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Dynamics of McMullen maps and Thurston-type theorems for rational maps with rotation domains

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Contents

Chapter 1

Main theorems

1.1 Abstract

The thesis mainly consists of two subjects:

• Dynamics of McMullen maps.

In this part, we study the local connectivity of Julia sets for rational maps. We develop Yoccoz puzzle techniques to study McMullen maps and show that the boundary of the basin of infinity is always a Jordan curve if the Julia set is not a Cantor set. This give a positive answer to a question posed by Devaney. We also show the Julia set of McMullen maps is locally connected except some special cases.

• Thurston's theory on characterization of rational maps and extensions.

For this subject, we establish a 'Decomposition Theorem':

Every non-parabolic branched covering can be decomposed along a stable multicurve into finitely many Siegel maps or Thurston maps, such that the combinatorics and rational realizations of these resulting maps essentially dominate the original one.

These resulting maps can be considered as the renormalizations of the original map. The motivation to establish such a theorem is to prove a Thurston-type theorem for rational maps with Herman rings. The Decomposition Theorem implies:

Thurston-type theorems for rational maps with Herman rings can be reduced to Thurston-type theorems for rational maps with Siegel disks.

According to Shishikura, a rational map with Herman rings admits finitely many rational maps, with Siegel disks or without rotation domains, as renormalizations. The Decomposition Theorem extends this philosophy beyond rational maps.

The Decomposition Theorem enables us to extend Thurston's Theorem to many poscritically infinite cases and give characterizations of rational maps with attracting cycles, Siegel disks and Herman rings. On the other hand, it allows us to construct many branched coverings without Thurston obstructions but not equivalent to rational maps.

Besides these two parts, the thesis also includes two short articles. One concerns the classification of rational maps admitting meromorphic line fields (Chapter 3) while the other concerns the parameter plane of a special family of rational maps (Chapter 4).

All of these parts are self-contained.

1.2 Dynamics of McMullen maps

The local connectivity of Julia sets for rational maps is a central problem in complex dynamical systems. It is well studied for classical type of rational maps, for example: hyperbolic and semihyperbolic maps, geometrically finite maps, see [CJY],[M1],[TY]. In polynomial case, it is also known a lot, see [DH2],[GS],[Kiwi],[Ly],[M2]. For quadratic polynomials, Yoccoz proved that the Julia set is locally connected provided that all periodic points are repelling and the map is not infinitely renormalizable, see [Hu],[M2]. Douady exhibited striking example of infinitely renormalizable quadratic polynomial with non locally connected Julia set, see [M2]. For general polynomial with connected Julia sets and without irrationally neutral cycles, Kiwi shows in [Kiwi] that the local connectivity of Julia set is equivalent to the non existence of wandering continua.

The powerful tool to study the local connectivity of Julia sets for polynomials is the so-called 'Branner-Hubbard-Yoccoz puzzle' techniques, which is introduced by Branner-Hubbard and Yoccoz, [BH]. It has a natural way of construction, which is induced by finite periodic external rays together with an equipotential curve.

However, for general rational maps, things are different. The construction of Yoccoz puzzle becomes quite involved, even impossible. Up to now, the only known rational maps which admit Yoccoz puzzle structures are the cubic Newton maps, whose Yoccoz puzzles are constructed by Roesch. In [Ro1], by Yoccoz puzzle techniques, Roesch shows striking differences between rational maps and polynomials. The method also leads to the local connectivity of Julia sets except some specific cases.

In this part, we use Yoccoz puzzle to study another family of rational maps, known as McMullen maps, of the form

$$f_\lambda : z \mapsto z^n + \lambda/z^n, \quad \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\},\ n \geq 3.$$

Dynamics of this family have been studied by Devaney and his group, see [D1],[D2],[DK],[DLU].

The difference of the Yoccoz puzzles between cubic Newton maps ([Ro1]) and McMullen maps is as follows: For cubic Newton maps, the ingredient of the Yoccoz puzzle is an converging ray that intersects the Julia set in a countably many points while for McMullen maps, the element to construct Yoccoz puzzle is a Jordan curve that intersects the Julia set in a Cantor set of points. This kind of Jordan curve is induced by some particular angle and can be viewed as an extention of the corresponding external ray.

We denote by B_λ the immediate attractive basin of ∞. The topology of ∂B_λ is of special interest. Based on Yoccoz puzzle techniques and combinatorial and topological analysis, we prove:

Theorem 1.2.1. (Cantor or Jordan) *For any $n \geq 3$ and any complex parameter λ, if the Julia set $J(f_\lambda)$ is not a Cantor set, then ∂B_λ is a Jordan curve.*

This affirmatively answers a question posed by Devaney at the Snowbird Conference on the 25th Birthday of the Mandelbrot set, see [DK]. For the higher regularity of ∂B_λ, we show that ∂B_λ is a quasicircle except two special cases.

Theorem 1.2.2. *Suppose the Julia set $J(f_\lambda)$ is not a Cantor set, then ∂B_λ is a quasicircle if it contains neither parabolic point nor recurrent critical point.*

Here, a recurrent critical point c on the Julia set of a rational map f is a critical point such that $c \in \omega(c)$, where $\omega(c)$ is the ω-limit set of c, defined as $\{z \in \overline{\mathbb{C}};$ there exist $n_k \to \infty$ such that $z = \lim f^{n_k}(c)\}$. In fact, we can show that if ∂B_λ contains a parabolic point, then ∂B_λ is not a quasicircle by Leau-Fatou-Flower Theorem ([M2]). The question whether ∂B_λ is a quasicircle when ∂B_λ contains a recurrent critical point is still unknown.

For the topology of the Julia set, we show

Theorem 1.2.3. *Suppose f_λ has no Siegel disk and the Julia set $J(f_\lambda)$ is connected, then $J(f_\lambda)$ is locally connected in either of the following cases:*

1. *The critical orbit does not accumulate on the boundary ∂B_λ.*
2. *The map f_λ is neither renormalizable nor $*-$renormalizable.*
3. *The parameter λ is real and positive.*

Here are the definitions of renormalization and $*-$renormalization: If there exist a critical point c of f_λ, an integer $p \geq 1$ and two disks U and V containing c such that

$$\varepsilon f_\lambda^p : U \to V$$

is a quadratic like map whose Julia set is connected (here $\varepsilon \in \{\pm 1\}$ is a symbol), then we say f_λ is p-renormalizable at c if $\varepsilon = 1$ and f_λ is p-$*$-renormalizable at c if $\varepsilon = -1$.

Theorem 1.2.3 implies that the Julia set is locally connected except some special cases. In fact, it's stronger than the following statement:

Theorem 1.2.4. *Suppose f_λ has no Siegel disk and the Julia set $J(f_\lambda)$ is connected, then $J(f_\lambda)$ is locally connected if the critical orbit does not accumulate on the boundary ∂B_λ.*

Theorem 1.2.4 is an analogue of Roesch's Theorem [Ro1]:

Theorem 1.2.5. (Roesch) *A genuine cubic Newton map, without Siegel disks, has a locally connected Julia set provided that the orbit of the non-fixed critical point does not accumulate on the boundary of any invariant basin of attraction.*

1.3 Decomposition theorem and Thurston-type theorems

Let $f : S^2 \to S^2$ be an orientation preserving branched covering of degree at leat two. We denote by $\deg(f, x)$ the local degree of f at $x \in S^2$. The critical set Ω_f of f is defined by

$$\Omega_f = \{x \in S^2; \deg(f, x) > 1\},$$

and the postcritical set P_f of f is defined by

$$P_f = \overline{\bigcup_{n\geq 1} f^n(\Omega_f)}.$$

We say that f is postcritically finite (also called 'critically finite') if P_f is a finite set. Such a map is always called a Thurston map. For a Thurston map, we define a function $\nu_f : S^2 \to \mathbb{N} \cup \{\infty\}$ in the following way: For each $x \in S^2$, define $\nu_f(x)$ (may be ∞) as the least common multiple of the local degrees $\deg(f^n, y)$ for all $n > 0$ and all $y \in S^2$ such that $f^n(y) = x$. (Notice that $\nu_f(x) = 1$ if $x \notin P_f$). We call $\mathcal{O}_f = (S^2, \nu_f)$ the orbifold of f.

In 1980s, Thurston proved the following theorem:

Theorem 1.3.1. (Thurston) *Let $f : S^2 \to S^2$ be a critically finite branched covering. Suppose that $\mathcal{O}_f$ does not have signature $(2, 2, 2, 2)$. Then f is combinatorially equivalent to a rational function R if and only if for any f-stable multicurve Γ, we have $\lambda(\Gamma, f) < 1$. The rational function R is unique up to Möbius conjugation.*

The detailed proof of Thurston's theorem is given by Douady and Hubbard [DH1].

Thurston's theorem has connection with a number of related areas such as Teichmuller theory, quasiconformal surgery, dynamics of several complex variables, transversality, group theory, algorithm, etc.

There are many applications of Thurston's theorem. Here is an incomplete list: Douady's proof of monotonicity of entropy for unimodal maps [Dou2], Kiwi's characterization of polynomial laminations [Kiwi] (using previous work of Bielefield-Fisher-Hubbard [BFH] and Poirier [Poi]), Mikulich's classification of postcritically finite Newton maps, McMullen's work on rational quotients [McM1], Pilgrim-Tan's cut-and-paste surgery along arcs ([PT1]), Rees' descriptions of parameter spaces [Rees2], Rees, Shishikura and Tan's studies on matings of polynomials ([Rees1],[ST], [Tan1], [Tan2]), ...

Over the years, there are several various attempts to generalize Thurston's theorem beyond postcritically finite rational maps. For example, David Brown [Bro], supported by the previous work of Hubbard and Schleicher [HS], has succeeded in extending the theory to the uni-critical polynomials with an infinite postcritical set (but always with a connected Julia set), and pushed it even further to the infinite degree case, namely the exponential maps. We would also like to mention a recent work of Hubbard-Schleicher-Shishikura [HSS] extending Thurston's theorem to postcritically finite exponential maps. Cui-Tan[CT1], and Jiang-Zhang [JZ], independently, using different methods, extend Thurston's theorem to hyperbolic rational maps. Furthermore, Cui and Tan [CT2] extend Thurston's theorem to geometrically finite rational maps. Meanwhile, Zhang [Zh2] extends Thurston's theorem to a class of rational maps with Siegel disks.

In this work, we aim to extend Thurston's theorem to a large class of branched covering, namely 'non-parabolic' branched covering. Roughly speaking, a 'non-parabolic' branched covering is a proper branched covering for which each critical point either has finite orbit or is attracted to an attracting cycle, or is eventually mapped to the closure of some rotation domain (rotation disk or rotation annulus, formal definition can be found in Section 6.1). A non-parabolic map with rotation disks and poscritically finite outside the closure of these rotation disks is called a Siegel map. Our main result is

Theorem 1.3.2. (Decomposition Theorem) *Let (f, P) be a non-parabolic map, then there exist a (f, P)-stable multicurve Γ and a collection of Siegel maps or Thurston maps, say $\{(h_k, P_k), k \in \Lambda\}$, where Λ is a finite index set, such that*

1. (Combinatorial part) (f, P) has no Thurston obstructions if and only if $\lambda(\Gamma, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) has no Thurston obstructions.

2. (Surgery part) (f, P) is q.c-equivalent to a rational map if and only if $\lambda(\Gamma, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) is q.c-equivalent to a rational map.

3. (Analytic part) (f, P) is q.c-equivalent to a unique rational map up to Möbius conjugation if and only if $\lambda(\Gamma, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) q.c-equivalent to a unique rational map up to Möbius conjugation.

From the viewpoint of 'reduction', the theorem implies that Thurston-type Theorem for every non-parabolic branched covering can be reduced to Thurston-type Theorem for finitely many Siegel type branched coverings. In particular, Thurston-type Theorem for rational maps with Herman rings can be reduced to Thurston-type Theorem for rational maps with Siegel disks.

The 'Decomposition Theorem' provides a mechanism to produce Thurston type Theorems for non-parabolic maps. Thus it has many applications. For example, it can reduce Thurston-type Theorem for hyperbolic maps to Thurston's Theorem for postcritically finite maps (This is the idea of Cui-Tan's work [CT1]) and thus generalizes Cui-Tan and Jiang-Zhang's work. As another application, it enables us to give a characterization of a class of rational maps with Herman rings based on Zhang's work [Zh2], as follows:

Theorem 1.3.3. *Let (f, P) be a non-parabolic map, with only one rotation annulus cycle which is of period one and has rotation number of bounded type, and without rotation disk. Then (f, P) is c-equivalent to a rational map (R, Q) if and only if (f, P) has no Thurston obstructions. Moreover, the Lebesgue measure of the Julia set $J(R)$ is zero, and (R, Q) is unique up to Möbius conjugation.*

There is no reason to believe that the absence of Thurston obstruction is always equivalent to rational realization for postcritically infinite branched covering, even if the equivalence is true for hyperbolic case ([CT1], [JZ]), some Siegel cases [Zh2] and Herman cases (Theorem 1.3.3). The mating of two quadratic Siegel polynomials $f_\theta(z) = z^2 + c_\theta$ and $f_{-\theta}(z) = z^2 + c_{-\theta}$, where $c_\alpha = \frac{e^{2\pi i\alpha}}{2}(1 - \frac{e^{2\pi i\alpha}}{2})$, provides a non-parabolic map $g = f_\theta \sqcup f_{-\theta}$ for which the equivalence is false. As a supplement to the Decomposition Theorem, following the same idea as Shishikura's construction [Sh1] of rational maps with prescribed numbers of non-repelling cycles and Herman rings, we can construct many such examples by surgery:

Theorem 1.3.4. *Given nonnegative integers n_A, n_{RD}, n_{RA}, d satisfying*

$$n_A + n_{RD} + 2n_{RA} \leq 2d - 2,\ 1 \leq n_{RA} \leq d - 2,\ n_{RD} + n_{RA} \geq 2.$$

There exists a non-parabolic map (f, P) of degree d, such that

1. $n_A(f) = n_A, n_{RD}(f) = n_{RD}, n_{RA}(f) = n_{RA}$, and the rotation number of each rotation cycle is of bounded type.

2. (f, P) has no Thurston obstructions.

3. (f, P) is not c-equivalent to a rational map.

1.4 Other topics

A line field supported on a subset E of the complex sphere $\overline{\mathbb{C}}$ is the Beltrami differential $\mu = \mu(z)d\bar{z}/dz$ supported on E with $|\mu| = 1$. We say μ is measurable if $\mu(z)$ is a measurable function. Let f be a rational map of degree $\deg(f) \geq 2$. We say f admits an invariant line field if there is a measurable line field μ supported on a set in $\overline{\mathbb{C}}$ with positive measure such that $f^*\mu = \mu$ a.e. (refer to [McM1]).

A meromorphic line field is a line field of the form $\mu = \bar{\phi}/|\phi|$, where ϕ is a nonzero meromorphic quadratic differential defined on $\overline{\mathbb{C}}$. We say f admits a meromorphic invariant line there is a meromorphic line field μ such that $f^*\mu = \mu$.

In Chapter 3, we classify the rational maps admitting meromorphic line fields:

Theorem 1.4.1. *Let f be a rational map of degree $\deg(f) \geq 2$. Then f admits a meromorphic invariant line field if and only if f is conformally conjugate to one of the following maps:*

1. *Integral Lattès map.*
2. *Power map $z \mapsto z^d$, for $d \in \mathbb{Z}$ and $|d| \geq 2$.*
3. *$\pm T_n, n \geq 2$, where T_n is the n-th Chebyshef polynomial defined by $T_n(2\cos z) = 2\cos(nz)$.*

In Chapter 4, We consider a family of rational maps

$$T_\lambda(z) = \left(\frac{z^2 + \lambda - 1}{2z + \lambda - 2}\right)^2,$$

where λ is a complex parameter. This family is indeed the family of renormalization transformations of 2-dimensional diamond-like hierachical Potts models in statistical mechanics. In 1983, Derrida *et al* show that the Yang-Lee zeros of the $\lambda-$state Potts model on the diamond hierachical lattice are dense in the Julia set $J(T_\lambda)$ of the map T_λ (See [DDI]). Since then, much interest has been devoted to this family since it exhibits a connection between statistical mechanics and complex dynamics (See [EL],[O],[QG],[QL]).

For this family, note that when $\lambda = 0$, the map T_λ degenerates to the quadratic polynomial $T_0(z) = (z+1)^2/4$; when $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, 1 and ∞ are two supperattracting fixed points for the map T_λ while 0 is a critical value.

The non-escape locus $\mathcal{M}$ associated to this family is defined by:

$$\mathcal{M} = \{\lambda \in \mathbb{C}^*; T_\lambda^n(0) \nrightarrow_{n\to\infty} 1 \text{ and } T_\lambda^n(0) \nrightarrow_{n\to\infty} \infty\} \cup \{0\}.$$

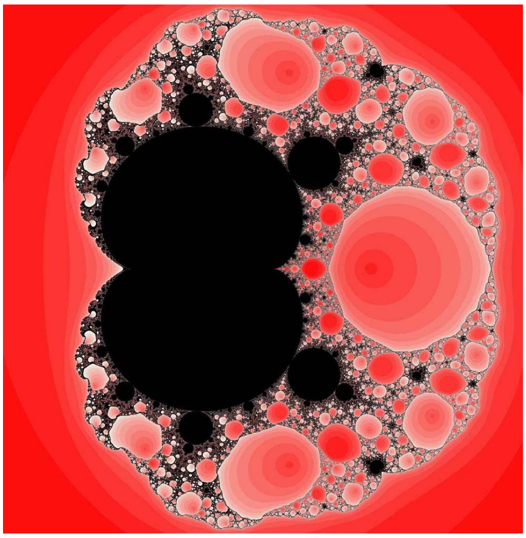

Figure 1.1: The non-escape locus $\mathcal{M}$

This part includes two results:

For the parameter plane, we have the following:

Theorem 1.4.2. *The non-escape locus $\mathcal{M}$ is connected.*

For the Julia set, we have the following:

Theorem 1.4.3. *If the Julia set $J(T_\lambda)$ is a quasi-circle, then the Hausdorff dimension of $J(T_\lambda)$ satisfies:*

$$HD(J(T_\lambda)) \leq 1 + |\phi_\lambda(0)|^{2/3},$$

where ϕ_λ is the Böttcher map of T_λ defined near the supperattracting fixed point 1.

The exponent $2/3$ in Theorem 1.4.3 is sharp.

Chapter 2
Background materials

This chapter presents some basic knowledge of complex analysis and conformal geometry, which are used in the thesis.

2.1 Spherical derivative

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function, $d(\cdot,\cdot)$ be the spherical metric. We define the spherical derivative of f by

$$\|f'(z)\| := \lim_{w\to z} \frac{d(f(w), f(z))}{d(w,z)} = \frac{1+|z|^2}{1+|f(z)|^2}|f'(z)|.$$

It's obvious that $\|f'(\cdot)\| : \overline{\mathbb{C}} \to [0,+\infty)$ is a continuous function. The area element of the sphere is $dS = \left(\frac{2}{1+|z|^2}\right)^2 dxdy$, by calculation

$$\int_{\overline{\mathbb{C}}} \|f'(z)\|^2 dS = \int_{\overline{\mathbb{C}}} f^*(dS) = \deg(f) \int_{\overline{\mathbb{C}}} dS = 4\pi \deg(f).$$

By mean value theorem, for any rational function f, there is $\xi \in \overline{\mathbb{C}}$, such that $\|f'(\xi)\| = \sqrt{\deg(f)}$.

The spherical derivative relates to the normality of rational family:

Theorem 2.1.1. *Let $\mathcal{F}$ be a family of rational maps. Then $\mathcal{F}$ is normal if and only if there is a constant $C = C(\mathcal{F})$ such that*

$$\|f'(z)\| \leq C, \ \forall (f,z) \in \mathcal{F} \times \overline{\mathbb{C}}.$$

As an immediate consequence, we have:

Corollary 2.1.1. *Let f be a rational map of degree at least two, then the Julia set $J(f)$ is not empty.*

Proof. If not, then $\mathcal{F} = \{f^n; n \geq 1\}$ is a normal family. By Theorem 2.1.1, there is a constant C such that for any $n \geq 1$ and any $z \in \overline{\mathbb{C}}$, $\|(f^n)'(z)\| \leq C$. On the other hand, by the mean value theorem, for any $n \geq 1$, there is $z_n \in \overline{\mathbb{C}}$, such that $\|(f^n)'(z_n)\| = \deg(f)^{n/2}$. Contradiction. $\square$

2.2 The modulus of an annulus

It's known that a two connected domain is conformally equivalent to $A_R = \{z \in \mathbb{C}; 1 < |z| < R\}$, $\mathbb{D} \setminus \{0\}$, or $\mathbb{C} \setminus \{0\}$. For the former case, we define the modulus of A_R to be $\frac{1}{2\pi} \log R$. In the latter two cases, we define the moduli of $\mathbb{D} \setminus \{0\}$ and $\mathbb{C} \setminus \{0\}$ to be ∞. The modulus is a conformal invariant.

Theorem 2.2.1. (McMullen Inequality) *Let U, V be two simply connected planner domains, such that $\overline{U} \subset V \neq \mathbb{C}$, and let $A = V \setminus \overline{U}$. Then the modulus of A and the Euclidean areas of U, V satisfy:*

$$e^{4\pi \mathrm{mod}(A)} \leq \frac{area(V)}{area(U)}.$$

Equality holds if and only if ∂U and ∂V are concentric circles.

Proof. We assume A is conformally isomorphic to $A_R = \{z \in \mathbb{C}; 1 < |z| < R\}$, then $\mathrm{mod}(A) = \frac{1}{2\pi} \log R$. Take a conformal map $\varphi : A_R \to A$, preserving the boundary order of A. Suppose that φ has Laurant expansion: $\varphi(z) = \sum_{n \in \mathbb{Z}} a_n z^n$. It follows from area formula that

$$area(V) = \pi \sum_{n \in \mathbb{Z}} n|a_n|^2 R^{2n}, \quad area(U) = \pi \sum_{n \in \mathbb{Z}} n|a_n|^2.$$

$$area(V) - R^2 area(U) = \pi \sum_{n \in \mathbb{Z}} n|a_n|^2 (R^{2n} - R^2) \geq 0. (*)$$

So we have

$$e^{4\pi \mathrm{mod}(A)} = R^2 \leq \frac{area(V)}{area(U)}.$$

From $(*)$ we see that equality holds if and only if for all $n \in \mathbb{Z} \setminus \{0, 1\}$, $a_n = 0$. In this case, $\varphi(z) = a_0 + a_1 z$ and ∂U,∂V are concentric circles. $\square$

Theorem 2.2.2. *1.* **(Monotonicity)** *Let A_1, A_2 be two annuli, $A_1 \subset A_2$, and A_1 separates the two boundary curves of A_2, then* $\mathrm{mod}(A_1) \leq \mathrm{mod}(A_2)$. *Equality holds if and only if $A_1 = A_2$.*

2. **(Subadditivity)** *Let A_1, A_2, A be annuli. $A_1, A_2 \subset A$, and $A_1 \cap A_2 = \emptyset$. We assume A_1 and A_2 separate the two boundary curves of A. Then* $\mathrm{mod}(A_1) + \mathrm{mod}(A_2) \leq \mathrm{mod}(A)$. *Equality holds if and only if A_1, A_2, A are standard annuli with $\overline{A_1 \cup A_2} = \overline{A}$.*

Proof. We only prove the 'monotonicity'. The proof of the 'subadditivity' follows from the same argument. Since the modulus is conformal invariant, we may assume $A_2 = \{1 < |z| < R\}$. We denote by U, V the simply connected planar domains bounded by the inner boundary and outer boundary of A_1. By McMullen Inequality

$$\mathrm{mod}(A_1) \leq \frac{1}{4\pi}\log\left(\frac{area(V)}{area(U)}\right) \leq \frac{1}{4\pi}\log\left(\frac{\pi R^2}{\pi}\right) = \mathrm{mod}(A_2).$$

The left equality holds if and only if the two boundaries of A_1 are concentric circles, and the middle equality implies that $A_1 = A_2$. $\square$

In the rest of this section, we introduce *Grötzsch constant* and discuss its relationship with modulus. This constant is called *Grötzsch defect* in Buff and Epstein's paper [BE].

We say that a compact set $K \subset \overline{\mathbb{C}}$ is *equatorial* if $\overline{\mathbb{C}} - K$ consists of two simply connected components, say U and V. Choose two base points p, q with $p \in U, q \in V$, and then take $\alpha \in Aut(\overline{\mathbb{C}})$, which maps p, q to $0, \infty$ respectively. Let $\varphi_0 : (\mathbb{D}, 0) \to (\alpha(U), 0)$ and $\varphi_\infty : (\overline{\mathbb{C}} - \overline{\mathbb{D}}, \infty) \to (\alpha(V), \infty)$ be two Riemann mappings. Suppose that

$$\varphi_0(z) = \sum_{n\geq 1} a_n z^n, z \in \mathbb{D}; \quad \varphi_\infty(\zeta) = \sum_{n\leq 1} b_n \zeta^n, \zeta \in \mathbb{C} - \overline{\mathbb{D}}.$$

We define the *Grötzsch constant* $\gamma(K, p, q)$ of K about p, q by:

$$\gamma(K, p, q) = \frac{1}{2\pi}\log\left|\frac{b_1}{a_1}\right|.$$

One may verify that $\gamma(K, p, q)$ is well-defined (independent of the choices of $\alpha, \varphi_0, \varphi_\infty$).

Example 2.2.1. *Let $\mathbb{S}$ the unit circle, choose two points $p, q \in \mathbb{C}$, with $|p| < 1, |q| > 1$, we have*

$$\gamma(\mathbb{S}, p, q) = \frac{1}{2\pi}\log\left(\frac{|p-q|^2}{(1-|p|^2)(|q|^2-1)}\right).$$

Theorem 2.2.3. *The Grötzsch constant satisfies:*

1. For any $\beta \in Aut(\overline{\mathbb{C}})$, $\gamma(\beta(K), \beta(p), \beta(q)) = \gamma(K, p, q)$.

2. $\gamma(K, p, q) \geq 0$. $\gamma(K, p, q) = 0$ if and only if there is $\beta \in Aut(\overline{\mathbb{C}})$, such that β maps $\mathbb{S}, 0, \infty$ to K, p, q, respectively.

3. If the interior of K is an annulus A, then $\gamma(K, p, q) \geq \mathrm{mod}(A)$. Equality holds if and only if there exist $\beta \in Aut(\overline{\mathbb{C}})$, $R > 1$, such that β maps $\{1 \leq |z| \leq R\}, 0, \infty$ to K, p, q, respectively.

4. (Reverse Grötzsch Inequality) For any $R > 1, 0 < r < 1$, let $A(R, r)$ be the annulus bounded by $\varphi_0(\{|z| = r\})$ and $\varphi_\infty(\{|\zeta| = R\})$, A_r be the annulus *bounded by* $\varphi_0(\{|z|=r\})$ *and* $\partial\alpha(U)$, A_R *be the annulus bounded by* $\varphi_\infty(\{|\zeta|=R\})$ *and* $\partial\alpha(V)$. *Then for any* $R>1, 0<r<1$, *we have*

$$\mathrm{mod}(A(R,r)) \le \mathrm{mod}(A_R)+\mathrm{mod}(A_r)+\gamma(K,p,q).$$

Equality holds for some pair (R,r) *if and only if there is* $\beta\in Aut(\overline{\mathbb{C}})$, $L\ge 1$ *such that* β *maps* $\{1\le|z|\le L\}, 0, \infty$ *to* K,p,q, *respectively.*

Remark 2.2.1. *The 'Reverse Grötzsch Inequality' is first introduced in [C], see also [CT1]. In [CT1], Cui Guizhen and Tan Lei use it as an analytic tool to prove a Thurston-type theorem for hyperbolic rational maps. In chaper 6, we will also use it to prove the 'Decomposition Theorem'.*

Proof. One may verify 1 by definition, we omit the details. The proofs of 2 and 3 are based on the area formula. Since $\gamma(K,p,q)$ is invariant under Möbius transformation, we may identify K,p,q with $\alpha(K),0,\infty$. It follows from area formula that

$$area(U)=\pi\sum_{n\ge1}n|a_n|^2,\quad area(\overline{\mathbb{C}}-V)=\pi\sum_{n\le1}n|b_n|^2.$$

Since $U\subset\overline{\mathbb{C}}-V$, $\pi|a_1|^2\le area(U)\le area(\overline{\mathbb{C}}-V)\le\pi|b_1|^2$. This means $|b_1|\ge|a_1|$. Equivalently, $\gamma(K,p,q)\ge0$. $|b_1|=|a_1|$ if and only if for any $n\ge2, a_n=0$, and for any $n\le-1$, $b_n=0$. So $\partial U=|a_1|\mathbb{S}, \partial V=|b_1|\mathbb{S}+b_0$. Since $U\cap V=\emptyset$, we have that $b_0=0$ and $\partial U=\partial V=|a_1|\mathbb{S}$.

If the interior of K is an annulus A, then it follows from McMullen Inequality that

$$\mathrm{mod}(A)\le\frac{1}{4\pi}\log\left(\frac{area(\overline{\mathbb{C}}-V)}{area(U)}\right)\le\frac{1}{4\pi}\log\left(\frac{\pi|b_1|^2}{\pi|a_1|^2}\right)=\gamma(K,p,q).$$

Equality $\mathrm{mod}(A)=\gamma(K,p,q)$ holds if and only if the two boundary curves of A are concentric circles, moreover $\varphi_0(z)=a_1z, \varphi_\infty(\zeta)=b_1\zeta+b_0$. One can verify that in this case, there is $\beta\in Aut(\overline{\mathbb{C}})$ such that β maps $\{1\le|z|\le e^{2\pi\mathrm{mod}(A)}\}, 0, \infty$ to K,p,q, respectively.

Here, we give two different proofs of the 'Reverse Grötzsch Inequality'.

The first is based on the McMullen Inequality. The bounded component of $\overline{\mathbb{C}}-\varphi_0(\{|z|=r\})$ is denoted by U_r, the bounded component of $\overline{\mathbb{C}}-\varphi_\infty(\{|\zeta|=R\})$ is denoted by V_R. It follows from area formula that

$$area(U_r)=\pi\sum_{n\ge1}n|a_n|^2r^{2n},\quad area(V_R)=\pi\sum_{n\le1}n|b_n|^2R^{2n}.$$

By McMullen Inequality,

$$\begin{aligned}\mathrm{mod}(A(R,r)) &\le \frac{1}{4\pi}\log\left(\frac{area(V_R)}{area(U_r)}\right)\le\frac{1}{4\pi}\log\left(\frac{\pi|b_1|^2R^2}{\pi|a_1|^2r^2}\right)\\ &= \mathrm{mod}(A_R)+\mathrm{mod}(A_r)+\gamma(K,p,q).\end{aligned}$$

One may easily verify the condition when the equality holds.

Here is another proof of the 'Reverse Grötzsch Inequality', based on Koebe distortion Theorem: we consider the function

$$f(R,r) = \mathrm{mod}(A(R,r)) - \mathrm{mod}(A_R) - \mathrm{mod}(A_r),\ 0 < r < 1, R > 1.$$

It follows from Grötzsch inequality that $f \geq 0$ and for any $0 < r_2 \leq r_1 < 1, R_2 \geq R_1 > 1$, we have $f(R_1, r_1) \leq f(R_2, r_2)$. This implies the limit $\lim_{R\to\infty, r\to 0} f(R,r)$ exists. By Koebe Theorem, when r is small enough, $\varphi_0(\{|z| = r\})$ looks like a round circle of radius $a_1 r$; when R is large enough, $\varphi_\infty(\{|\zeta| = R\})$ looks like a round circle of radius $b_1 R$. It turns out that

$$\begin{aligned} \mathrm{mod}(A(R,r)) &= \frac{1}{2\pi}\log\left|\frac{b_1 R(1+\mathcal{O}(1/R))}{a_1 r(1+\mathcal{O}(r))}\right| \\ &= \frac{1}{2\pi}\log\frac{R}{r} + \frac{1}{2\pi}\log\left|\frac{b_1}{a_1}\right| + \mathcal{O}(1/R) + \mathcal{O}(r). \end{aligned}$$

So we have

$$\lim_{R\to\infty, r\to 0} f(R,r) = \frac{1}{2\pi}\log\left|\frac{b_1}{a_1}\right| = \gamma(K,p,q).$$

This means, for any $0 < r < 1, R > 1$, $f(R,r) \leq \gamma(K,p,q)$. Moreover, the constant $\gamma(K,p,q)$ is sharp. □

2.3 Distortion Theorems

Here are some distortion theorems used in the thesis.

Theorem 2.3.1. (Koebe) *Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent function. Then for all $z \in \mathbb{D}$,*

$$|f'(0)|\frac{|z|}{(1+|z|)^2} \leq |f(z) - f(0)| \leq |f'(0)|\frac{|z|}{(1-|z|)^2}.$$

Let $U \subsetneq \mathbb{C}$ be a simply connected planar domain and $z \in U$. The shape of U about z is defined by:

$$\mathrm{Shape}(U,z) = \sup_{x\in\partial U}|x-z| / \inf_{x\in\partial U}|x-z|.$$

It's obvious that $\mathrm{Shape}(U,z) = \infty$ if and only if U is unbounded, and $\mathrm{Shape}(U,z) = 1$ if and only if U is a round disk centered at z. In all other cases, $1 < \mathrm{Shape}(U,z) < \infty$.

Let K be a connected and compact subset of U, containing at least two points. For any $z_1, z_2 \in K$, define the turning of K about z_1 and z_2 by:

$$\Delta(K; z_1, z_2) = \mathrm{diam}(K)/|z_1 - z_2|,$$

where $\mathrm{diam}(\cdot)$ is the Euclidean diameter. It's obvious that $1 \le \Delta(K; z_1, z_2) \le \infty$ and $\Delta(K; z_1, z_2) = \infty$ if and only if $z_1 = z_2$.

Theorem 2.3.2. *For $i \in \{1, 2\}$, let (V_i, U_i) be a pair of simply connected planar domains with $\overline{U_i} \subset V_i \subsetneq \mathbb{C}$. $g : V_1 \to V_2$ is a proper holomorphic map of degree D, U_1 is a component of $g^{-1}(U_2)$ and let $d = \deg(g|_{U_1})$. Then*

1. We have the following modular distortion:

$$d\mathrm{mod}(V_1 \setminus \overline{U_1}) \le \mathrm{mod}(V_2 \setminus \overline{U_2}) \le D\mathrm{mod}(V_1 \setminus \overline{U_1}).$$

2. Suppose further $\mathrm{mod}(V_2 \setminus \overline{U_2}) \ge m > 0$, then

2.1. (Shape distortion) There is a constant $C_0(D, m) > 0$ such that for all $z \in U_1$,

$$\mathrm{Shape}(U_1, z) \le C_0(D, m)\mathrm{Shape}(U_2, g(z)).$$

2.2. (Turning distortion) There is a constant $C_1(D, m) > 0$ such that for any connected and compact subset K of U_1 with $\#K \ge 2$ and any $z_1, z_2 \in K$,

$$\Delta(K; z_1, z_2) \le C_1(D, m)\Delta(g(K); g(z_1), g(z_2)).$$

Proof. The proof of the first statement (modular distortion) can be found in [KL].

