular values of f_λ, $\lambda > 0$ are in $\mathbb{R}$ and tend to either an attracting or rationally indifferent periodic point under iteration of f_λ^2. Therefore, the set $P(f_\lambda)$ of well defined forward orbits of singular values is in the Fatou set of f_λ for $\lambda > 0$. In particular, the point 0 is in the Fatou set $\mathcal{F}(f_\lambda)$ for $\lambda > 0$.*

Remark 5.2.3. *Note that $f_\lambda(iy) = \frac{y^m}{\sin^m y}$ and the image of any point on the imaginary axis is either infinity or a real number. By Theorem 5.2.2, each of the real numbers except at most two are in an attracting or a parabolic domain of f_λ corresponding to a real periodic point. Therefore, any Fatou component U of f_λ other than an attracting or parabolic domain (and their pre-images) intersects neither the real nor the imaginary axis. Thus, the component U is contained completely in one of the four quadrants of the complex plane.*

5.3 Dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$

The dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ is studied in this section. Theorem 5.3.1 proves non-existence of Baker domains in the Fatou set of $f_\lambda \in \mathcal{S}$ for $\lambda > 0$ though the function f_λ is not in the class B. The dynamics of f_λ is determined in Theorem 5.3.2.

Theorem 5.3.1. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$. Then, the Fatou set of f_λ has no Baker domain.*

Proof. Suppose, on the contrary that the Fatou set of f_λ has a Baker domain B of period p. All the singular values of f_λ are real by Proposition 5.1.2 and $f_\lambda(\mathbb{R}) = (0,\ \lambda]$. Therefore, $S_p(f_\lambda)$ is bounded for each $p > 1$ and the Fatou set of f_λ cannot have a Baker domain of period greater than 1 (c.f. Theorem 1.1.11). Therefore, $p = 1$ *i.e.*, B is an invariant Baker domain. By the definition of an invariant Baker domain, there is a point z^* in the boundary of B such that $\lim_{n\to\infty} f_\lambda^n(z) = z^*$ for all $z \in B$ and $f_\lambda(z^*)$ is not defined. Since the point at infinity is the only point in $\widehat{\mathbb{C}}$ where the function $f_\lambda(z)$ is not defined, $z^* = \infty$. Now, $\lim_{n\to\infty} f_\lambda^n(z) = \infty$ and $f_\lambda^n(z) \in B$ for all $n \in \mathbb{N}$ and $z \in B$ gives that the domain B is unbounded. Since $f_\lambda(\bar{z}) = \overline{f_\lambda(z)}$ for all $z \in \mathbb{C}$ and B is contained in one of the four quadrants by Remark 5.2.3, $\overline{B} = \{\bar{z} \in \mathbb{C}\ :\ z \in B\}$ is also an invariant Baker domain of f_λ. Clearly, one of B and $\overline{B}$ contains points with positive imaginary parts. Let it be B *i.e.*, $\Im(z) > 0$ for each $z \in B$.

We assert that the set $\{\Im(z)\ :\ z \in B\}$ is unbounded. To see it, let on the contrary that $\{\Im(z)\ :\ z \in B\}$ is bounded. Then $\{\Re(z)\ :\ z \in B\}$ must be unbounded as B is itself unbounded. Now, let $\{z_k\}_{k>0}$ be a sequence in B such that $\lim_{k\to\infty} |\Re(z_k)| = \infty$. Then $f_\lambda(z_k) = \frac{2^m z_k^m}{(e^{z_k} - e^{-z_k})^m} \to 0$ as $k \to \infty$ by Remark 5.1.1. The point 0 is in the attracting or parabolic domain for each $\lambda > 0$ by Remark 5.2.2. Let $N(0)$ be a neighbourhood of $z = 0$ completely lying in the Fatou set. Then, there is a natural number $\hat{k}$ such that $f_\lambda(z_k) \in N(0)$ for all $k > \hat{k}$. Consequently, z_k is in a Fatou component U such that $f_\lambda(U)$ is contained in an attracting domain or a parabolic domain and hence not in B for $k > \hat{k}$.

It contradicts the invariance of B. Thus the set $\{\Im(z) : z \in B\}$ is unbounded.

Let B be in the first quadrant of the plane. If B is assumed to lie in the second quadrant, the proof is similar. For $\theta \in (0, \frac{\pi}{2})$, let $S_\theta = \{z \in \mathbb{C} : \theta < Arg(z) < \frac{\pi}{2}\}$ and $S_{\theta'} = \{z \in \mathbb{C} : 0 < Arg(z) < \theta\}$ where $0 < Arg(z) < 2\pi$. Let $L_k = \{z \in \mathbb{C} : \Im(z) = \pi k\}$, $L_k^+ = \{z \in L_k : \Re(z) > 0\}$ and $L_k^- = \{z \in L_k : \Re(z) < 0\}$ for $k \in \mathbb{Z}$. We now show that the set $\{\Im(z) : z \in B \bigcap S_\theta\}$ is unbounded for each $\theta \in (0, \frac{\pi}{2})$. Suppose the set $\{\Im(z) : z \in B \bigcap S_{\theta'}\}$ is unbounded. Then a curve $\gamma : (0, \infty) \to \mathbb{C}$ can be found in $B \bigcap S_{\theta'}$ such that $\Im(z) < (\tan\theta)\Re(z)$ for all $z \in \gamma$ and $\Im(\gamma(t)) \to \infty$ as $t \to \infty$. Consequently, $\Re(\gamma(t)) \to \infty$ and $\left|\frac{\gamma(t)}{\sinh(\gamma(t))}\right| < 2\frac{|\Re(\gamma(t)) + i\Im(\gamma(t))|}{e^{\Re(\gamma(t))} - e^{-\Re(\gamma(t))}} < 2\frac{|(1+\tan\theta)\Re(\gamma(t))|}{e^{\Re(\gamma(t))} - e^{-\Re(\gamma(t))}} \to 0$ as $t \to \infty$. This gives that there is a $t_0 \in (0, \infty)$ such that $f_\lambda(\gamma(t)) \in N(0)$ for $t > t_0$. Consequently, the set $\{\gamma(t) : t > t_0\}$ is not in the Baker domain which is a contradiction. Therefore, the set $\{\Im(z) : z \in B \bigcap S_{\theta'}\}$ cannot be unbounded. Now, since $\{\Im(z) : z \in B\}$ is unbounded, the set $\{\Im(z) : z \in B \bigcap S_\theta\}$ is unbounded. In particular, there exists an integer k_0 such that the set $B \bigcap S_\theta$ intersects L_k^+ for all $k \geq k_0$. Choose θ in such a way that for all $\delta, \beta \in (\theta, \frac{\pi}{2})$, $|m(\delta - \beta)| < \frac{\pi}{4}$ where $f_\lambda(z) = \lambda\frac{z^m}{\sinh^m z}$. Note that

$$f_\lambda(x + i\pi k) = \lambda\frac{(x+i\pi k)^m}{\sinh^m(x+i\pi k)} = \begin{cases} -\lambda\dfrac{(x+i\pi k)^m}{\sinh^m x}, & \text{for odd } k \\[2ex] \lambda\dfrac{(x+i\pi k)^m}{\sinh^m x}, & \text{for even } k \end{cases} \tag{5.3}$$

Let $z_1 = x_1 + i\pi k,\ z_2 = x_2 + i\pi(k+1) \in B \bigcap S_\theta$ for some $k \geq k_0$. If $Arg(z_1) = \theta_1$ and $Arg(z_2) = \theta_2$, then $\theta_1,\ \theta_2 \in (\theta, \frac{\pi}{2})$ and $|Arg(z_1^m) - Arg(z_2^m)| = |m(\theta_1 - \theta_2)| < \frac{\pi}{4}$. Therefore, the two points z_1^m and z_2^m belong to two consecutive quadrants. This means either the real parts or the imaginary parts of z_1^m and z_2^m have same sign. Let the first possibility hold *i.e.*, $\frac{\Re(z_1^m)}{\Re(z_2^m)} > 0$. One of k and $k+1$ is even and another is odd. Also note that $\frac{\lambda}{\sinh^m x} > 0$ for $x > 0$. Using Equation (5.3), we have $\frac{\Re(f_\lambda(z_1))}{\Re(f_\lambda(z_2))} = -\frac{\Re(z_1^m)}{\Re(z_2^m)} < 0$. In other words, $\Re(f_\lambda(z_1))$ and $\Re(f_\lambda(z_2))$ have opposite sign. Thus $f_\lambda(B) = B$ intersects the imaginary axis which contradicts Remark 5.2.3. For $\frac{\Im(z_1^m)}{\Im(z_2^m)} > 0$, arguing similarly, we can get $\frac{\Im(f_\lambda(z_1))}{\Im(f_\lambda(z_2))} < 0$ which also results in a similar contradiction to Remark 5.2.3. Therefore, the Fatou set of f_λ does not contain any Baker domain. $\square$

Theorem 5.3.2. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$.*

1. *For $\lambda < \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the basin of attraction of the unique real attracting fixed point a_λ of f_λ.*

2. *For $\lambda = \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the parabolic basin corresponding to the unique real rationally indifferent fixed point x^* of f_λ.*

3. *For $\lambda > \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the basin of attraction or parabolic basin corresponding to a cycle of real 2-periodic points $\{a_{1\lambda},\ a_{2\lambda}\}$ of f_λ.*

Proof. By Remark 5.2.2, the set $P(f_\lambda) \setminus \{\infty\}$ is in the Fatou set of f_λ and the derived set $P(f_\lambda)' \subset \mathbb{R}$. The set $P(f_\lambda)'$ contains ∞ and at most finitely many pre-periodic points. By Remark 2 of Theorem 3 in [139], if for a meromorphic function f, f^n has finitely many limit functions on a wandering domain then each such limit function is constant and in the backward orbit of ∞. Therefore, if a point z_0 is in a wandering domain of f_λ, then every limit point of $\{f_\lambda^n(z_0)\}_{n>0}$ is ∞ (c.f. Theorem 1.1.12). In other words, $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$ if z_0 is in a wandering domain. Since $S_2(f_\lambda)$ is bounded, $f_\lambda^{2n}(z_0)$ cannot tend to ∞ as $n \to \infty$ by the main result of [108]. This is a contradiction. Therefore, the Fatou set of f_λ does not contain any wandering domain. We know that the boundary of a rotational domain of a meromorphic function f is contained in the closure of $P(f)$ (c.f. Theorem 7, [19]). Since the closure of $P(f_\lambda)$ intersects $\mathcal{J}(f_\lambda)$ at finitely many points, the Fatou set of f_λ does not contain any rotational domain. By Theorem 5.3.1, the Fatou set of f_λ also does not contain any Baker domain for $\lambda > 0$.

If U is an attracting domain or parabolic domain of period p and z_u is the corresponding attracting or rationally indifferent periodic point of f_λ, then there is a singular value s of f_λ

such that $f_\lambda^{np}(f_\lambda^k(s)) \to z_u$ as $n \to \infty$ for some $k,\ 0 < k \leq p$. Since all the singular values and their forward orbits (whenever defined) are in $\mathbb{R}$, z_u is real. Therefore, any attracting or parabolic domain of f_λ corresponds to a real attracting or rationally indifferent periodic point.

1. For $0 < \lambda < \lambda^*$, f_λ has only one real periodic point which is the attracting fixed point a_λ. Therefore, $\mathcal{F}(f_\lambda)$ is the basin of attraction of a_λ.

2. For $\lambda = \lambda^*$, f_λ has only one real periodic point which is the rationally indifferent fixed point x^*. Therefore, $\mathcal{F}(f_\lambda)$ is the parabolic basin corresponding to x^*.

3. For $\lambda > \lambda^*$, f_λ has a repelling fixed point r_λ and a cycle of real 2-periodic points $\{a_{1\lambda},\ a_{2\lambda}\}$ which is either attracting or rationally indifferent. Therefore, $\mathcal{F}(f_\lambda)$ is the attracting basin or parabolic basin corresponding to $\{a_{1\lambda},\ a_{2\lambda}\}$.

$\square$

Remark 5.3.1. *Consider the map $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(y) = \frac{y}{\sin y}$. Then $|\psi(y)| > 1$ for $y > 1$. Therefore, $|f_\lambda(iy)| = |\lambda \frac{(iy)^m}{(i \sin y)^m}| = |\lambda(\psi(y))^m| > \lambda$ for $\lambda > 0$ and for all $y > 1$. Since $\sec^2 y - 1 > 0$ in $(0,\ 1)$, the function $\mu(y) = \tan y - y$ is increasing and $\mu(y) > \mu(0) = 0$ for $y \in (0,\ 1)$. For $y \in (0,\ 1)$, $\mu(y) = \frac{\sin y - y\cos y}{\cos y} > 0$ gives $\sin y - y \cos y > 0$ because $\cos y > 0$. Therefore, $\psi'(y) = \frac{\sin y - y\cos y}{\sin^2 y} > 0$ and $\psi(y)$ is increasing in $(0,\ 1)$. Since $\psi(y) \to 1$ as $y \to 0$, $\psi(y) > 1$ for $y \in (0,\ 1)$. Hence $|f_\lambda(iy)| = |\lambda(\psi(y))^m| > \lambda$ for all $y > 0$. As the function $f_\lambda(z)$ is even, $|f_\lambda(iy)| > \lambda$ for all $y \in \mathbb{R}$ and $\lambda > 0$.*

5.4 Topology of the Fatou components

The present section deals with some topological issues pertaining to the Fatou components of f_λ. It is observed from Theorem 5.3.2 that the Fatou set of f_λ can contain components with period one and two. The connectivity of a periodic Fatou component of a meromorphic function is either 1, 2 or ∞ whereas the connectivity of a pre-periodic Fatou component can be any finite number [13]. In Theorem 5.4.1, it is proved that the Fatou set of $f_\lambda,\ 0<\lambda<\lambda^*$ is infinitely connected. The existence of pre-periodic Fatou components is established and the connectivity of all the Fatou components of f_λ is determined for $\lambda>\lambda^*$ in Theorem 5.4.2.

Theorem 5.4.1. *Let $f_\lambda \in \mathcal{S}$ and $0<\lambda<\lambda^*$. Then, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is connected. Further, the Fatou set $\mathcal{F}(f_\lambda)$ is infinitely connected.*

Proof. By Theorem 5.2.2(1), $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for $x \in \mathbb{R}$ and $0<\lambda<\lambda^*$ where a_λ is the attracting fixed point of f_λ. The Fatou set of f_λ is the attracting basin $A(a_\lambda) = \{z \in \mathbb{C} \ : \ f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}$ for $0<\lambda<\lambda^*$. Let $I(a_\lambda)$ be the immediate basin of attraction of a_λ. By definition, $I(a_\lambda)$ is a forward invariant connected subset of the Fatou set $\mathcal{F}(f_\lambda)$ containing a_λ. Note that $A(a_\lambda) = I(a_\lambda)$ if $I(a_\lambda)$ is backward invariant. Since $I(a_\lambda)$ is connected, in order to prove the connectedness of $\mathcal{F}(f_\lambda)$, it is sufficient to show that $I(a_\lambda)$ is backward invariant.

Let, if possible, V be a component of $f_\lambda^{-1}(I(a_\lambda))$ other than $I(a_\lambda)$. Since 0 is an omitted value of f_λ, there is only one singularity of f_λ^{-1} lying over 0 and that is logarithmic. This means that V contains an asymptotic path γ corresponding to the asymptotic value 0 and by Remark 5.1.1, the set $\{\Re(z) \ : \ z \in \gamma\}$ is unbounded. Therefore, the set $\{\Re(z) \ : \ z \in V\}$ is unbounded. The function f_λ is even and $f_\lambda(\bar{z}) = \overline{f_\lambda(z)}$ for all $z \in \mathbb{C}$. In view of Remark 5.2.3, it is assumed without loss of generality that, the set V is in the upper half-plane $\{z \in \mathbb{C} \ : \ \Im(z) > 0\}$. Let $\{w_n\}_{n>0}$ be a sequence on γ such that $\Re(w_n) \to \infty$ as $n \to \infty$. Then $\lim_{n\to\infty} f_\lambda(w_n) = 0$. Each of the vertical lines $l_n = \{z \in \mathbb{C} \ : \ \Re(z) = \Re(w_n) \text{ and } 0 \leq \Im(z) < \Im(w_n)\}$ joins a point of V and a point of $\mathbb{R} \bigcap I(a_\lambda)$ and we get that l_n intersects the boundary ∂V of V for each n. Let $z_n \in l_n \bigcap \partial V$.

Then $z_n \in \mathcal{J}(f_\lambda)$ and $\Im(z_n) < \Im(w_n)$ for all n. Further,

$$|f_\lambda(z_n)| = \lambda \left\{ \left(\frac{\Re(z_n)^2 + \Im(z_n)^2}{\sinh^2 \Re(z_n) + \sin^2 \Im(z_n)} \right)^{\frac{1}{2}} \right\}^m < \lambda \left\{ \left(\frac{\Re(w_n)^2 + \Im(w_n)^2}{\sinh^2 \Re(w_n) + \sin^2 \Im(z_n)} \right)^{\frac{1}{2}} \right\}^m \tag{5.4}$$

Since the sequence $\{\sin^2(\Im(z_n))\}_{n>0}$ is bounded, the right hand side of Equation (5.4) is equal to $|f_\lambda(w_n)|$ when $n \to \infty$. Therefore, $\lim_{n\to\infty} f_\lambda(z_n) = 0$. Let $D_r(0) = \{z \in \mathbb{C} \ : \ |z| < r\} \subset I(a_\lambda)$. Then, there exists an n_0 such that $f_\lambda(z_n) \in D_r(0)$ for all $n > n_0$. This means that z_n is in the Fatou set of f_λ for $n > n_0$ which is a contradiction. Therefore, each component of $f_\lambda^{-1}(I(a_\lambda))$ intersects $I(a_\lambda)$ and hence is a subset of $I(a_\lambda)$. Thus $I(a_\lambda)$ is backward invariant.

Since $\mathcal{F}(f_\lambda)$ is connected and contains an attracting fixed point, it is invariant. The connectivity of any invariant Fatou component of a meromorphic function is 1, 2 or ∞, 2 being the case when the component is an Herman ring (c.f. Theorem 1.1.17). Since the Fatou set $\mathcal{F}(f_\lambda)$ is an attracting domain for $0 < \lambda < \lambda^*$, the connectivity of $\mathcal{F}(f_\lambda)$ is either 1 or ∞. If possible, let $\mathcal{F}(f_\lambda)$ be simply connected. Then, the Julia set $\mathcal{J}(f_\lambda)$ is connected. As the point at ∞ and a pole w^* lying on the imaginary axis are in $\mathcal{J}(f_\lambda)$, there is an unbounded connected subset J_{w^*} of the Julia set containing w^*. Now, $\overline{-J_{w^*}} = \{z \in \mathbb{C} \ : \ -\overline{z} \in J_{w^*}\}$ is also in the Julia set by Proposition 5.1.1. Thus $J = J_{w^*} \bigcup \overline{-J_{w^*}}$ is in the Julia set and the set $\widehat{\mathbb{C}} \setminus J$ has at least two components each intersecting the Fatou set of f_λ. It contradicts the fact that $\mathcal{F}(f_\lambda)$ is connected. Therefore, $\mathcal{F}(f_\lambda)$ is infinitely connected for $0 < \lambda < \lambda^*$. $\square$

Remark 5.4.1. *The Fatou set of f_{λ^*} is the parabolic basin corresponding to the real rationally indifferent fixed point x^*. Since $f_{\lambda^*}(-x^*,\ x^*) \subset (x^*,\ \infty)$ and each petal corresponding to a rationally indifferent fixed point is forward invariant, the set $[0,\ \infty) \setminus \{x^*\}$ is in the same petal, say P. Further, $0 \in P$. Now, proceeding as in Theorem 5.4.1, it can be concluded that, the Fatou set $\mathcal{F}(f_{\lambda^*})$ is infinitely connected.*

Remark 5.4.2. *Since the Fatou set is connected with connectivity greater than three for $0 < \lambda \leq \lambda^*$, singleton components of $\mathcal{J}(f_\lambda)$ are dense in $\mathcal{J}(f_\lambda)$ (c.f. Theorem 1.1.22).*

It is seen in Theorem 5.4.1 that the Fatou set of f_λ is connected and hence unbounded for $0 < \lambda < \lambda^*$. The next proposition shows that, there are at least three Fatou components of f_λ two of which are unbounded for $\lambda > \lambda^*$.

Proposition 5.4.1. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > \lambda^*$. If U^+, U^- and U_0 denote the Fatou components containing $(a_{2\lambda}, +\infty)$, $(-\infty,\ -a_{2\lambda})$ and 0 respectively where $\{a_{1\lambda}, a_{2\lambda}\}$ is the 2-cycle of real periodic points of f_λ, then the Fatou components U^+, U^- and U_0 are mutually disjoint. Further, the components U^+ and U^- are unbounded.*

Proof. Observe that both U^+ and U^- are mapped into U_0 and U_0 is mapped into U^+ by f_λ for $\lambda > \lambda^*$. Since U_0 and U^+ form a cycle of 2-periodic Fatou components, $U_0 \neq U^+$. If U_0 intersects U^-, then $U_0 = U^-$ will become invariant which is not true. Therefore, U_0 is different from U^+ and U^-. If U^+ and U^- are the same component of $\mathcal{F}(f_\lambda)$, then $U^+ = U^-$ intersects the imaginary axis. Then, since all the points in the imaginary axis are mapped onto $\mathbb{R}\setminus(-\lambda,\ \lambda) \subset (-\infty,\ -a_{2\lambda}) \bigcup (a_{2\lambda},\ +\infty)$, the points of the set $U^+ \bigcap \{iy\ :\ y \in \mathbb{R}\}$ are mapped into U^+ and consequently, U^+ is invariant leading to a contradiction. Therefore, U_0, U^+ and U^- are mutually disjoint components of $\mathcal{F}(f_\lambda)$ for $\lambda > \lambda^*$. The components U^- and U^+ are unbounded by definition. $\square$

Theorem 5.4.2. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > \lambda^*$. Then, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ contains infinitely many pre-periodic components and each component of $\mathcal{F}(f_\lambda)$ is simply connected.*

Proof. It is clear from Theorem 5.2.2 that the point $0 \in \mathcal{F}(f_\lambda)$ for all λ. Let U_0 be the Fatou component containing 0. If $\lambda > \lambda^*$ and $\{a_{1\lambda},\ a_{2\lambda}\}$ is the 2-cycle of real periodic points of f_λ, then by Theorem 5.2.2, $(-\infty,\ -a_{2\lambda})$ and $(a_{2\lambda},\ +\infty)$ are in the Fatou set of f_λ. Let U^- and U^+ be the Fatou components of f_λ containing $(-\infty,\ -a_{2\lambda})$ and $(a_{2\lambda},\ +\infty)$

respectively. If a pre-image of a point of U^- lies in U^-, then $U^- \bigcap f_\lambda(U^-) \neq \emptyset$ which shows that $U^- = f_\lambda(U^-)$ since $f_\lambda(U^-)$ is connected. This means that U^- is forward invariant. But it is already known that U^- is not forward invariant. Therefore, no pre-image of any point of U^- lies in U^-. In other words, U^- is not backward invariant. Since none of U_0 and U^+ is mapped into U^- by f_λ, each component of $f_\lambda^{-1}(U^-)$ is different from U_0 and U^+ and consequently, is a pre-periodic Fatou component. Repeating the same arguments for each component of $f_\lambda^{-1}(U^-)$ and continuing the process, we can find infinitely many pre-periodic Fatou components.