Proof of the Shape distortion. The proof presented here is borrowed from Zhai's Thesis [Zhai]. Fix some point $z \in U_1$, take two Riemann mappings $\phi_1 : (V_1, z) \to (\mathbb{D}, 0)$ and $\phi_2 : (V_2, g(z)) \to (\mathbb{D}, 0)$, then the map $G = \phi_2 \circ g \circ \phi_1^{-1} : \mathbb{D} \to \mathbb{D}$ is a proper map, and $G(0) = 0$. By modular distortion, we have $\mathrm{mod}(V_1 \setminus \overline{U_1}) \ge m/D$. By Koebe Theorem and Grötzsch Theorem, there are two constants $C_1 = C_1(m/D)$ and $C_2 = C_2(m)$ such that

$$C_1^{-1}\mathrm{Shape}(\phi_1(U_1), 0) \le \mathrm{Shape}(U_1, z) \le C_1\mathrm{Shape}(\phi_1(U_1), 0),$$
$$C_2^{-1}\mathrm{Shape}(\phi_2(U_2), 0) \le \mathrm{Shape}(U_2, g(z)) \le C_2\mathrm{Shape}(\phi_2(U_2), 0).$$

In the following, we will show that there is a constant $C_3 = C_3(D) > 0$ such that

$$\mathrm{Shape}(\phi_1(U_1), 0) \le C_3\mathrm{Shape}(\phi_2(U_2), 0).$$

Let

$$L_1 = \max\nolimits_{x \in \partial\phi_1(U_1)} |x|, \; l_1 = \min\nolimits_{x \in \partial\phi_1(U_1)} |x|,$$
$$L_2 = \max\nolimits_{y \in \partial\phi_2(U_2)} |y|, \; l_2 = \min\nolimits_{y \in \partial\phi_2(U_2)} |y|.$$

By Schwarz Lemma, $l_1 \geq l_2$. If $L_2 \geq \frac{1}{2}$, then

$$\text{Shape}(\phi_1(U_1), 0) = \frac{L_1}{l_1} \leq \frac{1}{l_1} \leq \frac{2L_2}{l_2} = 2\text{Shape}(\phi_2(U_2), 0).$$

Now we consider the case $L_2 \leq \frac{1}{2}$. Let $\mathbb{D}_{2L_2} = \{z \in \mathbb{C}; |z| < 2L_2\}$ and W be the connected component of $G^{-1}(\mathbb{D}_{2L_2})$ that contains 0. By the maximum modulus principle, W is simply connected. Let $\varphi : (W, 0) \to (\mathbb{D}, 0)$ be a Riemann mapping.

Then the map $F = \frac{1}{2L_2} G \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ is a proper map, with $F(0) = 0$. Since $F \circ \varphi \circ \phi_1(U_1) = \frac{1}{2L_2}\phi_2(U_2)$ and $\max_{w \in \partial F \circ \varphi \circ \phi_1(U_1)} |w| = \frac{1}{2}$, by the previous argument,

$$\text{Shape}(\varphi \circ \phi_1(U_1), 0) \leq 2\text{Shape}(\phi_2(U_2), 0).$$

Notice that $\text{mod}(W \backslash \phi_1(U_1)) \geq \frac{1}{2\pi} \log 2$, we have that $\text{mod}(W \backslash \overline{U_2}) \geq \frac{1}{2\pi D} \log 2$. By Koebe Theorem and Grötzsch Theorem, there is a constant $C = C(D)$ such that

$$\text{Shape}(\phi_1(U_1), 0) \leq C\text{Shape}(\varphi \circ \phi_1(U_1), 0).$$

Then the conclusion follows immediately.

Proof of the Turning distortion. We assume that $g(z_1) \neq g(z_2)$. For else, $\Delta(g(K); g(z_1), g(z_2)) = \infty$ and the conclusion follows. Let $\rho(x, y)$ be the hyperbolic distance in V_2 and let B_1, B_2 be the hyperbolic disks both centered at $g(z_1)$, with radii $\max_{\zeta \in g(K)} \rho(g(z_1), \zeta)$ and $\rho(g(z_1), g(z_2))$ respectively. Let $\varphi : V_2 \to D$ be the Riemann mapping with $\varphi(g(z_1)) = 0$ and let $W = \varphi(U_2)$. Since $\text{mod}(\mathbb{D} \setminus \overline{W}) = \text{mod}(V_2 \setminus \overline{U}_2) \geq m$, we conclude by Grötzsch Theorem that there is a constant $r(m) \in (0, 1)$ such that $W \subset D_{r(m)}$, here we use D_r to denote the disk $\{z; |z| < r\}$.

Note that $\varphi(B_1), \varphi(B_2)$ are two round disks centered at 0, say D_R and D_r respectively. By Koebe distortion, there exist three constants $C_1(m), C_2(m), C_3(m) > 0$ such that

$$\text{Shape}(B_1, g(z_1)) \leq C_1(m), \ \text{Shape}(B_2, g(z_1)) \leq C_2(m),$$

$$R/r \leq C_3(m) \max_{\zeta \in g(K) \cap \partial B_1} |g(z_1) - \zeta| / |g(z_1) - g(z_2)| \leq C_3(m)\Delta(g(K); g(z_1), g(z_2)).$$

For $i \in \{1, 2\}$, let W_i be the component of $g^{-1}(B_i)$ that contains z_1. By the Maximum Principle, W_1 and W_2 are simply connected. We may assume that $K \subset \overline{W}_1$ (for else, we can replace B_1 by $\widehat{B}_1$, a hyperbolic disk centered at $g(z_1)$ with radius $\varepsilon + \max_{\zeta \in g(K)} \rho(g(z_1), \zeta)$, where ε is a small positive constant, and then let $\varepsilon \to 0^+$). Thus $\text{diam}(K) \leq \text{diam}(W_1) \leq 2 \sup_{\zeta \in \partial W_1} |\zeta - z_1|$. Consider the location of z_2, by Maximum Principle, either $z_2 \in \partial W_2$ or $z_2 \in U_1 \setminus \overline{W}_2$. In either case, $|z_1 - z_2| \geq \inf_{\zeta \in \partial W_2} |\zeta - z_1|$. Thus by Shape distortion,

$$\begin{aligned}
\Delta(K; z_1, z_2) &\leq 2 \sup_{\zeta \in \partial W_1} |\zeta - z_1| / \inf_{\zeta \in \partial W_2} |\zeta - z_1| \\
&= 2\text{Shape}(W_1, z_1)\text{Shape}(W_2, z_1)Q(W_1, W_2, z_1) \\
&\leq C_1(D, m)\text{Shape}(B_1, g(z_1))\text{Shape}(B_2, g(z_1))Q(W_1, W_2, z_1) \\
&\leq C_2(D, m)Q(W_1, W_2, z_1)
\end{aligned}$$

where $Q(W_1, W_2, z_1) = \inf_{\zeta \in \partial W_1} |\zeta - z_1| / \sup_{\zeta \in \partial W_2} |\zeta - z_1|$. In the following, to finish, we show that there is a constant $c(m) > 0$ such that

$$Q(W_1, W_2, z_1) \leq c(m)\Delta(g(K); g(z_1), g(z_2)).$$

In fact, we just need consider the case $Q(W_1, W_2, z_1) > 1$. In this case, the annulus $W_1 \setminus \overline{W}_2$ contains the round annulus $\{w \in \mathbb{C}; \sup_{\zeta \in \partial W_2} |\zeta - z_1| < |w - z_1| < \inf_{\zeta \in \partial W_1} |\zeta - z_1|\}$. It turns out that

$$\begin{aligned}
\frac{1}{2\pi} \log Q(W_1, W_2, z_1) &\leq \text{mod}(W_1 \setminus \overline{W}_2) \leq \text{mod}(B_1 \setminus \overline{B}_2) = \frac{1}{2\pi} \log \frac{R}{r} \\
&\leq \frac{1}{2\pi} \log \big(C_3(m)\Delta(g(K); g(z_1), g(z_2))\big)
\end{aligned}$$

The conclusion follows. □

2.4 Quasiconformal maps

Let $f : X \to Y$ be a homeomorphism between two Riemann surfaces. We say that f is a K-quasiconformal map ($K \geq 1$) if in the distribution sense, $\partial f/\partial \bar{z}, \partial f/\partial z \in L^2_{loc}(X)$, and the Beltrami coefficient

$$\mu_f(z) := \frac{\partial f/\partial \bar{z}}{\partial f/\partial z}$$

satisfies $\|\mu_f\|_\infty \leq k$, where $k = (K-1)/(K+1)$.

A mapping f is 1-quasiconformal map if and only if f is a conformal map in the normal sense.

Theorem 2.4.1. (Measurable Riemann Mapping Theorem) *For any $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$, there is a unique quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$, which fixes $0, 1, \infty$ and satisfies $\mu_\phi = \mu$.*

Moreover, for any $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty \leq 1$, there exist a unique family of quasiconformal maps $\phi_t : \mathbb{C} \to \mathbb{C}, |t| < 1$, which fix $0, 1, \infty$ and satisfy $\mu_{\phi_t} = t\mu$. Then $\phi_t(z)$ is holomorphic with respect to $t \in \mathbb{D}$ for each $z \in \mathbb{C}$.

A quasicircle is the image of the unit circle $\mathbb{S}$ under a quasiconformal map.

Theorem 2.4.2. (Ahlfors) *A Jordan curve $S \subset \mathbb{C}$ is a quasicircle if and only if there is a constant $C > 0$, such that for any $p, q \in S$, we have*

$$\min\{\operatorname{diam}(S^+_{pq}), \operatorname{diam}(S^-_{pq})\} \leq C|p - q|,$$

where diam *is the Euclidean diameter and S^+_{pq}, S^-_{pq} are two components of $S - \{p, q\}$.*

Given a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$, Astala [Ast] showed that the Hausdorff dimension of the quasicircle $\phi(\mathbb{S})$ is less than $1 + \|\mu_\phi\|_\infty$. He also conjectured that the upper bounded can be improved by $1 + \|\mu_\phi\|^2_\infty$. This conjecture is resolved by Smirnov [Smi].

Theorem 2.4.3. (Smirnov) *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a quasiconformal map, then the Hausdorff dimension of $\phi(\mathbb{S})$ is bounded above by $1 + \|\mu_\phi\|^2_\infty$.*

2.5 Holomorphic motion

Let $\mathbb{D}$ be the unit disk and E be a subset of $\overline{\mathbb{C}}$.

Definition 2.5.1. *A map $h : \mathbb{D} \times E \to \overline{\mathbb{C}}$ is called a holomorphic motion of E parameterized by $\mathbb{D}$ and with base point 0 if*

1. *$h(0, z) = z$ for all $z \in E$,*
2. *For every $c \in \mathbb{D}$, $z \mapsto h(c, z)$ is injective on E, and*
3. *For every $z \in E$, $c \mapsto h(c, z)$ is holomorphic for $c \in \mathbb{D}$.*

In fact, $\mathbb{D}$ can be replaced by any simply connected hyperbolic domain. Here is the well-known 'Holomorphic Motion Theorem' see [Slo], [GJW]:

Theorem 2.5.1. (Slodkowski) *Let E be a compact subset of $\overline{\mathbb{C}}$, $h : \mathbb{D} \times E \to \overline{\mathbb{C}}$ be a holomorphic motion of E parameterized by $\mathbb{D}$ and with base point* 0. *Then there is a holomorphic motion $H : \mathbb{D} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, which extends h. Moreover, for any fixed $c \in \mathbb{D}$, $H(c, \cdot) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a quasiconformal map, with dilatation*

$$K(H(c, \cdot)) \leq \frac{1 + |c|}{1 - |c|}.$$

We can use Slodkowski's Theorem and Smirnov's Theorem to estimate the Hausdorff dimension of Julia set when it is a quasicircle. The following example illustrates how the methodology works.

Example 2.5.1. *Given a quadratic polynomial* $f_c(z) = z^2 + c$*, suppose that* c *lies in the cardioid* $\heartsuit$ *of the Mandelbrot set. Then the Julia set* $J(f_c)$ *is a quasicircle and* $J(f_0) = \mathbb{S}$*. By the characterization of stability ([McM1], Theorem 4.2), there is a holomorphic motion* $h : \heartsuit \times J(f_0) \to \mathbb{C}$ *of* $J(f_0) = \mathbb{S}$ *parameterized by* $\heartsuit$ *and with base point* 0*, such that* $h(c, \mathbb{S}) = J(f_c)$ *for all* $c \in \heartsuit$*. By Slodkowski's Theorem,* h *admits an extension* $H : \heartsuit \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$*, and for any* $c \in \heartsuit$*,* $H(c, \cdot)$ *is a quasiconformal map with dilatation*

$$K(H(c,\cdot)) \leq \exp(d_{\heartsuit}(0,c)) = \frac{1 + |1 - \sqrt{1-4c}|}{1 - |1 - \sqrt{1-4c}|},$$

where $d_{\heartsuit}$ *is the hyperbolic metric in* $\heartsuit$*. Notice that* $H(c, \mathbb{S}) = J(f_c)$ *for all* $c \in \heartsuit$*, it follows from Smirnov's Theorem that the Hausdorff dimension* d_c *of* $J(f_c)$ *satisfies:*

$$d_c \leq 1 + |1 - \sqrt{1-4c}|^2, c \in \heartsuit.$$

This implies $d_c = 1 + \mathcal{O}(|c|^2)$ *when* $|c|$ *is small. One may compare this estimate with Ruelle's expansion of* d_c *when* $|c|$ *is small ([Ru]):*

$$d_c = 1 + \frac{|c|^2}{4\log 2} + o(|c|^2).$$

Example 2.5.2. *In chapter 4, we consider the quadratic family:*

$$t_\lambda(z) = \Big(\frac{z-1+\lambda}{z-1}\Big)^2, \ \lambda \in \mathbb{C} - \{0\}.$$

By the same method, we can show that when the Julia set $J(t_\lambda)$ *is a quasicircle, the Hausdorff dimension* d_λ *of* $J(t_\lambda)$ *satisfies:*

$$d_\lambda \leq 1 + |\phi_\lambda(0)|^{2/3},$$

where ϕ_λ *is the Böttcher map of* $T_\lambda = t_\lambda \circ t_\lambda$ *defined near the supperattracting fixed point 1. For more details, see the proof of Theorem 4.1.2.*

2.6 Extremal quasiconformal conjugacy

Let $f_c(z) = z^2 + c$ and M be the Mandelbrot set, B_c be the Böttcher coordinate of f_c defined in a neighborhood of ∞. Douady and Hubbard [DH2] showed that the map defined by

$$\Phi : \begin{cases} \mathbb{C} - M \to \mathbb{C} - \overline{\mathbb{D}} \\ c \mapsto B_c(c) \end{cases}$$

is a conformal isomorphism.

Let $\rho(\cdot,\cdot)$ be the hyperbolic distance in $\mathbb{C}-M$. For any $c_1, c_2 \in \mathbb{C}-M$, one may verify that

$$\tanh\Big(\frac{\rho(c_1,c_2)}{2}\Big) = \left|\frac{\mathrm{Log}\Phi(c_1) - \mathrm{Log}\Phi(c_2)}{\mathrm{Log}\Phi(c_1) + \overline{\mathrm{Log}\Phi(c_2)}}\right|,$$

where $\mathrm{Log}z := \log|z| + \arg z$ and the branch of Log is chosen such that $|\arg\Phi(c_1) - \arg\Phi(c_2)| \le \pi$. Let $\mathcal{Q}(c_1,c_2)$ be the set of all quasiconformal conjugacies between f_{c_1} and f_{c_2}. That is, for any $\phi \in \mathcal{Q}(c_1,c_2)$, $\phi : \mathbb{C} \to \mathbb{C}$ is a quasiconformal map and $\phi \circ f_{c_1} = f_{c_2} \circ \phi$.

Theorem 2.6.1. *Given $c_1, c_2 \in \mathbb{C}-M$, there is $\phi \in \mathcal{Q}(c_1,c_2)$ such that*

$$\|\mu_\phi\| = \inf\{\|\mu_\varphi\|; \varphi \in \mathcal{Q}(c_1,c_2)\} = \tanh\Big(\frac{\rho(c_1,c_2)}{2}\Big).$$

Proof. Step 1. For any $\varphi \in \mathcal{Q}(c_1,c_2)$,

$$\|\mu_\varphi\| \ge \tanh\Big(\frac{\rho(c_1,c_2)}{2}\Big).$$

We may assume $c_1 \neq c_2$, otherwise the conclusion follows immediately. First notice that $f_{c_1}^*(\mu_\varphi) = \mu_\varphi$. For $\lambda \in \mathbb{D}$, let φ_λ solve the Beltrami equation

$$\frac{\partial\varphi_\lambda}{\partial\bar z}\Big/\frac{\partial\varphi_\lambda}{\partial z} = \lambda\mu_\varphi/\|\mu_\varphi\|,$$

with $0, 1, \infty$ fixed. φ_λ is holomorphic with respect to $\lambda \in \mathbb{D}$ and $\varphi_0 = \mathrm{id}$.

The map $\varphi_\lambda \circ f_{c_1} \circ \varphi_\lambda^{-1}$ is the quadratic polynomial of the form $a(\lambda)z^2 + b(\lambda)$. Let $\phi_\lambda = a(\lambda)\varphi_\lambda$, then $\phi_\lambda \circ f_{c_1} \circ \phi_\lambda^{-1}$ is the quadratic polynomial $z^2 + c(\lambda)$, where $c : \mathbb{D} \to \mathbb{C}-M$ is a holomorphic map, with $c(0) = c_1$, $c(\|\mu_\varphi\|) = c_2$. By Schwarz Lemma, we have

$$\rho(c_1,c_2) = \rho(c(0), c(\|\mu_\varphi\|)) \le d_{\mathbb{D}}(0, \|\mu_\varphi\|) = \log\frac{1+\|\mu_\varphi\|}{1-\|\mu_\varphi\|}.$$

Equivalently,

$$\|\mu_\varphi\| \ge \tanh\Big(\frac{\rho(c_1,c_2)}{2}\Big).$$

Step 2. For any $c_* \in \mathbb{C}-M$ with $\arg\Phi(c_1) = \arg\Phi(c_*)$, there is $\zeta \in \mathcal{Q}(c_1,c_*)$, such that

$$\|\mu_\zeta\| = \tanh\Big(\frac{\rho(c_1,c_*)}{2}\Big) = \left|\frac{\log|\Phi(c_1)| - \log|\Phi(c_*)|}{\log|\Phi(c_1)| + \log|\Phi(c_*)|}\right|.$$

The proof is based on Thurston algorithm and the fact that $J(f_{c_1})$ has zero Lebesgue measure. Here is the detail:

For $i \in \{1, *\}$, let B_i be the Böttcher coordinate of f_{c_i}, $\Omega_i = \{z \in F(f_{c_i}); |B_i(z)| > |B_i(c_i)| = |\Phi(c_i)|\}$ and $D_i = \{w \in \mathbb{C}; |w| > |\Phi(c_i)|\}$. We define a quasiconformal homeomorphism $\delta : \Omega_1 \to \Omega_*$ such that the following diagram commutes

$$\begin{tikzcd} \Omega_1 \arrow[r, "\delta"] \arrow[d, "B_1"'] & \Omega_* \arrow[d, "B_*"] \\ D_1 \arrow[r, "re^{i\theta} \mapsto r^\alpha e^{i\theta}"'] & D_* \end{tikzcd}$$

where α satisfies $|\Phi(c_1)|^\alpha = |\Phi(c_*)|$. We can extend δ to a quasiconformal map $\zeta_0 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Then we can get a quasiconformal map ζ_1 such that $f_{c_*} \circ \zeta_1 = \zeta_0 \circ f_{c_1}$ and $\zeta_1|_{\Omega_1} = \delta$. By Thurston algorithm, there is a sequence of quasiconformal maps $\zeta_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f_{c_*} \circ \zeta_{n+1} = \zeta_n \circ f_{c_1}$ for $n \geq 0$ and ζ_{n+1} is isotopic to ζ_n rel $f_{c_1}^{-n}(\Omega_1)$. One can verify that

$$\text{ess.sup}_{z \in f_{c_1}^{-n-1}(\Omega_1)} |\mu_{\zeta_{n+1}}(z)| = \text{ess.sup}_{z \in f_{c_1}^{-n}(\Omega_1)} |\mu_{\zeta_n}(z)|.$$

For $n \geq 1$, the quasiconformal dilatation $K(\zeta_n)$ of ζ_n is bounded above by $K(\zeta_0)$, so $\{\zeta_n\}$ is a normal family. This implies that there is a limit map $\zeta_\infty = \lim \zeta_n$, which is in fact a q.c conjugacy between f_{c_1} and f_{c_*}. Since $J(f_{c_1})$ has zero Lebesgue measure, the Beltrami coefficient of ζ_∞ satisfies

$$\begin{aligned} \|\mu_{\zeta_\infty}\| &= \text{ess.sup}_{z \in F(f_{c_1})} |\mu_{\zeta_\infty}(z)| = \lim_{n \to \infty} \text{ess.sup}_{z \in f_{c_1}^{-n}(\Omega_1)} |\mu_{\zeta_\infty}(z)| \\ &= \lim_{n \to \infty} \text{ess.sup}_{z \in f_{c_1}^{-n}(\Omega_1)} |\mu_{\zeta_n}(z)| = \text{ess.sup}_{z \in \Omega_1} |\mu_{\zeta_0}(z)| \\ &= \text{ess.sup}_{z \in \Omega_1} |\mu_\delta(z)| = \left| \frac{\alpha - 1}{\alpha + 1} \right| = \left| \frac{\log|\Phi(c_1)| - \log|\Phi(c_*)|}{\log|\Phi(c_1)| + \log|\Phi(c_*)|} \right|. \end{aligned}$$

Step 3. The proof of the theorem.

We may replace c_2, φ by c_*, ζ_∞ respectively in Step 1. Then the map $c : \mathbb{D} \to \mathbb{C} - M$ satisfies: $c(0) = c_1, c(\|\mu_{\zeta_\infty}\|) = c_*$. By Step 2,

$$\rho(c_1, c_*) = \rho(c(0), c(\|\mu_{\zeta_\infty}\|)) = d_{\mathbb{D}}(0, \|\mu_{\zeta_\infty}\|).$$

Thus $c : \mathbb{D} \to \mathbb{C} - M$ is a covering map. One can write explicitly

$$c(\lambda) = \Phi^{-1}(\Phi(c_1)^{\frac{1+\lambda}{1-\lambda}}), \lambda \in \mathbb{D}.$$

For any $c_2 \in \mathbb{C} - M$, we can chose $\lambda \in c^{-1}(c_2)$ such that $d_{\mathbb{D}}(0, \lambda) = \rho(c_1, c_2)$. The quasiconformal map ϕ_λ conjugate f_{c_1} to f_{c_2}, and

$$\|\mu_{\phi_\lambda}\| = |\lambda| = \tanh\Big(\frac{\rho(c_1, c_2)}{2}\Big),$$

as required. □

Remark 2.6.1. *Given two quadratic polynomials $f_{c_i}(z) = z^2 + c_i$, $i = 1, 2$. Suppose that c_1 and c_2 lie in the same hyperbolic component of the Mandelbrot set M and neither of f_{c_i} is postcritically finite. Let $\lambda(c_i)$ be the multiplier of the attracting cycle of f_{c_i}, $\mathcal{Q}(c_1, c_2)$ be the set of all quasiconformal conjugacies between f_{c_1} and f_{c_2}. Then by the same method (based on Thurston algorithm and the fact that $J(f_{c_i})$ has zero Lebesgue measure) as in Theorem 2.6.1, one can show that there is $\phi \in \mathcal{Q}(c_1, c_2)$ such that*

$$\|\mu_\phi\| = \inf\{\|\mu_\varphi\|; \varphi \in \mathcal{Q}(c_1, c_2)\} = \left|\frac{\mathrm{Log}\lambda(c_1) - \mathrm{Log}\lambda(c_2)}{\mathrm{Log}\lambda(c_1) + \overline{\mathrm{Log}\lambda(c_2)}}\right|,$$

where the branch of Log *is chosen such that* $|\arg\lambda(c_1) - \arg\lambda(c_2)| \leq \pi$.

Chapter 3

On meromorphic invariant line fields

3.1 Introduction

A line field supported on a subset E of the complex sphere $\overline{\mathbb{C}}$ is the Beltrami differential $\mu = \mu(z)d\bar{z}/dz$ supported on E with $|\mu| = 1$. We say μ is measurable if $\mu(z)$ is a measurable function. Let f be a rational map of degree $\deg(f) \geq 2$. We say f admits an invariant line field if there is a measurable line field μ supported on a set in $\overline{\mathbb{C}}$ with positive measure such that $f^*\mu = \mu$ a.e. (refer to [McM1]).

We are mostly interested in the invariant line fields which are carried on the Julia sets for rational maps. One example ever known is so called 'integral Lattès map', which is constructed via torus endomorphism. The construction is as follows. Let $X = \mathbb{C}/\Lambda$ be a complex torus, and α be a complex number such that $|\alpha| > 1$ and $\alpha\Lambda \subset \Lambda$. The multiplication by α induces an endomorphism $F : X \to X$. Let $\wp : X \to \overline{\mathbb{C}}$ be the Weierstrass function. Since $\wp(-z) = \wp(z)$, the endomorphism F can induce a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f(\wp(z)) = \wp(F(z))$. Such a map f is called a Lattès map. If α is an integer, then F admits an invariant line field on X. This line field has the form $\mathrm{e}^{\mathrm{i}\theta}d\bar{z}/dz$ and can descend to an invariant line field for f (see [McM1]). On the other hand, we can verify that if f admits an invariant line field, then this line field can lift to an invariant line field for F and α turns out to be an integer. In this case, we say f is an integral Lattès map.

One of the central problems in complex dynamics is the following:

Conjecture. (No invariant line fields) *A rational map f of degree $\deg(f) \geq 2$ carries no invariant line fields on its Julia set, except when f is an integral Lattès map.*

The conjecture implies the density of hyperbolic maps in the space Rat_d of all rational maps of degree d (see [McM1]). Much study has been devoted to special families of rational maps, especially quadratic polynomials of the form $f_c(z) = z^2 + c$ for $c \in \mathbb{C}$. However, even for the quadratic family, the conjecture is still unsolved.

Fortunately, if we require the line field μ to be 'good', we can classify all the rational maps which leave μ invariant. Here, a 'good' line field means that it can be written in the form $\mu = \bar{\phi}/|\phi|$, where ϕ is a nonzero meromorphic quadratic differential defined on $\overline{\mathbb{C}}$. In this case, formally, we call μ a meromorphic line field, dual to ϕ. Correspondingly, we say f admits a meromorphic invariant line field if $f^*\mu = \mu$.

Now we can formulate our main theorem:

Theorem 3.1.1. *Let f be a rational map of degree $\deg(f) \geq 2$. Then f admits a meromorphic invariant line field if and only if f is conformally conjugate to one of the following maps:*

1. *Integral Lattès map.*
2. *Power map $z \mapsto z^d$, for $d \in \mathbb{Z}$ and $|d| \geq 2$.*
3. *$\pm T_n, n \geq 2$, where T_n is the n-th Chebyshef polynomial defined by $T_n(2\cos z) = 2\cos(nz)$.*

This theorem is deeply inspired by a theorem in [McM1] which states that if a rational map f admits an invariant line field which is holomorphic on a nonempty open set contained in the Julia set, then f is an integral Lattès map. Moreover, three examples are provided in [McM1]. One is the power map $z \mapsto z^d$, for which the line field is dual to dz^2/z^2; another is the integral Lattès map, for which the line field is dual to

$$\frac{dz^2}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)};$$

the third is the quadratic polynomial $f(z) = z^2 - 2$, for which the line field is dual to $dz^2/(z^2-4)$. So it is a natural question to figure out whether these are all examples which admit meromorphic invariant line fields. These examples motivate our study.

It is interesting to compare our classification theorem with another trichotomy theorem from the viewpoint of 'permutable maps'. Motivated by [M3], we call a rational map f is permutable if it commutes with another rational map g, $f \circ g = g \circ f$, where both f and g have degree at least two, and no iterate of f is equal to an iterate of g.

Theorem 3.1.2. (Ritt and Eremenko) *A rational map f of degree $\deg(f) \geq 2$ is permutable if and only if it is a finite quotient of an affine map; that is if and only if it is either a Lattès, Chebyshef, or power map.*

This theorem was first proved by Ritt [Ritt] in 1923, and by Eremenko [Ere] using a quite different method in 1989. For higher dimensional analogues, see [DS].

This chapter has been published as [W]. The same result is obtained by Rempe and van Strien in [RvS] using the orbifold theory, where they use it to prove the absence of invariant line fields on the Julia set of a class of transcendental meromorphic functions.

3.2 Proof of the Main Theorem

First, we need some notations. Let $\mathcal{M}(\overline{\mathbb{C}})$ be the set of all meromorphic quadratic differentials defined on $\overline{\mathbb{C}}$. For $\phi \in \mathcal{M}(\overline{\mathbb{C}})$, let $\mathcal{Z}(\phi)$ and $\mathcal{P}(\phi)$ be the zero set and the pole set of ϕ respectively. The order of $\phi \in \mathcal{M}(\overline{\mathbb{C}})$ at a point z_0, denoted by $\mathrm{ord}_{z_0}(\phi)$, is defined as follows. If z_0 is a zero of ϕ of order n, set $\mathrm{ord}_{z_0}(\phi) = n$; if z_0 is a pole of ϕ of order n, set $\mathrm{ord}_{z_0}(\phi) = -n$; for other cases, z_0 is called a regular point of ϕ, set $\mathrm{ord}_{z_0}(\phi) = 0$. For a rational map f, let $C(f)$ be the set of all critical points,

$$P(f) = \overline{\bigcup_{c \in C(f), n>0} f^n(c)}$$

be the postcritical set. The backward orbit of a point z, under iteration of f is denoted by $\mathrm{orb}^-(z) = \bigcup_{n \geq 0} f^{-n}(z)$. Let $\deg(f, z)$ be the local degree of f at z.

Proof of the Main Theorem. Let f be a rational map of degree $\deg(f) \geq 2$ and $\mu = \bar{\phi}/|\phi|$ be a meromorphic invariant line field of f for some $\phi \in \mathcal{M}(\overline{\mathbb{C}})$.

The 'if' part of the theorem is easy to verify. The proof for the 'only if' part is organized in five steps:

Step 1. $f^*(\bar{\phi}/|\phi|) = \bar{\phi}/|\phi|$ *if and only if there is a positive constant* C *such that* $f^*\phi = C\phi$. *This constant* C *is uniquely determined by* f. *Moreover, any other meromorphic invariant line field of* f *must have the form* $e^{i\theta}\mu$ *for some* $\theta \in \mathbb{R}$.

Note that the relation $f^*(\bar{\phi}/|\phi|) = \bar{\phi}/|\phi|$ is equivalent to

$$f^*\phi/\phi = |f^*\phi|/|\phi|. \tag{3.1}$$

This indicates that the well-defined holomorphic map $f^*\phi/\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ takes only positive value, thus equation (3.1) holds if and only if $f^*\phi/\phi$ is a positive constant by open map theorem.

Now suppose $\mu_i = \bar{\phi}_i/|\phi_i|$ $(i = 1, 2)$ are two meromorphic invariant line fields for f. Above argument shows $f^*\phi_i = C_i\phi_i,\ i = 1, 2$. Since ϕ_1/ϕ_2 is a well-defined holomorphic map from $\overline{\mathbb{C}}$ to itself, denoted by R, the relation

$$\frac{f^*\phi_1}{f^*\phi_2} = \frac{C_1\phi_1}{C_2\phi_2}$$

implies that $R \circ f = (C_1/C_2)R$. Comparing the degree of $R \circ f$ and f, we conclude R is a nonzero complex constant, and $C_1 = C_2$. Therefore μ_1 is identical to μ_2 up to a rotation.

From now on, we may write ϕ, C as ϕ_f, C_f, since they are determined by f. To find all rational maps which admit meromorphic invariant line fields is equivalent to find all solutions $(f, \phi_f, C_f) \in \mathrm{Rat}_2^+ \times \mathcal{M}(\overline{\mathbb{C}}) \times \mathbb{R}^+$ to the indeterminate equation

$$f^*\phi_f = C_f\phi_f,$$

where Rat_2^+ is the space of all rational maps of degree at least two. In local coordinate, $\phi_f = \phi_f(z)dz^2$, the indeterminate equation has the form

$$\phi_f(f(z))f'(z)^2 = C_f\phi_f(z). \tag{3.2}$$

Moreover, for any $z \in \overline{\mathbb{C}}$, comparing the order of $f^*\phi_f$ and ϕ_f at the point z, we have the following identity

$$\mathrm{ord}_z\big(f^*\phi_f\big) = \deg(f, z)\big(2 + \mathrm{ord}_{f(z)}(\phi_f)\big) - 2 = \mathrm{ord}_z(\phi_f). \tag{3.3}$$

Step 2. $\mathcal{Z}(\phi_f) = \emptyset$.

For else, let $z_0 \in \mathcal{Z}(\phi_f) \neq \emptyset$. We can conclude from equation (3.2) that $\mathrm{orb}^-(z_0) \subset \mathcal{Z}(\phi_f)$. Since $\mathcal{Z}(\phi_f)$ is a discrete subset of $\overline{\mathbb{C}}$, $\#\mathrm{orb}^-(z_0) < \infty$ and $\#f^{-1}\big(\mathrm{orb}^-(z_0)\big) \geq \#\mathrm{orb}^-(z_0)$. On the other hand,

$$f^{-1}\big(\mathrm{orb}^-(z_0)\big) = \bigcup_{n\geq 1} f^{-n}(z_0) \subset \mathrm{orb}^-(z_0).$$

So we have $f^{-1}\big(\mathrm{orb}^-(z_0)\big) = \mathrm{orb}^-(z_0)$. It is easy to see that all points in $\mathrm{orb}^-(z_0)$ are superattracting periodic points. If $\#\mathrm{orb}^-(z_0) \geq 3$, by Montel's Theorem, the set $\bar{\mathbb{C}}\backslash\mathrm{orb}^-(z_0)$ is completely invariant and lies in the Fatou set $F(f)$. This indicates that the Julia set $J(f) = \emptyset$, which is a contradiction. Thus $\#\mathrm{orb}^-(z_0)$ can only be 1 or 2.

If $\#\mathrm{orb}^-(z_0) = 2$, then f is conformally conjugate to the power map $z \mapsto z^d$, for some $d \in \mathbb{Z}$. But as is known that any meromorphic invariant line field of the power map must be dual to Cdz^2/z^2 (see the previous and Step 1), which has no zeros. So this leads to a contradiction.

If $\#\mathrm{orb}^-(z_0) = 1$, then $f^{-1}(z_0) = \{z_0\}$, $\deg(f, z_0) = \deg(f)$. By identity (3.3), we have

$$\mathrm{ord}_{z_0}(f^*\phi_f) = \deg(f)\big(\mathrm{ord}_{z_0}(\phi_f) + 2\big) - 2 = \mathrm{ord}_{z_0}(\phi_f).$$

But this is also impossible since $\mathrm{ord}_{z_0}(\phi_f) \geq 1, \deg(f) \geq 2$.

Step 3. f is critically finite, that is $\#P(f) < \infty$. Moreover $P(f) = \mathcal{P}(\phi_f)$.

For any $c \in C(f)$, equation (3.2) implies that $f(c) \in \mathcal{P}(\phi_f)$. For else c will be a zero of ϕ_f, which is already ruled out in step 2. Replacing f by f^n, we have $f^n(c) \in \mathcal{P}(\phi_f)$, thus

$$\bigcup_{n\geq 0} f^n(C(f)) \subset \mathcal{P}(\phi_f).$$

This means f is critically finite, since $\mathcal{P}(\phi_f)$ is a finite set. Moreover $P(f) \subset \mathcal{P}(\phi_f)$.

If $\mathcal{P}(\phi_f)\backslash P(f) \neq \emptyset$, taking $z_0 \in \mathcal{P}(\phi_f)\backslash P(f)$, we have from equation (3.2) that

$$\mathrm{orb}^-(z_0) \subset \mathcal{P}(\phi_f), \quad \#\mathrm{orb}^-(z_0) = \infty,$$

which is a contradiction. This ends the proof of step 3.