Let U be any Fatou component of f_λ. Suppose, on the contrary that U is multiply connected. Let γ be a simple closed curve in U such that the bounded component $B(\gamma^c)$ of $\gamma^c = \widehat{\mathbb{C}} \setminus \gamma$ intersects the Julia set $\mathcal{J}(f_\lambda)$. Set $B_j = f_\lambda^j(B(\gamma^c))$ for $j \in \mathbb{N}$. If $B(\gamma^c)$ does not contain a pole of f_λ, then $f_\lambda(z)$ is analytic on $\overline{B(\gamma^c)}$, the closure of $B(\gamma^c)$ and $B_1 = f_\lambda(B(\gamma^c))$ is bounded. Further, the function $f_\lambda(z)$ maps the interior of $B(\gamma^c)$ (which intersects the Julia set) into the interior of B_1 and, by the complete invariance of $\mathcal{J}(f_\lambda)$, it follows that $B_1 \bigcap \mathcal{J}(f_\lambda) \neq \emptyset$. If B_1 does not contain any pole of f_λ, then consider $B_2 = f_\lambda(B_1)$ and repeat the above arguments. Since the pre-images of all the poles of f_λ are dense in $\mathcal{J}(f_\lambda)$, $B(\gamma^c)$ contains a point $\tilde{w}$ such that $f_\lambda^{n^*}(\tilde{w})$ is a pole of f_λ for a natural number n^*. Let n^* be the least of such natural numbers. Then, the set B_{n^*} contains a pole. Since all the poles of f_λ are on the imaginary axis, the boundary of B_{n^*} intersects the imaginary axis. Therefore, the set $B_{n^*+1} = f_\lambda(B_{n^*})$ contains a neighbourhood of ∞ and the unboundedness of U^+ and U^- gives that B_{n^*+1} intersects both U^+ and U^-. Further, the f_λ-image of $\partial(B_{n^*})$ intersect at least one of U^+ or U^- by Remark 5.3.1. This implies that, there are some points in B_{n^*} which are mapped into U^+ and others into U^- by $f_\lambda(z)$. There are Fatou components V^+ and V^- in B_{n^*} which are mapped into U^+ and U^- respectively (c.f. Theorem 1.1.6). Being mapped to two different Fatou components, V^+ and V^- are also different. Note that $\partial B_{j+1} \subseteq f_\lambda(\partial B_j)$ for $j = 1,\ 2,\ 3, ...,\ n^*$. Therefore, $\partial(B_{n^*}) \subseteq$

$f_\lambda^{n^*-1}(\partial B_1) \subseteq f_\lambda^{n^*}(\gamma) \subset \mathcal{F}(f_\lambda)$. Consequently, $f_\lambda(\partial(B_{n^*})) \subset \mathcal{F}(f_\lambda)$ and either lies in U^+ or U^-. Let $f_\lambda(\partial B_{n^*}) \subset U^-$. Then f_λ maps V^+ into U^+ and the set $U^+ \setminus f_\lambda(V^+)$ contains at most two elements (c.f. Theorem 1.1.6). Now, U^+ contains infinitely many critical values of f_λ and each of these critical values (except possibly two) has at least one pre-image in V^+. Therefore V^+ contains infinitely many critical points of f_λ. These infinitely many critical points necessarily form an unbounded subset of V^+ which contradicts to the fact that $V^+ \subset B_{n^*}$ is bounded. Therefore, U is simply connected. $\square$

A comparison between the dynamics of $\lambda\frac{\sinh^m z}{z^m}$, $\lambda\tanh(e^z)$ and $\lambda(e^z+1+\frac{1}{e^z+1})$ is given in the Table 5.1.

Dynamics of $f_\lambda(z) = \lambda\frac{z^m}{\sinh^m z},\ \lambda \neq 0,\ m \in \mathbb{N}$	**Dynamics of** $g_\lambda(z) = \lambda\tanh(e^z),\ \lambda \neq 0$	**Dynamics of** $h_\lambda(z) = \lambda J(e^z+1),\ \lambda > 0$
The order of f_λ is 1.	The order of g_λ is ∞.	The order of h_λ is 1.
f_λ is not of bounded type.	g_λ is of finite type.	h_λ is of finite type.
f_λ has infinitely many critical values all lying in $\{x \in \mathbb{R} : \lvert x\rvert > \lvert\lambda\rvert\}$.	g_λ has no critical values.	h_λ has one critical value -2λ.
f_λ has one asymptotic value 0.	g_λ has three asymptotic values $0,\ \lambda$ and $-\lambda$.	h_λ has one asymptotic value 2λ.
f_λ is not periodic.	g_λ is periodic with period $2\pi i$.	h_λ is periodic with period $2\pi i$.
f_λ is even.	g_λ is neither even nor odd.	h_λ is neither even nor odd.
Bifurcation in the dynamics of f_λ occurs at two critical parameters $\lambda_f^* > 0$ and $-\lambda_f^*$. For $m = 1,\ 2,\ 3$, λ_f^* is approximately equal to $3.3198,\ 2.1772$ and 1.7926 respectively.	Bifurcation in the dynamics of g_λ occurs at one critical parameter $\lambda_g^* \approx -3.2946$.	Bifurcation in the dynamics of h_λ occurs at one critical parameter $\lambda_h^* \approx 0.27$.
The Fatou set of f_λ is infinitely connected for $\lvert\lambda\rvert < \lambda^*$.	The Fatou set of g_λ has infinitely many components and each is simply connected for $\lambda < \lambda_g^*$.	The Fatou set of h_λ is infinitely connected for $0 < \lambda < \lambda_h^*$.
The Julia set of f_λ has a dense subset of singleton components for $\lvert\lambda\rvert < \lambda^*$.	The Julia set of g_λ has no singleton components for $\lambda < \lambda_g^*$.	The Julia set of h_λ has a dense subset of singleton components for $0 < \lambda < \lambda_h^*$.
The Fatou set of f_λ has infinitely many components and each component is simply connected for $\lvert\lambda\rvert > \lambda^*$.	The Fatou set of g_λ has only one component and it is infinitely connected for $\lambda > \lambda_g^*$.	The Fatou set of h_λ is empty set for $\lambda > \lambda_h^*$.

Table 5.1: Comparison between the dynamics of $\lambda\frac{z^m}{\sinh^m z}$, $\lambda\tanh(e^z)$ and $\lambda J(e^z+1)$.

Chapter 6

Dynamics of certain real meromorphic functions

The dynamics of certain real meromorphic functions (which takes real values only on the real line) are investigated in this chapter. If a real meromorphic function $\psi(z)$ maps the upper half-plane onto the upper half-plane, then its poles a_k $(\pm k = 0,\ 1,\ 2, ...)$ are all real and simple, and it may be represented in the form $\psi(z) = a + bz + \sum\limits_{k=\alpha}^{\beta} A_k\Big(\dfrac{1}{a_k - z} - \dfrac{1}{a_k}\Big)$ $(-\infty \leq \alpha < \beta \leq \infty)$, where $b \geq 0$, a is real, $A_k \geq 0$ $(\pm k = 0,\ 1,\ 2, ...)$ and the series $\sum_{k=\alpha}^{\beta} \frac{A_k}{a_k^2}$ converges [85]. Note that the function $\psi(z)$ is a rational function if and only if both α and β are finite. Let $H^+ = \{z \in \mathbb{C} \ : \ \Im(z) > 0\}$ and $H^- = \{z \in \mathbb{C} \ : \ \Im(z) < 0\}$. We consider only the transcendental real meromorphic functions satisfying $f(H^+) \subseteq H^+$. For the dynamical study of functions for which $f(H^+) \subseteq H^-$, the results obtained in this chapter follow with simple modifications using the fact that $f^2(H^+) \subseteq H^+$ and $f^2(H^-) \subseteq H^-$. Define

$$\mathfrak{R} \equiv \left\{ f(z) : \quad \begin{array}{l} \text{(i) } f(z) = \displaystyle\sum_{k=-\infty}^{\infty} A_k \left(\frac{1}{a_k - z} - \frac{1}{a_k} \right) \\ \text{(ii) } A_k > 0,\ a_k \neq 0 \text{ for } k \in \mathbb{Z} \\ \text{(iii) } \displaystyle\sum_{k=-\infty}^{\infty} \frac{A_k}{a_k^2} \text{ converges} \end{array} \right\}.$$

Note that, if $f \in \mathfrak{R}$, then the point $z = 0$ is a fixed point and $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$. The class $\mathfrak{R}$ is closed under addition and the function $\lambda f \in \mathfrak{R}$ for $f \in \mathfrak{R}$ and $\lambda > 0$.

If $a_{-k} = -a_k$ and $A_k = A_{-k}$ for all $k \in \mathbb{N}$ and $A_0 = 0$ for a function $f \in \mathcal{R}$, then $A_{-k}\left(\frac{1}{a_{-k}-z} - \frac{1}{a_{-k}}\right) + A_k\left(\frac{1}{a_k-z} - \frac{1}{a_k}\right) = A_k\left(\frac{1}{-a_k-z} - \frac{1}{-a_k}\right) + A_k\left(\frac{1}{a_k-z} - \frac{1}{a_k}\right) = A_k \frac{2z}{a_k^2-z^2}$. In this case, the function $f(z)$ assumes the form $\sum_{k=1}^{\infty} \frac{A_k z}{a_k^2-z^2}$.

Let

$$\mathcal{R}^* \equiv \left\{ f(z) : \begin{array}{l} \text{(i) } f(z) = \displaystyle\sum_{k=1}^{\infty} \frac{A_k z}{a_k^2 - z^2} \\ \text{(ii) } A_k > 0,\ a_k \neq 0 \text{ for } k \in \mathbb{N} \\ \text{(iii) } \displaystyle\sum_{k=1}^{\infty} \frac{A_k}{a_k^2} \text{ converges} \end{array} \right\}.$$

Observe that $\mathcal{R}^* \subset \mathcal{R}$ and the functions in $\mathcal{R}^*$ preserve the imaginary axis. Important examples of functions in $\mathcal{R}^*$ include $\tan z = \displaystyle\sum_{k=1}^{\infty} \frac{z}{(\frac{2k-1}{2}\pi)^2 - z^2}$, $\dfrac{3}{z} - \dfrac{z \sin z}{\sin z - z\cos z} = \displaystyle\sum_{k=1}^{\infty} \frac{2z}{a_k^2 - z^2}$ where a_k's are positive roots of $\tan z = z$ and $\dfrac{1}{2i} + \dfrac{1}{z} + \dfrac{1}{i(e^{iz}-1)} = \displaystyle\sum_{k=1}^{\infty} \frac{2z}{4k^2\pi^2 - z^2}$.

If f is a transcendental real meromorphic function, then either (i) $\mathcal{J}(f) = \mathbb{R}\bigcup\{\infty\}$, (ii) $\mathcal{J}(f)$ is an unbounded Cantor set or (iii) $\mathcal{J}(f)$ is the complement of an interval in $\mathbb{R}\bigcup\{\infty\}$ [11]. However, the nature of the Fatou sets and related topological issues remain unexplored. In the present chapter, we investigate the change in the nature of the Fatou set of transcendental real meromorphic functions in the family

$$\mathcal{S} = \{h_{a,b,c}(z) = a + bz - \frac{c}{z} + f(z) \ : \ a,\ b,\ c \in \mathbb{R},\ b,\ c \geq 0 \text{ and } f \in \mathcal{R}\}.$$

A number of examples are discussed. Finally, the dynamics of functions in the one parameter family $\{a + \tan z \ : \ a \in \mathbb{C}\}$ is explored though these functions are not real meromorphic.