By the Riemann-Roch theorem,

$$\deg(\phi_f) = \#\mathcal{Z}(\phi_f) - \#\mathcal{P}(\phi_f) = -\#\mathcal{P}(\phi_f) = -4.$$

This means that ϕ_f has four poles (counting the multiplicity). Since f is critically finite, each periodic cycle of f is either repelling or superattracting (See [McM1] or [M1]).

Step 4. If f has no superattracting cycle, then f is an integral Lattès map.

The proof in this step is due to McMullen, compare [McM1]. For completeness, we include it here.

We first show that ϕ_f has four simple poles. That is, up to a constant,

$$\phi_f = \frac{dz^2}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}.$$

Indeed, if ϕ_f has a pole p_0 of order two or more, that is $\mathrm{ord}_{p_0}(\phi_f) \leq -2$, then we can conclude from the identity (3.3) by induction that for any $z \in \mathrm{orb}^-(p_0)$, $\mathrm{ord}_z(\phi_f) \leq -2$, therefore $\mathrm{orb}^-(p_0) \subset \mathcal{P}(\phi_f)$. The similar argument as in step 2 indicates that $f^{-1}\big(\mathrm{orb}^-(p_0)\big) = \mathrm{orb}^-(p_0)$ and $\#\mathrm{orb}^-(p_0) = 1$ or 2. It turns out that f is conjugate to a power map or a polynomial. In either case, f has a superattracting cycle. But this will contradict the assumption.

Now we consider the orbifold $\mathcal{O}_f$ of f. Recall that the orbifold $\mathcal{O}_f$ of the critically finite map f is a pair $(\overline{\mathbb{C}}, N_f)$, where $N_f : \overline{\mathbb{C}} \to \mathbb{N}\cup\{\infty\}$ takes values greater than one only on a discrete set of $\overline{\mathbb{C}}$. It is defined as follows:

(a) $N_f(x) = 1$, when $x \in \overline{\mathbb{C}}\backslash P(f)$,

(b) $N_f(x)$ is the least common multiple of the local degrees in the set $\{\deg(f^n, y); f^n(y) = x, n \geq 1\}$ for $x \in P(f)$,

(c) $N_f(x) = \infty$ if the local degrees in the set $\{\deg(f^n, y); f^n(y) = x, n \geq 1\}$ are unbounded.

We claim that $\mathcal{O}_f = (\overline{\mathbb{C}}, (2,2,2,2))$. In fact, it's easy to see that $N_f(x) = 1$ when $x \in \overline{\mathbb{C}} \backslash P(f)$. For $x \in P(f)$, it is obvious $N_f(x) \geq 2$. Let $z \in f^{-n}(x)$ for $n \geq 1$. Note that $(f^n)^*\phi_f = C_f^n \phi_f$, by identity (3.3), we have

$$\mathrm{ord}_z\big((f^n)^*\phi_f\big) = \deg(f^n, z)\big(2 + \mathrm{ord}_x(\phi_f)\big) - 2 = \mathrm{ord}_z(\phi_f).$$

Since $P(f) = \mathcal{P}(\phi_f)$ and every pole of ϕ_f is simple in this case, $\mathrm{ord}_x(\phi_f) = -1$. Therefore $\mathrm{ord}_z(\phi_f)$ has only two choices 0 or -1 and $\deg(f^n, z)$ can only be 1 or 2. Thus $N_f(x) \leq 2$. The above argument shows that for any $x \in P(f)$, $N_f(x) = 2$. This proves the claim.

By Theorem A.5 in [McM1], if $\mathcal{O}_f = (\overline{\mathbb{C}}, (2,2,2,2))$, then f is a Lattès map. By assumption, f admits a meromorphic invariant line field, so f is an integral Lattès map. Moreover, since ϕ_f is integrable over $\overline{\mathbb{C}}$, we have $C_f = \deg(f)$ from the following identity

$$\int_{\overline{\mathbb{C}}} |f^*\phi_f| = \deg(f) \int_{\overline{\mathbb{C}}} |\phi_f|.$$

Step 5. If f has a superattracting cycle, then f is either conjugate to a power map or conjugate to a Chebyshef polynomial.

Let z_0 be a superattracting point of f with period p. From the identity (3.3), we have

$$\mathrm{ord}_{z_0}\big((f^p)^*\phi_f\big) = \deg(f^p, z_0)\big(2 + \mathrm{ord}_{z_0}(\phi_f)\big) - 2 = \mathrm{ord}_{z_0}(\phi_f).$$

Since $\deg(f^p, z_0) \geq 2$, we have $\mathrm{ord}_{z_0}(\phi_f) = -2$. Thus z_0 is a pole of ϕ_f of order two. Moreover by identity (3.3) and induction, all preimages of z_0 are poles of ϕ_f of order two. There are two possibilities:

(P1) z_0 is a fixed point of f and $f^{-1}(z_0) = \{z_0\}$.

(P2) z_0 is of period two and $f^{-1}(z_0) = \{\zeta\}$, $f^{-1}(\zeta) = \{z_0\}$.

For (P1), there are two choices for ϕ_f up to a constant:

Case 1. $\phi_f = \dfrac{dz^2}{(z - z_0)^2(z - z_1)^2}$,

Case 2. $\phi_f = \dfrac{dz^2}{(z - z_0)^2(z - z_1)(z - z_2)}$.

In case 1, take $\gamma \in \mathbf{Aut}(\overline{\mathbb{C}})$, the automorphism group of $\overline{\mathbb{C}}$, such that $\gamma(0) = z_1$, $\gamma(\infty) = z_0$. Then $\gamma^*\phi_f = C dz^2/z^2$ for some constant C and $F = \gamma^{-1} \circ f \circ \gamma$ is a polynomial such that $F^*(\gamma^*\phi_f) = C_f \gamma^*\phi_f$. By conjugation, we may assume f is a polynomial and $\phi_f = dz^2/z^2$. The equation $f^*\phi_f = C_f \phi_f$ is equivalent to

$$(f'(z)/f(z))^2 = C_f/z^2. \tag{3.4}$$

Comparing the leading coefficients in both sides of (3.4), we have $C_f = \deg(f)^2$. It's easy to find the general polynomial solution $f(z) = Az^d$, where A is a nonzero complex constant and $d = \deg(f)$. In this case, f is conjugate to a power map.

In case 2, take $\gamma \in \mathbf{Aut}(\overline{\mathbb{C}})$, such that $\gamma(\infty) = z_0,\ \gamma(-2) = z_1,\ \gamma(2) = z_2$. It is easy to show $\gamma^*\phi_f = Cdz^2/(z^2-4)$ for some constant C and $F = \gamma^{-1} \circ f \circ \gamma$ is a polynomial. Thus as in case 1 we assume that f is a polynomial and $\phi_f = dz^2/(z^2-4)$. The equation $f^*\phi_f = C_f\phi_f$ is equivalent to

$$f'(z)^2(z^2-4) = C_f(f(z)^2-4). \tag{3.5}$$

We want to find all polynomial solutions to this equation. First note that $f(2) = 2$ or -2 if we set $z = 2$. Comparing the leading coefficients in both sides, we have $C_f = \deg(f)^2$. To solve the equation (3.5), we need a little trick. Let

$$z = w + \frac{1}{w}, \quad f(z) = \varphi(w) + \frac{1}{\varphi(w)},$$

where $\varphi(w)$ is required to be a holomorphic map, it need only to be defined in some open set U in $\mathbb{C}$. Indeed, since the map $w \mapsto w + w^{-1}$ is locally injective when $|w|$ is large, we can always do this. Calculation shows

$$f'(z) = w\frac{\varphi'(w)}{\varphi(w)}\frac{\varphi(w)-\varphi(w)^{-1}}{w-w^{-1}}.$$

Then the equation (3.5) becomes to

$$w\frac{\varphi'(w)}{\varphi(w)} = \pm n, \quad w \in U,$$

where $n = \deg(f)$. This new equation has general solution $\varphi(w) = Cw^n$ or $\varphi(w) = (Cw^n)^{-1}$ for some undeterminate constant C, so we have

$$f\Big(w + \frac{1}{w}\Big) = Cw^n + \frac{1}{Cw^n}, \quad w \in U.$$

This relation in fact holds for all $w \in \bar{\mathbb{C}}$ by identity theorem of holomorphic maps. If $f(2) = 2$, then $C = 1$, in this case $f(2\cos z) = 2\cos(nz)$ if we write $w = \mathrm{e}^{\mathrm{i}z}$, so $f = T_n$. If $f(2) = -2$, then $C = -1$, in this case $f = -T_n$. Therefore in case 2, f is conjugate to a Chebyshef polynomial T_n or $-T_n$.

Up to now, the only remaining case is (P2). In this case, we can easily show that $C_f = \deg(f)^2, \phi_f = dz^2/\big((z-z_0)^2(z-\zeta)^2\big)$ and f is conjugate to the power map $z \mapsto z^d$, for $d \in \mathbb{Z}$ and $d \leq -2$. We omit the details here.

The proof is completed.

Remark 3.2.1. *For convenience, we list all solutions to the indeterminate equation* $f^*\phi_f = C_f\phi_f$ *in the following table:*

f *is conjugate to*	ϕ_f	C_f	$\mathcal{O}_f$
Integral Lattès map	$\dfrac{dz^2}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}$	$\deg(f)$	$(\overline{\mathbb{C}}, (2,2,2,2))$
Power map	$\dfrac{dz^2}{(z-z_0)^2(z-z_1)^2}$	$\deg(f)^2$	$(\overline{\mathbb{C}}, (\infty,\infty))$
$\pm$*Chebyshef polynomial*	$\dfrac{dz^2}{(z-z_0)^2(z-z_1)(z-z_2)}$	$\deg(f)^2$	$(\overline{\mathbb{C}}, (2,2,\infty))$

We can see that for all cases $\mathcal{O}_f$ *is a parabolic orbifold,* $\sqrt{|\phi_f|}$ *is an orbifold metric on* $\mathcal{O}_f$*.*

Chapter 4

A Non-escape Locus

4.1 Introduction

It is well known that the famous Mandelbrot set of the quadratic polynomials $f_c(z) = z^2 + c$ is defined by

$$M = \{c \in \mathbb{C}; f_c^n(0) \text{ remains bounded as } n \to \infty\}.$$

The Mandelbrot set is the connected locus for the quadratic family. It is a central object of study in complex dynamics since it exhibits a rich geometric and combinatorial structure, with many intriguing details and many remaining mysteries. One of most interesting results about M is that it is a connected set, which was obtained by Douady and Hubbard in 1982 by constructing a Riemann mapping from the exterior of M to $\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}$, see [DH3]. This result leads to numerous further study of M, especially the study of its topological and combinatorial properties, using the method of 'parameter external rays' and 'puzzle' techniques.

In this chapter, we deal with a family of rational maps

$$T_\lambda(z) = \left(\frac{z^2 + \lambda - 1}{2z + \lambda - 2}\right)^2,$$

where λ is a complex parameter. This family is indeed the family of renormalization transformations of 2-dimensional diamond-like hierachical Potts models in statistical mechanics. In 1983, Derrida *et al* show that the Yang-Lee zeros of the $\lambda-$state Potts model on the diamond hierachical lattice are dense in the Julia set $J(T_\lambda)$ of the map T_λ (See [DDI]). Since then, much interest has been devoted to this family since it exhibits a connection between statistical mechanics and complex dynamics (See [EL],[O],[QG],[QL]).

For this family, note that when $\lambda = 0$, the map T_λ degenerates to the quadratic polynomial $T_0(z) = (z + 1)^2/4$; when $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, 1 and ∞ are two superattracting fixed points for the map T_λ while 0 is a critical value.

The non-escape locus $\mathcal{M}$ (an analogue of Mandelbrot set) associated to this family is defined by:

$$\mathcal{M} = \{\lambda \in \mathbb{C}^*; T_\lambda^n(0) \nrightarrow_{n\to\infty} 1 \text{ and } T_\lambda^n(0) \nrightarrow_{n\to\infty} \infty\} \cup \{0\}.$$

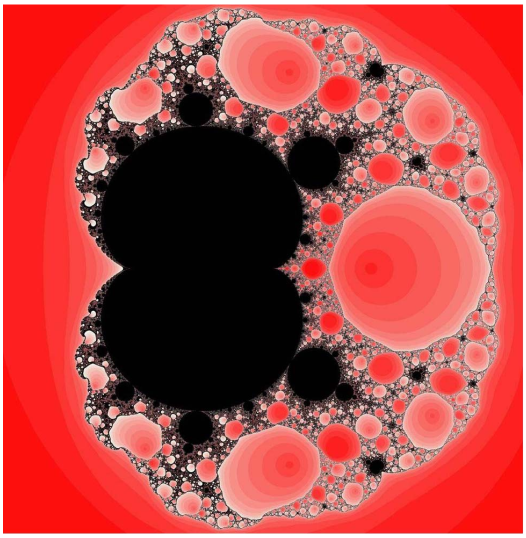

Figure 4.1: The parameter plane for T_λ ($\lambda \in \mathbb{C}$)

Figure 4.1 shows the picture of the non-escape locus $\mathcal{M}$ in the parameter plane for this family. The non-escape locus $\mathcal{M}$ can be identified as the complex plane minus infinitely many 'bubbles', which we will call 'capture domains' formally. An elementary property of $\mathcal{M}$ is that it is compact and symmetric about the real axis. Moreover, the intersection $\mathcal{M} \cap \mathbb{R}$ is contained in the closed interval $[0, 3]$, with the boundary points 0 and 3 lying in $\mathcal{M}$ (See [QL]). Many small copies of quadratic Mandelbrot set M are visible in the parameter plane. It is an amazing fact that the small copies of quadratic Mandelbrot set M are dense in $\mathcal{M}$. This is the a philosophy of 'universality of the Mandelbrot set', which was proved by McMullen, see [McM2].

For the non-escape locus $\mathcal{M}$, we have the following:

Theorem 4.1.1. *The non-escape locus $\mathcal{M}$ is connected.*

For the Julia set, we have the following:

Theorem 4.1.2. *If the Julia set $J(T_\lambda)$ is a quasi-circle, then the Hausdorff dimension of $J(T_\lambda)$ satisfies:*

$$HD(J(T_\lambda)) \leq 1 + |\phi_\lambda(0)|^{2/3},$$

where ϕ_λ is the Böttcher map of T_λ defined near the superattracting fixed point 1.

This chapter is organized as follows: in Section 4.2, we discuss the location of the critical points and decompose the parameter plane into the non-escape locus $\mathcal{M}$ plus infinitely many capture domains; in Section 4.3, we parameterize all the capture domains by constructing a Riemann mapping from every capture domain to the unit disk $\mathbb{D}$ and prove Theorem 4.1.1; in Section 4.4, we prove Theorem 4.1.2 by using Smirnov's Theorem and Slodkowski's Theorem.

This chapter has been published as [WQYQG]. I felt depressed when I found that some of our results had already been included in Aspenberg and Yampolsky's paper [AY] after I finished writing this part two years ago. Anyway, there are some differences between our argument and Aspenberg-Yampolsky's, so I include it here as a part of the thesis.

This chapter in fact deals with a special quadratic family. For more discussions of the quadratic family, see also [T] and [Rees3].

4.2 Critical points and capture domains

For $\lambda \in \mathbb{C}^*$, let $C(T_\lambda)$ be the set of all critical points for T_λ. Easy calculation shows

$$T'_\lambda(z) = \frac{4(z-1)(z+\lambda-1)(z^2+\lambda-1)}{(2z+\lambda-2)^3}.$$

Thus we have

$$C(T_\lambda) = \{1, \infty, 1-\lambda, \pm\sqrt{1-\lambda}, 1-\lambda/2\}.$$

Moreover $T_\lambda^{-1}(\infty) = \{\infty, 1-\lambda/2\}, T_\lambda^{-1}(0) = \{\pm\sqrt{1-\lambda}\}$. Let $\mathcal{A}_\lambda(1)$ and $\mathcal{A}_\lambda(\infty)$ be the immediate basins of attraction for the superattracting fixed points 1 and ∞ respectively.

First, we introduce an interesting property of the map T_λ:

Splitting Principle *For $\lambda \in \mathbb{C}^*$, we have $T_\lambda = t_\lambda \circ t_\lambda$, where*

$$t_\lambda(z) = \Big(\frac{z-1+\lambda}{z-1}\Big)^2.$$

It's easy to see that t_λ permutes 1 and ∞, $t_\lambda^{-1}(\mathcal{A}_\lambda(\infty)) = \mathcal{A}_\lambda(1)$. The orbits of $\pm\sqrt{1-\lambda}$ and $1-\lambda$ under iterations of T_λ in fact lie interlacedly in the same orbit under iterations of t_λ:

$$\pm\sqrt{1-\lambda} \mapsto 1-\lambda \mapsto 0 \mapsto (1-\lambda)^2 \mapsto \cdots$$

Lemma 4.2.1. *The Julia set $J(T_\lambda)$ is connected for all $\lambda \in \mathbb{C}$.*

Proof. If $\lambda = 0$, $T_0(z) = (z+1)^2/4$ is conformally conjugate to $z \mapsto z^2 + 1/4$, whose Julia set is connected. If $\lambda \neq 0$, by Splitting Principle, $J(T_\lambda) = J(t_\lambda)$. It is known from Yin (See [Yin]) that the Julia set for a quadratic rational map is either connected or a Cantor set, thus $J(t_\lambda)$ is connected since t_λ has two superattracting periodic points 1 and ∞. □

Lemma 4.2.2. *For $\lambda \in \mathbb{C}^*$, the following conditions are equivalent.*

1. *$J(T_\lambda)$ is a quasicircle.*
2. *$\sqrt{1-\lambda} \in \mathcal{A}_\lambda(1)$.*
3. *$-\sqrt{1-\lambda} \in \mathcal{A}_\lambda(1)$.*
4. *$0 \in \mathcal{A}_\lambda(1)$.*
5. *$1-\lambda \in \mathcal{A}_\lambda(\infty)$.*
6. *$1-\lambda/2 \in \mathcal{A}_\lambda(\infty)$.*

Proof. First we show $1 \Rightarrow 2+3+4+5+6$. Suppose $J(T_\lambda)$ is a quasicircle, the Fatou set $F(T_\lambda)$ decomposes into two completely invariant components $\mathcal{A}_\lambda(1)$ and $\mathcal{A}_\lambda(\infty)$. It is obvious that $1-\lambda/2 \in \mathcal{A}_\lambda(\infty)$ and $\pm\sqrt{1-\lambda}$ lie in the same Fatou component. By Riemann-Hurwitz formula, $\{\pm\sqrt{1-\lambda}, 0\} \subset \mathcal{A}_\lambda(1)$ and $1-\lambda \in \mathcal{A}_\lambda(\infty)$.

It is obvious that $2 \Rightarrow 4,\ 3 \Rightarrow 4$ since $T_\lambda(\pm\sqrt{1-\lambda}) = \{0\}$ and T_λ fixes $\mathcal{A}_\lambda(1)$. $5 \Rightarrow 4$ follows from the fact that $t_\lambda(\mathcal{A}_\lambda(\infty)) = \mathcal{A}_\lambda(1)$ and $t_\lambda(1-\lambda) = 0$.

Now we show $4 \Rightarrow 1$. Suppose $0 \in \mathcal{A}_\lambda(1)$. Since $T_\lambda^{-1}(0) = \{\pm\sqrt{1-\lambda}\}$, the pair $\{\pm\sqrt{1-\lambda}\}$ has two possibilities of location: either both lie in $\mathcal{A}_\lambda(1)$ or only one lies in $\mathcal{A}_\lambda(1)$. By Lemma 4.2.1, every Fatou component is simply connected. We see that the latter is ruled out by Riemann-Hurwitz formula. Thus $\{\pm\sqrt{1-\lambda}\} \subset \mathcal{A}_\lambda(1)$. Moreover $C(T_\lambda) \cap \mathcal{A}_\lambda(1) = \{1, \pm\sqrt{1-\lambda}\}$ and $\mathcal{A}_\lambda(1)$ is completely invariant.

Note that $1-\lambda \in T_\lambda^{-1}(\mathcal{A}_\lambda(\infty))$, since $T_\lambda(1-\lambda) = t_\lambda(0) \in \mathcal{A}_\lambda(\infty)$. For the critical points $1-\lambda$ and $1-\lambda/2$, either both lie in $\mathcal{A}_\lambda(\infty)$ or at most one lies in $\mathcal{A}_\lambda(\infty)$. Also by Riemann-Hurwitz formula, the former is the only choice. So we have $\{1-\lambda, 1-\lambda/2\} \subset \mathcal{A}_\lambda(\infty)$ and $\mathcal{A}_\lambda(\infty)$ is completely invariant. Thus $F(T_\lambda) = \mathcal{A}_\lambda(1) \cup \mathcal{A}_\lambda(\infty)$. Since T_λ is a hyperbolic map, $J(T_\lambda)$ is a quasicircle. Thus $4 \Rightarrow 1$.

To conclude, we show $6 \Rightarrow 5$. Suppose $1-\lambda/2 \in \mathcal{A}_\lambda(\infty)$, this means $\mathcal{A}_\lambda(\infty)$ is completely invariant since $T_\lambda^{-1}(\infty) = \{\infty, 1-\lambda/2\}$. It turns out that $1-\lambda \in \mathcal{A}_\lambda(\infty)$ by Riemann-Hurwitz formula. □

Lemma 4.2.3. *For any $\lambda \in \mathbb{C}^*$, we have $0 \notin \mathcal{A}_\lambda(\infty)$ and $1-\lambda \notin \mathcal{A}_\lambda(1)$.*

Proof. If $0 \in \mathcal{A}_\lambda(\infty)$, then any critical point in the set $\{\pm\sqrt{1-\lambda}, 1-\lambda/2\}$ will lie in some component of $T_\lambda^{-1}(\mathcal{A}_\lambda(\infty))$. Since every Fatou component of $F(T_\lambda)$ is simply connected by Lemma 4.2.1, this is impossible by Riemann-Hurwitz formula.

Since $t_\lambda^{-1}(0) = \{1-\lambda\}$ and $t_\lambda^{-1}(\mathcal{A}_\lambda(\infty)) = \mathcal{A}_\lambda(1)$, $0 \notin \mathcal{A}_\lambda(\infty)$ indicates $1-\lambda \notin \mathcal{A}_\lambda(1)$. □

Since t_λ permutes 1 and ∞, we can describe the non-escape locus in another way:

$$\mathcal{M} = \{\lambda \in \mathbb{C}^*; t_\lambda^{2n}(0) \nrightarrow_{n\to\infty} 1 \text{ and } t_\lambda^{2n+1}(0) \nrightarrow_{n\to\infty} 1\} \cup \{0\}$$

Now we consider the parameters outside of the non-escape locus $\mathcal{M}$. Let λ be such a parameter, the critical value 0 is eventually mapped into $\mathcal{A}_\lambda(1)\cup\mathcal{A}_\lambda(\infty)$ under iterations of t_λ. This observation leads to the following definition:

Definition 4.2.1. *Let*

$$\mathcal{H}_0=\{\lambda\in\mathbb{C}^*; 0\in\mathcal{A}_\lambda(1)\},$$
$$\mathcal{H}_n=\{\lambda\in\mathbb{C}^*; t_\lambda^n(0)\in\mathcal{A}_\lambda(1), t_\lambda^{n-1}(0)\notin\mathcal{A}_\lambda(\infty)\}, n\geq 1,$$

a component of $\mathcal{H}_n$ is called a capture domain of level n for $n\geq 0$.

Lemma 4.2.4. *The parameter plane has the following decomposition:*

$$\mathbb{C}=\mathcal{M}\sqcup\big(\sqcup_{n\geq 0}\mathcal{H}_n\big),$$

where $\sqcup$ denotes the union of mutually disjoint sets.

Proof. First we show that the sets in $\{\mathcal{H}_n; n\geq 0\}$ are mutually disjoint. If not, suppose $\lambda\in\mathcal{H}_n\cap\mathcal{H}_m$ for $m>n\geq 0$. Then by definition

$$t_\lambda^n(0)\in\mathcal{A}_\lambda(1),\ t_\lambda^m(0)\in\mathcal{A}_\lambda(1),\ t_\lambda^{m-1}(0)\notin\mathcal{A}_\lambda(\infty).$$

We see that $t_\lambda^{m-n}(\mathcal{A}_\lambda(1))=\mathcal{A}_\lambda(1)$, so $m-n$ is even and $m-n\geq 2$. It follows that $t_\lambda^{m-1}(0)=t_\lambda^{m-n-1}(t_\lambda^n(0))\in\mathcal{A}_\lambda(\infty)$. But this is a contradiction.

Now we prove that for any $\lambda\in\mathbb{C}\setminus\mathcal{M}$, λ must lie in $\mathcal{H}_n$ for some $n\geq 0$. Indeed, by definition of $\mathcal{M}$, for any $\lambda\in\mathbb{C}\setminus\mathcal{M}$, either $t_\lambda^{2n}(0)\to 1$ or $t_\lambda^{2n+1}(0)\to 1$ as $n\to\infty$. So there is a minimal integer $m\geq 0$ such that $t_\lambda^m(0)\in\mathcal{A}_\lambda(1)$. If $m=0$, then $\lambda\in\mathcal{H}_0$. If $m=1$, Lemma 4.2.3 shows that $0\notin\mathcal{A}_\lambda(\infty)$, so $\lambda\in\mathcal{H}_1$. If $m\geq 2$, we can conclude that $t_\lambda^{m-1}(0)\notin\mathcal{A}_\lambda(\infty)$, for else $t_\lambda^{-1}(\mathcal{A}_\lambda(\infty))=\mathcal{A}_\lambda(1)$ indicates $t_\lambda^{m-2}(0)\in\mathcal{A}_\lambda(1)$, which will contradict the choice of m. Thus in this case we also have $\lambda\in\mathcal{H}_m$. $\square$

Remark 4.2.1. *It is easy to verify that for $n\geq 1$,*

$$\begin{aligned}\mathcal{H}_{2n}&=\{\lambda\in\mathbb{C}^*; T_\lambda^n(0)\in\mathcal{A}_\lambda(1), T_\lambda^{n-1}(0)\notin\mathcal{A}_\lambda(1)\}\\&=\{\lambda\in\mathbb{C}^*; T_\lambda^{n+1}(1-\lambda)\in\mathcal{A}_\lambda(\infty), T_\lambda^n(1-\lambda)\notin\mathcal{A}_\lambda(\infty)\},\\\mathcal{H}_{2n-1}&=\{\lambda\in\mathbb{C}^*; T_\lambda^n(0)\in\mathcal{A}_\lambda(\infty), T_\lambda^{n-1}(0)\notin\mathcal{A}_\lambda(\infty)\}\\&=\{\lambda\in\mathbb{C}^*; T_\lambda^n(1-\lambda)\in\mathcal{A}_\lambda(1), T_\lambda^{n-1}(1-\lambda)\notin\mathcal{A}_\lambda(1)\}.\end{aligned}$$

Lemma 4.2.5. *For any $\lambda\in\mathbb{C}^*\setminus\mathcal{H}_0$, we have*

$$\mathcal{A}_\lambda(1)\cap C(T_\lambda)=\{1\},\ \mathcal{A}_\lambda(\infty)\cap C(T_\lambda)=\{\infty\}.$$

Proof. It is obvious that $1-\lambda/2\notin\mathcal{A}_\lambda(1)$. By Lemma 4.2.2, $\mathcal{A}_\lambda(1)$ does not contain $\sqrt{1-\lambda}$ or $-\sqrt{1-\lambda}$ while $\mathcal{A}_\lambda(\infty)$ does not contain $1-\lambda$ or $1-\lambda/2$. By Lemma 4.2.3, $\{\pm\sqrt{1-\lambda}\}\cap\mathcal{A}_\lambda(\infty)=\emptyset$ and $1-\lambda\notin\mathcal{A}_\lambda(1)$. The conclusion follows. $\square$

4.3 Proof of Theorem 4.1.1

In this section, we parameterize all the capture domains in the parameter plane. We show that every capture domain is simply connected by constructing a Riemann mapping from the capture domain to the unit disk $\mathbb{D}$, and this will lead to the connectivity of the non-escape locus $\mathcal{M}$.

To prove every capture domain of level $n \geq 1$ is conformally equivalent to the unit disk $\mathbb{D}$, we use the method of quasiconformal surgery and holomorphic motion theorem (See Proposition 4.3.2). The method of quasiconformal surgery is classic, which was first used by Douady and Hubbard to parameterize the hyperbolic components of the quadratic Mandelbrot set M (See [DH2]). This method is developed by Roesch to study the parameter plane of cubic Newton maps and McMullen maps, see [Ro2],[Ro3].

However, this method cannot be applied to $\mathcal{H}_0$, since the Julia set for the map in $\mathcal{H}_0$ is a quasicircle and there is no way to construct quasiconformal deformation. To deal with $\mathcal{H}_0$, we divide the proof into several steps. First, for two maps $t_{\lambda_1}, t_{\lambda_2}$ in $\mathcal{H}_0$ satisfying an 'argument relation', we construct a quasiconformal conjugacy between the two maps elaborately using the philosophy of so called 'bootstrap argument' (See Lemma 4.3.2). This construction shows us an important relation between the Beltrami coefficient of the quasiconformal map and the Green functions of t_{λ_1} and t_{λ_2}. Furthermore, we will see later that this quasiconformal map has an extremal property: its Beltrami coefficient achieves the minimal norm (See the concluding remark of this section).

On the other hand, we show that the essential norm of the Beltrami coefficient can be bounded below by a constant depending on the Poincaré distance between λ_1 and λ_2 in $\mathcal{H}_0$ (See Lemma 4.3.3). In this way, the Beltrami coefficient acts like a bridge connecting the Green functions with the Poincaré distance. This yields an inequality between the two objects, which play a crucial role in the parameterization for $\mathcal{H}_0$ (See Proposition 4.3.1). It is an amusing fact that once we prove $\mathcal{H}_0$ is conformally equivalent to the punctured disk $\mathbb{D}^*$, we see that the inequality is actually an identity (See the concluding remark of this section). This is again a philosophy of 'bootstrap argument'.

As a consequence, we show the non-escape locus $\mathcal{M}$ has capacity equal to 2.

In the following, we always use $\mathbb{D}$ to denote the unit disk. Let $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ be the punctured disk and $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$. For a hyperbolic Riemann surface S, let $d_S(z_1, z_2)$ be the hyperbolic distance for a pair $(z_1, z_2) \in S \times S$.

For $\lambda \in \mathbb{C}^*$, the map T_λ has two superattracting fixed points 1 and ∞. The Green functions $G_\lambda : \mathcal{A}_\lambda(1) \to (0, \infty]$ and $G_\lambda^\infty : \mathcal{A}_\lambda(\infty) \to (0, \infty]$ are defined as follows:

$$G_\lambda(z) = -\lim_{k\to\infty} 2^{-k} \log |T_\lambda^k(z) - 1|,\ z \in \mathcal{A}_\lambda(1),$$
$$G_\lambda^\infty(z) = \lim_{k\to\infty} 2^{-k} \log |T_\lambda^k(z)|,\ z \in \mathcal{A}_\lambda(\infty).$$

It is known that G_λ and G_λ^∞ are continuous and satisfy:

$$\begin{aligned} G_\lambda \circ T_\lambda(z) &= 2G_\lambda(z),\ z \in \mathcal{A}_\lambda(1), \\ G_\lambda^\infty \circ T_\lambda(z) &= 2G_\lambda^\infty(z),\ z \in \mathcal{A}_\lambda(\infty). \end{aligned}$$

Lemma 4.3.1. *For $\lambda \in \mathbb{C}^*$, the Green functions satisfy:*

$$\begin{aligned} G_\lambda \circ t_\lambda(z) &= G_\lambda^\infty(z),\ z \in \mathcal{A}_\lambda(\infty), \\ G_\lambda^\infty \circ t_\lambda(z) &= 2G_\lambda(z),\ z \in \mathcal{A}_\lambda(1). \end{aligned}$$

Proof. First suppose $\lambda \in \mathbb{C}^* \setminus \mathcal{H}_0$. By Lemma 4.2.5, the only critical point of T_λ that lies in $\mathcal{A}_\lambda(1)$ is 1 itself, thus the Böttcher map $\phi_\lambda : \mathcal{A}_\lambda(1) \to \mathbb{D}$ defined by $\phi_\lambda(z) = \lim_{k\to\infty} (T_\lambda^k(z) - 1)^{2^{-k}}$ is a conformal isomorphism. Similarly, the only critical point of T_λ that lies in $\mathcal{A}_\lambda(\infty)$ is ∞ itself, the Böttcher map $\phi_\lambda^\infty : \mathcal{A}_\lambda(\infty) \to \bar{\mathbb{C}}\setminus\bar{\mathbb{D}}$ defined by $\phi_\lambda^\infty(z) = \lim_{k\to\infty} (T_\lambda^k(z))^{2^{-k}}$ is also a conformal isomorphism. Since $t_\lambda : \mathcal{A}_\lambda(\infty) \to \mathcal{A}_\lambda(1)$ is a proper map of degree one, the Böttcher maps satisfy:

$$\begin{aligned} \phi_\lambda \circ t_\lambda(z) &= (\phi_\lambda^\infty(z))^{-1},\ z \in \mathcal{A}_\lambda(\infty), \\ \phi_\lambda^\infty \circ t_\lambda(z) &= (\phi_\lambda(z))^{-2},\ z \in \mathcal{A}_\lambda(1). \end{aligned}$$

Thus the Green functions satisfy:

$$\begin{aligned} G_\lambda \circ t_\lambda(z) &= -\log |\phi_\lambda \circ t_\lambda(z)| = \log |\phi_\lambda^\infty(z)| = G_\lambda^\infty(z),\ z \in \mathcal{A}_\lambda(\infty), \\ G_\lambda^\infty \circ t_\lambda(z) &= \log |\phi_\lambda^\infty \circ t_\lambda(z)| = -2\log |\phi_\lambda(z)| = 2G_\lambda(z),\ z \in \mathcal{A}_\lambda(1). \end{aligned}$$

Now suppose $\lambda \in \mathcal{H}_0$, the Böttcher maps ϕ_λ and ϕ_λ^∞ can be defined in neighborhoods of 1 and ∞ respectively, say $U_\lambda(1)$ and $U_\lambda(\infty)$. We may assume that $U_\lambda(1)$ and $U_\lambda(\infty)$ are small enough such that $t_\lambda : U_\lambda(\infty) \to U_\lambda(1)$ is a conformal isomorphism. Thus we have

$$\phi_\lambda \circ t_\lambda(z) = (\phi_\lambda^\infty(z))^{-1},\ G_\lambda \circ t_\lambda(z) = G_\lambda^\infty(z),\ z \in U_\lambda(\infty)$$

Let $GO_\lambda(1) = \bigcup_{n\geq 0} T_\lambda^{-n}(1)$ and $GO_\lambda(\infty) = \bigcup_{n\geq 0} T_\lambda^{-n}(\infty)$ be the grand orbits of 1 and ∞ respectively. It is easy to check that $t_\lambda^{-1}(GO_\lambda(1)) = GO_\lambda(\infty), t_\lambda^{-1}(GO_\lambda(\infty)) = GO_\lambda(1)$. Note that G_λ is harmonic in $\mathcal{A}_\lambda(1)\setminus GO_\lambda(1)$ and G_λ^∞ is harmonic in $\mathcal{A}_\lambda(\infty)\setminus GO_\lambda(\infty)$. The function $G_\lambda \circ t_\lambda$

is harmonic in $t_\lambda^{-1}(\mathcal{A}_\lambda(1)\backslash GO_\lambda(1)) = \mathcal{A}_\lambda(\infty)\backslash GO_\lambda(\infty)$. Comparing $G_\lambda \circ t_\lambda$ and G_λ^∞, we see that both $G_\lambda \circ t_\lambda$ and G_λ^∞ are equal to ∞ in $GO_\lambda(\infty)$, harmonic in $\mathcal{A}_\lambda(\infty)\backslash GO_\lambda(\infty)$. Since the two coincide in $U_\lambda(\infty)$, by identity theorem of harmonic functions, we have $G_\lambda \circ t_\lambda(z) = G_\lambda^\infty(z),\ z \in \mathcal{A}_\lambda(\infty)$. It follows that

$$G_\lambda^\infty \circ t_\lambda(z) = G_\lambda \circ T_\lambda(z) = 2G_\lambda(z),\ z \in \mathcal{A}_\lambda(1).$$

□

Lemma 4.3.2. *Let $\lambda_1, \lambda_2 \in \mathcal{H}_0$ with $\arg\phi_{\lambda_1}(0) = \arg\phi_{\lambda_2}(0)$, then there is a quasiconformal map h with $0, 1, \infty$ fixed such that $h \circ t_{\lambda_1} = t_{\lambda_2} \circ h$. Moreover, the Beltrami coefficient μ of h satisfies*

$$\|\mu\|_\infty = \left|\frac{G_{\lambda_1}(0) - G_{\lambda_2}(0)}{G_{\lambda_1}(0) + G_{\lambda_2}(0)}\right|.$$

Proof. For $\lambda \in \mathcal{H}_0$ and $n \in \mathbb{Z}$, let $E_\lambda(n)$ be the component of $\{z \in \mathcal{A}_\lambda(1); G_\lambda(z) > 2^{-n}G_\lambda(0)\}$ that contains 1 and $E_\lambda^\infty(n)$ be the component of $\{z \in \mathcal{A}_\lambda(\infty); G_\lambda^\infty(z) > 2^{-n}G_\lambda^\infty(1-\lambda)\}$ that contains ∞. By Lemma 4.3.1, we can verify by induction that:

- For $n \in \mathbb{Z}$, $E_\lambda(n) \subset\subset E_\lambda(n+1),\ E_\lambda^\infty(n) \subset\subset E_\lambda^\infty(n+1)$.
- $\bigcup_{n\geq 0} E_\lambda(n) = \mathcal{A}_\lambda(1),\ \bigcup_{n\geq 0} E_\lambda^\infty(n) = \mathcal{A}_\lambda(\infty)$.
- $E_\lambda(n)$ and $E_\lambda^\infty(n)$ are simply connected for all $n \in \mathbb{Z}$.
- $t_\lambda : E_\lambda^\infty(n) \to E_\lambda(n)$ is a proper map of degree one (if $n \leq 0$) or two (if $n \geq 1$) while $t_\lambda : E_\lambda(n+1) \to E_\lambda^\infty(n)$ is a proper map of degree two for all $n \in \mathbb{Z}$.

Given $\lambda_1, \lambda_2 \in \mathcal{H}_0$ with $\arg\phi_{\lambda_1}(0) = \arg\phi_{\lambda_2}(0)$, we define a quasiconformal map $\delta : \mathbb{D}_{|\phi_{\lambda_1}(0)|} \to \mathbb{D}_{|\phi_{\lambda_2}(0)|}$ by $\delta(re^{i\theta}) = r^\alpha e^{i\theta}$, where α satisfies $|\phi_{\lambda_1}(0)|^\alpha = |\phi_{\lambda_2}(0)|$. In complex coordinate, $\delta(z) = z^{(\alpha+1)/2}\bar{z}^{(\alpha-1)/2}$. The Beltrami coefficient μ_δ of δ satisfies

$$\mu_\delta(z) = \frac{\alpha-1}{\alpha+1}\frac{z}{\bar{z}},\ \|\mu_\delta\|_\infty = \left|\frac{\alpha-1}{\alpha+1}\right| = \left|\frac{G_{\lambda_1}(0) - G_{\lambda_2}(0)}{G_{\lambda_1}(0) + G_{\lambda_2}(0)}\right|.$$

It is easy to check that $\delta(z^2) = \delta(z)^2$ for $z \in \mathbb{D}_{|\phi_{\lambda_1}(0)|}$.

We first construct three quasiconformal maps $h_0 : E_{\lambda_1}(0) \to E_{\lambda_2}(0)$, $h_0^\infty : E_{\lambda_1}^\infty(0) \to E_{\lambda_2}^\infty(0)$ and $h_1 : E_{\lambda_1}(1) \to E_{\lambda_2}(1)$ such that $h_1|_{E_{\lambda_1}(0)} = h_0$ and the following diagram is commutative.

$$\begin{tikzcd}
E_{\lambda_1}(1) \arrow[r, "t_{\lambda_1}"] \arrow[d, "h_1"'] & E_{\lambda_1}^\infty(0) \arrow[r, "t_{\lambda_1}"] \arrow[d, "h_0^\infty"'] & E_{\lambda_1}(0) \arrow[r, "\phi_{\lambda_1}"] \arrow[d, "h_0"'] & \mathbb{D}_{|\phi_{\lambda_1}(0)|} \arrow[d, "\delta"'] \\
E_{\lambda_2}(1) \arrow[r, "t_{\lambda_2}"'] & E_{\lambda_2}^\infty(0) \arrow[r, "t_{\lambda_2}"'] & E_{\lambda_2}(0) \arrow[r, "\phi_{\lambda_2}"'] & \mathbb{D}_{|\phi_{\lambda_2}(0)|}
\end{tikzcd}$$

Note that $\phi_{\lambda_i} : E_{\lambda_i}(0) \to \mathbb{D}_{|\phi_{\lambda_i}(0)|}$ is a conformal isomorphism for $i \in \{1, 2\}$, we can define a quasiconformal map $h_0 : E_{\lambda_1}(0) \to E_{\lambda_2}(0)$ as follows:

$$h_0(z) = \phi_{\lambda_2}^{-1} \circ \delta \circ \phi_{\lambda_1}(z), \ z \in E_{\lambda_1}(0).$$

By construction, $h_0(1) = 1$ and $h_0 \circ T_{\lambda_1}(z) = T_{\lambda_2} \circ h_0(z)$ for $z \in E_{\lambda_1}(0)$. Moreover, the map h_0 can be extended to a homeomorphism from $\overline{E_{\lambda_1}(0)}$ to $\overline{E_{\lambda_2}(0)}$ with the boundary point 0 fixed.

Since $t_{\lambda_i} : E^\infty_{\lambda_i}(0) \to E_{\lambda_i}(0)$ is a proper map of degree one for $i \in \{1, 2\}$, the map $h^\infty_0 : E^\infty_{\lambda_1}(0) \to E^\infty_{\lambda_2}(0)$ with ∞ fixed can be defined by

$$h^\infty_0(z) = t_{\lambda_2}^{-1} \circ h_0 \circ t_{\lambda_1}(z), \ z \in E^\infty_{\lambda_1}(0).$$

Now we define h_1. Since $t_{\lambda_i} : E_{\lambda_i}(1)\setminus\{1\} \to E^\infty_{\lambda_i}(0)\setminus\{\infty\}$ is a covering map of degree two, we can get a lifting of h^∞_0, say $h_1 : E_{\lambda_1}(1) \setminus \{1\} \to E_{\lambda_2}(1) \setminus \{1\}$, such that $h_1(0) = 0$ and $h^\infty_0 \circ t_{\lambda_1}(z) = t_{\lambda_2} \circ h_1(z), \ z \in E_{\lambda_1}(1) \setminus \{1\}$. By continuity, we can define $h_1(1) = 1$. Now we show $h_1|_{E_{\lambda_1}(0)} = h_0$. First note that both $h_1|_{E_{\lambda_1}(0)}$ and h_0 are liftings of h_0 via branch covering maps T_{λ_1} and T_{λ_2}. That is, the following diagram is commutative.

$$\begin{tikzcd}
E_{\lambda_1}(0) \arrow[r, "F"] \arrow[d, "T_{\lambda_1}"'] & E_{\lambda_2}(0) \arrow[d, "T_{\lambda_2}"] \\
E_{\lambda_1}(-1) \arrow[r, "h_0"'] & E_{\lambda_2}(-1)
\end{tikzcd}$$

where $F \in \{h_1|_{E_{\lambda_1}(0)}, h_0\}$. Since $T_{\lambda_i} : E_{\lambda_i}(0) \setminus \{1\} \to E_{\lambda_i}(-1) \setminus \{1\}$ is a covering map of degree two and both h_1 and h_0 fix the boundary point 0 on $\partial E_{\lambda_1}(0)$, we conclude $h_1|_{E_{\lambda_1}(0)} = h_0$ by uniqueness of lifting and continuity.

Suppose for some $n \geq 0$, we already get quasiconformal maps $h_n : E_{\lambda_1}(n) \to E_{\lambda_2}(n)$, $h^\infty_n : E^\infty_{\lambda_1}(n) \to E^\infty_{\lambda_2}(n)$ and $h_{n+1} : E_{\lambda_1}(n+1) \to E_{\lambda_2}(n+1)$ such that $h_{n+1}|_{E_{\lambda_1}(n)} = h_n$ and the right part of the following diagram is commutative.

$$\begin{tikzcd}
E_{\lambda_1}(n+2) \arrow[r, "t_{\lambda_1}"] \arrow[d, "h_{n+2}"'] & E^\infty_{\lambda_1}(n+1) \arrow[r, "t_{\lambda_1}"] \arrow[d, "h^\infty_{n+1}"'] & E_{\lambda_1}(n+1) \arrow[r, "t_{\lambda_1}"] \arrow[d, "h_{n+1}"'] & E^\infty_{\lambda_1}(n) \arrow[r, "t_{\lambda_1}"] \arrow[d, "h^\infty_n"'] & E_{\lambda_1}(n) \arrow[d, "h_n"'] \\
E_{\lambda_2}(n+2) \arrow[r, "t_{\lambda_2}"'] & E^\infty_{\lambda_2}(n+1) \arrow[r, "t_{\lambda_2}"'] & E_{\lambda_2}(n+1) \arrow[r, "t_{\lambda_2}"'] & E^\infty_{\lambda_2}(n) \arrow[r, "t_{\lambda_2}"'] & E_{\lambda_2}(n)
\end{tikzcd}$$

We want to get extensions of h^∞_n and h_{n+1}, denoted by h^∞_{n+1} and h_{n+2} respectively, such that the left part of the diagram is commutative.

We first construct h^∞_{n+1}. Since $t_{\lambda_i} : E^\infty_{\lambda_i}(n+1) \setminus \{1-\lambda_i\} \to E_{\lambda_i}(n+1) \setminus \{0\}$ is a covering map of degree two for $i \in \{1, 2\}$, there is a lifting of h_{n+1}, say

$h_{n+1}^\infty : E_{\lambda_1}^\infty(n+1) \setminus \{1-\lambda_1\} \to E_{\lambda_2}^\infty(n+1) \setminus \{1-\lambda_2\}$, such that $h_{n+1}^\infty(\infty) = \infty$ and $h_{n+1} \circ t_{\lambda_1}(z) = t_{\lambda_2} \circ h_{n+1}^\infty(z)$ for $z \in E_{\lambda_1}^\infty(n+1) \setminus \{1-\lambda_1\}$. By continuity, we can define $h_{n+1}^\infty(1-\lambda_1) = 1-\lambda_2$. Now we show $h_{n+1}^\infty|_{E_{\lambda_1}^\infty(n)} = h_n^\infty$. By assumption $h_{n+1}|_{E_{\lambda_1}(n)} = h_n$, we have

$$h_n \circ t_{\lambda_1}(z) = t_{\lambda_2} \circ F(z),\ z \in E_{\lambda_1}^\infty(n),\ F \in \{h_{n+1}^\infty|_{E_{\lambda_1}^\infty(n)}, h_n^\infty\}.$$

Since $t_{\lambda_i} : E_{\lambda_i}^\infty(n) \setminus \{1-\lambda_i\} \to E_{\lambda_i}(n) \setminus \{0\}$ is a covering map of degree one (if $n = 0$) or two (if $n \geq 1$) for $i \in \{1, 2\}$ and $h_{n+1}^\infty(\infty) = h_n^\infty(\infty) = \infty$, we conclude $h_{n+1}^\infty|_{E_{\lambda_1}^\infty(n)} = h_n^\infty$ by uniqueness of lifting and continuity.

We then construct h_{n+2}. Since $t_{\lambda_i} : E_{\lambda_i}(n+2) \setminus \{1\} \to E_{\lambda_i}^\infty(n+1) \setminus \{\infty\}$ is a covering map of degree two for $i \in \{1, 2\}$, h_{n+1}^∞ can be lifted to h_{n+2} such that $h_{n+2}(0) = 0$. We have just proved that $h_{n+1}^\infty|_{E_{\lambda_1}^\infty(n)} = h_n^\infty$, thus both $h_{n+2}|_{E_{\lambda_1}(n+1)}$ and h_{n+1} satisfy

$$h_n^\infty \circ t_{\lambda_1}(z) = t_{\lambda_2} \circ F(z),\ z \in E_{\lambda_1}(n+1),\ F \in \{h_{n+2}|_{E_{\lambda_1}(n+1)}, h_{n+1}\}.$$

Again by uniqueness of lifting and continuity, we have $h_{n+2}|_{E_{\lambda_1}(n+1)} = h_{n+1}$.

By induction, we finally get two sequences of quasiconformal maps $\{h_n : E_{\lambda_1}(n) \to E_{\lambda_2}(n);\ n \geq 0\}$ and $\{h_n^\infty : E_{\lambda_1}^\infty(n) \to E_{\lambda_2}^\infty(n);\ n \geq 0\}$ such that h_{n+1} and h_{n+1}^∞ are extensions of h_n and h_n^∞ respectively for $n \geq 0$.

Up to now, we can define two quasiconformal maps $\beta : \mathcal{A}_{\lambda_1}(1) \to \mathcal{A}_{\lambda_2}(1)$ and $\gamma : \mathcal{A}_{\lambda_1}(\infty) \to \mathcal{A}_{\lambda_2}(\infty)$ such that $\beta|_{E_{\lambda_1}(n)} = h_n, \gamma|_{E_{\lambda_1}^\infty(n)} = h_n^\infty$ for all $n \geq 0$. It's easy to check

- $\gamma \circ t_{\lambda_1}(z) = t_{\lambda_2} \circ \beta(z),\ z \in \mathcal{A}_{\lambda_1}(1)$,
- $\beta \circ t_{\lambda_1}(z) = t_{\lambda_2} \circ \gamma(z),\ z \in \mathcal{A}_{\lambda_1}(\infty)$,
- The Beltrami coefficients of β and γ, say μ_β and μ_γ, satisfy $\|\mu_\beta\|_\infty = \|\mu_\gamma\|_\infty = \|\mu_\delta\|_\infty$.

By Lemma 4.2.2, the Julia sets $J(t_{\lambda_1})$ and $J(t_{\lambda_2})$ are quasicircles, thus we can get extensions of β and γ, denoted by $\bar\beta$ and $\bar\gamma$ respectively, such that $\bar\beta : \mathcal{A}_{\lambda_1}(1) \cup J(t_{\lambda_1}) \to \mathcal{A}_{\lambda_2}(1) \cup J(t_{\lambda_2})$ and $\bar\gamma : \mathcal{A}_{\lambda_1}(\infty) \cup J(t_{\lambda_1}) \to \mathcal{A}_{\lambda_2}(\infty) \cup J(t_{\lambda_2})$ are homeomorphisms (See [Ahl]). By continuity, we have

$$\bar\gamma \circ t_{\lambda_1}(z) = t_{\lambda_2} \circ \bar\beta(z),\ \bar\beta \circ t_{\lambda_1}(z) = t_{\lambda_2} \circ \bar\gamma(z),\quad z \in J(t_{\lambda_1}). \tag{4.1}$$

In the following, we show $\bar\beta|_{J(t_{\lambda_1})} = \bar\gamma|_{J(t_{\lambda_1})}$. Let $\mathrm{Fix}(R)$ be the set of all fixed points for a rational map R. First note that for $\lambda \in \mathcal{H}_0$, the set $\mathrm{Fix}(t_\lambda)$ consists of three repelling fixed points and $\mathrm{Fix}(T_\lambda) = \mathrm{Fix}(t_\lambda) \cup \{1, \infty\}$. The maps $\bar\beta$ and $\bar\gamma$ satisfy

$$\tau \circ T_{\lambda_1}(z) = T_{\lambda_2} \circ \tau(z),\ z \in J(t_{\lambda_1}),\ \tau \in \{\bar\beta, \bar\gamma\}.$$

Thus both $\bar\beta$ and $\bar\gamma$ map $\mathrm{Fix}(T_{\lambda_1}) \cap J(t_{\lambda_1}) = \mathrm{Fix}(t_{\lambda_1})$ onto $\mathrm{Fix}(T_{\lambda_2}) \cap J(t_{\lambda_2}) = \mathrm{Fix}(t_{\lambda_2})$. By (4.1), we have $\bar\beta|_{\mathrm{Fix}(t_{\lambda_1})} = \bar\gamma|_{\mathrm{Fix}(t_{\lambda_1})}$. Let $e(z) = \bar\beta^{-1} \circ \bar\gamma(z)$

for $z \in J(t_{\lambda_1})$. The map $e : J(t_{\lambda_1}) \to J(t_{\lambda_1})$ is an orientation preserving homeomorphism with three fixed points and satisfies

$$e \circ t_{\lambda_1} \circ e(z) = t_{\lambda_1}(z),\ z \in J(t_{\lambda_1}). \tag{4.2}$$

We show e is in fact the identity map by induction. For $p \in \mathrm{Fix}(t_{\lambda_1})$, $e(p) = p$. Suppose $t_{\lambda_1}^{-1}(p) = \{p, q\}$, by (4.2), we have $t_{\lambda_1}^{-1}(p) = \{p, q\} = \{e(p), e(q)\}$, thus $e(q) = q$. Assume that for some $n \geq 1$, $e|_{t_{\lambda_1}^{-n}(p)} = \mathrm{id}$. For any $q \in t_{\lambda_1}^{-(n+1)}(p) \backslash t_{\lambda_1}^{-n}(p)$, there is $p_n \in t_{\lambda_1}^{-n}(p)$, such that $q \in t_{\lambda_1}^{-1}(p_n) = \{q, q'\}$. By (4.2), we have $\{q, q'\} = \{e(q), e(q')\}$. Since e is orientation preserving, the triples $\{q, q', p_n\}$ and $\{e(q), e(q'), e(p_n) = p_n\}$ have the same cyclic order, thus $e(q) = q,\ e(q') = q'$. In this way, we have $e|_{t_{\lambda_1}^{-(n+1)}(p)} = \mathrm{id}$. By induction, $e|_{\bigcup_{n\geq 0} t_{\lambda_1}^{-n}(p)} = \mathrm{id}$. Since $J(t_{\lambda_1}) = \overline{\bigcup_{n\geq 0} t_{\lambda_1}^{-n}(p)}$, we conclude that e is the identity map by continuity. This means $\bar{\beta}$ and $\bar{\gamma}$ coincide on the Julia set $J(t_{\lambda_1})$.

Now we define

$$h(z) = \begin{cases} \bar{\beta}(z), & z \in \mathcal{A}_{\lambda_1}(1) \bigcup J(t_{\lambda_1}), \\ \bar{\gamma}(z), & z \in \mathcal{A}_{\lambda_1}(\infty). \end{cases}$$

The map $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a homeomorphism with $0, 1$ and ∞ fixed such that

$$h \circ t_{\lambda_1}(z) = t_{\lambda_2} \circ h(z),\ z \in \mathbb{C}.$$

By construction, $h|_{F(t_{\lambda_1})} : F(t_{\lambda_1}) \to F(t_{\lambda_2})$ is a quasiconformal map. Since the Julia set $J(t_{\lambda_1})$ is a quasicircle which is quasiconformally removable, the map $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is actually a quasiconformal map. Moreover, the Beltrami coefficient μ of h satisfies

$$|\mu(z)| = \|\mu\|_\infty = \left| \frac{G_{\lambda_1}(0) - G_{\lambda_2}(0)}{G_{\lambda_1}(0) + G_{\lambda_2}(0)} \right|,\ \text{a.e. } z \in \mathbb{C}.$$

□

Lemma 4.3.3. *For $\lambda_1, \lambda_2 \in \mathbb{C}^*$, suppose there is a quasiconformal map h such that $h \circ t_{\lambda_1} = t_{\lambda_2} \circ h$, then either $\lambda_1 = \lambda_2 \in \partial\mathcal{M}$ or λ_1 and λ_2 lie in the same component of $\mathbb{C}\backslash\partial\mathcal{M}$. In the latter case, suppose λ_1 and λ_2 lie in the component $\mathcal{U}$ of $\mathbb{C}\backslash\partial\mathcal{M}$, then*

$$\|\mu\|_\infty \geq \frac{\exp(d_{\mathcal{U}}(\lambda_1, \lambda_2)) - 1}{\exp(d_{\mathcal{U}}(\lambda_1, \lambda_2)) + 1},$$

where μ is the Beltrami coefficient of h.

Proof. First note that h fixes $0, 1$ and ∞. Since t_{λ_1} and t_{λ_2} are quasiconformally conjugate via h, the Beltrami coefficient μ of h satisfies $t^*_{\lambda_1}(\mu) = \mu$.

If $\|\mu\|_\infty = 0$, then h is the identity map and $\lambda_1 = \lambda_2$. The conclusion follows.

In the following, we assume $\|\mu\|_\infty > 0$. For $c \in \mathbb{D}$, let f_c solve the Beltrami equation

$$\frac{\partial f_c}{\partial \bar{z}} / \frac{\partial f_c}{\partial z} = c\mu/\|\mu\|_\infty$$

with $0, 1, \infty$ fixed. f_c is holomorphic with respect to $c \in \mathbb{D}$ and $f_0 = \mathrm{id}$.

The map $R_c = f_c \circ t_{\lambda_1} \circ f_c^{-1}$ is a rational map since it preserves the standard complex structure. It is of degree two and satisfies the following properties:

- R_c is holomorphic with respect to $c \in \mathbb{D}$ and $R_0 = t_{\lambda_1}$;
- R_c permutes 1 and ∞. Moreover, 1 is a pole of R_c of order two;
- R_c has a zero of order two.

It is easy to check that

$$R_c(z) = f_c \circ t_{\lambda_1} \circ f_c^{-1}(z) = \Big(\frac{z - 1 + \lambda(c)}{z - 1}\Big)^2 = t_{\lambda(c)}(z),$$

where $c \mapsto \lambda(c)$ is a holomorphic map from $\mathbb{D}$ to $\mathbb{C}$ and $\lambda(0) = \lambda_1$. Since

$$t_{\lambda(c)} \circ f_c(1 - \lambda_1) = f_c \circ t_{\lambda_1}(1 - \lambda_1) = f_c(0) = 0,$$

we have $1 - \lambda(c) = f_c(1 - \lambda_1)$. Thus λ has the expression $\lambda(c) = 1 - f_c(1 - \lambda_1)$. It is obvious that $\lambda(\|\mu\|_\infty) = \lambda_2$ since $f_{\|\mu\|_\infty} = h$.

First suppose $\lambda_1 \in \partial\mathcal{M}$. Note that $t_{\lambda(c)}$ has the same dynamical property as t_{λ_1} for all $c \in \mathbb{D}$, the map $\lambda : \mathbb{D} \to \mathbb{C}$ can not take values outside of the non-escape locus $\mathcal{M}$. By open map theorem, λ is a constant map. In particular, $\lambda_1 = \lambda_2$.

Now suppose $\lambda_1 \in \mathbb{C}\backslash\partial\mathcal{M}$. In this case, the image $\lambda(\mathbb{D})$ under the map $\lambda : \mathbb{D} \to \mathbb{C}$ has no intersection with $\partial\mathcal{M}$ by the previous argument, thus $\lambda(\mathbb{D})$ must be contained in some component $\mathcal{U}$ of $\mathbb{C}\backslash\partial\mathcal{M}$ ($\mathcal{U}$ is a hyperbolic Riemann surface since $\partial\mathcal{M} \cap \mathbb{R}$ contains at least three points: $0, 3$ and $32/27$. See [QL]). Since the map $\lambda : \mathbb{D} \to \mathcal{U}$ is holomorphic, by Schwarz lemma,

$$d_{\mathcal{U}}(\lambda_1, \lambda_2) = d_{\mathcal{U}}(\lambda(0), \lambda(\|\mu\|_\infty)) \le d_{\mathbb{D}}(0, \|\mu\|_\infty) = \log\frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}.$$

Thus we have

$$\|\mu\|_\infty \ge \frac{\exp(d_{\mathcal{U}}(\lambda_1, \lambda_2)) - 1}{\exp(d_{\mathcal{U}}(\lambda_1, \lambda_2)) + 1}.$$

□

Lemma 4.3.4. *$\mathcal{H}_0$ is connected and $\mathcal{H}_0 \cup \{\infty\}$ contains a neighborhood of ∞.*

Proof. First we make a coordinate change for the family $\{T_\lambda; \lambda \in \mathcal{H}_0\}$. Let $\lambda = \nu^{-3}$ and $\varphi_\nu(z) = \nu^2(z-1)$. We can conjugate T_λ to a new map $S_\nu = \varphi_\nu \circ T_\lambda \circ \varphi_\nu^{-1}$. Calculation shows

$$S_\nu(\zeta) = \frac{\zeta^2(2\nu + 4\nu^2\zeta + \zeta^2)}{(1+2\nu\zeta)^2}.$$

To prove $\mathcal{H}_0 \cup \{\infty\}$ contains a neighborhood of ∞ is equivalent to prove $0 \in \mathcal{A}_\lambda(1)$ when $|\lambda|$ is large. It's also equivalent to prove

$$-\nu^2 = \varphi_\nu(0) \in \varphi_\nu(\mathcal{A}_\lambda(1)) = \mathcal{A}_\nu(0)$$

when $|\nu|$ is small, where $\mathcal{A}_\nu(0)$ is the immediate basin of attraction for the superattracting fixed point $\zeta = 0$ of the map S_ν. By continuity, there is a small positive number δ such that when $|\nu| < \delta$, we have $S_\nu(\mathbb{D}_{1/2}) \subset \mathbb{D}_{1/2}$. This means $\mathbb{D}_{1/2}$ lies in the Fatou set $F(S_\nu)$. Thus $-\nu^2 \in \mathbb{D}_{1/2} \subset \mathcal{A}_\nu(0)$ when $|\nu|$ is small.

To conclude, we show $\mathcal{H}_0$ is connected. The map $\Phi : \mathcal{H}_0 \cup \{\infty\} \to \mathbb{D}$ defined by $\Phi(\lambda) = \phi_\lambda(0)$ for $\lambda \in \mathcal{H}_0$ and $\Phi(\infty) = 0$ is holomorphic and locally injective in a neighborhood $\mathcal{V}_\infty$ of ∞ (See the proof of Proposition 4.3.1). Thus the image $\Phi(\mathcal{H}_0 \cup \{\infty\})$ contains a neighborhood of 0. For any $\lambda \in \mathcal{H}_0$, there is $\lambda_0 \in \mathcal{V}_\infty \setminus \{\infty\}$ such that $\arg \phi_{\lambda_0}(0) = \arg \phi_\lambda(0)$. By Lemma 4.3.2, t_λ and t_{λ_0} are quasiconformally conjugate. By Lemma 4.3.3, λ and λ_0 lie in the same component of $\mathcal{H}_0$. Thus $\mathcal{H}_0$ is connected and it is the unbounded component of $\mathbb{C} \setminus \partial\mathcal{M}$. □

Proposition 4.3.1. *The map $\Phi : \mathcal{H}_0 \to \mathbb{D}^*$ defined by $\Phi(\lambda) = \phi_\lambda(0)$ is a conformal isomorphism, where ϕ_λ is the Böttcher map for T_λ defined near the fixed point 1.*

Proof. We develop two methods to prove the proposition. The first is to prove that the map Φ is a proper map of degree one while the second is to prove the map Φ preserves the Poincaré metrics. Lemma 4.3.2 and Lemma 4.3.3 play a crucial role in both proofs.

First note that the map $\Phi : \mathcal{H}_0 \to \mathbb{D}^*$ has Laurent expansion

$$\Phi(\lambda) = -2\lambda^{-1} + O(\lambda^{-2})$$

when $|\lambda|$ is large and ∞ is a removable singularity for Φ. Thus we can define $\Phi(\infty) = 0$ such that $\Phi : \mathcal{H}_0 \cup \{\infty\} \to \mathbb{D}$ is holomorphic and locally injective near ∞. Moreover, there exist a neighborhood $\mathcal{V}_\infty$ of ∞ and $\varepsilon \in (0,1)$ such that $\Phi : \mathcal{V}_\infty \to \mathbb{D}_\varepsilon$ is biholomorphic. We may assume $\partial\mathcal{V}_\infty$ is an analytic simple curve by choosing ε small enough.

Proof 1. Given any $\lambda \in \mathcal{H}_0 \setminus \mathcal{V}_\infty$, there is a unique $\lambda_0 \in \partial\mathcal{V}_\infty$ such that $\arg\phi_{\lambda_0}(0) = \arg\phi_\lambda(0)$. It follows from Lemma 4.3.2 and Lemma 4.3.3 that

$$\frac{\exp(d_{\mathcal{H}_0}(\lambda,\lambda_0))-1}{\exp(d_{\mathcal{H}_0}(\lambda,\lambda_0))+1} \le \left|\frac{G_\lambda(0)-G_{\lambda_0}(0)}{G_\lambda(0)+G_{\lambda_0}(0)}\right| = \left|\frac{G_\lambda(0)-\log(1/\varepsilon)}{G_\lambda(0)+\log(1/\varepsilon)}\right|.$$

From this inequality and by continuity of the function $\lambda \mapsto G_\lambda(0) = -\log|\Phi(\lambda)|$, we conclude $G_\lambda(0) \le \log(1/\varepsilon)$ for all $\lambda \in \mathcal{H}_0 \setminus \mathcal{V}_\infty$. Thus we have

$$\frac{\exp(d_{\mathcal{H}_0}(\lambda,\partial\mathcal{V}_\infty))-1}{\exp(d_{\mathcal{H}_0}(\lambda,\partial\mathcal{V}_\infty))+1} \le \frac{\exp(d_{\mathcal{H}_0}(\lambda,\lambda_0))-1}{\exp(d_{\mathcal{H}_0}(\lambda,\lambda_0))+1} \le \frac{\log(1/\varepsilon)-G_\lambda(0)}{\log(1/\varepsilon)+G_\lambda(0)},$$

where $d_{\mathcal{H}_0}(\lambda,\partial\mathcal{V}_\infty) = \inf_{\zeta\in\partial\mathcal{V}_\infty} d_{\mathcal{H}_0}(\lambda,\zeta)$. This inequality indicates

$$G_\lambda(0) \le \log(1/\varepsilon)\exp(-d_{\mathcal{H}_0}(\lambda,\partial\mathcal{V}_\infty)),\ \ \lambda \in \mathcal{H}_0 \setminus \mathcal{V}_\infty.$$

From this we know that when $\lambda \to \lambda^* \in \mathbb{C}\cap\partial\mathcal{H}_0$, we have $d_{\mathcal{H}_0}(\lambda,\partial\mathcal{V}_\infty) \to \infty$, $G_\lambda(0) \to 0$ and $|\Phi(\lambda)| \to 1$. This implies that $\Phi : \mathcal{H}_0 \cup \{\infty\} \to \mathbb{D}$ is a proper map. Since $\Phi^{-1}(0) = \{\infty\}$ and Φ is locally injective near ∞, Φ is a conformal isomorphism.

Proof 2. Given $\lambda_0 \in \mathcal{V}_\infty \setminus \{\infty\}$, there is an analytic arc γ in $\mathcal{V}_\infty$ passing through λ_0 such that $\arg\phi_{\lambda_0}(0) = \arg\phi_\lambda(0)$ for all $\lambda \in \gamma$. By Lemma 4.3.2 and Lemma 4.3.3, we have

$$\frac{\exp(d_{\mathcal{H}_0}(\lambda,\lambda_0))-1}{\exp(d_{\mathcal{H}_0}(\lambda,\lambda_0))+1} \le \left|\frac{G_\lambda(0)-G_{\lambda_0}(0)}{G_\lambda(0)+G_{\lambda_0}(0)}\right| = \frac{|\log\Phi(\lambda)-\log\Phi(\lambda_0)|}{|\log|\Phi(\lambda)|+\log|\Phi(\lambda_0)||},\ \ \lambda\in\gamma \tag{4.3}$$

Taking a limit $\lambda \to \lambda_0$ along γ, we have

$$\begin{aligned}\lim_{\lambda\to\lambda_0}\frac{1}{|\lambda-\lambda_0|}\frac{\exp(d_{\mathcal{H}_0}(\lambda,\lambda_0))-1}{\exp(d_{\mathcal{H}_0}(\lambda,\lambda_0))+1} &= \frac{\rho_{\mathcal{H}_0}(\lambda_0)}{2},\\ \lim_{\lambda\to\lambda_0}\frac{1}{|\lambda-\lambda_0|}\frac{|\log\Phi(\lambda)-\log\Phi(\lambda_0)|}{|\log|\Phi(\lambda)|+\log|\Phi(\lambda_0)||} &= \frac{|\Phi'(\lambda_0)|}{2|\Phi(\lambda_0)|\log(1/|\Phi(\lambda_0)|)},\end{aligned}$$

where $ds_{\mathcal{H}_0} = \rho_{\mathcal{H}_0}(\lambda)|d\lambda|$ is the Poincaré metric on $\mathcal{H}_0$. By (4.3), we have

$$ds_{\mathcal{H}_0}(\lambda_0) = \rho_{\mathcal{H}_0}(\lambda_0)|d\lambda| \le \frac{|\Phi'(\lambda_0)||d\lambda|}{|\Phi(\lambda_0)|\log(1/|\Phi(\lambda_0)|)} = \Phi^*(ds_{\mathbb{D}^*})(\lambda_0).$$

On the other hand, by Schwarz Lemma, $\Phi^*(ds_{\mathbb{D}^*})(\lambda_0) \le ds_{\mathcal{H}_0}(\lambda_0)$. Thus we have $\Phi^*(ds_{\mathbb{D}^*})(\lambda_0) = ds_{\mathcal{H}_0}(\lambda_0)$. This indicates the map $\Phi : \mathcal{H}_0 \to \mathbb{D}^*$ is a covering map. Note that $\mathcal{H}_0$ is not simply connected since ∞ is a cusp, $\mathcal{H}_0$ is conformally equivalent to $\mathbb{D}^*$ and Φ is a proper map (See [F]). Since $\Phi : \mathcal{H}_0 \to \mathbb{D}^*$ has Laurent expansion $\Phi(\lambda) = -2\lambda^{-1} + O(\lambda^{-2})$ near ∞, Φ is a conformal isomorphism. $\square$

Corollary 4.3.1. *The non-escape locus $\mathcal{M}$ has logarithmic capacity equal to 2.*

Proof. We know from Proposition 4.3.1 that the map $\Phi : \mathcal{H}_0 \to \mathbb{D}^*$ is a conformal isomorphism, thus the Green function for $\mathcal{H}_0$ is $G_\lambda(0) = -\log|\Phi(\lambda)|$, whose asymptotic behavior at ∞ is of the form

$$G_\lambda(0) = \log|\lambda| + \gamma + o(1),$$

where $\gamma = -\log 2$ is the Robin constant. Since $\mathbb{C} \cap \partial\mathcal{H}_0$ is the outer boundary of $\mathcal{M}$, the capacity of $\mathcal{M}$ is equal to $\mathrm{e}^{-\gamma} = 2$. □

Proposition 4.3.2. *Let $\mathcal{H}$ be a component of $\mathcal{H}_n$ for $n \geq 1$, the map $\Phi_{\mathcal{H}}$: $\mathcal{H} \to \mathbb{D}$ defined by $\Phi_{\mathcal{H}}(\lambda) = \phi_\lambda(t_\lambda^n(0))$ is a conformal isomorphism, where ϕ_λ is the Böttcher map for T_λ defined throughout $\mathcal{A}_\lambda(1)$.*

Proof. The proof will be based on the following claim:

Claim :*There is a holomorphic map $\lambda : \mathbb{D} \to \mathcal{H}$ such that $\Phi_{\mathcal{H}}(\lambda(\zeta)) = \zeta$ for all $\zeta \in \mathbb{D}$.*

Once the claim is proved, we see that $\Phi_{\mathcal{H}}$ is surjective and admits a global inverse map λ. Thus $\Phi_{\mathcal{H}} : \mathcal{H} \to \mathbb{D}$ is in fact a conformal isomorphism. Now we will prove the claim via quasiconformal surgery and holomorphic motion theorem.