6.1 Dynamics of $h_{a,b}$ and h_a

In this section, the change in the Fatou set of $h_{a,b} \equiv h_{a,b,c}$ with $c = 0$ and $h_a \equiv h_{a,b,c}$ with $b = c = 0$ are investigated where $h_{a,b,c} \in \mathcal{S}$. Define $p^* = h'_{a,b}(0) = b + \sum_{k=-\infty}^{\infty} \frac{A_k}{a_k^2}$.

Proposition 6.1.1. *Let $h_{a,b}(z) = a + bz + f(z)$ for $a \in \mathbb{R}$, $b \geq 0$ and $f \in \mathcal{R}$. Then*

1. *The Fatou set $\mathcal{F}(h_{a,b})$ contains neither a wandering domain nor a rotational domain.*

2. *Each attracting or rationally indifferent periodic point of $h_{a,b}(z)$ is a fixed point. Further, if the function $h_{a,b}(z)$ preserves the imaginary axis, then the fixed point lies on the imaginary axis.*

3. *The function $h_{a,b}(z)$ has at most one real attracting or rationally indifferent fixed point.*

4. *If each real fixed point of $h_{a,b}(z)$ is repelling, then the Fatou set $\mathcal{F}(h_{a,b})$ of $h_{a,b}$ is $H^+ \bigcup H^-$ or a completely invariant Baker domain. In particular, the conclusion remains true if $b \geq 1$.*

5. *For $a = 0$, the Fatou set $\mathcal{F}(h_{a,b})$ is an attracting domain if $p^* < 1$ and a parabolic domain if $p^* = 1$.*

6. *If $|h'_{a,b}(x)| > 1$ for $x \in \mathbb{R}$ whenever $h'_{a,b}(x)$ is defined and if the set $\{|x_1 - x_2| \,:\, h_{a,b}(x_1) = \infty = h_{a,b}(x_2)$ and $h_{a,b}(z)$ has no pole in $(x_1,\ x_2)\}$ is bounded, then $\mathcal{F}(h_{a,b}) = H^+ \bigcup H^-$.*

7. *If the Fatou set $\mathcal{F}(h_{a,b})$ is connected, then it is infinitely connected.*

Proof. As both H^+ and H^- are completely invariant under $h_{a,b}$, H^+ and H^- must be in the Fatou set $\mathcal{F}(h_{a,b})$ by Theorem 1.1.1. Therefore, the Fatou set $\mathcal{F}(h_{a,b})$ has either two completely invariant Fatou component namely H^+ and H^- or it is connected.

1. If there is a wandering domain or a rotational domain in $\mathcal{F}(h_{a,b})$, then there must be infinitely many Fatou components which is not possible. Therefore, the Fatou set $\mathcal{F}(h_{a,b})$ contains neither a wandering domain nor a rotational domain.

2. Let w_0 be an attracting or rationally indifferent periodic point of $h_{a,b}(z)$ with prime period p. Then, there are at least p number of p-periodic components in the Fatou set of $h_{a,b}$. But the Fatou set $\mathcal{F}(h_{a,b})$ is either connected or consists of two completely invariant components. Therefore, $p = 1$ and w_0 is a fixed point of $h_{a,b}(z)$.

   Let $w_0 \in \mathbb{C} \setminus \{iy \ : \ y \in \mathbb{R}\}$ be an attracting or a rationally indifferent fixed point of $h_{a,b}(z)$. Then each neighbourhood $N(w_0)$ of w_0 intersects at least one of H^+ or H^-. Let $N(w_0) \bigcap H^+ \neq \emptyset$. Since $H^+ \subset \mathcal{F}(h_{a,b})$, $\lim_{n\to\infty} h_{a,b}^n(z) = w_0$ for all $z \in H^+$. If $z \in i\mathbb{R}^+ = \{iy \ : \ y > 0\}$, then $h_{a,b}^n(z) \in i\mathbb{R}^+$ for all $n \in \mathbb{N}$. This gives that $\lim_{n\to\infty} h_{a,b}^n(z) \neq w_0$ for $z \in i\mathbb{R}^+$ leading to a contradiction. A contradiction can be arrived by similar arguments if $N(w_0) \bigcap H^- \neq \emptyset$. Hence, we conclude that the fixed point w_0 is on the imaginary axis.

3. Any two real attracting or rationally indifferent fixed points give rise to more than one Fatou component intersecting H^+ which is not possible because $H^+ \subset \mathcal{F}(h_{a,b})$. Therefore, the function $h_{a,b}(z)$ has at most one real attracting or rationally indifferent fixed point.

4. If each real fixed point of $h_{a,b}(z)$ is repelling, then it follows from (2) of this proposition that $h_{a,b}(z)$ has no real attracting or rationally indifferent periodic point. Since both H^+ and H^- are in the Fatou set, $\mathcal{J}(h_{a,b}) \subseteq \mathbb{R}$. If the Fatou set $\mathcal{F}(h_{a,b})$ is not equal to $H^+ \bigcup H^-$, then $\mathcal{F}(h_{a,b}) \bigcap \mathbb{R} \neq \emptyset$ and the Fatou set $\mathcal{F}(h_{a,b})$ is connected. Now, if the Fatou set is an attracting domain or a parabolic domain, then it must correspond to a real attracting or rationally indifferent fixed point since $h_{a,b}(\mathbb{R}) \subseteq \mathbb{R}$. It is not possible. Therefore, the Fatou set $\mathcal{F}(h_{a,b})$ is not an attracting domain or a parabolic domain. The Fatou set is non-empty and does not contain any rotational domain or wandering domain. Therefore, the Fatou set $\mathcal{F}(h_{a,b})$ is a completely invariant Baker domain.

For $b \geq 1$, $h'_{a,b}(z) = b + \sum_{k=-\infty}^{+\infty} \frac{A_k}{(a_k - z)^2} > 1$ when $z \in \mathbb{R}$ and each real fixed point of $h_{a,b}(z)$ is repelling. Thus, the above conclusion remains true.

5. For $a = 0$, the point $z = 0$ is a fixed point of $h_{a,b}(z)$. The fixed point $z = 0$ is attracting for $p^* < 1$ and rationally indifferent for $p^* = 1$. Now for $p^* < 1$, any neighbourhood of 0 intersects both H^+ and H^- which means that the Fatou set of $h_{a,b}$ is the attracting domain of 0 when $p^* < 1$. Similarly, the Fatou set of $h_{a,b}$ is the parabolic domain corresponding to the rationally indifferent fixed point 0 for $p^* = 1$.

6. Let, on the contrary $\mathcal{F}(h_{a,b}) \bigcap \mathbb{R} \neq \emptyset$. Suppose $I = (c,\ d) \subset \mathbb{R} \subset \mathcal{F}(h_{a,b})$. The function $h_{a,b}(z)$ is increasing in each interval between consecutive poles because $|h'_{a,b}(z)| > 1$ for $z \in \mathbb{R}$ whenever $h'_{a,b}(z)$ is defined. Then, $h_{a,b}(d) - h_{a,b}(c) = k(d - c)$ for some $k > 1$. If $h_{a,b}(I)$ does not contain a pole of $h_{a,b}(z)$, then $h^2_{a,b}(d) - h^2_{a,b}(c) = k^2(d - c)$. Observe that $k^n \to \infty$ as $n \to \infty$ and the set $\{|x_1 - x_2| \ :\ h_{a,b}(x_1) = \infty = h_{a,b}(x_2)$ and $h_{a,b}(z)$ has no pole in $(x_1,\ x_2)\}$ is bounded. Repeating this process, it is seen that there is a natural number n_0 such that $h^{n_0}_{a,b}(I)$ contains a pole of $h_{a,b}(z)$ which is a contradiction to the complete invariance of the Fatou set. Therefore, $\mathcal{F}(h_{a,b}) = H^+ \bigcup H^-$.

7. Let $\mathcal{F}(h_{a,b})$ be connected. Then $\mathcal{F}(h_{a,b}) \bigcap \mathbb{R} \neq \emptyset$. Suppose that $I = (p,\ q)$ where p and q are two consecutive poles of $h_{a,b}$. Observe that $\lim_{x \to p^+} h_{a,b}(x) = -\infty$ and $\lim_{x \to q^-} h_{a,b}(x) = +\infty$ because $h_{a,b}(z)$ is increasing in I. Therefore, $h_{a,b}(I) = \mathbb{R}$ and by the complete invariance of the Fatou set, it follows that the interval I intersects the Fatou set. Consequently, the Julia set $\mathcal{J}(h_{a,b})$ consists of infinitely many maximally connected sets and $\mathcal{F}(h_{a,b})$ is infinitely connected.

$\square$

Remark 6.1.1. *Proposition 6.1.1(1), (3), (4), (6) and (7) remain true for every real meromorphic function.*

Let $h_a(z) = a + f(z)$ for $f \in \mathcal{R}$ and $a \in \mathbb{R}$. Then $h'_a(z) = f'(z) > 0$ for all $z \in \mathbb{R}$. Let

$$J = \{x \in \mathbb{R} \ : \ 0 < f'(x) < 1\}.$$

If p and q are two consecutive poles of $f(z)$, then $f'(z)$ is continuous in (p, q) and undefined at p and q. For next two theorems, let $\varphi(x) = x - f(x)$ for $x \in \mathbb{R}$. If the set J is empty, then $h'_a(x) = f'(x) \geq 1$ for all $x \in \mathbb{R}$ whenever $f'(x)$ is defined and the dynamics of h_a is discussed in the following theorem.

Theorem 6.1.1. *Let $h_a(z) = a + f(z)$ for $f \in \mathcal{R}$ and $a \in \mathbb{R}$, and $J^* = \{x \in \mathbb{R} \ : \ f'(x) = 1\}$. Suppose that the set $J = \{x \in \mathbb{R} \ : \ 0 < f'(x) < 1\}$ is empty.*

1. *If $J^* = \emptyset$ or $a \notin \varphi(J^*)$, then the Fatou set $\mathcal{F}(h_a)$ is either $H^+ \bigcup H^-$ or a completely invariant Baker domain.*

2. *If $J^* \neq \emptyset$ and $a \in \varphi(J^*)$, then the Fatou set $\mathcal{F}(h_a)$ is a parabolic domain corresponding to a real rationally indifferent fixed point.*

Proof. The Fatou set $\mathcal{F}(h_a)$ does not contain any wandering or rotational domains by Proposition 6.1.1(1).

1. If $J^* = \emptyset$, then $h'_a(x) > 1$ for $x \in \mathbb{R}$ whenever $h'_a(x)$ is defined and consequently, each real fixed point of $h_a(z)$ is repelling. If $a \notin \varphi(J^*)$, then $h_a(x) \neq x$ for $x \in J^*$. This means that each real fixed point of h_a lie in $\mathbb{R} \setminus J^*$ and consequently, is repelling. Now, applying Proposition 6.1.1(4) it is concluded that, the Fatou set is either $H^+ \bigcup H^-$ or a completely invariant Baker domain.

2. Let $J^* \neq \emptyset$ and $a \in \varphi(J^*)$. Then, there is a point $x_a \in J^*$ such that $\varphi(x_a) = x_a - f(x_a) = a$. This means that $h_a(x_a) = x_a$ and $h'_a(x_a) = f'(x_a) = 1$. Any neighbourhood of the rationally indifferent fixed point x_a intersects H^+ and H^-. Therefore, the Fatou set $\mathcal{F}(h_a)$ is the parabolic domain corresponding to x_a.