Given any $\lambda_0 \in \mathcal{H}$, let W_{λ_0} be the component of $t_{\lambda_0}^{-1}(\mathcal{A}_{\lambda_0}(1))$ other than $\mathcal{A}_{\lambda_0}(\infty)$. By definition of $\mathcal{H}$, $t_{\lambda_0}^{n-1}(0) \in W_{\lambda_0}$. The Böttcher map ϕ_{λ_0} can be defined in the whole basin $\mathcal{A}_{\lambda_0}(1)$ since there is no critical point in $\mathcal{A}_{\lambda_0}(1)$ other than 1 (By Lemma 4.2.5). Let $\zeta_0 = \phi_{\lambda_0}(t_{\lambda_0}^n(0))$. For $\kappa > 0$, let $D(\zeta_0, \kappa) = \{\zeta \in \mathbb{D}; d_{\mathbb{D}}(\zeta, \zeta_0) < \kappa\}$ be the hyperbolic disk centered at ζ_0 with radius κ, $W_{\lambda_0,\kappa} = (t_{\lambda_0}|_{W_{\lambda_0}})^{-1} \circ \phi_{\lambda_0}^{-1}(D(\zeta_0, \kappa))$ be the relatively compact subset of W_{λ_0}. For any $\zeta \in D(\zeta_0, \kappa)$, we will define a map $\delta_\zeta : W_{\lambda_0,\kappa} \to \phi_{\lambda_0}^{-1}(D(\zeta_0, \kappa))$ satisfying the following properties:

- $\delta_{\zeta_0}(z) = t_{\lambda_0}(z)$ for all $z \in W_{\lambda_0,\kappa}$;
- $\delta_\zeta(t_{\lambda_0}^{n-1}(0)) = \phi_{\lambda_0}^{-1}(\zeta)$;
- $\delta_\zeta : W_{\lambda_0,\kappa} \to \phi_{\lambda_0}^{-1}(D(\zeta_0, \kappa))$ is a quasiconformal map for any fixed ζ;
- δ_ζ is holomorphic with respect to $\zeta \in D(\zeta_0, \kappa)$ for any fixed $z \in W_{\lambda_0,\kappa}$.

Let $E = \partial D(\zeta_0, \kappa) \cup \{\zeta_0\}$. To construct such a map δ_ζ, we first define a map $h : D(\zeta_0, \kappa) \times E \to \mathbb{D}$ as follows:

$$h(\zeta, z) = \begin{cases} z, & z \in \partial D(\zeta_0, \kappa), \\ \zeta, & z = \zeta_0. \end{cases}$$

It is easy to check that

- $h(\zeta_0, z) = z, \ z \in E$,

- for every fixed $\zeta \in D(\zeta_0, \kappa)$, $z \mapsto h(\zeta, z)$ is injective on E,
- for every fixed $z \in E$, $\zeta \mapsto h(\zeta, z)$ is holomorphic in $D(\zeta_0, \kappa)$.

Thus $h : D(\zeta_0, \kappa) \times E \to \mathbb{D}$ is a holomorphic motion parameterized by $D(\zeta_0, \kappa)$ with base point ζ_0. By Slodkowski's theorem (See Theorem 2.5.1 or [Slo]), there is a holomorphic motion $H : D(\zeta_0, \kappa) \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ extending h. Moreover, for any fixed $\zeta \in D(\zeta_0, \kappa)$, $H(\zeta, \cdot) : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a quasiconformal map with dilatation $K(H(\zeta, \cdot)) \leq \exp(d_{D(\zeta_0,\kappa)}(\zeta_0, \zeta))$. In particular, $H(\zeta_0, \cdot)$ is the identity map. Let

$$\delta_\zeta(z) = \phi_{\lambda_0}^{-1} \circ H(\zeta, \phi_{\lambda_0} \circ t_{\lambda_0}(z))$$

for $z \in W_{\lambda_0,\kappa}$. It is easy to check that δ_ζ satisfies the required properties.

Now we define a quasiregular map

$$L_\zeta(z) = \begin{cases} \delta_\zeta(z), & z \in W_{\lambda_0,\kappa}, \\ t_{\lambda_0}(z), & z \in \bar{\mathbb{C}} \setminus W_{\lambda_0,\kappa}. \end{cases}$$

Let σ be the standard complex structure, we can construct a complex structure σ_ζ invariant under L_ζ as follows:

$$\sigma_\zeta = \begin{cases} (t_{\lambda_0}^m)^*(\delta_\zeta^*\sigma), & \text{in } t_{\lambda_0}^{-m}(W_{\lambda_0,\kappa}) \text{ for } m \geq 0, \\ \sigma, & \text{in } \bar{\mathbb{C}} \setminus \bigcup_{m\geq 0} t_{\lambda_0}^{-m}(W_{\lambda_0,\kappa}). \end{cases}$$

Let μ_ζ be the Beltrami coefficient for the complex structure σ_ζ. Since L_ζ is equal to t_{λ_0} outside of $W_{\lambda_0,\kappa}$, it is easy to check that

$$\|\mu_\zeta\|_\infty \leq \frac{K(H(\zeta, \cdot)) - 1}{K(H(\zeta, \cdot)) + 1} \leq \frac{\exp(d_{D(\zeta_0,\kappa)}(\zeta_0, \zeta)) - 1}{\exp(d_{D(\zeta_0,\kappa)}(\zeta_0, \zeta)) + 1} < 1$$

for all $\zeta \in D(\zeta_0, \kappa)$. Moreover, μ_ζ is holomorphic with respect to ζ in the distribution sense by Slodkowski's theorem. By Ahlfors-Bers Theorem, there is a quasiconformal map $f_\zeta : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ with 0,1 and ∞ fixed such that $f_\zeta^*(\sigma) = \sigma_\zeta$. f_ζ is holomorphic with respect to ζ and $f_{\zeta_0} = \mathrm{id}$.

The map $R_\zeta = f_\zeta \circ L_\zeta \circ f_\zeta^{-1}$ is a rational map since it preserves the standard complex structure σ. It is of degree two and satisfies the following properties:

- R_ζ is holomorphic with respect to $\zeta \in D(\zeta_0, \kappa)$ and $R_{\zeta_0} = t_{\lambda_0}$;
- R_ζ permutes 1 and ∞. Moreover, 1 is a pole of R_ζ of order two;
- R_ζ has a zero of order two.

From these information, we conclude

$$R_\zeta(z) = f_\zeta \circ L_\zeta \circ f_\zeta^{-1}(z) = \Big(\frac{z - 1 + \lambda_\kappa(\zeta)}{z - 1}\Big)^2 = t_{\lambda_\kappa(\zeta)}(z) \tag{4.4}$$

where $\zeta \mapsto \lambda_\kappa(\zeta)$ is holomorphic for $\zeta \in D(\zeta_0, \kappa)$ and $\lambda_\kappa(\zeta_0) = \lambda_0$. From (4.4), we get

$$\lambda_\kappa(\zeta) = 1 - f_\zeta(1 - \lambda_0),\ \zeta \in D(\zeta_0, \kappa).$$

This relation indicates that the map λ_κ is determined by a slice of holomorphic motion.

It is easy to check that the map $f_\zeta : \mathcal{A}_{\lambda_0}(1) \to \mathcal{A}_{\lambda_\kappa(\zeta)}(1)$ is a conformal isomorphism and the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{A}_{\lambda_0}(1) \arrow[r, "T_{\lambda_0}"] \arrow[d, "f_\zeta"'] & \mathcal{A}_{\lambda_0}(1) \arrow[d, "f_\zeta"] \\
\mathcal{A}_{\lambda_\kappa(\zeta)}(1) \arrow[r, "T_{\lambda_\kappa(\zeta)}"'] & \mathcal{A}_{\lambda_\kappa(\zeta)}(1)
\end{tikzcd}$$

Thus the Böttcher maps for T_{λ_0} and $T_{\lambda_\kappa(\zeta)}$ satisfy the following relation

$$\phi_{\lambda_0}(z) = \phi_{\lambda_\kappa(\zeta)} \circ f_\zeta(z), \quad z \in \mathcal{A}_{\lambda_0}(1).$$

For any $\zeta \in D(\zeta_0, \kappa)$, by definition of L_ζ, we see that $t^n_{\lambda_\kappa(\zeta)}(0) \in \mathcal{A}_{\lambda_\kappa(\zeta)}(1)$ and $t^{n-1}_{\lambda_\kappa(\zeta)}(0) \notin \mathcal{A}_{\lambda_\kappa(\zeta)}(\infty)$. Thus $\lambda_\kappa(D(\zeta_0, \kappa)) \subset \mathcal{H}_n$. In fact, $\lambda_\kappa(D(\zeta_0, \kappa))$ is contained in $\mathcal{H}$ since $\lambda_\kappa(D(\zeta_0, \kappa))$ is connected and $\lambda_\kappa(\zeta_0) = \lambda_0 \in \mathcal{H} \cap \lambda_\kappa(D(\zeta_0, \kappa))$. For $\zeta \in D(\zeta_0, \kappa)$, we have

$$\begin{aligned}
\Phi_{\mathcal{H}}(\lambda_\kappa(\zeta)) &= \phi_{\lambda_\kappa(\zeta)}(t^n_{\lambda_\kappa(\zeta)}(0)) = \phi_{\lambda_\kappa(\zeta)} \circ f_\zeta \circ L^n_\zeta \circ f^{-1}_\zeta(0) \\
&= \phi_{\lambda_\kappa(\zeta)} \circ f_\zeta \circ \delta_\zeta \circ t^{n-1}_{\lambda_0}(0) = \phi_{\lambda_0} \circ \delta_\zeta \circ t^{n-1}_{\lambda_0}(0) = \zeta.
\end{aligned}$$

Note that for $\kappa_1 > \kappa_2 > 0$, both λ_{κ_1} and λ_{κ_2} are local inverse of $\Phi_{\mathcal{H}}$ such that $\lambda_{\kappa_1}(\zeta_0) = \lambda_{\kappa_2}(\zeta_0) = \lambda_0$, thus we have $\lambda_{\kappa_1}|_{D(\zeta_0, \kappa_2)} = \lambda_{\kappa_2}$. Since $\bigcup_{\kappa>0} D(\zeta_0, \kappa) = \mathbb{D}$, there is a holomorphic map $\lambda : \mathbb{D} \to \mathcal{H}$ such that $\lambda|_{D(\zeta_0,\kappa)} = \lambda_\kappa$ for all $\kappa > 0$ and $\Phi_{\mathcal{H}}(\lambda(\zeta)) = \zeta$ for all $\zeta \in \mathbb{D}$. This ends the proof of the claim. $\square$

Proof of Theorem 4.1.1. First note that $\mathcal{M}$ is compact. To prove $\mathcal{M}$ is connected is equivalent to prove every connected component of $\bar{\mathbb{C}} \setminus \mathcal{M}$ is simply connected. By Lemma 4.2.4, a connected component of $\bar{\mathbb{C}} \setminus \mathcal{M}$ is either $\mathcal{H}_0 \cup \{\infty\}$ or a capture domain of level $n \geq 1$. These components are simply connected by Proposition 4.3.1 and Proposition 4.3.2.

$\square$

An concluding remark: extremal property of Beltrami coefficient

At the end of this section, we give a remark about Lemma 4.3.2. Given $\lambda_1, \lambda_2 \in \mathcal{H}_0$ with $\arg \phi_{\lambda_1}(0) = \arg \phi_{\lambda_2}(0)$, let $QC(\lambda_1, \lambda_2)$ be the set of all quasiconformal maps $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $\varphi \circ t_{\lambda_1} = t_{\lambda_2} \circ \varphi$. We will show in the following that the map h constructed in Lemma 4.3.2 has the following extremal property:

$$\|\mu\|_\infty = \inf\{\|\mu_\varphi\|_\infty;\ \varphi \in QC(\lambda_1, \lambda_2)\},$$

where μ and μ_φ are the Beltrami coefficients of h and $\varphi \in QC(\lambda_1, \lambda_2)$ respectively.

Indeed, from Proposition 4.3.1 we know that the map $\Phi : \mathcal{H}_0 \to \mathbb{D}^*$ is a conformal isomorphism. We may assume $|\Phi(\lambda_1)| \leq |\Phi(\lambda_2)|$, thus we have

$$d_{\mathcal{H}_0}(\lambda_1, \lambda_2) = d_{\mathbb{D}^*}(\Phi(\lambda_1), \Phi(\lambda_2)) = \int_{|\Phi(\lambda_1)|}^{|\Phi(\lambda_2)|} \frac{dr}{r\log(1/r)} = \log \frac{G_{\lambda_1}(0)}{G_{\lambda_2}(0)}.$$

By Lemma 4.3.3, for any $\varphi \in QC(\lambda_1, \lambda_2)$,

$$\|\mu_\varphi\|_\infty \geq \frac{\exp(d_{\mathcal{H}_0}(\lambda_1, \lambda_2)) - 1}{\exp(d_{\mathcal{H}_0}(\lambda_1, \lambda_2)) + 1} = \left| \frac{G_{\lambda_1}(0) - G_{\lambda_2}(0)}{G_{\lambda_1}(0) + G_{\lambda_2}(0)} \right| = \|\mu\|_\infty.$$

Here is a more general result on extremal quasiconformal conjugacy: Let $\mathcal{H}$ be a capture domain of level $n \geq 0$, $\lambda_1, \lambda_2 \in \mathcal{H}$ be two parameters such that each t_{λ_i} is not postcritically finite. We still use $QC(\lambda_1, \lambda_2)$ to denote the set of all quasiconformal maps $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $\varphi \circ t_{\lambda_1} = t_{\lambda_2} \circ \varphi$. Then there exists $\phi \in QC(\lambda_1, \lambda_2)$ such that

$$\|\mu_\phi\| = \inf\{\|\mu_\varphi\|; \varphi \in \mathcal{Q}(\lambda_1, \lambda_2)\} = \left| \frac{\mathrm{Log}\phi_{\lambda_1}(t^n_{\lambda_1}(0)) - \mathrm{Log}\phi_{\lambda_2}(t^n_{\lambda_2}(0))}{\mathrm{Log}\phi_{\lambda_1}(t^n_{\lambda_1}(0)) + \overline{\mathrm{Log}\phi_{\lambda_2}(t^n_{\lambda_2}(0))}} \right|,$$

where the Log is chosen such that $|\arg \phi_{\lambda_1}(t^n_{\lambda_1}(0)) - \arg \phi_{\lambda_2}(t^n_{\lambda_2}(0))| \leq \pi$.

4.4 Proof of Theorem 4.1.2

It is known from the previous section that when $\lambda \in \mathcal{H}_0$, $J(t_\lambda)$ is a quasicircle. There are two natural questions:

1. *What's the asymptotic behavior of $J(t_\lambda)$ when $|\lambda| \to \infty$?*
2. *How to estimate the Hausdorff dimension of $J(t_\lambda)$ for all $\lambda \in \mathcal{H}_0$?*

It is observed by Hu and Lin (See [HL]) that $J(t_\lambda)$ becomes larger and more circular as the real parameter $\lambda \to \infty$. In 1995, Osbaldestin shows that the Hausdorff dimension $HD(J(t_\lambda))$ of $J(t_\lambda)$ has the following expansion (see [O]):

$$HD(J(t_\lambda)) = 1 + \frac{|\lambda|^{-2/3}}{4\log 2} + \mathcal{O}(|\lambda|^{-1})$$

when the real parameter $\lambda \to \infty$. This expansion also holds for $\lambda \in \mathbb{C}$ when $|\lambda|$ is large.

Theorem 4.1.2 provides an answer to the second question. In the following, we prove Theorem 4.1.2. It's equivalent to prove the following inequality:

$$HD(J(t_\lambda)) \leq 1 + |\Phi(\lambda)|^{2/3}, \quad \lambda \in \mathcal{H}_0$$

where $\Phi : \mathcal{H}_0 \to \mathbb{D}^*$ is the map that constructed in Proposition 4.3.1. The proof of Theorem 4.1.2 will be based on Smirnov's theorem ([Smi], See also Theorem 2.4.3) and Slodkowski's theorem ([Slo], See also Theorem 2.5.1).

Proof of Theorem 4.1.2. Let ℓ be a quasicircle, the dilatation $K[\ell]$ of ℓ is defined by

$$K[\ell] = \inf\{K(\varphi);\ \varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}} \text{ is a quasiconformal map such that } \varphi(\mathbb{S}^1) = \ell\}.$$

By Smirnov's theorem, we have

$$HD(J(t_\lambda)) \le 1 + \left(\frac{K[J(t_\lambda)] - 1}{K[J(t_\lambda)] + 1}\right)^2,\ \lambda \in \mathcal{H}_0.$$

In the following, we prove

$$K[J(t_\lambda)] \le \frac{1 + |\Phi(\lambda)|^{1/3}}{1 - |\Phi(\lambda)|^{1/3}},\ \lambda \in \mathcal{H}_0.$$

Recall that under the coordinate change in Lemma 4.3.4, the family $\{T_\lambda; \lambda \in \mathcal{H}_0\}$ becomes to $\{S_\nu; \nu \in \mathcal{V}_0^* = \mathcal{V}_0 \backslash \{0\}\}$, where $\mathcal{V}_0$ is a neighborhood of 0 such that the map $\nu \mapsto \lambda = \nu^{-3}$ is a proper map of degree three from $\mathcal{V}_0^*$ to $\mathcal{H}_0$. It is easy to check that under coordinate change, $K[J(t_\lambda)] = K[J(S_\nu)]$, $HD(J(t_\lambda)) = HD(J(S_\nu))$. Let $\varphi : \mathcal{V}_0 \to \mathbb{D}$ be a Riemann mapping with $\varphi(0) = 0$. The map Φ and φ satisfy

$$\Phi(\lambda) = \Phi(\nu^{-3}) = e^{i\theta}\varphi(\nu)^3,\ \nu \in \mathcal{V}_0,$$

where θ is a real constant.

Given a pair of compact sets (X, Y), recall that the Hausdorff distance $\sigma_H(X, Y)$ between X and Y is defined by

$$\sigma_H(X, Y) = \max\{\max_{x\in X} \sigma(x, Y),\ \max_{y\in Y} \sigma(X, y)\},$$

where $\sigma(\cdot, \cdot)$ denotes the spherical distance.

We claim $\sigma_H(J(S_\nu), \mathbb{S}^1) \to 0$ as $\nu \to 0$. Indeed, using basic analysis, we can verify that for any $\varepsilon \in (0, 1)$, there is a small positive number δ such that when $|\nu| < \delta$, we have $S_\nu(\mathbb{D}_{1-\varepsilon}) \subset \mathbb{D}_{1-\varepsilon}$, $S_\nu(\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_{1+\varepsilon}) \subset \bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_{1+\varepsilon}$. Thus both $\mathbb{D}_{1-\varepsilon}$ and $\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_{1+\varepsilon}$ lie in the Fatou set $F(S_\nu)$. It turns out that $\sigma_H(J(S_\nu), \mathbb{S}^1) \le 2\varepsilon$. This means the Julia set $J(S_\nu)$ moves continuously at $\nu = 0$ in Hausdorff topology.

It is obvious that the Julia set $J(S_\nu)$ moves continuously on $\mathcal{V}_0^*$ in Hausdorff topology. Thus by adding an new map $S_0(\zeta) = \zeta^4$ to the family $\{S_\nu; \nu \in \mathcal{V}_0^*\}$, we see that the Julia set $J(S_\nu)$ moves continuously on $\mathcal{V}_0$ in Hausdorff topology. By characterizations of stability (see [McM1], Theorem 4.2), the Julia set

$J(S_\nu)$ moves holomorphically on $\mathcal{V}_0$. So there is a holomorphic motion $h : \mathcal{V}_0 \times \mathbb{S}^1 \to \bar{\mathbb{C}}$ parameterized by $\mathcal{V}_0$ with base point 0 such that $h(0, \cdot) = \mathrm{id}$ and $h(\nu, \mathbb{S}^1) = J(S_\nu)$ for all $\nu \in \mathcal{V}_0$. By Slodkowski's theorem (see [Slo]), there is a holomorphic motion $H : \mathcal{V}_0 \times \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ extending h. Moreover for any fixed $\nu \in \mathcal{V}_0$, $H(\nu, \cdot) : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a quasiconformal map with dilatation $K(H(\nu, \cdot)) \leq \exp(d_{\mathcal{V}_0}(0, \nu))$. Since $H(\nu, \mathbb{S}^1) = J(S_\nu)$, we have

$$\begin{aligned} K[J(t_\lambda)] &= K[J(S_\nu)] \leq K(H(\nu, \cdot)) \leq \exp(d_{\mathcal{V}_0}(0, \nu)) \\ &= \exp(d_{\mathbb{D}}(0, \varphi(\nu))) = \frac{1 + |\varphi(\nu)|}{1 - |\varphi(\nu)|} = \frac{1 + |\Phi(\lambda)|^{1/3}}{1 - |\Phi(\lambda)|^{1/3}}. \end{aligned}$$

□

A concluding remark We remark that the exponent $2/3$ in Theorem 4.1.2 is sharp in the following sense:

$$\max\{t; HD(J(t_\lambda)) \leq 1 + |\Phi(\lambda)|^t, \forall \lambda \in \mathcal{H}_0\} = \frac{2}{3}.$$

Moreover,

$$\lim_{\lambda \to \infty} \frac{HD(J(t_\lambda)) - 1}{|\Phi(\lambda)|^{2/3}} = \frac{1}{2^{8/3} \log 2}.$$

This follows from Osbaldestin's result on the asymptotic behavior of $HD(J(t_\lambda))$ near ∞.

Chapter 5

Dynamics of McMullen maps

5.1 Introduction

The local connectivity of Julia sets for rational maps is a central problem in complex dynamical systems. It is well studied for classical type of rational maps, for example: hyperbolic and semihyperbolic maps, geometrically finite maps, see [CJY],[M2],[TY]. In polynomial case, it is also known a lot, see [DH2],[GS],[Kiwi],[Ly],[M2]. For quadratic polynomials, Yoccoz proved that the Julia set is locally connected provided that all periodic points are repelling and the map is not infinitely renormalizable, see [Hu],[M2]. Douady exhibited striking example of infinitely renormalizable quadratic polynomial with non locally connected Julia set, see [M2]. For general polynomial with connected Julia sets and without irrationally neutral cycles, Kiwi shows in [Kiwi] that the local connectivity of Julia set is equivalent to the non existence of wandering continua.

The powerful tool to study the local connectivity of Julia sets for polynomials is the so-called 'Branner-Hubbard-Yoccoz puzzle' techniques, which is introduced by Branner-Hubbard and Yoccoz, [BH]. It has a natural way of construction, which is induced by finite periodic external rays together with an equipotential curve.

However, for general rational maps, things are different. The construction of Yoccoz puzzle becomes quite involved, even impossible. Up to now, the only known rational maps which admit Yoccoz puzzle structures are the cubic Newton maps, whose Yoccoz puzzles are constructed by Roesch. In [Ro1], by Yoccoz puzzle techniques, Roesch shows striking differences between rational maps and polynomials. The method also leads to the local connectivity of Julia sets except some specific cases.

In this article, we exhibit Yoccoz puzzle structure for another family of rational maps, known as McMullen maps, of the form

$$f_\lambda : z \mapsto z^n + \lambda/z^n, \quad \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\},\ n \geq 3.$$

Dynamics of this family have been studied by Devaney and his group, see [D1],[D2],[DK],[DLU].

The difference of Yoccoz puzzle between cubic Newton maps and McMullen maps is as follows: For cubic Newton maps, the ingredient of the Yoccoz puzzle is an converging ray that intersects the Julia set in a countably many points while for McMullen maps, the element to construct Yoccoz puzzle is a Jordan curve (will be called ‘cut ray’) that intersects the Julia set in a Cantor set of points. This kind of Jordan curve is induced by some particular angle and can be viewed as an extention of the corresponding external ray (see Section 5.3).

We denote by B_λ the immediate attractive basin of ∞. The topology of ∂B_λ is of special interest. Based on Yoccoz puzzle techniques and combinatorial and topological analysis, we prove:

Theorem 5.1.1. (Cantor or Jordan) *For any $n \geq 3$ and any complex parameter λ, if the Julia set $J(f_\lambda)$ is not a Cantor set, then ∂B_λ is a Jordan curve.*

This affirmatively answers a question posed by Devaney at the Snowbird Conference on the 25th Birthday of the Mandelbrot set, see [DK]. For the higher regularity of ∂B_λ, we show that ∂B_λ is a quasicircle except two special cases.

Theorem 5.1.2. *Suppose the Julia set $J(f_\lambda)$ is not a Cantor set, then ∂B_λ is a quasicircle if it contains neither parabolic point nor recurrent critical point.*

Here, a recurrent critical point c on the Julia set of a rational map f is a critical point such that $c \in \omega(c)$, where $\omega(c)$ is the ω-limit set of c, defined as $\{z \in \overline{\mathbb{C}};$ there exist $n_k \to \infty$ such that $z = \lim f^{n_k}(c)\}$. It follows from Proposition 5.7.5 that if ∂B_λ contains a parabolic point, then ∂B_λ is not a quasicircle by Leau-Fatou-Flower Theorem, see [M2]. The question whether ∂B_λ is a quasicircle when ∂B_λ contains a recurrent critical point is still unknown.

For the topology of the Julia set, we show

Theorem 5.1.3. *Suppose f_λ has no Siegel disk and the Julia set $J(f_\lambda)$ is connected, then $J(f_\lambda)$ is locally connected in either of the following cases:*

1. *The critical orbit does not accumulate on the boundary ∂B_λ.*
2. *The map f_λ is neither renormalizable nor $*-$renormalizable.*
3. *The parameter λ is real and positive.*

Here, the definitions of *renormalization* and $*-$*renormalization* can be found in Section 5.5. Theorem 5.1.3 implies that the Julia set is locally connected except some special cases. In fact, it's stronger than the following statement:

Theorem 5.1.4. *Suppose f_λ has no Siegel disk and the Julia set $J(f_\lambda)$ is connected, then $J(f_\lambda)$ is locally connected if the critical orbit does not accumulate on the boundary ∂B_λ.*

Theorem 5.1.4 is an analogue of Roesch's Theorem [Ro1]:

Theorem 5.1.5. (Roesch) *A genuine cubic Newton map, without Siegel disks, has a locally connected Julia set provided that the orbit of the non-fixed critical point does not accumulate on the boundary of any invariant basin of attraction.*

This chapter is organized as follows:

In Section 5.2, we present some basic results on McMullen maps.

In Section 5.3, we construct the 'cut rays', a kind of Jordan curves that cut the Julia set into two different parts. We first construct a Cantor set of angles on the unit circle which is used to generate 'cut rays'. Then we discuss the construction of 'cut rays' based on the work of Devaney.

In Section 5.4, basic knowledge of Yoccoz puzzles, graphs and tableaux are presented. The aim of this section is to find a Yoccoz puzzle with a non-degenerate critical annulus (See Section 5.4.2). A natural construction of 'modified puzzle piece' is discussed (See Section 5.4.3).

In Section 5.5, we discuss the renormalizations of McMullen maps from the viewpoint of puzzle piece.

In Section 5.6, we present a criterion of local connectivity. We introduce a '**BD** condition' on the boundary of immediate basin of attraction. Such condition can be considered as 'local semi-hyperbolicity'. We show that '**BD** condition' implies good topology.

In Section 5.7, we study the local connectivity of ∂B_λ in all possible cases and show that ∂B_λ enjoys higher regularity except two special cases.

In Section 5.8, we study the local connectivity of the Julia set $J(f_\lambda)$ based on the 'Characterization of Local Connectivity' and the 'Shrinking Lemma'.

5.2 Preliminaries and Notations

In this section, we present some basic results and notations for the family of rational maps:

$$f_\lambda(z) = z^n + \lambda/z^n$$

where $\lambda \in \mathbb{C}^*$ and $n \geq 3$. This kind of map is known as 'McMullen map' since it is first studied by McMullen. McMullen proved that when $|\lambda|$ is small enough, the Julia set of $z \mapsto z^2 + \lambda/z^3$ is Cantor set of circles, see [McM3].

For any $\lambda \in \mathbb{C}^*$, the map f_λ has a superattracting fixed point at ∞. The immediate basin of ∞ is denoted by B_λ, and the component of $f_\lambda^{-1}(B_\lambda)$ that contains 0 is denoted by T_λ. The set of all critical points of f_λ is $\{0, \infty\} \cup C_\lambda$, where $C_\lambda = \{\sqrt[2n]{\lambda}\omega; \omega^{2n} = 1\}$. Besides ∞, there are only two critical values for f_λ: $v_\lambda^+ = 2\sqrt{\lambda}$ and $v_\lambda^- = -2\sqrt{\lambda}$. In fact, there is only one critical orbit (up to a sign). Let $P(f_\lambda) = \overline{\bigcup_{n\geq 1} f_\lambda^k(C_\lambda)} \cup \{\infty\}$ be the postcritical set.

The Böttcher map ϕ_λ for f_λ is defined in a neighborhood of ∞ by $\phi_\lambda(z) = \lim_{k\to\infty}(f_\lambda^k(z))^{n^{-k}}$. The Böttcher map is unique if we require $\phi'_\lambda(\infty) = 1$. It is known that the Böttcher map ϕ_λ can be extended to a domain $\mathrm{Dom}(\phi_\lambda) \subset B_\lambda$ such that $\phi_\lambda : \mathrm{Dom}(\phi_\lambda) \to \{z \in \bar{\mathbb{C}} : |z| > R\}$ is a conformal isomorphism for some smallest number $R \geq 1$. In particular, if B_λ contains no critical point other than ∞, then $\mathrm{Dom}(\phi_\lambda) = B_\lambda$; if B_λ contains a critical point $c \in \{0\} \cup C_\lambda$, then by 'The Escape Trichotomy' (Theorem 5.2.1), the Julia set $J(f_\lambda)$ is a Cantor set.

The Green function $G_\lambda : B_\lambda \to (0, \infty]$ is defined by

$$G_\lambda(z) = \lim_{k\to\infty} n^{-k} \log |f_\lambda^k(z)|.$$

By definition, $G_\lambda(f_\lambda(z)) = nG_\lambda(z)$ for $z \in B_\lambda$ and $G_\lambda(z) = \log |\phi_\lambda(z)|$ for $z \in \mathrm{Dom}(\phi_\lambda)$. The Green function G_λ can be extended to $A_\lambda = \bigcup_{k\geq 0} f_\lambda^{-k}(B_\lambda)$ by defining

$$G_\lambda(z) = n^{-k}G_\lambda(f_\lambda^k(z)) \text{ for } z \in f_\lambda^{-k}(B_\lambda).$$

In the following, for a set E in $\bar{\mathbb{C}}$ and $a \in \mathbb{C}$, let $aE = \{az; z \in E\}$, $a + E = \{a + z; z \in E\}$, $E^* = \{\bar{z}; z \in E\}$, $\bar{E}$ be the closure of E and $\mathrm{int}(E)$ be the interior of E.

Lemma 5.2.1. (Symmetry of the Dynamical Plane) *Let ω satisfy $\omega^{2n} = 1$, then*

1. $\omega J(f_\lambda) = J(f_\lambda)$.
2. $G_\lambda(\omega z) = G_\lambda(z)$ *for* $z \in A_\lambda$.
3. $\omega\mathrm{Dom}(\phi_\lambda) = \mathrm{Dom}(\phi_\lambda)$, *and* $\phi_\lambda(\omega z) = \omega\phi_\lambda(z)$ *for* $z \in \mathrm{Dom}(\phi_\lambda)$.

Proof. For 1, since $A_\lambda = \{z \in \bar{\mathbb{C}}; f_\lambda^k(z)$ tends to infinity as $k \to \infty\}$ and $f_\lambda^k(\omega z) = \pm f_\lambda^k(z)$ for $k \geq 1$, $f_\lambda^k(\omega z)$ tends to infinity if and only if $f_\lambda^k(z)$ tends to infinity as $k \to \infty$. Thus $\omega A_\lambda = A_\lambda$. The conclusion follows from the fact that $J(f_\lambda) = \partial A_\lambda$.

2. By the definition of G_λ.

3. Since $\mathrm{Dom}(\phi_\lambda)$ is the connected component of $\{z \in B_\lambda; G_\lambda(z) > \log R\}$ that contains ∞, we conclude that $\omega\mathrm{Dom}(\phi_\lambda) = \mathrm{Dom}(\phi_\lambda)$. Note that $\phi_\lambda(\omega z)$ and $\omega\phi_\lambda(z)$ are two Riemann mappings from $\mathrm{Dom}(\phi_\lambda)$ onto $\{z \in \bar{\mathbb{C}}; |z| > R\}$ with the same derivative at ∞, we have $\phi_\lambda(\omega z) = \omega\phi_\lambda(z)$ by the uniqueness of Riemann mapping theorem. □

The Mandelbrot set for this family is defined by

$$M = \{\lambda \in \mathbb{C}^*;\ f_\lambda^k(v_\lambda^+) \text{ does not tend to infinity as } k \to \infty\}.$$

Lemma 5.2.2. (Symmetry of the Parameter Plane) *The Mandelbrot set M satisfies:*

1. $M^* = M$.

2. $\nu M = M$ with $\nu^{n-1} = 1$.

3. For any line $\ell \in \{\varepsilon\mathbb{R}; \varepsilon^{2n-2} = 1\}$, M is symmetric about ℓ.

Proof. 1. Since $\overline{f_\lambda(\bar{z})} = f_{\bar{\lambda}}(z)$, the critical orbit of f_λ and the critical orbit of $f_{\bar{\lambda}}$ are symmetric under the map $z \to \bar{z}$. They either both remain bounded or both tend to infinity. Thus $M^* = M$.

2. Let $\nu = e^{2\pi i/(n-1)}$ and $\varphi(z) = e^{\pi i/(n-1)}z$. For $k \geq 1$,

$$\varphi^{-1} \circ f_{\nu\lambda}^k \circ \varphi(z) = \begin{cases} (-1)^k f_\lambda^k(z), & n \text{ odd}, \\ f_\lambda^k(z), & n \text{ even}. \end{cases}$$

Thus the critical orbit of f_λ tends to infinity if and only if the critical orbit of $f_{\nu\lambda}$ tends to infinity. Equivalently, $\lambda \in M$ if and only if $\nu\lambda \in M$.

3. It follows from 1 and 2. □

From Lemma 5.2.2, f_λ and $f_{\lambda e^{2\pi i/(n-1)}}$ have the same dynamical property and their Julia sets are identical up to a rotation. Thus the fundamental domain of the parameter plane is $\{\lambda \in \mathbb{C}^*; \arg\lambda \in [0, \frac{2\pi}{n-1})\}$.

The following theorem due to Devaney, Look and Uminsky gives a classification of Julia sets with different topological type, see [DLU].

Theorem 5.2.1. (Devaney-Look-Uminsky) *We have the following 'Escape Trichotomy':*

1. If $v_\lambda^+ \in B_\lambda$, then $J(f_\lambda)$ is a Cantor set.

2. If $v_\lambda^+ \in T_\lambda \neq B_\lambda$, then $J(f_\lambda)$ is a Cantor set of circles.

3. If $f_\lambda^k(v_\lambda^+) \in T_\lambda \neq B_\lambda$ for some $k \geq 1$, then $J(f_\lambda)$ is a Sierpiński curve, which is locally connected.

In all other cases, the critical orbits remain bounded and the Julia set $J(f_\lambda)$ is connected.

For $n \geq 3$, it is known that the unbounded component of $\mathbb{C}^* - M$ consists of the parameters for which the Julia set is a Cantor set, this region is called *Cantor set locus*, see Figure 5.1. The component of $\mathbb{C}^* - M$ that contains a punctured neighborhood of 0 is the region where the Julia set $J(f_\lambda)$ is a Cantor set of circles; this is the *McMullen domain*, as it is McMullen who first discovered this type of Julia set. The complement of these two regions is

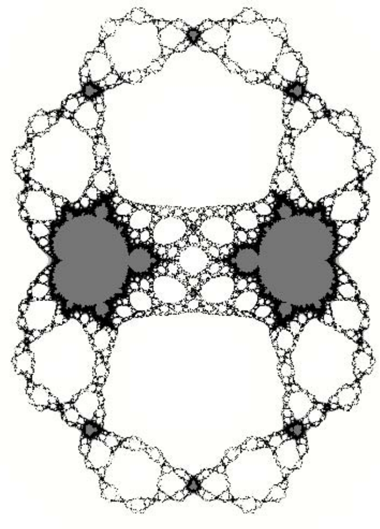

Figure 5.1: Parameter plane for McMullen maps when $n = 3$.

the *connected locus*. The small copies of quadratic Mandelbrot set correspond to the renormalizable parameters while the 'holes' in the connected locus are always called *Sierpiński holes* according to Devaney. These regions correspond to the parameters for which the Julia set is a Sierpiński curve.

We will see later that when the critical orbit tends to ∞, the boundary ∂B_λ is a quasicircle if it is connected. So this case is already well studied. For this reason, throughout the paper, all discussions are based on the following:

Hypothesis: *The critical orbit remain bounded, or equivalently,* $C_\lambda \cap A_\lambda = \emptyset$.

At the end of this section, we give some notations. We restrict our attention to the parameters $\lambda \in \mathcal{H} = \{\lambda \in \mathbb{C}^*; \arg\lambda \in (0, \frac{2\pi}{n-1})\}$ because of the symmetry of the parameter plane. The real positive parameters will be considered separately in Section 5.7.3.

Let $c_0 = c_0(\lambda) = \sqrt[2n]{\lambda}$ be the critical point that lies on $\mathbb{R}^+$ when $\lambda \in \mathbb{R}^+$ and varies analytically as λ ranges over $\mathcal{H}$. Let $c_k = c_0 e^{k\pi i/n}$ for $1 \leq k \leq 2n-1$. The critical points c_k with k even are mapped to $v_\lambda^+ = 2\sqrt{\lambda}$ while the critical points c_k with k odd are mapped to $v_\lambda^- = -2\sqrt{\lambda}$.

Let $\ell_k = c_k\mathbb{R}^+$ be the straight line connecting the origin with ∞ and passing through c_k for $0 \leq k \leq 2n-1$. We call ℓ_k a *critical ray*. The closed sector bounded by ℓ_k and ℓ_{k+1} is denoted by S_k for $0 \leq k \leq n$. Define $S_{-k} = -S_k$ for $1 \leq k \leq n-1$. So the sectors are arranged in counterclockwise order about the origin as $S_0, S_1, \cdots, S_n, S_{-1}, \cdots, S_{-(n-1)}$.

The critical value v_λ^+ always lies in S_0 since $\arg c_0 < \arg v_\lambda^+ < \arg c_1$ for all

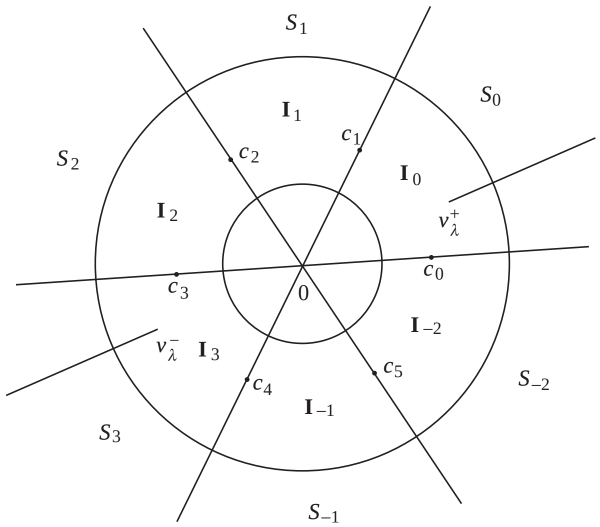

Figure 5.2: Sectors in the dynamical plane when $n = 3$.

$\lambda \in \mathcal{H}$. Correspondingly, the critical value v_λ^- lies in S_n. It's easy to check that the image of ℓ_k under f_λ is a straight ray connecting one of the critical values to ∞, this ray is called *critical value ray*. As a consequence, f_λ maps the interior of each sectors of $\{S_{\pm 1}, \cdots, S_{\pm(n-1)}\}$ univalently onto a region Υ_λ which can be identified as the complex sphere $\overline{\mathbb{C}}$ minus two critical value rays.

Let $\mathcal{P}$ denote the set of all components of $\bigcup_{k\geq 0} f_\lambda^{-k}(B_\lambda)$. For $U \in \mathcal{P}$ and $v > 0$, let $\mathbf{e}(U, v) = \{z \in U; G_\lambda(z) = v\}$ be the equipotential curve. The annulus bounded by $\mathbf{e}(B_\lambda, v)$ and $\mathbf{e}(T_\lambda, v)$ is denoted by $\mathbf{A}_v$. We may choose v large enough such that $\partial\mathbf{A}_v$ intersects with every critical ray at exactly two points. The bounded component and unbounded component of $\overline{\mathbb{C}} \setminus \mathbf{e}(B_\lambda, v)$ are denoted by $\mathbf{V}(v)$ and $\mathbf{U}(v)$, respectively.

Now, we define radial rays of U for every $U \in \mathcal{P} \setminus \{B_\lambda\}$. Under Hypothesis, we see that there is a unique Riemann mapping $\phi_{T_\lambda} : T_\lambda \to \mathbb{D}$ such that

$$\phi_{T_\lambda}(z)^{-n} = \phi_\lambda(f_\lambda(z)),\ z \in T_\lambda;\ \phi'_{T_\lambda}(0) = 1/\sqrt[n]{\lambda}.$$

The radial ray $R_{T_\lambda}(\theta)$ of angle θ is defined as $\phi_{T_\lambda}^{-1}((0,1)e^{2\pi i\theta})$. For $U \in \mathcal{P} \setminus \{B_\lambda, T_\lambda\}$, there is a smallest integer $k \geq 1$ such that $f_\lambda^k : U \to T_\lambda$ is a conformal map. The radial ray $R_U(\theta)$ is defined as the pull back of $R_{T_\lambda}(\theta)$ under f_λ^k.

Let $\mathbb{I} = \{0, n, \pm 1, \ldots, \pm(n-1)\}$ be an index set. $\mathbf{I}_k = \overline{\mathbf{A}_v} \cap S_k$ for $k \in \mathbb{I}$ and $\mathbf{I} = \bigcup_{k \in \mathbb{I}\setminus\{0,n\}} \mathbf{I}_k$. The set of all points whose orbits remain in $\mathbf{I}$ under all iterations of f_λ is denoted by Λ_λ. Obviously $\Lambda_\lambda = \bigcap_{k\geq 0} f_\lambda^{-k}(\mathbf{I})$ and Λ_λ is a subset of the Julia set $J(f_\lambda)$.

For any $k \in \mathbb{I} \setminus \{0, n\}$, $f_\lambda : \mathrm{int}(S_k) \to \Upsilon_\lambda$ is a conformal map, and the inverse is denoted by $h_k : \Upsilon_\lambda \to \mathrm{int}(S_k)$.

Given a point $z \in \Lambda_\lambda$, suppose $f_\lambda^k(z) \in S_{s_k}$ for $k \geq 0$, define the itinerary of z by $\mathbf{s}_\lambda(z) = (s_0, s_1, s_2, \cdots)$. The itinerary is always well defined in the set Λ_λ since if some iteration $f_\lambda^k(z)$ lies on the boundary of two adjacent sectors, then the next iteration $f_\lambda^{k+1}(z)$ will lie inside $S_0 \cup S_n$.

Let $\Sigma = \{\mathbf{s} = (s_0, s_1, s_2, \cdots); s_k \in \mathbb{I}\backslash\{0, n\}$ for every $k \geq 0\}$ be the space of one-sided sequences of the symbols $\pm 1, \dots, \pm(n-1)$. For $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \Sigma$, the shift map $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(\mathbf{s}) = (s_1, s_2, \cdots)$. We denote $(s_0, s_1, s_2, \cdots)$ by $(\overline{s_0, \cdots, s_{p-1}})$ if $s_{k+p} = s_k$ for $k \geq 0$.

It's obvious that $\mathbf{s}_\lambda(f_\lambda(z)) = \sigma(\mathbf{s}_\lambda(z))$ for $z \in \Lambda_\lambda$.

Lemma 5.2.3. *The set Λ_λ is a Cantor set and the itinerary map $\mathbf{s}_\lambda : \Lambda_\lambda \to \Sigma$ is bijective.*

Proof. First note that for any $\lambda \in \mathcal{H}$, $\mathbf{I}$ is a compact subset of Υ_λ. With respect to the hyperbolic metric of Υ_λ and by Schwarz Lemma, there is a number $\delta \in (0,1)$ such that for any $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \Sigma$ and any $m \geq 0$,

$$\text{Hyper.diam}\Big(\bigcap_{0\leq k\leq m} f_\lambda^{-k}(\mathbf{I}_{s_k})\Big) \leq \text{Hyper.diam}(\mathbf{I}) \cdot \delta^m.$$

Thus $\bigcap_{k\geq 0} f_\lambda^{-k}(\mathbf{I}_{s_k})$ consists of a single point, say $z_{\mathbf{s}}$. This implies that Λ_λ is a Cantor set and the map $\mathbf{s}_\lambda : \Lambda_\lambda \to \Sigma$ defined by $\mathbf{s}_\lambda(z_{\mathbf{s}}) = \mathbf{s}$ is bijective. $\square$

5.3 Cut Rays in the Dynamical Plane

In this section, we will construct the 'cut rays', a kind of Jordan curves that cut the Julia set into two different parts. The construction is due to Devaney [D2]. We give some more properties which will be used in our paper.

We first construct a Cantor set of angles on the unit circle which is used to generate 'cut rays' in [D2]. These angles can be considered as a combinatorial invariant when the parameter λ ranges over $\mathcal{H}$.

To begin with, we identify the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ with $(0,1]$. We say three angles satisfy $t_1 \leq t_2 \leq t_3$ on $\mathbb{S}$ if t_1, t_2, t_3 are in the counterclockwise order.

5.3.1 A Cantor set on the unit circle

In the following, we construct a subset Θ of (0,1]. The set Θ is a Cantor set and is used to generate 'cut rays' in the next section.

First, define a map $\tau : (0,1] \to (0,1]$ by $\tau(\theta) = n\theta \bmod 1$. Let $\Theta_k = (\frac{k}{2n}, \frac{k+1}{2n}]$ for $0 \leq k \leq n$ and $\Theta_{-k} = \Theta_k + \frac{1}{2}$ for $1 \leq k \leq n-1$. Obviously, $(0,1] = \bigcup_{k\in\mathbb{I}} \Theta_k$.

Define a map $\chi : \mathbb{I} \to \mathbb{N}$ by

$$\chi(k) = \begin{cases} k, & \text{if } 0 \le k \le n, \\ n-k, & \text{if } -(n-1) \le k \le -1. \end{cases}$$

For $k \in \mathbb{I}$, we have

$$\tau(\Theta_k) \supset \begin{cases} \bigcup_{j=1}^{n-1} \Theta_j, & \text{if } \chi(k) \text{ is even}, \\ \bigcup_{j=1}^{n-1} \Theta_{-j}, & \text{if } \chi(k) \text{ is odd}. \end{cases}$$

For $\theta \in (0,1]$, suppose $\tau^k(\theta) \in \Theta_{s_k}$ for $k \ge 0$, define the itinerary $\mathbf{s}(\theta)$ of θ by $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$.

Let Θ be the set of all angles $\theta \in (0,1]$ whose orbit remains in $\mathcal{E} = \bigcup_{k=1}^{n-1}(\Theta_k \cup \Theta_{-k})$ under all iterations of τ. The set Θ can be written as $\Theta = \bigcap_{k\ge 0} \tau^{-k}(\mathcal{E}) = \bigcap_{k \ge 0} \tau^{-k}(\overline{\mathcal{E}})$. One can easily verify that Θ is a Cantor set.

The image of Θ under the itinerary map is denoted by $\Sigma_0 = \{\mathbf{s}(\theta); \theta \in \Theta\}$. One can easily verify that Σ_0 is a subspace of Σ that consists of all elements $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \Sigma$ such that for $k \ge 0$, if $\chi(s_k)$ is even, then $s_{k+1} \in \{1, \cdots, n-1\}$; if $\chi(s_k)$ is odd, then $s_{k+1} \in \{-1, \cdots, -(n-1)\}$.

The itinerary map $\mathbf{s} : \Theta \to \Sigma_0$ is bijective since for any $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \Sigma_0$, the intersection $\bigcap_{k \ge 0} \tau^{-k}(\Theta_{s_k})$ consists of a single point. In the following, we first construct an inverse map for $\mathbf{s}$ (Lemma 5.3.1).

Let $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \Sigma$, define a map $\kappa : \Sigma \to (0,1]$ by

$$\kappa(\mathbf{s}) = \frac{1}{2}\left(\frac{\chi(s_0)}{n} + \sum_{k \ge 1} \frac{|s_k|}{n^{k+1}}\right).$$

Lemma 5.3.1. $\kappa(\Sigma) = \Theta$ *and* $\kappa(\mathbf{s}(\theta)) = \theta$ *for all* $\theta \in \Theta$.

Proof. First, we show $\kappa(\mathbf{s}(\theta)) = \theta$ for $\theta \in \Theta$. Let $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$ and $\hat{\theta} = \kappa(\mathbf{s}(\theta))$. It suffices to show $\mathbf{s}(\hat{\theta}) = \mathbf{s}(\theta)$ since $\mathbf{s} : \Theta \to \Sigma_0$ is bijective.

It follows that $\hat{\theta} \in \Theta_{s_0}$ since

$$\frac{\chi(s_0)}{2n} < \hat{\theta} \le \frac{1}{2}\left(\frac{\chi(s_0)}{n} + \sum_{k \ge 1} \frac{n-1}{n^{k+1}}\right) = \frac{\chi(s_0)}{2n} + \frac{1}{2n}.$$

For $k \ge 1$,

$$\tau^k(\hat{\theta}) = \begin{cases} \frac{1}{2}(\chi(s_0) + |s_1| + \cdots + |s_{k-1}|) + \frac{1}{2}\sum_{j \ge k} \frac{|s_j|}{n^{j-k+1}}, & \text{if } n \text{ is odd}, \\ \frac{|s_{k-1}|}{2} + \frac{1}{2}\sum_{j \ge k} \frac{|s_j|}{n^{j-k+1}}, & \text{if } n \text{ is even}. \end{cases}$$

Since $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots) \in \Sigma_0$, we have for $j \ge 1$,

$$\frac{|s_j|}{2} = \begin{cases} \frac{1}{2}(\chi(s_j) - \chi(s_{j-1})) \bmod 1, & \text{if } n \text{ is odd}, \\ \frac{1}{2}\chi(s_j) \bmod 1, & \text{if } n \text{ is even}. \end{cases}$$

and
$$\frac{\chi(s_{j-1})}{2}+\frac{|s_j|}{2n}=\frac{\chi(s_j)}{2n} \bmod 1.$$

Thus we have

$$\tau^k(\hat{\theta})=\frac{\chi(s_{k-1})}{2}+\frac{1}{2}\sum_{j\geq k}\frac{|s_j|}{n^{j-k+1}}=\frac{\chi(s_k)}{2n}+\frac{1}{2}\sum_{j\geq k+1}\frac{|s_j|}{n^{j-k+1}}.$$

This means $\tau^k(\hat{\theta})\in\Theta_{s_k}$ for $k\geq 1$. So θ and $\hat{\theta}$ have the same itinerary.

In the following, we show $\kappa(\Sigma)=\Theta$. First, by the previous argument, $\Theta=\kappa(\Sigma_0)\subset\kappa(\Sigma)$. Conversely, for any $\mathbf{s}=(s_0,s_1,s_2,\cdots)\in\Sigma$, there is a unique sequence of symbols $\varepsilon_1,\varepsilon_2,\cdots\in\{\pm1\}$, such that $\mathbf{s}^*=(s_0,\varepsilon_1 s_1,\varepsilon_2 s_2,\cdots)\in\Sigma_0$. Thus $\kappa(\mathbf{s})=\kappa(\mathbf{s}^*)\in\Theta$. □

Remark 5.3.1. *For any* $\mathbf{s}=(s_0,s_1,s_2,\cdots)\in\Sigma$, *one can verify that*
$$\kappa^{-1}(\kappa(\mathbf{s}))=\{(s_0,\pm s_1,\pm s_2,\cdots)\}.$$

Lemma 5.3.2. *The set* Θ *satisfies:*
1. $\tau(\Theta)=\Theta$.
2. $\Theta+\frac{1}{2}=\Theta$.
3. *Periodic angles are dense in* Θ.

Proof. 1. It is obvious that $\tau(\Theta)\subset\Theta$. τ is surjective since $\tau^{-1}(\theta)\cap\mathcal{E}\neq\emptyset$ for all $\theta\in\Theta$.

2. First note that $\mathcal{E}+\frac{1}{2}=\mathcal{E} \bmod 1$. For $k\geq 1$, since $\tau^k(\theta+\frac{1}{2})=\tau^k(\theta)$ when n is even and $\tau^k(\theta+\frac{1}{2})=\tau^k(\theta)+\frac{1}{2}$ when n is odd, we have $\tau^k(\theta+\frac{1}{2})\in\mathcal{E}$ if and only if $\tau^k(\theta)\in\mathcal{E}$. Thus $\theta\in\Theta$ if and only if $\theta+\frac{1}{2}\in\Theta$.

3. Let $\theta\in\Theta$ with itinerary $\mathbf{s}(\theta)=(s_0,s_1,s_2,\cdots)$. For any $k\geq 1$, either $(\overline{s_0,\cdots,s_k})\in\Sigma_0$, or there is a symbol $s_{k+1}^*\in\{\pm1,\cdots,\pm(n-1)\}$ such that $(\overline{s_0,\cdots,s_k,s_{k+1}^*})\in\Sigma_0$. If $(\overline{s_0,\cdots,s_k})\in\Sigma_0$, let $\theta_k=\kappa((\overline{s_0,\cdots,s_k}))$. Else, let $\theta_k=\kappa((\overline{s_0,\cdots,s_k,s_{k+1}^*}))$. It's obvious that θ_k is periodic. By Lemma 5.3.1, $\theta_k\in\Theta$ and
$$|\theta-\theta_k|\leq C(n)n^{-k}(\ \to 0\ \text{ as }\ k\to\infty),$$
where $C(n)$ is a constant, depending only on n. This implies that periodic angles are dense in Θ. □

Remark 5.3.2. *The Hausdorff dimension of* Θ *is* $\frac{\log(n-1)}{\log n}$.

For $\lambda\in\mathcal{H}$ and $k\in\mathbb{I}$, let $\Theta_k^\lambda=\Theta_k+\frac{\arg c_0(\lambda)}{2\pi}=\Theta_k+\frac{\arg\lambda}{4n\pi} \bmod 1$. Recall that for $\lambda\in\mathcal{H}$, $\arg\lambda\in(0,\frac{2\pi}{n-1})$. It's easy to check that

$$\tau(\Theta_k^\lambda)\supset\begin{cases}\bigcup_{j=1}^{n-1}\Theta_j^\lambda, & \text{if } \chi(k) \text{ is even},\\ \bigcup_{j=1}^{n-1}\Theta_{-j}^\lambda, & \text{if } \chi(k) \text{ is odd}.\end{cases}$$

As before, we define Θ^λ as the set of all angles in $(0, 1]$ whose orbits remain in $\mathcal{E}^\lambda = \bigcup_{k=1}^{n-1}(\Theta_k^\lambda \cup \Theta_{-k}^\lambda)$ under all iterations of τ. Thus $\Theta^\lambda = \bigcap_{k\geq 0} \tau^{-k}(\mathcal{E}^\lambda)$. For $\theta \in (0, 1]$, suppose $\tau^k(\theta) \in \Theta_{s_k}^\lambda$ for $k \geq 0$, define the itinerary of θ by $\mathbf{s}^\lambda(\theta) = (s_0, s_1, s_2, \cdots)$. It's easy to show that the itinerary map $\mathbf{s}^\lambda : \Theta^\lambda \to \Sigma_0$ is bijective.

Lemma 5.3.3. (Combinatorial invariant) *$\Theta^\lambda = \Theta$ and for any $\theta \in \Theta$, $\mathbf{s}^\lambda(\theta) = \mathbf{s}(\theta)$.*

Proof. It suffices to show that if $\mathbf{s}^\lambda(\alpha) = \mathbf{s}(\beta)$ for $\alpha \in \Theta^\lambda$ and $\beta \in \Theta$, then $\alpha = \beta$.

First note that $\Theta_k^\lambda \cap \Theta_k \neq \emptyset$ for any $k \in \mathbb{I}$. Suppose $\mathbf{s}^\lambda(\alpha) = \mathbf{s}(\beta) = (s_0, s_1, s_2, \cdots)$, and let $A_m = \bigcap_{0\leq k\leq m} \tau^{-k}(\Theta_{s_k}^\lambda \cap \Theta_{s_k})$ for $m \geq 0$. By induction argument, we see that A_m is a connected interval of the form $(a_m, b_m]$ with $a_{m+1} > a_m, b_{m+1} < b_m$ and $n(b_{m+1} - a_{m+1}) = b_m - a_m$ for $m \geq 0$. Thus $A_{m+1} \subset \overline{A_{m+1}} \subset A_m$ and $\bigcap_{k\geq 0} A_m = \bigcap_{k\geq 0} \overline{A_m}$ consists of a single point, say θ. On the other hand,

$$\{\theta\} = \bigcap_{k\geq 0} A_m = \Big(\bigcap_{k\geq 0} \tau^{-k}(\Theta_{s_k}^\lambda)\Big) \bigcap \Big(\bigcap_{k\geq 0} \tau^{-k}(\Theta_{s_k})\Big) = \{\alpha\} \cap \{\beta\}.$$

Thus we have $\alpha = \beta = \theta$. $\square$

5.3.2 Cut rays

In this section, for any $\lambda \in \mathcal{H}$ and any $\theta \in \Theta$, we will construct a Jordan curve, say Ω_λ^θ, that cuts the dynamical plane of f_λ into two parts. The curve will meet the Julia set $J(f_\lambda)$ in a Cantor set of points. This kind of Jordan curve Ω_λ^θ will be called a cut ray of angle θ. In the following, we devote to construct such rays, but with a slightly different presentation from Devaney's in [D2].

Recall that the itinerary map $\mathbf{s}_\lambda : \Lambda_\lambda \to \Sigma$ is bijective from a Cantor set onto a symbolic space. We first extend the definition of $\mathbf{s}_\lambda$ to a larger set. Let $\mathbf{E}_\lambda = \bigcap_{k\geq 0} f_\lambda^{-k}(\bigcup_{j\in\mathbb{I}\setminus\{0,n\}} S_j)$ be the set of all points in the dynamical plane whose orbits remain in $\bigcup_{j\in\mathbb{I}\setminus\{0,n\}} S_j$ under all iterations of f_λ. The map $\mathbf{s}_\lambda : \mathbf{\Lambda}_\lambda \to \Sigma$ can be extended to $\mathbf{s}_\lambda : \mathbf{E}_\lambda \to \Sigma$ as follows: For any $z \in \mathbf{E}_\lambda$, suppose $f_\lambda^k(z) \in S_{s_k}$ for $k \geq 0$, then the itinerary of z is defined by $\mathbf{s}_\lambda(z) = (s_0, s_1, s_2, \cdots)$. One can see that the map $\mathbf{s}_\lambda : \mathbf{E}_\lambda \to \Sigma$ is not well-defined for the points that are eventually mapped to ∞. For example, under this definition, the itinerary of 0 or ∞ can be defined as any element of Σ. Even though there is some confusion of definition on the set $\mathbf{E}_\lambda \cap \cup_{k\geq 0} f_\lambda^{-k}(\infty)$, it's allowed to define the itinerary of $p \in \mathbf{E}_\lambda \cap \cup_{k\geq 0} f_\lambda^{-k}(\infty)$ as any element $(s_0, s_1, s_2, \cdots) \in \Sigma$ provided that $f_\lambda^k(p) \in S_{s_k}$ for any $k \geq 0$.

Given an angle $\theta \in \Theta$ with itinerary $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$, it's easy to check that when n is odd, $\mathbf{s}(\theta + 1/2) = (-s_0, -s_1, -s_2, \cdots) = -\mathbf{s}(\theta)$; when n is even, $\mathbf{s}(\theta + 1/2) = (-s_0, s_1, s_2, \cdots)$. We consider the set of all points in $\mathbf{E}_\lambda$ whose itinerary is of the form $(s_0, \pm s_1, \pm s_2, \cdots)$. This set is denoted by ω_λ^θ. By definition, ω_λ^θ contains 0 and ∞. Moreover,

$$\omega_\lambda^\theta = \{z \in \mathbf{E}_\lambda; \mathbf{s}_\lambda(z) = (s_0, \pm s_1, \pm s_2, \cdots)\} = \{z \in \mathbf{E}_\lambda; \kappa(\mathbf{s}_\lambda(z)) = \theta\}.$$

According to Devaney, the set ω_λ^θ is called a full ray of angle θ. Let $\Omega_\lambda^\theta = \omega_\lambda^\theta \cup \omega_\lambda^{\theta+1/2}$, we call the set Ω_λ^θ a cut ray of angle θ (or $\theta + 1/2$). It's obvious that

$$\Omega_\lambda^\theta = \{z \in \mathbf{E}_\lambda; \mathbf{s}_\lambda(z) = (\pm s_0, \pm s_1, \pm s_2, \cdots)\} = \bigcap_{k \geq 0} f_\lambda^{-k}(S_{s_k} \cup S_{-s_k}).$$

We first give an intuitionistic description of the cut ray Ω_λ^θ. For $m \geq 0$, let

$$\Omega_{\lambda,m}^\theta = \bigcap_{0 \leq k \leq m} f_\lambda^{-k}(S_{s_k} \cup S_{-s_k}).$$

Note that the set $\Omega_{\lambda,0}^\theta$ is a union of two closed sectors S_{s_0} and S_{-s_0}. $\Omega_{\lambda,1}^\theta$ is a string of four closed disks that lie inside $\Omega_{\lambda,0}^\theta$. Inductively, $\Omega_{\lambda,m}^\theta$ is a string of 2^{m+1}closed disks that are contained in $\Omega_{\lambda,m-1}^\theta$, and each of these disks meets exactly two others at the preimages of ∞. Hence $\Omega_{\lambda,m}^\theta$ is a connected and compact set. One can show that $\Omega_{\lambda,m}^\theta$ converges to $\Omega_\lambda^\theta = \cap_{k \geq 0} \Omega_{\lambda,k}^\theta$ in Hausdorff topology as $m \to \infty$ (This is because a shrinking sequence of compact sets always converges in Hausdorff topology). Roughly speaking, the set $\Omega_{\lambda,m}^\theta$ becomes thinner when m becomes larger and $\Omega_{\lambda,m}^\theta$ finally shrinks to Ω_λ^θ. So it is believed that Ω_λ^θ is a Jordan curve(A rigorous proof of this fact will be given in Proposition 5.3.3).

By construction, the cut ray satisfies:

- $\Omega_\lambda^\theta = -\Omega_\lambda^\theta$.
- $\Omega_\lambda^\theta \setminus \{0, \infty\}$ is contained in the interior of $S_{s_0} \cup S_{-s_0}$.
- $f_\lambda : \Omega_\lambda^\theta \to \Omega_\lambda^{\tau(\theta)}$ is a two-to-one map.

Lemma 5.3.4. *Let $\lambda \in \mathcal{H}$, then there is a constant $v > 0$ such that for any $\theta \in \Theta$,*

$$\overline{R_\lambda(\theta)} \cap \mathbf{U}(v) = \{z \in \mathbf{E}_\lambda \cap \mathbf{U}(v); \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\}.$$

Proof. The proof is based on $\phi_\lambda'(\infty) = 1$ and Lemma 5.3.3. We omit the detail since it's easy. $\square$

Proposition 5.3.1. *For any $\lambda \in \mathcal{H}$ and any $\theta \in \Theta$, the external ray $R_\lambda(\theta)$ lands at ∂B_λ and $\overline{R_\lambda(\theta)} = \{z \in \mathbf{E}_\lambda; \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\}$.*

Proof. Suppose $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$. Let $\ell_\lambda(v,\theta) = \{z \in R_\lambda(\theta); v \leq G_\lambda(z) \leq nv\}$ be the portion of $R_\lambda(\theta)$ that lies between two equipotential curves $\mathbf{e}(B_\lambda, v)$ and $\mathbf{e}(B_\lambda, nv)$. By Lemma 5.3.4, we may assume v large enough such that for any $\beta \in \Theta$, $\overline{R_\lambda(\beta)} \cap \mathbf{U}(v) = \{z \in \mathbf{E}_\lambda \cap \mathbf{U}(v); \mathbf{s}_\lambda(z) = \mathbf{s}(\beta)\}$. By pulling back $\ell_\lambda(v, \tau(\theta))$ by f_λ^{-1} to S_{s_0}, we can extend the portion of $\overline{R_\lambda(\theta)}$, say $\gamma_0 = \overline{R_\lambda(\theta)} \cap \mathbf{U}(v)$, to a longer one $\gamma_1 = h_{s_0}(\ell_\lambda(v, \tau(\theta))) \cup \gamma_0$. Obviously, $\gamma_1 \subset S_{s_0} \cap \overline{R_\lambda(\theta)}$. Continuing inductively, suppose we have already constructed a portion γ_k of $\overline{R_\lambda(\theta)}$, then we add a segment $h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta)))$ to γ_k and get $\gamma_{k+1} = \gamma_k \cup h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta)))$. By construction, one can check that $h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta))) \subset S_{s_0} \cap \overline{R_\lambda(\theta)}$ and for any $z \in h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta)))$, $\mathbf{s}_\lambda(z) = (s_0, s_1, s_2, \cdots)$. It turns out that $R_\lambda(\theta) \subset \{z \in \mathbf{E}_\lambda; \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\}$ and

$$R_\lambda(\theta) \setminus \gamma_0 = \bigcup_{k \geq 0} h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta))).$$

In the following, we show the external ray $R_\lambda(\theta)$ lands at ∂B_λ. Since $h_k : \Upsilon_\lambda \to \Upsilon_\lambda$ contracts the hyperbolic metric ρ_λ of Υ_λ for any $k \in \mathbb{I} \setminus \{0, n\}$, there is a constant $\delta \in (0,1)$ such that

$$\rho_\lambda(h_k(x), h_k(y)) \leq \delta \rho_\lambda(x,y), \quad \forall x, y \in \overline{\mathbf{V}(nv)} \cap \big(\cup_{j \in \mathbb{I} \setminus \{0,n\}} S_j\big), \forall k \in \mathbb{I} \setminus \{0, n\}.$$

Thus with respect to the hyperbolic metric of Υ_λ, we have

$$\text{Hyper.length}\big(h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta)))\big) = \mathcal{O}(\delta^k).$$

This implies that $R_\lambda(\theta) \setminus \gamma_0$ has finite hyperbolic length in Υ_λ, thus the external ray $R_\lambda(\theta)$ lands at ∂B_λ. Let $p_\lambda(\theta)$ be the landing point. It's easy to check that $\mathbf{s}_\lambda(p_\lambda(\theta)) = \mathbf{s}(\theta)$ and $p_\lambda(\theta) \in \partial B_\lambda \cap \Lambda_\lambda$.

To finish, we show $\overline{R_\lambda(\theta)} \supset \{z \in \mathbf{E}_\lambda; \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\}$. For any $x \in \{z \in \mathbf{E}_\lambda; \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\} \setminus \{\infty\}$, we consider the orbit of x.

If the orbit of x remains bounded, then by Lemma 5.2.3, we have $x \in \Lambda_\lambda$. Since $\mathbf{s}_\lambda|_{\Lambda_\lambda} : \Lambda_\lambda \to \Sigma$ is bijective and $\mathbf{s}_\lambda(x) = \mathbf{s}_\lambda(p_\lambda(\theta)) = \mathbf{s}(\theta)$, we conclude $x = p_\lambda(\theta) \in \overline{R_\lambda(\theta)}$.

If the orbit of x tends to ∞, then by Lemma 5.3.4, there is an integer $M \geq 1$ such that $f_\lambda^M(z) \in R_\lambda(\tau^M(\theta))$. Note that for any $k \geq 1$, $h_{s_{k-1}}(R_\lambda(\tau^k(\theta))) = R_\lambda(\tau^{k-1}(\theta))$ and $h_{s_{k-1}}(f_\lambda^k(z)) = f_\lambda^{k-1}(z)$, we have $z \in R_\lambda(\theta)$. Thus $\overline{R_\lambda(\theta)} \supset \{z \in \mathbf{E}_\lambda; \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\}$. □

Proposition 5.3.2. *For any $\lambda \in \mathcal{H}$ and any $\theta \in \Theta$ with itinerary $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$, the cut ray Ω_λ^θ satisfies*

1. Ω_λ^θ meets the Julia set $J(f_\lambda)$ in a Cantor set of points. More precisely, $\Omega_\lambda^\theta \cap J(f_\lambda) = (\kappa \circ \mathbf{s}_\lambda|_{\Lambda_\lambda})^{-1}(\{\theta, \theta + \frac{1}{2}\})$.

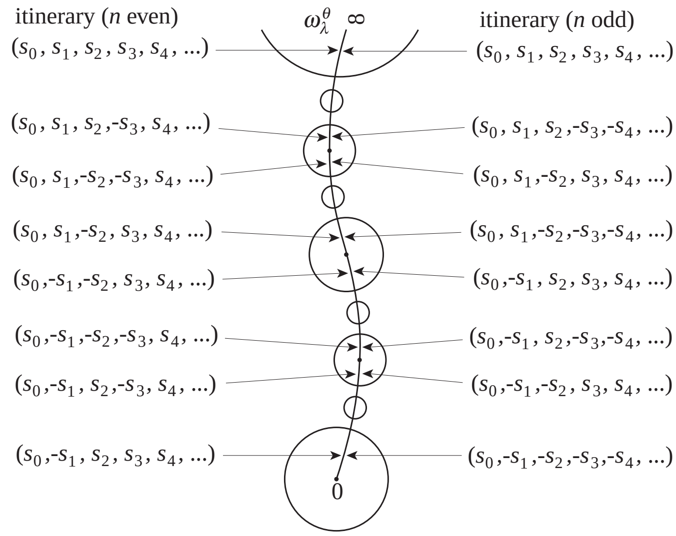

Figure 5.3: Combinatorial structure of a full ray ω_λ^θ with $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$.