$\square$

If the set J is non-empty then it can be written as a union of countably many intervals *i.e.*, $J = \bigcup_{n \in K} J_n$ where $K \subseteq \mathbb{Z}$ is some index set. Note that $f'(x) = 1$ only when x belongs to the boundary of some J_n and $\varphi'(x) = 1 - f'(x) \geq 0$ for $x \in J_n$. Therefore, $\varphi(x)$ is increasing in J_n for all $n \in K$. Set $I_n = \varphi(J_n)$ and $I = \bigcup_{n \in K} I_n$. The values of a at the end points of I_n are the critical parameters at which a change in dynamics of h_a occurs as given in the following theorem. Let A^0 and ∂A denote the interior and the boundary of a set A respectively.

Theorem 6.1.2. *Let $h_a(z) = a + f(z)$ for $f \in \mathcal{R}$ and $a \in \mathbb{R}$. If the set J is non-empty, then*

1. *For $a \in \bigcup_{n \in K} I_n^0$, the Fatou set $\mathcal{F}(h_a)$ is the attracting domain corresponding to a real attracting fixed point x_a of $h_a(z)$.*

2. *For $a \in \bigcup_{n \in K} \partial I_n$, the Fatou set $\mathcal{F}(h_a)$ is the parabolic domain corresponding to the real rationally indifferent fixed point n_a of $h_a(z)$.*

3. *For $a \in \mathbb{R} \setminus \overline{\bigcup_{n \in K} I_n}$, the Fatou set $\mathcal{F}(h_a)$ is either $H^+ \bigcup H^-$ or a completely invariant Baker domain.*

Proof. 1. If $a \in \bigcup_{n \in K} I_n^0$, then $a \in I_n^0$ for some $n \in K$. Since the function $\varphi(x)$ is increasing in J_n for all n, $\varphi(J_n^0) = I_n^0$ and there exists a point $x_a \in J_n$ such that $\varphi(x_a) = a$. This means that $h_a(x_a) = x_a$ and $0 < h_a'(x_a) < 1$. Therefore, x_a is an attracting fixed point of $h_a(z)$. The immediate basin of attraction of x_a intersects H^+ and H^-. Thus, the Fatou set $\mathcal{F}(h_a)$ is the attracting domain corresponding to the real attracting fixed point x_a.

2. If $a \in \bigcup_{n \in K} \partial I_n$, then $a \in \partial I_n$ for some $n \in K$. Since the function $\varphi(x)$ is increasing in J_n for all n, $\varphi(\partial J_n) = \partial I_n$ and there exists a n_a on an end point of J_n for some n such that $\varphi(n_a) = a$. This means that $h_a(n_a) = n_a$ and $h'_a(n_a) = 1$. Therefore, n_a is a rationally indifferent fixed point of $h_a(z)$. The parabolic domain corresponding to n_a intersects both H^+ and H^-. Thus, the Fatou set $\mathcal{F}(h_a)$ is the parabolic domain corresponding to n_a.

3. If $a \in \mathbb{R} \setminus \overline{\bigcup_{n \in K} I_n}$ and $\mathcal{F}(h_a) \neq H^+ \bigcup H^-$, then $\mathcal{F}(h_a) \bigcap \mathbb{R} \neq \emptyset$ and the Fatou set $\mathcal{F}(h_a)$ is connected. Each real periodic point of h_a is repelling. Since $h_a(\mathbb{R}) \subset \mathbb{R}$, if the Fatou set is an attracting domain or parabolic domain, then it must correspond to a real attracting or rationally indifferent fixed point which is not possible. There are no rotational domain or wandering domain in the Fatou set of h_a by Proposition 6.1.1. Therefore, the Fatou set $\mathcal{F}(h_a)$ is a completely invariant Baker domain.

$\square$

Remark 6.1.2. *Let $I_n \bigcap I_m \neq \emptyset$ for some $n \neq m$ and $a \in I_n \bigcap I_m$. Then $J_m \neq J_n$ and $\varphi(J_m) = \varphi(J_n)$. This gives that, there exist two points $x_a \in J_m$ and $\tilde{x_a} \in J_n$ such that $\varphi(x_a) = \varphi(\tilde{x_a}) = a$. Consequently, $h_a(x_a) = x_a$ and $h_a(\tilde{x_a}) = \tilde{x_a}$. Further, $0 < h'_a(x_a) = f'(x_a) < 1$ as $x_a \in J_m$ and similarly, $0 < h'_a(\tilde{x_a}) < 1$. Therefore, x_a and $\tilde{x_a}$ are two distinct real attracting fixed points of $h_a(z)$ which is a contradiction to Proposition 6.1.1(3). Hence, $I_n \bigcap I_m = \emptyset$ for $m \neq n$.*

6.2 Dynamics of $h_{b,c}$ and h_b

In this section, we determine the dynamics of the functions $h_{a,b,c} \in \mathcal{S}$ for various possible values of b and c with $a = 0$ where $f \in \mathcal{R}^*$ is bounded on the imaginary axis. Theorem 6.2.1 deals with the dynamics of functions $h_{a,b,c}(z) = a + bz - \frac{c}{z} + f(z)$ for $a = 0,\ b \geq 0,\ c > 0$. The dynamics of functions $h_{a,b,c} \in \mathcal{S}$ is described where $a = c = 0$ and $b > 0$ in Theorem 6.2.4.

Firstly, we find a sufficient condition for real meromorphic functions $f \in \mathcal{R}^*$ to be bounded on the imaginary axis.

Proposition 6.2.1. *Let $f(z) = \sum_{k=1}^{\infty} \frac{A_k z}{a_k^2 - z^2}$ be a function in $\mathcal{R}^*$. If $\{A_k\}_{k=1}^{\infty}$ is bounded and $\frac{a_k^2}{A_k} > k^2$ where $k > k_0$ for some $k_0 \in \mathbb{N}$, then the function $f(z)$ is bounded on the imaginary axis.*

Proof. Note that $f(iy) = \sum_{k=1}^{\infty} \frac{A_k iy}{a_k^2 - (iy)^2} = \sum_{k=1}^{\infty} \frac{A_k iy}{a_k^2 + y^2} = \sum_{k=1}^{k_0} \frac{A_k iy}{a_k^2 + y^2} + \sum_{k=k_0+1}^{\infty} \frac{A_k iy}{a_k^2 + y^2}$ for $y \in \mathbb{R}$. Since $f(-iy) = -f(iy)$ for all $y \in \mathbb{R}$, it is sufficient to prove that $f(iy)$ is bounded on $\mathbb{R}^+$.

Clearly, the function $\sum_{k=1}^{k_0} \frac{A_k iy}{a_k^2 + y^2}$ is bounded for $y \in \mathbb{R}^+$ and it remains to show that the function $S(y) = \sum_{k=k_0+1}^{\infty} \frac{A_k iy}{a_k^2 + y^2} = \sum_{k=k_0+1}^{\infty} \frac{iy}{(a_k^2/A_k) + (y^2/A_k)}$ is bounded for $y \in \mathbb{R}^+$. Since $\{A_k\}_{k=1}^{\infty}$ is bounded and $\frac{a_k^2}{A_k} > k^2$ for all $k > k_0$, $\frac{a_k^2}{A_k} + \frac{y^2}{A_k} > k^2 + My^2$ for some $M > 0$. It now follows that $k^2 + My^2 > k^2 + My^2 - 1 > 0$ for all $k > k_0$ and consequently, $|S(y)| < \sum_{k=k_0+1}^{\infty} \frac{y}{k^2 + My^2 - 1}$ for $y > 0$. Now, $|S(y)| < S_1(y) + S_2(y)$ where $S_1(y) = \sum_{k_0 < k < y} \frac{y}{k^2 + My^2 - 1}$ and $S_2(y) = \sum_{k \geq y} \frac{y}{k^2 + My^2 - 1}$. Set $M_0 = \sqrt{1/M}$ and $\delta = \max\{k_0,\ M_0\} + 1$. As the function $S(y)$ is continuous and hence bounded in $[0,\ \delta]$, it is sufficient to prove that $S(y)$ is bounded on $(\delta,\ +\infty)$.

Now $\frac{y}{k^2 + My^2 - 1} < \frac{y}{My^2 - 1} = \frac{y}{M(y^2 - (1/M))}$ and it gives $S_1(y) < \sum_{k_0 < k < y} \frac{y}{M(y^2 - M_0^2)}$. Therefore, $S_1(y) < \sum_{k_0 < k < y} \frac{y + M_0}{M(y - M_0)(y + M_0)} < \sum_{k_0 < k < y} \frac{1}{M(y - M_0)}$. As there are $[y] - k_0$ terms in this series, $S_1(y) < \frac{[y] - k_0}{M(y - M_0)} < \frac{y + 1}{M(y - M_0)}$. The function $\frac{y+1}{M(y - M_0)}$ is bounded on $(\delta,\ +\infty)$. Hence, the function $S_1(y)$ is bounded on $(\delta,\ +\infty)$. Note that, for $y \leq k_0$, there are no terms in the series $S_1(y)$ and $S_1(y) = 0$.

For $y \in (\delta,\ +\infty)$, $My^2 > M\delta^2 > MM_0^2 = 1$ and $My^2 - 1 > -k$ which gives that $k^2 + My^2 - 1 > k^2 - k > 0$. Therefore, $\frac{y}{k^2 + My^2 - 1} < \frac{y+1}{k^2 - k}$ and $S_2(y) < \sum_{k \geq y} \frac{y+1}{k^2 - k}$. If n is the smallest integer satisfying $n \geq y$, then $\sum_{k=n}^{\infty} \frac{1}{k^2 - k} = \sum_{k=n}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{n-1}$ which gives $S_2(y) < \frac{y+1}{n-1}$. Now, $n \geq y > \delta > 1$ implies $n - 1 \geq y - 1$ and $S_2(y) \leq \frac{y+1}{y-1}$. The function $\frac{y+1}{y-1}$ is bounded on $(\delta,\ +\infty)$ and we conclude that $S_2(y)$ is bounded on $(\delta,\ +\infty)$. $\square$

Theorem 6.2.1. *Let $h_{b,c} = bz - \frac{c}{z} + f(z)$ where $b \geq 0,\ c > 0$ and $f \in \mathcal{R}^*$ be bounded on the imaginary axis. Then,*

1. *For $b \in [0,\ 1)$, the Fatou set $\mathcal{F}(h_{b,c}) = H^+ \bigcup H^-$, where H^+ and H^- are the attracting domains corresponding to a conjugate pair of attracting fixed points z_a and $\overline{z_a}$ respectively.*

2. *For $b\ \geq\ 1$, the Fatou set $\mathcal{F}(h_{b,c})$ consists of Baker domains.*

Proof. As the function $h_{b,c}(z)$ preserves the positive imaginary axis, the function $\phi_{b,c}(y) = \frac{h_{b,c}(iy)}{i}$ is a non-negative real valued continuous function on $\mathbb{R}^+ \bigcup \{0\}$. A point y_0 is fixed point of $\phi_{b,c}(y)$ if and only if iy_0 is a fixed point of $h_{b,c}(z)$. Let $\phi(y) = \frac{f(iy)}{i}$ for $y \in \mathbb{R}$. Then $\phi(y)$ is bounded on the positive real axis and $\phi_{b,c}(y) = \frac{1}{i}(biy - \frac{c}{iy} + f(iy)) = by + \frac{c}{y} + \phi(y)$.