2. Ω_λ^θ meets the Fatou set $F(f_\lambda)$ in a countable union of external rays and radial rays, together with the preimages of ∞ that lie in the closure of these rays. More precisely,

$$\Omega_\lambda^\theta \cap B_\lambda = R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2}) \cup \{\infty\}$$

$$\Omega_\lambda^\theta \cap T_\lambda = \begin{cases} h_{-s_0}(R_\lambda(\tau(\theta))) \cup h_{s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup \{0\}, & \text{if } n \text{ is odd,} \\ h_{s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup h_{-s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup \{0\}, & \text{if } n \text{ is even.} \end{cases}$$

For any $U \in \mathcal{P} \setminus \{B_\lambda, T_\lambda\}$ with $U \cap \Omega_\lambda^\theta \neq \emptyset$, U is of the form $h_{b_0} \circ \cdots \circ h_{b_{k-1}}(T_\lambda)$, where $k \geq 1$ and $(b_0, \cdots, b_{k-1}) \in \{(\pm s_0, \cdots, \pm s_{k-1})\}$. Moreover

$$\begin{aligned} &\Omega_\lambda^\theta \cap U = h_{b_0} \circ \cdots \circ h_{b_{k-1}}(\Omega_\lambda^{\tau^k(\theta)} \cap T_\lambda) \\ = &\begin{cases} h_{b_0} \circ \cdots \circ h_{b_{k-1}}\Big(h_{-s_k}(R_\lambda(\tau^{k+1}(\theta))) \cup h_{s_k}(R_\lambda(\tau^{k+1}(\theta) + \frac{1}{2})) \cup \{0\}\Big), & \text{if } n \text{ is odd,} \\ h_{b_0} \circ \cdots \circ h_{b_{k-1}}\Big(h_{-s_k}(R_\lambda(\tau^{k+1}(\theta) + \frac{1}{2})) \cup h_{s_k}(R_\lambda(\tau^{k+1}(\theta) + \frac{1}{2})) \cup \{0\}\Big), & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

See Figure 5.3 for the combinatorial structure of a part of a cut ray.

Proof. 1. For $z \in \Omega_\lambda^\theta$, first note that $z \in \Omega_\lambda^\theta \cap J(f_\lambda)$ if and only if the orbit of z remains bounded, if and only if $z \in \Lambda_\lambda$ and $\mathbf{s}_\lambda(z) \in \{(\pm s_0, \pm s_1, \pm s_2, \cdots)\} = \kappa^{-1}(\{\theta, \theta + \frac{1}{2}\})$. Thus we have $\Omega_\lambda^\theta \cap J(f_\lambda) = (\kappa \circ \mathbf{s}_\lambda|_{\Lambda_\lambda})^{-1}(\{\theta, \theta + \frac{1}{2}\})$.

2. Let U be a Fatou component such that $U \cap \Omega_\lambda^\theta \neq \emptyset$. Then by 1, U is eventually mapped onto B_λ.

Case 1: $U = B_\lambda$**.** By Proposition 5.3.1, $\Omega_\lambda^\theta \cap B_\lambda \supset R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2}) \cup \{\infty\}$. On the other hand, for any $z \in (\Omega_\lambda^\theta \setminus B_\lambda) \setminus \{\infty\}$, there is a integer $M \geq 1$ such that $f_\lambda^M(z) \in \mathbf{U}(v)$, where v is a positive constant chosen by Lemma 5.3.4. Since $\mathbf{s}_\lambda(f_\lambda^M(z)) \in \{(\pm s_M, \pm s_{M+1}, \pm s_{M+2}, \cdots)\}$, we conclude that the itinerary of $f_\lambda^M(z)$ must be identical as an itinerary of some angle $\beta \in \Theta$. Thus

$$\mathbf{s}_\lambda(f_\lambda^M(z)) = \begin{cases} (s_M, s_{M+1}, \cdots) \text{ or } (-s_M, -s_{M+1}, \cdots), & \text{if } n \text{ is odd,} \\ (s_M, s_{M+1}, \cdots), & \text{if } n \text{ is even.} \end{cases}$$

Case 1.1: n **is odd.** By Proposition 5.3.1, $f_\lambda^M(z) \in R_\lambda(\tau^M(\theta)) \cup R_\lambda(\tau^M(\theta) + \frac{1}{2})$. Note that $f_\lambda^{-1}(R_\lambda(\tau^M(\theta))) \cap (S_{s_{M-1}} \cup S_{-s_{M-1}}) \cap B_\lambda = R_\lambda(\tau^{M-1}(\theta))$, $f_\lambda^{-1}(R_\lambda(\tau^M(\theta)+\frac{1}{2})) \cap (S_{s_{M-1}} \cup S_{-s_{M-1}}) \cap B_\lambda = R_\lambda(\tau^{M-1}(\theta)+\frac{1}{2})$. We conclude $f_\lambda^{M-1}(z) \in R_\lambda(\tau^{M-1}(\theta)) \cup R_\lambda(\tau^{M-1}(\theta) + \frac{1}{2})$. It turns out that $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$ by induction. So in this case, $\Omega_\lambda^\theta \cap B_\lambda = R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2}) \cup \{\infty\}$.

Case 1.2: n **is even.** By Proposition 5.3.1, $f_\lambda^M(z) \in R_\lambda(\tau^M(\theta))$. Since $f_\lambda^{-1}(R_\lambda(\tau^M(\theta))) \cap (S_{s_{M-1}} \cup S_{-s_{M-1}}) \cap B_\lambda = R_\lambda(\tau^{M-1}(\theta)) \cup R_\lambda(\tau^{M-1}(\theta) + \frac{1}{2})$, we have $f_\lambda^{M-1}(z) \in R_\lambda(\tau^{M-1}(\theta)) \cup R_\lambda(\tau^{M-1}(\theta) + \frac{1}{2})$. If $M = 1$, then $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$ and the proof is done. If $M > 1$, then we claim $f_\lambda^{M-1}(z) \in R_\lambda(\tau^{M-1}(\theta))$. This is because $f_\lambda^{-1}(R_\lambda(\tau^{M-1}(\theta)+\frac{1}{2})) \cap (S_{s_{M-2}} \cup S_{-s_{M-2}}) \cap B_\lambda = \emptyset$. Again by induction, we have $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$ in this case.

Case 2: $U = T_\lambda$**.** In this case, if n is odd, then $f_\lambda(\Omega_\lambda^\theta \cap T_\lambda \cap S_{s_0}) = \Omega_\lambda^{\tau(\theta)} \cap B_\lambda \cap S_{-s_1} = R_\lambda(\tau(\theta)+\frac{1}{2}) \cup \{\infty\}$ and $f_\lambda(\Omega_\lambda^\theta \cap T_\lambda \cap S_{-s_0}) = \Omega_\lambda^{\tau(\theta)} \cap B_\lambda \cap S_{s_1} = R_\lambda(\tau(\theta)) \cup \{\infty\}$. So $\Omega_\lambda^\theta \cap T_\lambda = h_{-s_0}(R_\lambda(\tau(\theta))) \cup h_{s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup \{0\}$; if n is even, then $f_\lambda(\Omega_\lambda^\theta \cap T_\lambda \cap S_{s_0}) = f_\lambda(\Omega_\lambda^\theta \cap T_\lambda \cap S_{-s_0}) = \Omega_\lambda^{\tau(\theta)} \cap B_\lambda \cap S_{-s_1} = R_\lambda(\tau(\theta)+\frac{1}{2}) \cup \{\infty\}$. So $\Omega_\lambda^\theta \cap T_\lambda = h_{s_0}(R_\lambda(\tau(\theta))+\frac{1}{2}) \cup h_{-s_0}(R_\lambda(\tau(\theta)+\frac{1}{2})) \cup \{0\}$.

Case 3: $U \in \mathcal{P} \setminus \{B_\lambda, T_\lambda\}$**.** In this case, there is a smallest integer $k \geq 1$ such that $f_\lambda^k(U) = T_\lambda$. Since $f_\lambda^k : U \to T_\lambda$ is a conformal map and for any $0 \leq j \leq k-1$, $f_\lambda^j(U)$ lies inside some sector S_{k_j}, we conclude U must take the form $h_{b_0} \circ \cdots \circ h_{b_{k-1}}(T_\lambda)$ for some $(b_0, \cdots, b_{k-1}) \in \{(\pm s_0, \cdots, \pm s_{k-1})\}$. By pulling back $f_\lambda^k(U \cap \Omega_\lambda^\theta) = \Omega_\lambda^{\tau^k(\theta)} \cap T_\lambda$ via f_λ^k, we have $\Omega_\lambda^\theta \cap U = h_{b_0} \circ \cdots \circ h_{b_{k-1}}(\Omega_\lambda^{\tau^k(\theta)} \cap T_\lambda)$. The conclusion follows by case 2. □

Proposition 5.3.3. *For any $\lambda \in \mathcal{H}$ and any $\theta \in \Theta$, the cut ray Ω_λ^θ is a Jordan curve.*

Proof. Suppose $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$. For $k \geq 0$, define

$$\widehat{\Omega}_{\lambda,0}^{\tau^k(\theta)} = \Omega_\lambda^{\tau^k(\theta)} \cup \mathbf{I}_{s_k} \cup \mathbf{I}_{-s_k}, \quad \widehat{\Omega}_{\lambda,k}^\theta = \bigcap_{0 \leq j \leq k} f_\lambda^{-j}(\widehat{\Omega}_{\lambda,0}^{\tau^j(\theta)}).$$

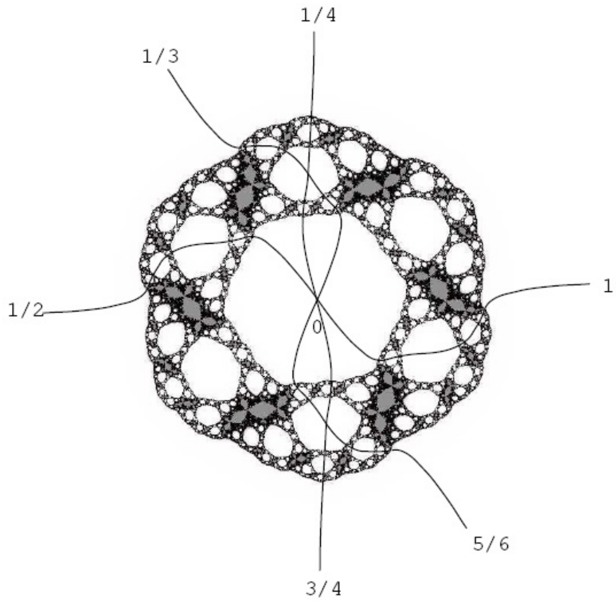

Figure 5.4: Cut rays with angles $1/4, 1/3, 1/2$ when $n = 3$.

The set $\widehat{\Omega}_{\lambda,k}^{\theta}$ is connected and compact and it contains $\Omega_{\lambda}^{\theta}$. It's easy to check that $\widehat{\Omega}_{\lambda,k}^{\theta} \supset \widehat{\Omega}_{\lambda,k+1}^{\theta}$ and $\bigcap_{k\geq 0} \widehat{\Omega}_{\lambda,k}^{\theta} = \Omega_{\lambda}^{\theta}$. Let D_k^+ be the component of $\overline{\mathbb{C}} \setminus \widehat{\Omega}_{\lambda,k}^{\theta}$ that contains v_{λ}^+ and D_k^- be the component of $\overline{\mathbb{C}} \setminus \widehat{\Omega}_{\lambda,k}^{\theta}$ that contains v_{λ}^-. Let $D_{\infty}^+ = \bigcup_{k\geq 0} D_k^+$ and $D_{\infty}^- = \bigcup_{k\geq 0} D_k^-$, then $D_{\infty}^+ \cup D_{\infty}^- \cup \Omega_{\lambda}^{\theta} = \overline{\mathbb{C}}$.

We first construct a Cantor set on $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. Let $E_1 = (5/24, 13/24), E_2 = (17/24, 25/24)$ be two open intervals on $\mathbb{S}$ and ζ be the map $t \mapsto 3t \bmod \mathbb{Z}$. By definition, $\zeta(E_i) \supset \overline{E_1 \cup E_2}$. Let $T_k = \bigcap_{0\leq j\leq k} \zeta^{-j}(E_1 \cup E_2)$. Then $T_k \supset T_{k+1}$ and T_k has 2^{k+1} components. The intersection $\bigcap_{k\geq 0} T_k$ is denoted by T_{∞}. Since $T_{\infty} = \bigcap_{k\geq 0} \zeta^{-k}(E_1 \cup E_2) = \bigcap_{k\geq 0} \zeta^{-k}(\overline{E_1} \cup \overline{E_2})$, we conclude that T_{∞} is a Cantor set.

Now, we define two sequences of Jordan curves $\{\gamma_k^+ : \mathbb{S} \to \partial D_k^+\}$, $\{\gamma_k^- : \mathbb{S} \to \partial D_k^-\}$ in the following way: for $k \geq 0$,

1. $\gamma_{k+1}^+|_{\mathbb{S}\setminus T_k} = \gamma_k^+|_{\mathbb{S}\setminus T_k} = \gamma_k^-|_{\mathbb{S}\setminus T_k} = \gamma_{k+1}^-|_{\mathbb{S}\setminus T_k}$.
2. $\gamma_k^+(\mathbb{S} \setminus T_k) = \Omega_{\lambda}^{\theta} \cap \partial D_k^+ = \Omega_{\lambda}^{\theta} \cap \partial D_k^- = \gamma_k^-(\mathbb{S} \setminus T_k)$.
3. $\gamma_k^+(T_k) = \partial D_k^+ \setminus \Omega_{\lambda}^{\theta}$, $\gamma_k^-(T_k) = \partial D_k^- \setminus \Omega_{\lambda}^{\theta}$.

In the following, we show that each sequence of maps $\{\gamma_k^+ : \mathbb{S} \to \partial D_k^+\}$, $\{\gamma_k^- : \mathbb{S} \to \partial D_k^-\}$ converges in the spherical metric. By construction, $\gamma_{k+1}^+|_{\mathbb{S}\setminus T_k} = \gamma_k^+|_{\mathbb{S}\setminus T_k}$ and for any component W of T_k, $\gamma_{k+1}^+(W)$ and $\gamma_k^+(W)$ are contained in the same component of $\bigcap_{0\leq j\leq k} f_{\lambda}^{-j}(\mathbf{I}_{s_j} \cup \mathbf{I}_{-s_j})$. Since the spherical metric and the hyperbolic metric are comparable in any compact subset of Υ_λ, we conclude by Lemma 5.2.3 that

$$\max_{t\in\mathbb{S}} \operatorname{dist}_{\overline{\mathbb{C}}}\big(\gamma_{k+1}^+(t), \gamma_k^+(t)\big) = \mathcal{O}(\delta^k),$$

where $\operatorname{dist}_{\overline{\mathbb{C}}}$ is the spherical metric and $\delta \in (0,1)$ is a constant. Thus the sequence $\{\gamma_k^+\}$ has a limit map $\gamma_\infty^+ : \mathbb{S} \to \partial D_\infty^+$ which is continuous and surjective. Similarly, the sequence $\{\gamma_k^-\}$ also has a limit map $\gamma_\infty^- : \mathbb{S} \to \partial D_\infty^-$, continuous and surjective. The limit maps γ_∞^+ and γ_∞^- satisfy $\gamma_\infty^+|_{\mathbb{S}\setminus T_\infty} = \gamma_\infty^-|_{\mathbb{S}\setminus T_\infty}$. By continuity, γ_∞^+ and γ_∞^- are identical on $\mathbb{S}$. This implies that $\partial D_\infty^+ = \partial D_\infty^- = \Omega_\lambda^\theta$ and Ω_λ^θ is locally connected.

To finish, we show Ω_λ^θ is Jordan curve. Let $\Phi : \mathbb{D} \to D_\infty^+$ be a Riemann mapping. Since ∂D_∞^+ is locally connected, Φ has an extension from $\overline{\mathbb{D}}$ to $\overline{D_\infty^+}$. If two distinct radical segments $\Phi((0,1)e^{2\pi i\theta_1})$ and $\Phi((0,1)e^{2\pi i\theta_2})$ converge to the same point p, then the Jordan curve $\Phi((0,1)e^{2\pi i\theta_1}) \cup \Phi((0,1)e^{2\pi i\theta_2}) \cup \{\Phi(0), p\}$ separates a section of the boundary ∂D_∞^+ from D_∞^-. But this is a contradiction since D_∞^+ and D_∞^- share a common boundary. □

Proposition 5.3.4. *For $\lambda \in \mathcal{H}$ and $\theta \in \Theta$, all periodic points on $\Omega_\lambda^\theta \cap J(f_\lambda)$ are repelling.*

Proof. Suppose $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$. Let $z \in \Omega_\lambda^\theta \cap J(f_\lambda)$ be a periodic point, with period p. Then the itinerary of z is of the form $(\overline{a_0, a_1, \cdots, a_{p-1}})$, where $a_j \in \{\pm s_j\}$ for $0 \le j \le p-1$. Let $a_k = a_{k \bmod p}$ for $k \ge 0$ and $\mathbf{I}_{a_0\cdots a_s} = \bigcap_{0\le k\le s} f_\lambda^{-k}(\mathbf{I}_{a_k})$. By Lemma 5.2.3, the hyperbolic diameter of $\mathbf{I}_{a_0\cdots a_s}$ is $O(\delta^s)$ when s is large. So we can choose N large enough such that $f_\lambda^p : \operatorname{int}(\mathbf{I}_{a_0\cdots a_N}) \to \operatorname{int}(\mathbf{I}_{a_p\cdots a_N}) = \operatorname{int}(\mathbf{I}_{a_0\cdots a_{N-p}})$ is a conformal map. Since $z \in \operatorname{int}(\mathbf{I}_{a_0\cdots a_N}) \subset \mathbf{I}_{a_0\cdots a_N} \subset \operatorname{int}(\mathbf{I}_{a_0\cdots a_{N-p}})$, we conclude $|(f_\lambda^p)'(z)| > 1$ by Schwarz Lemma. Thus z is a repelling periodic point. □

Proposition 5.3.2 tells us the combinatorial structure of the cut ray Ω_λ^θ. The following proposition shows that the iterated preimages of Ω_λ^θ have the same combinatorial structure as Ω_λ^θ provided that Ω_λ^θ doesn't meet the critical orbit.

Proposition 5.3.5. *For $\lambda \in \mathcal{H}$ and $\theta \in \Theta$, suppose the cut ray Ω_λ^θ doesn't meet the critical orbit, then for any $\alpha \in \bigcup_{k\ge 0} \tau^{-k}(\theta)$, there is a unique ray ω_λ^α such that*

1. *ω_λ^α is a continuous curve connecting 0 with ∞.*
2. *$\omega_\lambda^{\alpha+1/2} = -\omega_\lambda^\alpha$.*
3. *$f_\lambda(\omega_\lambda^\alpha) = \omega_\lambda^{\tau(\alpha)} \cup \omega_\lambda^{\tau(\alpha)+1/2}$.*
4. *$\omega_\lambda^\alpha \cap B_\lambda = R_\lambda(\alpha) \cup \{\infty\}$.*

For this reason, we still call ω_λ^α a full ray of angle α, and $\Omega_\lambda^\alpha = \omega_\lambda^\alpha \cup \omega_\lambda^{\alpha+1/2}$ a cut ray of angle α(or $\alpha + \frac{1}{2}$).

Proof. The proof is based on induction argument. Suppose $\alpha \in \bigcup_{k\geq 0} \tau^{-k}(\theta)$ is an angle such that the full ray ω_λ^α and the cut ray Ω_λ^α satisfy 1,2,3,4. Then for $\beta \in \tau^{-1}(\alpha)$, we define ω_λ^β by lifting Ω_λ^α in the following way:

$$f_\lambda(\omega_\lambda^\beta) = \Omega_\lambda^\alpha, \quad \omega_\lambda^\beta \cap B_\lambda = R_\lambda(\beta) \cup \{\infty\}.$$

This ray ω_λ^β is unique since we require $\omega_\lambda^\beta \cap B_\lambda = R_\lambda(\beta) \cup \{\infty\}$. Also by uniqueness of lifting maps, we conclude $\omega_\lambda^{\beta+\frac{1}{2}} = -\omega_\lambda^\beta$ by the fact $R_\lambda(\beta + \frac{1}{2}) = -R_\lambda(\beta)$ and $\Omega_\lambda^\alpha = -\Omega_\lambda^\alpha$.

In the following, we show that ω_λ^β connects ∞ and 0. If not, then ω_λ^β must be a curve connecting ∞ with itself, hence a Jordan curve. This implies that ω_λ^β doesn't meet with 0. Since $\Omega_\lambda^\alpha = -\Omega_\lambda^\alpha$, all curves in the set $\mathcal{C} = \{e^{k\pi i/n}\omega_\lambda^\beta,\ H_\lambda(e^{k\pi i/n}\omega_\lambda^\beta); 0 \leq k < 2n\}$ are preimages of Ω_λ^α, where $H_\lambda(z) = \sqrt[n]{\lambda}/z$. Since Ω_λ^α doesn't meet the critical orbit, we conclude that for any $\gamma_1, \gamma_2 \in \mathcal{C}$ with $\gamma_1 \neq \gamma_2$, γ_1 and γ_2 are disjoint outside $\{0, \infty\}$. This means $\#\mathcal{C} = 4n$. But this is a contradiction since the degree of f_λ is $2n$. $\square$

Recall that for any $\theta \in \Theta$ with itinerary $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$, the cut ray Ω_λ^θ contains at least two points: 0 and ∞, and $\Omega_\lambda^\theta \setminus \{0, \infty\}$ is contained in the interior of $S_{s_0} \cup S_{-s_0}$. Now given two angles $\alpha, \beta \in \Theta$ with $\Omega_\lambda^\alpha \neq \Omega_\lambda^\beta$, suppose $\mathbf{s}(\alpha) = (s_0^\alpha, s_1^\alpha, s_2^\alpha, \cdots)$, $\mathbf{s}(\beta) = (s_0^\beta, s_1^\beta, s_2^\beta, \cdots)$. Let $\mathbf{J}(\alpha, \beta)$ be the first integer $k \geq 0$ such that $|s_k^\alpha| \neq |s_k^\beta|$. Note that the intersection $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta$ consists of at least two points 0 and ∞. If furthermore $\mathbf{J}(\alpha, \beta) = 0$, then $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta = \{0, \infty\}$. The following proposition tells us the number of intersection points in general case.

Proposition 5.3.6. *Let $\alpha, \beta \in \Theta$ with $\Omega_\lambda^\alpha \neq \Omega_\lambda^\beta$, then the intersection $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta$ consists of $2^{\mathbf{J}(\alpha,\beta)+1}$ points.*

Proof. We consider the orbit of $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta$ under f_λ:

$$\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta \to \Omega_\lambda^{\tau(\alpha)} \cap \Omega_\lambda^{\tau(\beta)} \to \cdots \to \Omega_\lambda^{\tau^{\mathbf{J}(\alpha,\beta)}(\alpha)} \cap \Omega_\lambda^{\tau^{\mathbf{J}(\alpha,\beta)}(\beta)}$$

Note that for any $0 \leq k \leq \mathbf{J}(\alpha, \beta) - 1$, $f_\lambda : \Omega_\lambda^{\tau^k(\alpha)} \cap \Omega_\lambda^{\tau^k(\beta)} \to \Omega_\lambda^{\tau^{k+1}(\alpha)} \cap \Omega_\lambda^{\tau^{k+1}(\beta)}$ is a two-to-one map, thus we have

$$\#(\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta) = 2\#(\Omega_\lambda^{\tau(\alpha)} \cap \Omega_\lambda^{\tau(\beta)}) = \cdots = 2^{\mathbf{J}(\alpha,\beta)}\#(\Omega_\lambda^{\tau^{\mathbf{J}(\alpha,\beta)}(\alpha)} \cap \Omega_\lambda^{\tau^{\mathbf{J}(\alpha,\beta)}(\beta)}) = 2^{\mathbf{J}(\alpha,\beta)+1}.$$

$\square$

Remark 5.3.3. *From the proof of Proposition 5.3.6, we know that any two different cut rays Ω_λ^α and Ω_λ^β intersect at the preimages of ∞. More precisely, $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta \subset \bigcup_{0 \leq k \leq \mathbf{J}(\alpha,\beta)+1} f_\lambda^{-k}(\infty)$, and for $2 \leq k \leq \mathbf{J}(\alpha, \beta) + 1$, the intersection $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta \cap f_\lambda^{-(k-1)}(0)$ consists of 2^{k-1} points.*

5.4 Puzzles, Graphs and Tableaux

5.4.1 The Yoccoz Puzzle

Let $X_\lambda = \bar{\mathbb{C}} \backslash \{z \in B_\lambda; G_\lambda(z) \geq 1\} = \mathbf{V}(1)$. Given N periodic angles $\theta_1, \cdots, \theta_N$ that lie in different periodic cycles of Θ, let

$$g_\lambda(\theta_1, \cdots, \theta_N) = \bigcup_{k \geq 0} \Big(\Omega_\lambda^{\tau^k(\theta_1)} \cup \cdots \cup \Omega_\lambda^{\tau^k(\theta_N)} \Big).$$

Obviously, $g_\lambda(\theta_1, \cdots, \theta_N)$ is f_λ-invariant. The graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ generated by $\theta_1, \cdots, \theta_N$ is defined as following:

$$\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N) = \partial X_\lambda \cup \Big(X_\lambda \cap g_\lambda(\theta_1, \cdots, \theta_N) \Big).$$

The Yoccoz Puzzle induced by the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is constructed in the following way: The Yoccoz Puzzle of depth zero consists of all connected components of $X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$, and each component is called a puzzle piece of depth zero. The Yoccoz Puzzle of greater depth can be constructed by induction: If $P_d^{(1)}, \cdots, P_d^{(m)}$ are the puzzle pieces of depth d, then the connected components of the set $f_\lambda^{-1}(P_d^{(i)})$ are the puzzle pieces $P_{d+1}^{(j)}$ of depth $d+1$. One can verify that the puzzle pieces of depth d consists of all connected components of $f_\lambda^{-d}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N))$ and each puzzle piece is a disk.

To make the puzzle well-defined, we should avoid the situation that the critical orbits touch the set $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. If the critical orbits touch the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$, we say the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is touchable. In this case, since there are infinitely many periodic angles in Θ, we can change the N-tuple $(\theta_1, \cdots, \theta_N)$ to another N-tuple $(\theta_1', \cdots, \theta_N')$ to make the graph not touchable. So in the following discussion, we always assume the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is not touchable.

Let J_0 be the set of all points on the Julia set $J(f_\lambda)$ whose orbits eventually meet the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. Then $J_0 = \bigcup_{k \geq 0} f_\lambda^{-k}(\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N) \cap J(f_\lambda))$. For any $z \in \overline{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, there is a unique sequence of puzzle pieces $P_0(z) \supset P_1(z) \supset P_2(z) \supset \cdots$ which contain z. By Proposition 5.3.4, if f_λ has a non-repelling cycle in $\mathbb{C}$, say $\mathcal{C} = \{z, f_\lambda(z), \cdots, f_\lambda^p(z) = z\}$, then this cycle must avoid the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. This implies that $\mathcal{C} \subset \overline{\mathbb{C}} \setminus (A_\lambda \cup J_0)$. Thus for any $d \geq 0$ and any $x \in \mathcal{C}$, the puzzle piece $P_d(x)$ is well defined.

Lemma 5.4.1. *Suppose the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is not touchable, then for any $z \in \overline{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, the puzzle pieces satisfy:*

$$-P_0(z) = P_0(-z); \quad \omega P_d(z) = P_d(\omega z), \ \omega^{2n} = 1, \ d \geq 1.$$

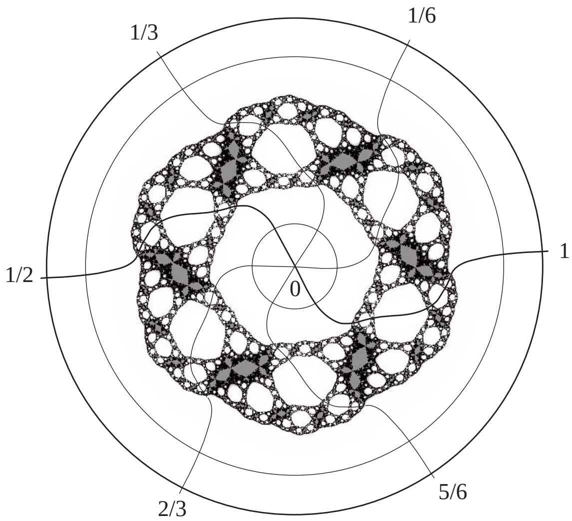

Figure 5.5: A graph with Yoccoz puzzle to depth one ($n = 3$ and $\mathbf{G}_\lambda = \mathbf{G}_\lambda(1/2)$).

Proof. By definition of the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ and the symmetry of the Green function $G_\lambda : A_\lambda \to (0, +\infty]$ (See Lemma 5.2.1), we have $X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N) = -X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. Thus $-P_0(z) = P_0(-z)$. Suppose that for some $d \geq 0$,

$$f_\lambda^{-d}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)) = -f_\lambda^{-d}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)).$$

Since $f_\lambda(\omega z) = \pm f_\lambda(z)$ and $G_\lambda(\omega z) = G_\lambda(z)$, we have $f_\lambda(z) \in f_\lambda^{-d}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N))$ if and only if $f_\lambda(\omega z) \in f_\lambda^{-d}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N))$. Thus

$$f_\lambda^{-(d+1)}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)) = \omega f_\lambda^{-(d+1)}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)).$$

The conclusion follows by induction. □

Lemma 5.4.2. *Suppose the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is not touchable, then for any $d \geq 0$ and any puzzle piece P_d of depth d, the intersection $\overline{P}_d \cap J(f_\lambda)$ is connected.*

Proof. It's equivalent to prove that every connected component of $\overline{\mathbb{C}} \setminus (\overline{P}_d \cap J(f_\lambda))$ is simply connected. Since the Julia set $J(f_\lambda)$ is connected, every component of $\overline{\mathbb{C}} \setminus (\overline{P}_d \cap J(f_\lambda))$ that lies inside P_d is simply connected. So we just need consider the components of $\overline{\mathbb{C}} \setminus (\overline{P}_d \cap J(f_\lambda))$ that intersect with ∂P_d. Note that the puzzle piece P_d is bounded by finitely many cut rays, say $\Omega_\lambda^{\beta_1}, \cdots, \Omega_\lambda^{\beta_s}$, together with finitely many equipotential curves $\mathbf{e}(U_1, v), \cdots, \mathbf{e}(U_t, v)$. By the structure of cut rays (Proposition 5.3.2), there is exactly one component of

$\overline{\mathbb{C}} \setminus (\overline{P}_d \cap J(f_\lambda))$ that intersects with the boundary ∂P_d. This component is the union of $\overline{\mathbb{C}} \setminus \overline{P}_d$ and countably many Fatou components that intersect with the cut rays $\Omega_\lambda^{\beta_1}, \cdots, \Omega_\lambda^{\beta_s}$. Thus it's also simply connected. $\square$

5.4.2 Admissible graphs

Given a point $z \in \overline{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, the difference set $A_d(z) = P_d(z) \setminus \overline{P_{d+1}(z)}$ is an annulus, either degenerate or of positive modulus. Here, d is called the depth of $A_d(z)$. For $d \geq 1$ and $c \in C_\lambda$, the annulus $A_d(z)$ is called off-critical, c-critical or c-semi-critical if $P_d(z)$ contains no critical points, $P_{d+1}(z)$ contains the critical point c or $A_d(z)$ contains the critical point c, respectively.

Since the critical annuli play a crucial rule in our discussion, we will devote ourself to finding a graph such that with respect to the Yoccoz puzzle induced by such a graph, the critical annulus $A_d(c)$ is non-degenerate for some $d \geq 1$. By Lemma 5.4.1, if some critical annulus $A_d(c)$ of depth $d \geq 1$ is non-degenerate, then all critical annuli of the same depth are non-degenerate. The graph that satisfies this property is of special favourite.

Definition 5.4.1. *We say the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is admissible if with respect to the Yoccoz puzzle induced by $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$, there exists a non-degenerate critical annulus $A_d(c)$ for some critical point $c \in C_\lambda$ and some depth $d \geq 1$. Else, we say the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is non-admissible.*

The following remark tells us that a graph may be non-admissible in some cases.

Remark 5.4.1. *There exist non-admissible graphs. For example, for any $n \geq 3$, suppose f_λ is 1-renormalizable at c_0(See Section 5.5 for definition), then the graph $\mathbf{G}_\lambda(1)$ is non-admissible since $A_d(c_0)$ is degenerate for all depths $d \geq 1$, see Figure 5.5.*

However, even if there are non-admissible graphs, we can always find an admissible graph by an elaborate choice. The aim of this section is to prove the existence of admissible graphs for $n \geq 3$.

Proposition 5.4.1. *For any $n \geq 3$ and any $\lambda \in \mathcal{H}$, if f_λ is not critically finite, then there always exists an admissible graph.*

The proof is divided into three lemmas: Lemma 5.4.3, Lemma 5.4.4 and Lemma 5.4.5. In fact, these lemmas enable us to prove much more: when $n \geq 5$, there exist infinitely many admissible graphs without the assumption of the critical finiteness of f_λ (See Lemma 5.4.5); when $n = 4$, there exists at least one admissible graph without the assumption of the critical finiteness of f_λ (See Lemma 5.4.4); when $n = 3$, there exists at least one admissible graph except some particular critically finite cases (See Lemma 5.4.3).

Lemma 5.4.3. *When $n = 3$, there exists an admissible graph except when the critical orbit of f_λ eventually lands at a repelling cycle of period one or two. More precisely,*

1. If neither $\mathbf{G}_\lambda(1/4)$ nor $\mathbf{G}_\lambda(1/2)$ is touchable, then at least one of the graphs $\mathbf{G}_\lambda(1/4)$, $\mathbf{G}_\lambda(1/2)$, $\mathbf{G}_\lambda(1/4, 1/2)$ is admissible.

2. If $\mathbf{G}_\lambda(1/2)$ is touchable, then either $\mathbf{G}_\lambda(1/4)$ is admissible or the critical orbit of f_λ eventually lands at a repelling cycle of period two.

3. If $\mathbf{G}_\lambda(1/4)$ is touchable, then either $\mathbf{G}_\lambda(1/2)$ is admissible or the critical orbit of f_λ eventually lands at a repelling fixed point.

Proof. First note that

$$f_\lambda^{-1}(\Omega_\lambda^{1/4}) = \Omega_\lambda^{1/12} \cup \Omega_\lambda^{1/4} \cup \Omega_\lambda^{5/12},\ f_\lambda^{-1}(\Omega_\lambda^{1/2}) = \Omega_\lambda^{1/6} \cup \Omega_\lambda^{1/3} \cup \Omega_\lambda^{1/2}.$$

1. In this case, the full rays $\omega_\lambda^{1/12}$ and $\omega_\lambda^{1/6}$ decompose S_0 into four domains: D_1, D_2, D_3 and D_4, see Figure 6. If neither $\mathbf{G}_\lambda(1/4)$ nor $\mathbf{G}_\lambda(1/2)$ is touchable, then the graphs $\mathbf{G}_\lambda(1/4)$, $\mathbf{G}_\lambda(1/2)$, $\mathbf{G}_\lambda(1/4, 1/2)$ are all well-defined. Now, we consider the location of the critical value v_λ^+, there are four possibilities:

Case 1: $v_\lambda^+ \in D_1$**.** In this case, the annulus $A_0(v_\lambda^+) = P_0(v_\lambda^+) \setminus \overline{P_1(v_\lambda^+)}$ is non-degenerate with respect to the Yoccoz puzzle induced by either of the graphs $\mathbf{G}_\lambda(1/4)$, $\mathbf{G}_\lambda(1/2)$ and $\mathbf{G}_\lambda(1/4, 1/2)$. It turns out that the critical annulus $A_1(c)$ is non-degenerate for all $c \in C_\lambda$. Thus, in this case, all the graphs $\mathbf{G}_\lambda(1/4)$, $\mathbf{G}_\lambda(1/2)$, $\mathbf{G}_\lambda(1/4, 1/2)$ are admissible.

Case 2: $v_\lambda^+ \in D_2$**.** The annulus $A_0(v_\lambda^+) = P_0(v_\lambda^+) \setminus \overline{P_1(v_\lambda^+)}$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(1/4)$. So all critical annuli $A_1(c)$ are non-degenerate. Thus the graph $\mathbf{G}_\lambda(1/4)$ is admissible.