1. Since $b \in [0,\ 1)$ and $\phi(y)$ is bounded on $\mathbb{R}^+$, $\lim_{y \to +\infty} \phi_{b,c}(y) - y = \lim_{y \to +\infty}(b-1)y + \frac{c}{y} + \phi(y) = -\infty$ and $\lim_{y \to 0^+} \phi_{b,c}(y) - y = +\infty$. By the intermediate value theorem, there is a zero of $\phi_{b,c}(y) - y$ and hence a fixed point of $\phi_{b,c}(y)$ in $\mathbb{R}^+$. Consequently, there is a fixed point z_a of $h_{b,c}(z)$ in the positive imaginary axis. Any Fatou component of a meromorphic function containing a fixed point is either an attracting domain or a Siegel disc. Since $z_a \in H^+ \subset \mathcal{F}(h_{b,c})$, the Fatou component of $h_{b,c}$ containing H^+ is either an attracting domain or a Siegel disc. But $\mathcal{F}(h_{b,c})$ does not contain any Siegel disc. Therefore, H^+ is contained in the attracting basin of z_a. Since $h_{b,c}(\overline{z}) = \overline{h_{b,c}(z)}$, there is also a fixed point $\overline{z_a}$ of $h_{b,c}(z)$ in the negative imaginary axis and H^- is contained in the attracting basin of $\overline{z_a}$. Thus $\mathcal{F}(h_{b,c}) = H^+ \bigcup H^-$ where H^+ and H^- are the basins of attractions of a conjugate pair of attracting fixed points z_a and $\overline{z_a}$ respectively.

2. For $b \geq 1$, $\phi_{b,c}(y) - y = (b-1)y + \frac{c}{y} + \phi(y) > 0$ for all $y > 0$. This means that, $h_{b,c}(z)$ has no fixed point on $i\mathbb{R}^+$. As $h_{b,c}(-iy) = -h_{b,c}(iy)$ for $y \in \mathbb{R}$, the function $h_{b,c}(z)$

has also no fixed point on $i\mathbb{R}^-$. By Proposition 6.1.1(2), $h_{b,c}(z)$ has no attracting or rationally indifferent fixed point in $\mathbb{C}$. Therefore, the Fatou set $\mathcal{F}(h_{b,c})$ does not contain any attracting or parabolic domain. It follows from Proposition 6.1.1(1) that, $\mathcal{F}(h_{b,c})$ does not contain any wandering domain or rotational domain. Since $\mathcal{F}(h_{b,c})$ is non-empty, it contains Baker domains only.

□

Remark 6.2.1. *1. The second case in the above theorem where $b \geq 1$ remains true even if f is assumed to be unbounded on the imaginary axis.*

2. In Theorem 6.2.1(2), the Fatou set of $h_{b,c}$ is either a disjoint union of two completely invariant Baker domains H^+ and H^- or is a single completely invariant Baker domain. If the set $\{|x_1-x_2| \,:\, h_{b,c}(x_1) = h_{b,c}(x_2) = \infty$ and $h_{a,b}(z)$ has no pole in $(x_1,\, x_2)\}$ is bounded, it follows by Proposition 6.1.1(6) that $\mathcal{F}(h_{b,c}) = H^+ \bigcup H^-$ and each of H^+ and H^- is a completely invariant Baker domain.

Theorem 6.2.2. *Suppose $g_j, f_j \in \mathcal{R}^*$. Let $\lim_{y\to+\infty} \frac{g_j(iy)}{i} = l_j \neq 0$ and $\lim_{y\to+\infty} \frac{f_j(iy)}{i} = m_j \neq 0$ for each $j = 1,\ 2, ...,\ n$. Let $h(z) = \sum_{j=1}^{n}\{\alpha_j g_j(z) - \frac{\beta_j}{f_j(z)}\}$ where $\alpha_j > 0$ and $\beta_j \geq 0$. Assume that at least one β_j is not zero. Then the Fatou set of h is the union of two completely invariant attracting domains.*

Proof. Observe that $\Im(\alpha_j g_j(z))$ and $\Im\left(\frac{-\beta_j}{f_j(z)}\right)$ have the same sign for $z \in \mathbb{C}$ and each j if $\beta_j > 0$ and, the function $h(z) = \displaystyle\sum_{j=1}^{n}\left\{\alpha_j g_j(z) - \frac{\beta_j}{f_j(z)}\right\}$ is real meromorphic for $\alpha_j > 0$ and $\beta_j \geq 0$. Further, $h(-z) = -h(z)$ for $z \in \mathbb{C}$. Let $\mu_j(y) = \frac{g_j(iy)}{i}$ and $\varphi_j(y) = \frac{f_j(iy)}{i}$ for each j. Then $\phi(y) = \dfrac{h(iy)}{i} = \dfrac{1}{i}\displaystyle\sum_{j=1}^{n}\left\{\alpha_j g_j(iy) - \frac{\beta_j}{f_j(iy)}\right\} = \sum_{j=1}^{n}\left\{\frac{\alpha_j g_j(iy)}{i} - \frac{\beta_j}{i f_j(iy)}\right\} = \sum_{j=1}^{n}\left\{\alpha_j \mu_j(iy) + \frac{\beta_j}{\varphi_j(y)}\right\}$. Since $\displaystyle\lim_{y\to+\infty} \mu_j(y) = l_j \neq 0$ and $\displaystyle\lim_{y\to+\infty} \varphi_j(y) = m_j \neq 0$, it follows that $\displaystyle\lim_{y\to+\infty} \phi(y) = \sum_{j=1}^{n} \alpha_j l_j + \frac{\beta_j}{m_j} < \infty$ and consequently, $\displaystyle\lim_{y\to+\infty} \phi(y) - y = -\infty$. Since g_j

and f_j are bounded on the imaginary axis, $\lim_{y\to 0^+}\mu_j(y)=\lim_{y\to 0^+}\varphi_j(y)=0$ and consequently, $\lim_{y\to 0^+}\phi(y)-y=\lim_{y\to 0^+}\sum_{j=1}^{n}\left(\alpha_j\mu_j(y)+\frac{\beta_j}{\varphi_j(y)}-y\right)=+\infty$. By the intermediate value theorem, there is a $y_0\in\mathbb{R}^+$ such that $\phi(y_0)=y_0$ and consequently, $h(iy_0)=iy_0$. By similar arguments used in Theorem 6.2.1(1), it is concluded that the Fatou set of h is the union of two completely invariant attracting domains. □

Before describing the dynamics of $h_b(z)=bz+f(z)$, we present an important theorem regarding parabolic domains [18]. For each positive t, each positive integer p, and each $k\in\{0,\ 1,\ ...,\ p-1\}$, we define the sets $\Pi_k(t)=\{re^{i\theta}:\ r^p<t(1+\cos(p\theta));\ |2k\pi/p-\theta|<\pi/p\}$ and these sets are called petals. The line of symmetry of $\Pi_k(t)$ is the ray $\theta=2k\pi/p$ and is called the axis of the petal Π_k.

Theorem 6.2.3. *(The Petal theorem) Suppose that the analytic map ψ has a Taylor series expansion $\psi(z)=z-z^{p+1}+O(z^{2p+1})$ at the origin where $O(z^{2p+1})$ denotes terms having degree of z equal or higher than $2p+1$. Then for all sufficiently small t,*

1. *ψ maps each petal $\Pi_k(t)$ into itself.*
2. *$\psi^n(z)\to 0$ uniformly on each petal as $n\to\infty$.*
3. *arg $\psi^n(z)\to 2k\pi/p$ locally uniformly on Π_k as $n\to\infty$.*
4. *$|\psi(z)|<|z|$ on a neighbourhood of the axis of each petal.*
5. *$\psi:\ \Pi_k(t)\to\Pi_k(t)$ is conjugate to a translation.*

Remark 6.2.2. 1. *In the above theorem there are exactly p petals in the parabolic domain of f corresponding to the rationally indifferent fixed point 0.*

2. *Any two distinct petals are contained in two different Fatou components.*

Theorem 6.2.4. *Let $h_b(z) = bz + f(z)$ where $b \geq 0$ and $f \in \mathcal{R}^*$ be bounded on the imaginary axis. Then,*

1. *For $h'_b(0) < 1$, the Fatou set $\mathcal{F}(h_b)$ is the attracting basin of 0.*

2. *For $h'_b(0) = 1$, the Fatou set $\mathcal{F}(h_b)$ is the parabolic domain corresponding to the rationally indifferent fixed point 0. Further, the Fatou set $\mathcal{F}(h_b) = H^+ \bigcup H^-$.*

3. *For $h'_b(0) > 1$, the Fatou set $\mathcal{F}(h_b)$ is equal to $H^+ \bigcup H^-$. Further, H^+ and H^- are completely invariant basins of attraction if $0 \leq b < 1$ and each component of the Fatou set is a Baker domain if $b \geq 1$.*

Proof. Observe that $h_b(0) = 0$ and the fixed point $z = 0$ is attracting, rationally indifferent or repelling if $h'_b(0) < 1$, $= 1$ or > 1 respectively. As $h_b(H^+) \subseteq H^+$, H^+ is contained in a Fatou component and similarly, H^- is also contained in a Fatou component of h_b.

1. For $h'_b(0) < 1$, every neighbourhood of 0 intersects both H^+ and H^-. Therefore, the Fatou set of h_b contains both the half-planes and $\mathcal{F}(h_b) = A(0)$, the attracting basin of 0. In other words, the Fatou set is connected when $h'_b(0) < 1$.

2. For $h'_b(0) = 1$, any neighbourhood of 0 intersects both H^+ and H^-. Therefore, the Fatou set $\mathcal{F}(h_b)$ contains these two half-planes and $\mathcal{F}(h_b) = P(0)$, the parabolic domain corresponding to the rationally indifferent fixed point 0. Note that $h_b(z) = h_b(0)+h'_b(0)z+\frac{h''_b(0)}{2!}z^2+\frac{h'''_b(0)}{3!}z^3+...$ for z lying in a sufficiently small neighbourhood of 0. Observing that $h_b(0) = 0$, $h'_b(0) = b+f'(0) = b+\sum_{-\infty}^{\infty}\frac{A_k}{a_k^2} > 0$, $h''_b(0) = f''(0) = 0$, $h'''_b(0) = f'''(0) > 0$, $h^{iv}_b(0) = f^{iv}(0) = 0$ and $h^{v}_b(0) = f^{iv}(0) > 0$ and comparing this expansion with that of Petal theorem 6.2.3, we conclude that, there are two petals in $\mathcal{F}(h_b) = P(0)$ when $h'_b(0) = 1$. By Theorem 6.2.3 (3) the axes of the petals must be $i\mathbb{R}^+$ and $i\mathbb{R}^-$ because these remain invariant under h_b. Therefore, the Fatou set $\mathcal{F}(h_b)$ consists of two disjoint simply connected petals, namely H^+ and H^-.

3. Since $h_b'(0) > 1$, the point 0 is a repelling fixed point and there exists a $\delta > 0$ such that $|h_b'(z)| > 1$ for $z \in (0, i\delta)$. Consider $\phi_b(y) = by + \phi(y)$ where $\phi(y) = \frac{f(iy)}{i}$. Then $\phi_b'(y) - 1 > 0$ and $\phi_b(y) - y$ is increasing in $(0,\ \delta)$. Therefore, $\phi_b(\delta) - \delta > \phi_b(0) - 0 = 0$.

   For $0 \leq b < 1$, $\lim_{y\to+\infty} \phi_b(y) - y = \lim_{y\to+\infty}(b-1)y + \phi(y) = -\infty$ since $\phi(y)$ is bounded on the positive imaginary axis. Therefore, there is a fixed point of $\phi_b(y)$ in $(\delta,\ \infty)$. Consequently $h_b(z)$ has a fixed point z_a in $i\mathbb{R}^+$. A Fatou component of a meromorphic function containing a fixed point is either an invariant Siegel disc or an invariant attracting domain. The Fatou set of h_b does not contain any rotational domain. As $z_a \in H^+ \subset \mathcal{F}(h_b)$, it is concluded that H^+ is contained in an attracting basin. Since $h_b(\overline{z}) = \overline{h_b(z)}$ for $z \in \mathbb{C}$, the half-plane H^- is also contained in an attracting domain corresponding to the attracting fixed point $\overline{z_a}$ lying in the negative imaginary axis. Hence, the Fatou set $\mathcal{F}(h_b)$ is equal to $H^+ \bigcup H^-$ where H^+ and H^- are basins of attractions of a conjugate pair of attracting fixed points z_a and $\overline{z_a}$.

   For $b \geq 1$, $\phi_b(y) - y = (b-1)y + \phi(y) > 0$ for all $y \in \mathbb{R}^+$ which gives that, there is no fixed point of $\phi_b(y)$ in $\mathbb{R}^+$ and $\lim_{n\to\infty} \phi^n(y) = \infty$. Consequently, there is no fixed point of $h_b(z)$ in H^+ by Proposition 6.1.1(2) and $\lim_{n\to\infty} h_b^n(z) = \infty$ for all $z \in i\mathbb{R}^+$. Since $H^+ \subset \mathcal{F}(h_b)$, $\lim_{n\to\infty} h_b^n(z) = \infty$ for all $z \in H^+$. Further, the Fatou set $\mathcal{F}(h_b)$ does not contain any attracting domain, parabolic domain, wandering domain or any rotational domain by Proposition 6.1.1(1). Since the Fatou set is non-empty, each component of $\mathcal{F}(h_b)$ is a Baker domain.

$\square$

Corollary 6.2.1. *Let $f \in \mathcal{R}^*$ be bounded on the imaginary axis and consider the one parameter family $\mathcal{S} = \{f_\lambda \equiv \lambda f\ :\ \lambda \in \mathbb{R} \setminus \{0\}\}$. Taking $b = 0$ in Theorem 6.2.4, it is seen that, the Fatou set $\mathcal{F}(f) = A(0)$, the attracting basin of 0 for $0 < \lambda < \frac{1}{f'(0)}$. The Fatou set $\mathcal{F}(f) = P(0)$, the parabolic domain corresponding to rationally indifferent fixed* *point* 0*, with two invariant petals,* H^+ *and* H^- *for* $\lambda = \frac{1}{f'(0)}$ *and* $\mathcal{F}(f) = H^+ \bigcup H^-$*, the union of two invariant attracting domains for* $\lambda > \frac{1}{f'(0)}$*. Since* $f(-z) = -f(z)$ *for* $z \in \mathbb{C}$*,* f_λ *and* $f_{-\lambda}$ *are conformally conjugate and their dynamics are essentially same. Therefore, bifurcations in the dynamics of functions in* $\mathcal{S}$ *occur at parameter values* $\pm\frac{1}{f'(0)}$*.*

The above corollary for $f(z) = \tan z$ was proved by Linda and Keen [42].

6.3 Examples

Some examples of functions of the family $\mathcal{S}$ are discussed in this section illustrating the theorems proved in this chapter.

Example 1:

Let $h_a(z) = a + \lambda \tan z$ for $z \in \mathbb{C}$, $a \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$. The function $f(z) = \lambda \tan z = \lambda \sum_{k=1}^{\infty} \frac{z}{(\frac{2k-1}{2}\pi)^2 - z^2} \in \mathcal{R}^*$ and is bounded on the imaginary axis. Let $J = \{x \in \mathbb{R} \ : \ 0 < f'(x) < 1\}$ and $J^* = \{x \in \mathbb{R} \ : \ f'(x) = 1\}$. The set $\{|x_1 - x_2| \ : \ h_a(x_1) = h_a(x_2) = \infty$ and $h_{a,b}(z)$ has no pole in $(x_1,\ x_2)\}$ is bounded. Since h_a has only finitely many singular values, the Fatou set of h_a has no Baker domain.

For $0 < \lambda < 1$, the set J is non-empty and the dynamics of h_a follows from Theorem 6.1.2. If $J = \bigcup_{n \in K} J_n$ for some index set $K \subset \mathbb{Z}$ where J_n are intervals in $\mathbb{R}$, then set $I_n = \phi(J_n)$ where $\phi(x) = x - f(x)$. The Fatou set $\mathcal{F}(h_a)$ is the attracting domain corresponding to a real attracting fixed point x_a of $h_a(z)$ if $a \in \bigcup_{n \in K} I_n^0$ and the Fatou set $\mathcal{F}(h_a)$ is the parabolic domain corresponding to the real rationally indifferent fixed point n_a of $h_a(z)$ for $a \in \bigcup_{n \in K} \partial I_n$. For $a \in \mathbb{R} \setminus \overline{\bigcup_{n \in K} I_n}$, the Fatou set $\mathcal{F}(h_a)$ is either $H^+ \bigcup H^-$ or a completely invariant Baker domain. Since h_a has no Baker domains, $\mathcal{F}(h_a) = H^+ \bigcup H^-$.

For $\lambda = 1$, $h_a'(z) = \sec^2 z$. The set $J = \emptyset$ and $J^* = \{k\pi \ : \ k \in \mathbb{Z}\}$. Note that $\varphi(J^*) = J^*$ where $\varphi(x) = x - f(x)$ for $x \in \mathbb{R}$. The dynamics of $h_a(z)$ is given by Theorem 6.1.1. Therefore, the Fatou set $\mathcal{F}(h_a)$ is the parabolic domain corresponding to a rationally indifferent fixed point for $a \in J^*$. For $a \notin J^*$, the Fatou set is either $H^+ \bigcup H^-$

or a Baker domain. But Baker domains do not exist in $\mathcal{F}(h_a)$. Therefore, the Fatou set $\mathcal{F}(h_a) = H^+ \bigcup H^-$. Further, each component of the Fatou set is a completely invariant attracting domains for $a \notin J^*$.

For $\lambda > 1$, $|h'_a(z)| > 1$ for all $z \in \mathbb{R}$ and by Proposition 6.1.1(6), it is concluded that $\mathcal{F}(h_a) = H^+ \bigcup H^-$. Further, none of H^+ and H^- are Baker domains or parabolic domains (since, h_a has no rationally indifferent real fixed point). Therefore, each of H^+ and H^- is an attracting domain.

Example 2:

Let λ_n's be the roots of $\tan z - z = 0$ and $f(z) = \dfrac{3}{z} - \dfrac{z \sin z}{z \cos z - \sin z} = \displaystyle\sum_{k=1}^{\infty} \frac{2z}{\lambda_n^2 - z^2}$. It is easy to verify that $f(z)$ is bounded on the imaginary axis and $f \in \mathcal{R}^*$. Let $h_{b,c}(z) = bz - \frac{c}{z} + f(z)$ for $z \in \mathbb{C}$. It follows from Theorem 6.2.1 that, when $c > 0$, the Fatou set $\mathcal{F}(h_{b,c})$ is the union of two attracting domains $H^+ \bigcup H^-$ for $0 \leq b < 1$ and consists of Baker domains for $b \geq 1$.

Example 3:

Let $f(z) = \displaystyle\sum_{j=1}^{n} \alpha_j \tan(r_j z) - \beta_j \cot(s_j z)$ where $r_j,\ s_j,\ \alpha_j,\ \beta_j > 0,\ j = 1,\ 2,\ ...,\ n$. Then $\tan(r_j z) \in \mathcal{R}^*$ for each j and $\frac{\tan(r_j iy)}{i} = \tanh(r_j y) \to 1$ as $y \to +\infty$. Hence $\mathcal{F}(f)$ is union of two attracting domains by Theorem 6.2.2.

Example 4:

Let $f(z) = \displaystyle\sum_{j=1}^{n} \alpha_j g(r_j z) - \frac{\beta_j}{g(s_j z)}$ where $r_j,\ s_j,\ \alpha_j,\ \beta_j \in \mathbb{R}^+,\ j = 1, 2, ..., n$. The function $g(z) = -i\{\frac{1}{2} - \frac{1}{iz} + \frac{1}{e^{iz}-1}\} = \sum_{k=1}^{\infty} \frac{2z}{4k^2\pi^2 - z^2} \in \mathcal{R}^*$. Then $\frac{g(iy)}{i} = -\{\frac{1}{2} + \frac{1}{y} + \frac{e^y}{1-e^y}\} \to \frac{1}{2}$ as $y \to \infty$. Therefore, $\mathcal{F}(f)$ is union of two attracting domains by Theorem 6.2.2.

Example 5:

Let $\mathcal{F} = \{f_{\lambda_1,\lambda_2}(z) = \lambda_1 z + \lambda_2 \tan z \ :\ \lambda_1 \geq\ 0,\ \lambda_2 > 0\}$ be a two parameter family of transcendental meromorphic functions. Set $f'_{\lambda_1,\lambda_2}(0) = \lambda_1 + \lambda_2$. The function $\lambda_2 \tan z$ belongs to the class $\mathcal{R}^*$ and is bounded on the imaginary axis for each $\lambda_2 > 0$. It follows from the Theorem 6.2.4 that, a bifurcation occurs in the dynamics of functions in the family $\mathcal{F}$ as follows.

1. If $\lambda_1 + \lambda_2 < 1$, then the Fatou set of f_{λ_1,λ_2} is the attracting basin of 0.

2. If $\lambda_1 + \lambda_2 = 1$, then the Fatou set of f_{λ_1,λ_2} is the parabolic domain corresponding to the rationally indifferent fixed point 0.

3. If $\lambda_1 + \lambda_2 > 1$, then the Fatou set of f_{λ_1,λ_2} is the union of H^+ and H^- each of which is a completely invariant attracting domains if $\lambda_1 < 1$. For $\lambda_1 \geq 1$, $|f'_{\lambda_1,\lambda_2}(x)| > 1$ for $x \in \mathbb{R}$ and it follows from Proposition 6.1.1(6) that $\mathcal{F}(h_{a,b}) = H^+ \bigcup H^-$. Further, each component is a completely invariant Baker domain.

6.4 Dynamics of $a + \tan z$

Let $T_a(z) = a + \tan z$ for $z \in \mathbb{C}$ where $a \in \mathbb{C}$. For $a \in \mathbb{R}$, the function $T_a(z)$ is real meromorphic and its dynamics is discussed in Example 1. This section deals with the dynamics of $T_a(z)$ when $a \in \mathbb{C}\backslash\mathbb{R}$. The function $T_a(z)$ is not real meromorphic for $a \in \mathbb{C}\backslash\mathbb{R}$ and natural difficulties arise in the investigation of its dynamics. Since $-T_{-a}(-z) = -(-a+\tan(-z)) = T_a(z)$ for all z and $a \in \mathbb{C}$, the two functions T_a and T_{-a} are conformally conjugate and have essentially same dynamics. Therefore, we only determine the dynamics of T_a for $\Im(a) > 0$. The function $T_a(z)$ has no critical value and only two finite asymptotic values, namely $a + i$ and $a - i$. The Fatou set $\mathcal{F}(T_a)$ is shown to be an immediate basin of attraction for $a \in P_1 \bigcup P_2$ where $P_1 = \{x + iy \in \mathbb{C} \ : \ x \in \mathbb{R} \text{ and } y > 1\} \bigcup \{x + i \in \mathbb{C} \ : \ x \neq \frac{2k+1}{2}\pi \text{ for any } k \in \mathbb{Z}\}$ and $P_2 = \{\pi k + iy \in \mathbb{C} \ : \ k \in \mathbb{Z} \text{ and } y > 0\}$ in Theorem 6.4.1. Theorem 6.4.2 deals with the topology of Fatou components of T_a when $a \notin P_1 \bigcup P_2$ under certain extra conditions.

Proposition 6.4.1. *Let $a \in \mathbb{C}$ such that $\Im(a) > 0$ and $T_a(z) = a + \tan z$ for $z \in \mathbb{C}$. Then, the Fatou set $\mathcal{F}(T_a)$ of T_a contains a completely invariant component U_a and the singular* *value $a+i$ lies in U_a. Further, if the singular value $a-i$ lies in U_a, then $\mathcal{F}(T_a) = U_a$ and is an attracting domain corresponding to an attracting fixed point lying in H^+.*

Proof. For $\Im(a) > 0$, $T_a(H^+) \subset H^+$. Therefore, $T_a^n(H^+) \subseteq H^+$ for all n and it follows from Theorem 1.1.1 that, H^+ is contained in the Fatou set of T_a. Denote the Fatou component of T_a containing H^+ by U_a. Since H^+ is forward invariant under T_a and contains $a+i$, the component U_a is forward invariant and contains the singular value $a+i$. Observe that, for sufficiently small ϵ, there exists a $M(\epsilon) > 0$ such that each component of $T_a^{-1}(D_\epsilon(a+i))$ intersects $H_{M(\epsilon)} = \{z \in \mathbb{C} : \Im(z) > M(\epsilon)\}$. Since $H_{M(\epsilon)} \subset U_a$, the component U_a is backward invariant. Hence U_a is completely invariant.

Since the function T_a has finitely many singular values, the Fatou set $\mathcal{F}(T_a)$ does not contain any Baker domain (c.f. Theorem 1.1.11) or wandering domain [14]. If both the singular values of T_a lie in U_a, then the Fatou set of T_a does not contain any rotational domain and any attracting domain or parabolic domain different from U_a does not exist in $\mathcal{F}(T_a)$ (c.f. Theorem 1.1.7). Therefore, for any Fatou component U of T_a there is a $k \in \mathbb{Z}$ such that $T_a^k(U) \subseteq U_a$. The complete invariance of U_a gives that $k = 0$ and we have $\mathcal{F}(T_a) = U_a$. Now, it is clear that U_a is either an immediate basin of attraction or a parabolic domain corresponding to an attracting or rationally indifferent fixed point. Since $T_a(H^+) \subset H^+ \subset U_a$, the fixed point must lie in H^+ and be attracting. □

Remark 6.4.1. *It is clear from Proposition 6.4.1 that H^+ is contained in the completely invariant Fatou component U_a if $\Im(a) > 0$. Further, if $x \in \mathbb{R}$ and $x \neq \frac{2k+1}{2}\pi$ for any $k \in \mathbb{Z}$, then $T_a(x) \subset H^+ \subset U_a$ and the complete invariance of U_a gives that $\mathbb{R} \setminus \{\frac{2k+1}{2}\pi : k \in \mathbb{Z}\} \subset U_a$.*

Next theorem shows that the Fatou set of T_a is equal to the completely invariant attracting domain for certain values of a.

Theorem 6.4.1. *Let $a \in P_1 \bigcup P_2$ where $P_1 = \{x+iy \in \mathbb{C} : x \in \mathbb{R}$ and $y > 1\} \bigcup \{x+i \in$*

$\mathbb{C} \ : \ x \neq \frac{2k+1}{2}\pi$ *for any* $k \in \mathbb{Z}\}$ *and* $P_2 = \{\pi k + iy \in \mathbb{C} \ : \ k \in \mathbb{Z}$ *and* $y > 0\}$. *Then, the Fatou set* $\mathfrak{F}(T_a)$ *of* T_a *is an attracting domain corresponding to an attracting fixed point lying in* H^+.

Proof. In view of Proposition 6.4.1, it is sufficient to show that $a-i \in U_a$ for all $a \in P_1 \bigcup P_2$.

Case I: $a \in P_1$

If $a = r_a + is_a$ and $s_a > 1$, then $\Im(a-i) > 0$ and we have $a - i \in H^+ \subset U_a$. If $a = r_a + i$ and $r_a \neq \frac{2k+1}{2}\pi$ for any $k \in \mathbb{Z}$, then $a - i \in \mathbb{R} \setminus \{\frac{2k+1}{2}\pi \ : \ k \in \mathbb{Z}\} \subset U_a$ by Remark 6.4.1. Therefore, $a - i \in U_a$ whenever $a \in P_1$.

Case II: $a \in P_2$

In this case, $a = \pi k + is_a$ for some $k \in \mathbb{Z}$ and $s_a > 0$. Then $T_a(\pi k + iy) = \pi k + is_a + \tan(\pi k + iy) = \pi k + i(s_a + \tanh y)$. The function $\phi(y) = s_a + \tanh y$ is increasing in $\mathbb{R}$. Consider $\psi(y) = \phi(y) - y$ for $y \in \mathbb{R}$. Then $\psi'(y) = \text{sech}^2 y - 1 < 0$ for all nonzero y and the function $\psi(y)$ is decreasing in $\mathbb{R}$. Observe that $\lim_{y \to +\infty} \psi(y) = -\infty$ and $\lim_{y \to -\infty} \psi(y) = +\infty$. Further, $\psi(0) = s_a > 0$. Therefore, there is a unique $y_a > 0$ such that $\psi(y) < 0$ for $y > y_a$, $\psi(y) = 0$ for $y = y_a$ and $\psi(y) > 0$ for $y < y_a$. Hence, the function $\phi(y)$ has only one real fixed point y_a. Obviously, y_a is attracting. As $\phi(y) > y$ for $0 < y < y_a$ the sequence $\{\phi^n(y)\}_{n>0}$ is increasing and bounded above by y_a. Similarly, the sequence $\{\phi^n(y)\}_{n>0}$ is decreasing and bounded below by y_a for $y > y_a$. Therefore, by the monotone convergence theorem, $\lim_{n \to \infty} \phi^n(y) = y_a$ for $y > 0$. For $y < 0$, $\phi(y) > y$ and the sequence $\{\phi^n(y)\}_{n>0}$ is increasing. If $\phi^n(y) < 0$ for all n, then the sequence $\{\phi^n(y)\}_{n>0}$ must converge to a fixed point of $\phi(y)$ in negative real axis. But $\phi(y)$ has only one real fixed point y_a and that is positive leading to a contradiction. So, there is a n_y such that $\phi^{n_y}(y) > 0$. Since $\phi(y) > 0$ for all $y > 0$, $\phi^n(y) > 0$ for all $n \geq n_y$. Consequently, $\lim_{n \to \infty} \phi^n(y) = y_a$ for $y < 0$. We now conclude that $T_a(\pi k + iy_a) = \pi k + is_a + i \tanh y_a = \pi k + i(s_a + \tanh y_a) = \pi k + iy_a$, the fixed point $\pi k + iy_a$ is attracting and $\lim_{n \to \infty} T_a^n(z) = \pi k + iy_a$ for all $z = \pi k + iy$ where $k \in \mathbb{Z}$ and $y \in \mathbb{R}$. Both the singular values $\pi k + i(s_a + 1)$ and $\pi k + i(s_a - 1)$ of T_a lie on the line $h(k) = \{\pi k + iy \ : \ y \in \mathbb{R}\}$. Since $h(k) \bigcap H^+ \neq \emptyset$, $h(k) \subset U_a$ and $a - i \in U_a$. $\square$

Remark 6.4.2. *If $a \in \{\frac{2k+1}{2}\pi + i \ : \ k \in \mathbb{Z}\}$, then the singular value $a - i$ is a pole of T_a. Therefore $\mathcal{F}(T_a)$ does not contain any rotational domain (c.f. Theorem 1.1.7). Further, any attracting domain or parabolic domain different from U_a also does not exist in $\mathcal{F}(T_a)$. It follows by same arguments used in Proposition 6.4.1 that $\mathcal{F}(T_a) = U_a$ and U_a is an attracting domain corresponding to an attracting fixed point lying in H^+.*

Theorem 6.4.2. *Let $a \in \mathbb{C}$ such that $\Im(a) > 0$ and $T_a(z) = a + \tan z$ for $z \in \mathbb{C}$. If U_a is the completely invariant Fatou component of $T_a(z)$ containing H^+ and the singular value $a - i \notin U_a$, then each component of $\mathcal{F}(T_a)$ other than U_a is simply connected.*

Proof. Let U be a multiply connected component of $\mathcal{F}(T_a)$ different from U_a and γ be a Jordan curve in U such that the bounded component $B(\gamma^c)$ of γ^c intersects the Julia set $\mathcal{J}(T_a)$. Set $B_j = T_a^j(B(\gamma^c))$ for $j \in \mathbb{N}$. If B_1 contains no pole of T_a, then T_a is analytic on the closure of B_1 and the interior of B_1 is mapped into the interior of B_2 which means B_2 intersects the Julia set. If B_2 does not contain any pole of T_a, then repeat the arguments. As all the pre-images of poles are dense in the Julia set, we can find a natural number n^* such that B_{n^*} contains a pole of T_a and B_{n^*+1} contains a neighbourhood of infinity. Note that $\partial B_{j+1} \subseteq T_a(\partial B_j)$ for $j = 1, \ 2, \ 3, \ ..., \ n^*$.

Since $U \neq U_a$, the boundary of B_{n^*+1} does not lie in U_a and $a + i \in U_a \subset B_{n^*+1}$. The set B_{n^*} contains an asymptotic path corresponding to the asymptotic value $a + i$. Since there is only one singularity of T_a^{-1} over $a + i$ and that is logarithmic, the set B_{n^*} is unbounded [23] which is a contradiction. Thus, each component of $\mathcal{F}(T_a)$ different from U_a is simply connected. $\square$

Corollary 6.4.1. *Let $a \in \mathbb{C}$, $\Im(z) > 0$ and $T_a(z) = a + \tan z$ for $z \in \mathbb{C}$. Then $T_a(z)$ has no Herman ring.*

Proof. For $a \in \mathbb{R}$, the function $T_a(z)$ is real meromorphic and cannot contain any Herman ring by Proposition 6.1.1(1). If the singular value $a-i$ lies in U_a, then $\mathcal{F}(T_a)$ is a completely invariant attracting domain by Proposition 6.4.1 and cannot contain any Herman ring. If $a - i \notin U_a$, then U_a is completely invariant and cannot be a Herman ring. Each Fatou component other than U_a is simply connected by Theorem 6.4.2 and therefore, $T_a(z)$ has no Herman ring for $a \in \mathbb{C}$. $\square$

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Journals

1. **Dynamics of** $\lambda \tanh(e^z)$, *Discrete and Continuous Dynamical Systems - Series A*, 19, (September 2007), 1, 121-138.

2. **Dynamics of certain class of Critically Bounded Entire Transcendental Functions**, *Journal of Mathematical Analysis and Applications*, 329, (May 2007), 2, 1446-1459.

3. **Bifurcation and Chaotic Burst in the Dynamics of** λe^{bz+ce^z},*Journal of Combinatorics, Information and System Sciences*, 28-29 (2003-2004), 211-227.

4. **Chaotic Burst in the Dynamics of Critically Bounded Meromorphic Functions**, Under revision, *International Journal of Bifurcation and Chaos.*

5. **Dynamics of a class of critically unbounded meromorphic functions**, Under preparation.

Conference proceedings

1. Tarakanta Nayak, Siegel Discs in Complex Dynamics, *Proceedings of The Second National Conference on Non-linear Systems and Dynamics, (NCNSD-2005), 202-206* .

2. M. Guru Prem Prasad, Tarakanta Nayak and Ashis Kumar Roy, Exploding Julia sets in the Dynamics of $f_\lambda(z) = \lambda J_1(iz)/iz$, *Proceedings of The Third National Conference on Non-linear Systems and Dynamics, (NCNSD-2006), 171-174.*