Case 3: $v_\lambda^+ \in D_3$**.** The annulus $A_0(v_\lambda^+)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(1/4, 1/2)$. So all critical annuli $A_1(c)$ are non-degenerate and the graph $\mathbf{G}_\lambda(1/4, 1/2)$ is admissible.

Case 4: $v_\lambda^+ \in D_4$**.** Similar argument as above, we conclude the graph $\mathbf{G}_\lambda(1/2)$ is admissible.

2. In this case, the graph $\mathbf{G}_\lambda(1/4)$ is necessarily non-touchable. First note that the cut ray $\Omega_\lambda^{5/12}$ decomposes $\Omega_\lambda^{1/2}$ into four parts: $\Omega_\lambda^{1/2}(2, 2)$, $\Omega_\lambda^{1/2}(2, -2)$, $\Omega_\lambda^{1/2}(-2, 2)$ and $\Omega_\lambda^{1/2}(-2, -2)$, where

$$\Omega_\lambda^{1/2}(\varepsilon_0, \varepsilon_1) = \{z \in \Omega_\lambda^{1/2}; \mathbf{s}_\lambda(z) = (\varepsilon_0, \varepsilon_1, \pm 2, \pm 2, \cdots)\}, \quad (\varepsilon_0, \varepsilon_1) \in \{(\pm 2, \pm 2)\}.$$

Moreover, for any $z \in (\Omega_\lambda^{1/2}(2, 2) \cup \Omega_\lambda^{1/2}(-2, -2)) \cap J(f_\lambda)$, the annulus $A_0(z)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(1/4)$.

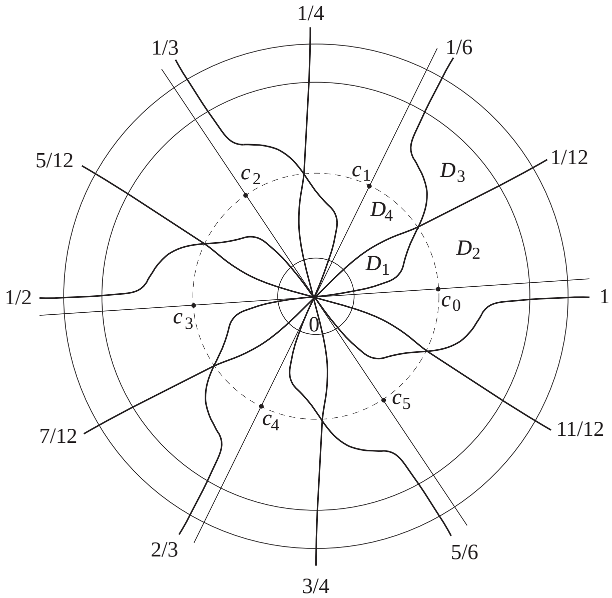

Figure 5.6: Candidates for admissible graph when $n = 3$.

Since $\mathbf{G}_\lambda(1/2)$ is touchable, there is an integer $p \geq 1$ and a critical point $c \in C_\lambda$ such that $f_\lambda^p(c) \in \Omega_\lambda^{1/2}$. Consider the itinerary of $f_\lambda^p(c)$, say $\mathbf{s}_\lambda(f_\lambda^p(c)) = (s_0, s_1, s_2, \cdots)$. There are two possibilities:

Case 1. There is an integer $n \geq 0$ such that $(s_n, s_{n+1}) = (2, 2)$ or $(-2, -2)$. In this case, $f_\lambda^{n+p}(c) \in (\Omega_\lambda^{1/2}(2, 2) \cup \Omega_\lambda^{1/2}(-2, -2)) \cap J(f_\lambda)$, thus the annulus $A_0(f_\lambda^{n+p}(c))$ is non-degenerate. It turns out that the critical annulus $A_{n+p}(c)$ is non-degenerate. So the graph $\mathbf{G}_\lambda(1/4)$ is admissible.

Case 2. For any integer $n \geq 0$, $(s_n, s_{n+1}) = (2, -2)$ or $(-2, 2)$. In this case, either $\mathbf{s}_\lambda(f_\lambda^p(c)) = (2, -2, 2, -2, \cdots) = (\overline{2, -2})$ or $\mathbf{s}_\lambda(f_\lambda^p(c)) = (-2, 2, -2, 2, \cdots) = (\overline{-2, 2})$. By Proposition 5.3.4, $f_\lambda^p(c)$ lies in a repelling cycle of period two.

3. The proof is similar as 2. In this case, the graph $\mathbf{G}_\lambda(1/2)$ is necessarily non-touchable. First note that the cut ray $\Omega_\lambda^{1/3}$ decomposes $\Omega_\lambda^{1/4}$ into four parts: $\Omega_\lambda^{1/4}(1, -1)$, $\Omega_\lambda^{1/4}(1, 1)$, $\Omega_\lambda^{1/4}(-1, -1)$ and $\Omega_\lambda^{1/4}(-1, 1)$, where

$$\Omega_\lambda^{1/4}(\varepsilon_0, \varepsilon_1) = \{z \in \Omega_\lambda^{1/4}; \mathbf{s}_\lambda(z) = (\varepsilon_0, \varepsilon_1, \pm 1, \pm 1, \cdots)\}, \quad (\varepsilon_0, \varepsilon_1) \in \{(\pm 1, \pm 1)\}.$$

Moreover, for any $z \in (\Omega_\lambda^{1/4}(1, -1) \cup \Omega_\lambda^{1/4}(-1, 1)) \cap J(f_\lambda)$, the annulus $A_0(z)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(1/2)$.

Since $\mathbf{G}_\lambda(1/4)$ is touchable, there is an integer $p \geq 1$ and a critical point $c \in C_\lambda$ such that $f_\lambda^p(c) \in \Omega_\lambda^{1/4}$. Consider the itinerary of $f_\lambda^p(c)$, say $\mathbf{s}_\lambda(f_\lambda^p(c)) = (s_0, s_1, s_2, \cdots)$. There are two possibilities:

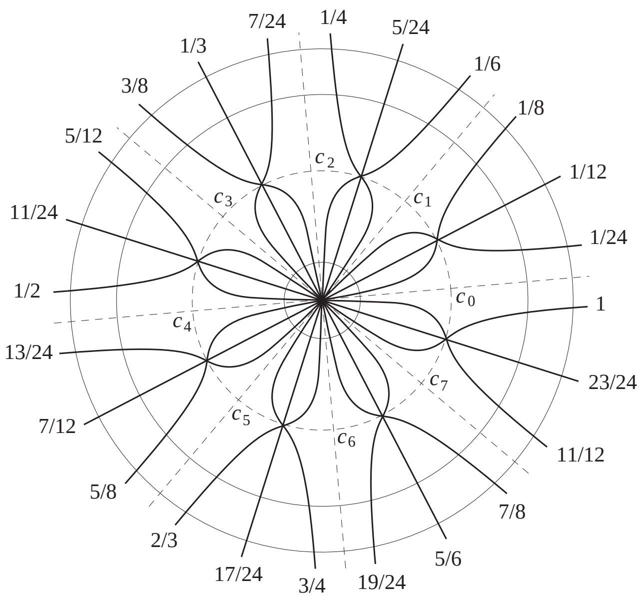

Figure 5.7: Candidates for admissible graph when $n = 4$.

Case 1. There is an integer $n \geq 0$ such that $(s_n, s_{n+1}) = (-1, 1)$ or $(1, -1)$. In this case, $f_\lambda^{n+p}(c) \in (\Omega_\lambda^{1/4}(1, -1) \cup \Omega_\lambda^{1/4}(-1, 1)) \cap J(f_\lambda)$, thus the annulus $A_0(f_\lambda^{n+p}(c))$ is non-degenerate. It turns out that the critical annulus $A_{n+p}(c)$ is non-degenerate. So the graph $\mathbf{G}_\lambda(1/2)$ is admissible.

Case 2. For any integer $n \geq 0$, $(s_n, s_{n+1}) = (1, 1)$ or $(-1, -1)$. In this case, either $\mathbf{s}_\lambda(f_\lambda^p(c)) = (1, 1, \cdots) = (\overline{1})$ or $\mathbf{s}_\lambda(f_\lambda^p(c)) = (-1, -1, \cdots) = (\overline{-1})$. By Proposition 5.3.4, $f_\lambda^p(c)$ is a repelling fixed point. $\square$

Lemma 5.4.4. *When $n = 4$, if $\mathbf{G}_\lambda(1/3)$ is not touchable, then $\mathbf{G}_\lambda(1/3)$ is admissible; if $\mathbf{G}_\lambda(1/3)$ is touchable, then $\mathbf{G}_\lambda(2/3, 1)$ is admissible.*

Proof. First note that $\mathbf{s}(1/3) = (2, 2, \cdots) = (\overline{2})$, $\mathbf{s}(2/3) = (-1, -1, \cdots) = (\overline{-1})$ and $\mathbf{s}(1) = (-3, -3, \cdots) = (\overline{-3})$. Thus $\Omega_\lambda^{1/3} \subset S_2 \cup S_{-2}$, $\Omega_\lambda^{2/3} \subset S_1 \cup S_{-1}$ and $\Omega_\lambda^1 \subset S_3 \cup S_{-3}$, see Figure 7. It's easy to verify

$$f_\lambda^{-1}(\Omega_\lambda^{1/3}) = \Omega_\lambda^{1/12} \cup \Omega_\lambda^{5/24} \cup \Omega_\lambda^{1/3} \cup \Omega_\lambda^{11/24}.$$

If the graph $\mathbf{G}_\lambda(1/3)$ is not touchable, then with respect to the Yoccoz puzzle induced by $\mathbf{G}_\lambda(1/3)$, the puzzle piece $P_1(v_\lambda^+)$ is a subset of the domain bounded by $\omega_\lambda^{5/24}$ and $\omega_\lambda^{23/24}$ together with the equipotential curves $\mathbf{e}(B_\lambda, 1/n)$ and $\mathbf{e}(T_\lambda, 1/n)$. Thus the annulus $A_0(v_\lambda^+)$ is non-degenerate. It turns out that all critical annuli $A_1(c)$ are non-degenerate. So the graph $\mathbf{G}_\lambda(1/3)$ is admissible. If the graph $\mathbf{G}_\lambda(1/3)$ is touchable, then there exist an integer $p \geq 1$ and a critical point $c \in C_\lambda$ such that $f_\lambda^p(c) \in \Omega_\lambda^{1/3}$. Note that the

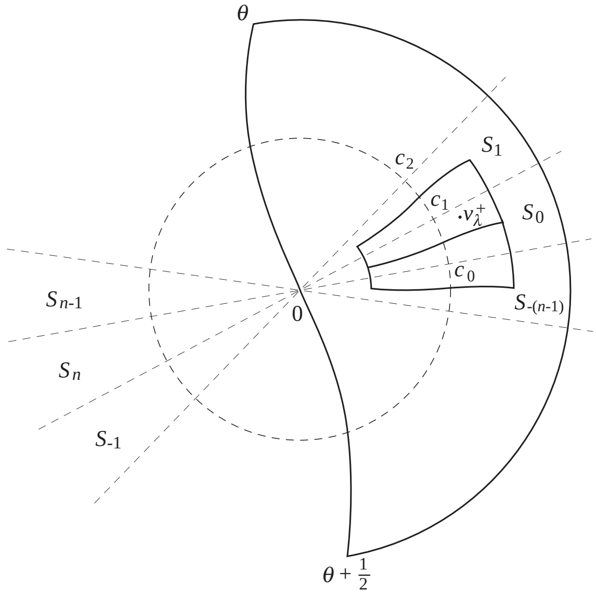

Figure 5.8: Candidates for admissible graph when $n \geq 5$.

preimage of $\Omega_\lambda^{2/3}$ that lies in $S_2 \cup S_{-2}$ is $\Omega_\lambda^{7/24}$ and the preimage of Ω_λ^1 that lies in $S_2 \cup S_{-2}$ is $\Omega_\lambda^{3/8}$. In this case, with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(2/3, 1)$, the puzzle piece $P_1(f_\lambda^p(c))$ is bounded by $\Omega_\lambda^{7/24}$ and $\Omega_\lambda^{3/8}$, thus the annulus $A_0(f_\lambda^p(c))$ is non-degenerate. So all critical annuli $A_p(c)$ are non-degenerate and the graph $\mathbf{G}_\lambda(2/3, 1)$ is admissible. $\square$

Lemma 5.4.5. *When $n \geq 5$, there are infinitely many periodic angles $\theta \in \Theta$ such that the graph $\mathbf{G}_\lambda(\theta)$ is admissible.*

Proof. Let $\widehat{\Theta} = \bigcup_{j \geq 0} \tau^{-j}\Big(\bigcup_{2 \leq k \leq n-2}(\Theta_k \cup \Theta_{-k})\Big)$ be the set of all angles in Θ whose orbits remain in $\bigcup_{2 \leq k \leq n-2}(\Theta_k \cup \Theta_{-k})$ under all iterations of τ and let $\widehat{\Theta}_{per}$ be the set of all periodic angles in $\widehat{\Theta}$. Similar argument as Lemma 5.3.2, we can show that $\widehat{\Theta}_{per}$ is a dense subset of $\widehat{\Theta}$. By Lemma 5.3.1, one can check that the set $\widehat{\Theta}_{per}$ can be written as

$$\widehat{\Theta}_{per} = \bigcup_{p \geq 1} \{\kappa(\mathbf{s}); \mathbf{s} = (\overline{s_0, \cdots, s_{p-1}}) \in \Sigma_0 \text{ and } s_0, \cdots, s_{p-1} \in \{\pm 2, \cdots, \pm(n-2)\}\}$$

and any angle $\theta \in \widehat{\Theta}_{per}$ is of the form

$$\theta = \frac{1}{2}\left(\frac{\chi(s_0)}{n} + \frac{|s_0|}{n(n^p - 1)} + \frac{n^p}{n^p - 1}\sum_{1 \leq k < p} \frac{|s_k|}{n^{k+1}}\right).$$

We can choose an angle $\theta \in \widehat{\Theta}_{per}$ such that the critical orbit avoids the graph $\mathbf{G}_\lambda(\theta)$(Note that there are infinitely many such choices of angle θ). Then with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(\theta)$, $\overline{P_1(v_\lambda^+)}$ is a proper subset of $P_0(v_\lambda^+)$, thus the graph $\mathbf{G}_\lambda(\theta)$ is admissible. See Figure 8. $\square$

In the rest of this section, we prove an important property of the cut rays that are used to generate admissible graphs. Let

$$\Theta_{ad} = \begin{cases} \{\frac{1}{4}, \frac{1}{2}\}, & n = 3, \\ \{\frac{1}{3}, \frac{2}{3}, 1\}, & n = 4, \\ \widehat{\Theta}_{per}, & n \geq 5. \end{cases}$$

Note that for any admissible graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ constructed by Lemma 5.4.3, Lemma 5.4.4 and Lemma 5.4.5, $\{\theta_1, \cdots, \theta_N\} \subset \Theta_{ad}$. In the following, we will prove

Proposition 5.4.2. *For any $\theta \in \Theta_{ad}$, the intersection $\Omega_\lambda^\theta \cap \partial B_\lambda$ consists of two points.*

The proof is based on the following

Lemma 5.4.6. *Suppose $\theta \in \Theta$ and θ satisfies one of the following conditions:*

C1. There are two sequences $\{\theta_k^+\}_{k\geq 1}, \{\theta_k^-\}_{k\geq 1} \subset \Theta$ such that for all $k \geq 1$, $\theta_k^- < \theta < \theta_k^+$ and $\mathbf{J}(\theta_k^+, \theta) = \mathbf{J}(\theta_k^-, \theta) \to \infty$ as $k \to \infty$.

C2. There is a sequence $\{\theta_k\}_{k\geq 1} \subset \Theta$ such that $\theta_1 < \theta_2 < \theta_3 < \dots$ (or $\theta_1 > \theta_2 > \theta_3 > \dots$) and $\mathbf{J}(\theta_k, \theta) = k$ for any $k \geq 1$.

Then the intersection $\Omega_\lambda^\theta \cap \partial B_\lambda$ consists of two points.

Proof. 1. Suppose θ satisfies C1 and $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$. By Proposition 5.3.6, the cut ray $\Omega_\lambda^{\theta_k^+}$ (or $\Omega_\lambda^{\theta_k^-}$) intersects with Ω_λ^θ at $2^{\mathbf{J}(\theta_k^+,\theta)+1}$ points, hence decomposes Ω_λ^θ into $2^{\mathbf{J}(\theta_k^+,\theta)+1}$ parts:

$$\Omega_\lambda^\theta(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{\mathbf{J}(\theta_k^+,\theta)}), (\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{\mathbf{J}(\theta_k^+,\theta)}) = (\pm s_0, \pm s_1, \cdots, \pm s_{\mathbf{J}(\theta_k^+,\theta)}),$$

where

$$\Omega_\lambda^\theta(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_p) := \{z \in \Omega_\lambda^\theta; \mathbf{s}_\lambda(z) = (\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_p, \pm s_{p+1}, \pm s_{p+2}, \cdots)\}.$$

By the structure of the cut rays (Proposition 5.3.2), since the angle θ satisfies condition C1, we conclude that among these $2^{\mathbf{J}(\theta_k^+,\theta)+1}$ parts, only two intersect with $\overline{B_\lambda}$: $\Omega_\lambda^\theta(s_0, s_1, \cdots, s_{\mathbf{J}(\theta_k^+,\theta)})$ and $\Omega_\lambda^\theta(-s_0, (-1)^n s_1, \cdots, (-1)^n s_{\mathbf{J}(\theta_k^+,\theta)})$. Moreover, for any $k \geq 1$

$$\Omega_\lambda^\theta \cap \overline{B_\lambda} \subset \Omega_\lambda^\theta(s_0, s_1, \cdots, s_{\mathbf{J}(\theta_k^+,\theta)}) \cup \Omega_\lambda^\theta(-s_0, (-1)^n s_1, \cdots, (-1)^n s_{\mathbf{J}(\theta_k^+,\theta)}).$$

It turns out that

$$\begin{aligned}\Omega_\lambda^\theta \cap \overline{B_\lambda} \;&\subset\; \bigcap_{k\geq 1}\Big(\Omega_\lambda^\theta(s_0, s_1, \cdots, s_{\mathbf{J}(\theta_k^+,\theta)}) \cup \Omega_\lambda^\theta(-s_0, (-1)^n s_1, \cdots, (-1)^n s_{\mathbf{J}(\theta_k^+,\theta)})\Big)\\ &= \{z \in \Omega_\lambda^\theta; \mathbf{s}_\lambda(z) = (s_0, s_1, s_2, \cdots) \text{ or } (-s_0, (-1)^n s_1, (-1)^n s_2, \cdots)\}\\ &= \overline{R_\lambda(\theta)} \cup \overline{R_\lambda(\theta+1/2)}.\end{aligned}$$

By Proposition 5.3.2, the intersection $\Omega_\lambda^\theta \cap \partial B_\lambda$ consists of two points. These two points are the landing points of the external rays $R_\lambda(\theta)$ and $R_\lambda(\theta+1/2)$.

2. Now we suppose θ satisfies C2 and $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$. We only prove the case when n is odd. The argument applies equally well to the case when n is even. Let $\{\theta_k\}_{k\geq 1} \subset \Theta$ be the sequence such that $\theta_1 < \theta_2 < \theta_3 < \dots$ and $\mathbf{J}(\theta_k, \theta) = k$ for any $k \geq 1$. The following facts are straightforward:

Fact 1. Let $z \in \Omega_\lambda^\theta$. If the itinerary $\mathbf{s}_\lambda(z)$ is of the form $(\varepsilon_0, \cdots, \varepsilon_k, s_{k+1}, s_{k+2}, \cdots)$ or $(\varepsilon_0, \cdots, \varepsilon_k, -s_{k+1}, -s_{k+2}, \cdots)$ for some $k \geq 0$, then z lies in the closure of some external ray or radial ray $R_U(\theta_U)$ for $U \in \mathcal{P}$. (By Proposition 5.3.2)

Fact 2. For any $k > 1$, $\overline{B_\lambda}$ has no intersection with any bounded component of $\overline{\mathbb{C}} \setminus \bigcup_{1\leq j\leq k} \Omega_\lambda^{\theta_j}$. (By Proposition 5.3.1 and 5.3.2)

Fact 3. The sections of Ω_λ^θ that intersect with the unbounded component of $\overline{\mathbb{C}} \setminus \bigcup_{1\leq j\leq k} \Omega_\lambda^{\theta_j}$ are as follows:

$$\Omega_\lambda^\theta(s_0, \cdots, s_k),\ \Omega_\lambda^\theta(-s_0, \cdots, -s_k),$$
$$\Omega_\lambda^\theta(s_0, \cdots, s_j, -s_{j+1}, \cdots, -s_k),\ \Omega_\lambda^\theta(-s_0, \cdots, -s_j, s_{j+1}, \cdots, s_k),\ 0 \leq j < k.$$

Let $\mathcal{E}_k$ be the collection of these sections.

By Fact 2 and Fact 3, we have $\overline{B_\lambda} \cap \Omega_\lambda^\theta \subset \bigcup_{E\in\mathcal{E}_k} E$ for any $k > 1$. It follows that $\overline{B_\lambda} \cap \Omega_\lambda^\theta \subset \bigcap_{k>1}\bigcup_{E\in\mathcal{E}_k} E = \{z \in \Omega_\lambda^\theta, \mathbf{s}_\lambda(z)$ is of the form $\pm\, \mathbf{s}(\theta)$ or $\pm\,(s_0, s_1, \cdots, s_k, -s_{k+1}, -s_{k+2}, \cdots)$ for some $k \geq 0\}$.

By Fact 1, for any $z \in \overline{B_\lambda} \cap \Omega_\lambda^\theta$, either $z \in \overline{R_\lambda(\theta)} \cup \overline{R_\lambda(\theta+1/2)}$ or there exist $U \in \mathcal{P} \setminus \{B_\lambda\}$ and an angle θ_U such that $z \in \overline{R_U(\theta_U)}$. In the following, we show that the latter is impossible. In fact, if $z \in \overline{B_\lambda} \cap \Omega_\lambda^\theta \cap \overline{R_U(\theta_U)}$, then $z \in \partial B_\lambda \cap \partial U$. Let $p \geq 0$ be the first integer such that $f_\lambda^p(U) = T_\lambda$.

After iterations, we see that $f_\lambda^p(z) \in \partial B_\lambda \cap \partial T_\lambda$ and $f_\lambda^p(z)$ is the landing point of the radial ray $R_{T_\lambda}(\theta_{T_\lambda}) = f_\lambda^p(R_U(\theta_U))$. On the other hand, $f_\lambda^{p+1}(z)$ is the landing point of the external ray $R_\lambda(\theta_\lambda) = f_\lambda^{p+1}(R_U(\theta_U))$. This implies $f_\lambda^p(z)$ is also a landing point of some external ray $R_\lambda(\beta)$, $\beta \in \tau^{-1}(\theta_\lambda)$. Since both $R_{T_\lambda}(\theta_{T_\lambda})$ and $R_\lambda(\beta)$ land at $f_\lambda^p(z)$, and $f_\lambda(R_{T_\lambda}(\theta_{T_\lambda})) = f_\lambda(R_\lambda(\beta)) = R_\lambda(\theta_\lambda)$, $f_\lambda^p(z)$ is necessarily a critical point in C_λ.

But this is a contradiction since for any $\alpha \in \Theta$, the cut ray Ω_λ^α avoids the critical set C_λ.

Now we are in the situation $\overline{B_\lambda} \cap \Omega_\lambda^\theta \subset \overline{R_\lambda(\theta)} \cup \overline{R_\lambda(\theta+1/2)}$ and the conclusion follows. $\square$

Proof of Proposition 5.4.2. It suffices to verify that for any $\theta \in \Theta_{ad}$, θ satisfies either C1 or C2 by Lemma 5.4.6.

When $n = 3$, $\mathbf{s}(1/4) = (\overline{1,-1})$, $\mathbf{s}(1/2) = (\overline{2})$. Define two sequences of angles $\{\alpha_k\}_{k\geq 1}, \{\beta_k\}_{k\geq 1} \subset \Theta$ such that:

$$\begin{aligned}
\mathbf{s}(\alpha_1) &= (1,-2,-1,1,-1,1,\cdots),\ \mathbf{s}(\beta_1) = (2,1,-1,2,2,2,\cdots),\\
\mathbf{s}(\alpha_2) &= (1,-1,2,1,-1,1,\cdots),\quad \mathbf{s}(\beta_2) = (2,2,1,-1,2,2,\cdots),\\
\mathbf{s}(\alpha_3) &= (1,-1,1,-2,-1,1\cdots),\ \mathbf{s}(\beta_3) = (2,2,2,1,-1,2,\cdots),\\
&\cdots
\end{aligned}$$

Then $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$ and $\mathbf{J}(\alpha_k, 1/4) = k$ for any $k \geq 1$; $\beta_1 < \beta_2 < \beta_3 < \cdots$ and $\mathbf{J}(\beta_k, 1/2) = k$. Thus both $1/4$ and $1/2$ satisfy condition C2.

When $n = 4$, $\mathbf{s}(1/3) = (\overline{2})$, $\mathbf{s}(2/3) = (\overline{-1})$, $\mathbf{s}(1) = (\overline{-3})$. Define three sequences of angles $\{\alpha_k\}_{k\geq 1}, \{\beta_k\}_{k\geq 1}, \{\gamma_k\}_{k\geq 1} \subset \Theta$ such that:

$$\begin{aligned}
\mathbf{s}(\alpha_1) &= (2,1,-2,2,2,\cdots),\ \mathbf{s}(\beta_1) = (-1,-3,-1,-1,\cdots), \mathbf{s}(\gamma_1) = (-3,-1,-3,-3,\cdots),\\
\mathbf{s}(\alpha_2) &= (2,2,1,-2,2,\cdots),\ \mathbf{s}(\beta_2) = (-1,-1,-3,-1,\cdots), \mathbf{s}(\gamma_2) = (-3,-3,-1,-3,\cdots),\\
\mathbf{s}(\alpha_3) &= (2,2,2,1,-2,\cdots),\ \mathbf{s}(\beta_3) = (-1,-1,-1,-3,\cdots), \mathbf{s}(\gamma_3) = (-3,-3,-3,-1,\cdots),\\
&\cdots
\end{aligned}$$

Then $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$ and $\mathbf{J}(\alpha_k, 1/3) = k$; $\beta_1 > \beta_2 > \beta_3 > \cdots$ and $\mathbf{J}(\beta_k, 2/3) = k$; $\gamma_1 < \gamma_2 < \gamma_3 < \cdots$ and $\mathbf{J}(\gamma_k, 1) = k$. Thus $1/3, 2/3, 1$ all satisfy condition C2.

When $n \geq 5$, we can prove that for any $\theta \in \widehat{\Theta} \supset \widehat{\Theta}_{per}$, θ satisfies condition C1, as follows. Suppose $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$. For any $k \geq 1$, we choose $s_k^-, s_k^+ \in \{\pm 1, \pm(n-1)\}$ and $s_{k+1}^-, s_{k+1}^+ \in \mathbb{I} \setminus \{0, n\}$ such that

(1) $|s_k^-| < |s_k| < |s_k^+|$,

(2) $(s_0, \cdots, s_{k-1}, s_k^-, s_{k+1}^-, s_{k+2}, \cdots), (s_0, \cdots, s_{k-1}, s_k^+, s_{k+1}^+, s_{k+2}, \cdots) \in \Sigma_0$. Let

$$\begin{aligned}
\theta_k^+ &= \kappa((s_0, \cdots, s_{k-1}, s_k^+, s_{k+1}^+, s_{k+2}, s_{k+3}, \cdots)),\\
\theta_k^- &= \kappa((s_0, \cdots, s_{k-1}, s_k^-, s_{k+1}^-, s_{k+2}, s_{k+3}, \cdots)).
\end{aligned}$$

It's easy to check that $\theta_k^- < \theta < \theta_k^+$ and $\mathbf{J}(\theta_k^+, \theta) = \mathbf{J}(\theta_k^-, \theta) = k \to \infty$ as $k \to \infty$. $\square$

5.4.3 Modified puzzle piece

Following the idea of 'thickened puzzle piece' in [M2] to study the quadratic Julia set, we construct the 'modified puzzle piece' for McMullen maps. The

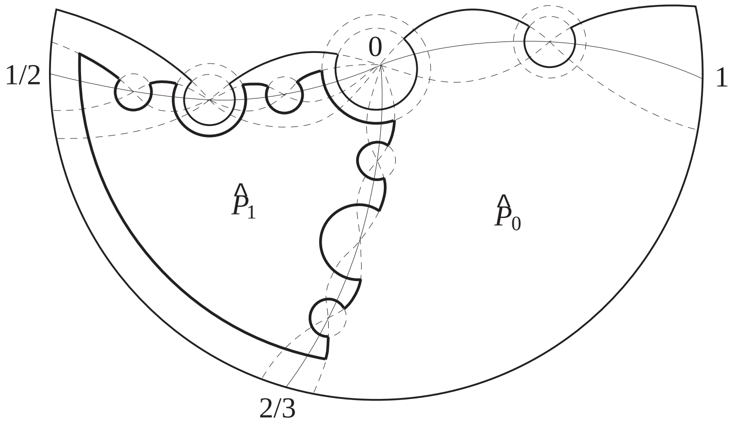

Figure 5.9: An example of 'modified puzzle pieces', to depth one.

'modified puzzle piece' can be applied to study the local connectivity of $J(f_\lambda)$ in the non-renormalizable case (See Lemma 5.7.1). It is also used to define renormalizations (See Remark 5.5.1).

Given an angle $\theta \in \Theta$ with itinerary $\mathbf{s}(\theta) = (s_0, s_1, s_2, \cdots)$, recall that the cut ray Ω_λ^θ is identified as $\Omega_\lambda^\theta = \bigcap_{k\geq 0} f_\lambda^{-k}(S_{s_k} \cup S_{-s_k})$. As is known that Ω_λ^θ can be approximated by the sequence of compact sets $\{\Omega_{\lambda,m}^\theta = \bigcap_{0\leq k\leq m} f_\lambda^{-k}(S_{s_k} \cup S_{-s_k})\}_{m\geq 0}$ in Hausdorff topology. Now we consider the set $\bar{\mathbb{C}} \setminus \Omega_{\lambda,m}^\theta$. The open set $\bar{\mathbb{C}} \setminus \Omega_{\lambda,m}^\theta$ consists of two connected components and the boundary of each component is a Jordan curve. Denote these two boundary curves by $\gamma_{\lambda,m}^1(\theta)$ and $\gamma_{\lambda,m}^2(\theta)$. Let $V_m(\theta) = \gamma_{\lambda,m}^1(\theta) \cap \gamma_{\lambda,m}^2(\theta)$ be the intersection of these two curves. It is obvious that $V_m(\theta)$ consists of finite points and $V_m(\theta) = \Omega_\lambda^\theta \cap \Big(\bigcup_{0\leq k\leq m+1} f_\lambda^{-k}(\infty)\Big)$. For any $v \in V_m(\theta)$, let $D(v)$ be the connected component of $\{z \in A_\lambda; G_\lambda(z) > 1\}$ that contains v. Obviously, $D(v)$ is a disk.

In the following, we construct 'modified puzzle piece'. For the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$, recall that each puzzle piece P_0 of depth zero is contained in a unique component of $\bar{\mathbb{C}} \setminus g_\lambda(\theta_1, \cdots, \theta_N)$. This component is simply connected and is denoted by Q_0. We may choose m large enough such that for any $\alpha, \beta \in \{\tau^k(\theta_j); 1 \leq j \leq N, k \geq 0\}$ with $\Omega_\lambda^\alpha \neq \Omega_\lambda^\beta$,

$$\Omega_{\lambda,m}^\alpha \cap \Omega_{\lambda,m}^\beta = \Omega_\lambda^\alpha \cap \Omega_\lambda^\beta.$$

The disk Q_0 is bounded by some collection of cut rays, say $\{\Omega_\lambda^\alpha; \alpha \in \Lambda(Q_0)\}$, where $\Lambda(Q_0)$ is an index set induced by Q_0. For any $\alpha \in \Lambda(Q_0)$, choose a curve $\gamma(\alpha) \in \{\gamma_{\lambda,m}^1(\alpha), \gamma_{\lambda,m}^2(\alpha)\}$ such that $\gamma(\alpha) \cap Q_0 = \emptyset$. Let $\widehat{Q}_0$ be the connected component of $\bar{\mathbb{C}} \setminus \bigcup_{\alpha\in\Lambda(Q_0)} \gamma(\alpha)$ that contains Q_0 and let $V(Q_0) = \bigcup_{\alpha\in\Lambda(Q_0)}(V_m(\alpha) \cap \partial Q_0)$. The modified puzzle piece $\widehat{P}_0$ of P_0 is defined as follows:

$$\widehat{P}_0 = \widehat{Q}_0 - \bigcup_{v \in V(Q_0)} \overline{D(v)}.$$

Roughly speaking, we can get $\widehat{P}_0$ from Q_0 by thickening Q_0 near $\partial Q_0 \setminus V(Q_0)$ and truncating Q_0 near the points in $V(Q_0)$. The puzzle piece P_0 is not contained in $\widehat{P}_0$, that is the reason why we call $\widehat{P}_0$ the 'modified puzzle piece' of P_0 rather than the 'thickened puzzle piece' of P_0.

The modified puzzle pieces of greater depth can be constructed by the usual inductive procedure: If $\widehat{P}_d^{(j)}$ is the modified puzzle piece of depth d, then each component of $f_\lambda^{-1}(\widehat{P}_d^{(j)})$ is the modified puzzle piece of depth $d+1$, see Figure 5.9.

The virtue of these modified puzzle pieces is: If a puzzle piece $P_d^{(j)}$ contains $P_{d+1}^{(k)}$, then the modified puzzle piece $\widehat{P}_d^{(j)}$ contains $\overline{\widehat{P}_{d+1}^{(k)}}$. This can be easily proved by induction. In other words, this construction replace all of our annuli by non-degenerate annuli.

For $z \in \overline{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, let $\widehat{P}_d(z)$ be the modified puzzle piece of $P_d(z)$. We will only make use of modified puzzle pieces which are small enough to satisfy the following added restriction: If $\widehat{P}_d(z)$ contains a critical point, then $P_d(z)$ must already contain this critical point. Note that if the graph $G_\lambda(\theta_1, \cdots, \theta_N)$ is not touchable, then this requirement is easily satisfied for any bounded value of depth d by choosing m large enough, and this will suffice for applications.

By construction, the puzzle piece $P_d(z)$ and the modified puzzle piece $\widehat{P}_d(z)$ satisfy the following relation:

$$\overline{P_d(z)} \subset \widehat{P}_d(z) \cup A_\lambda, \quad \bigcap_{d \geq 0} \overline{P_d(z)} \subset \bigcap_{d \geq 0} \widehat{P}_d(z).$$

The modified puzzle pieces also satisfy the following symmetry properties: For any $z \in \overline{\mathbb{C}} \setminus (A_\lambda \cup J_0)$,

$$-\widehat{P}_0(z) = \widehat{P}_0(-z); \quad \omega \widehat{P}_d(z) = \widehat{P}_d(\omega z), \ \omega^{2n} = 1, \ d \geq 1.$$

5.4.4 Tableaux

In this section, we present some basic knowledge of tableaux based on Milnor's Lecture [M2]. The applications of tableaux analysis combined with puzzle techniques can be found in [BH], [Hu], [M2], [PQRTY], [QY], [Ro1] and [RY] and many other papers.

Recall that J_0 is the set of all points on $J(f_\lambda)$ whose orbits eventually meet the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. For $x \in \overline{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, the tableau $T(x)$ is defined as two dimensional array $(P_{d,l}(x))_{d,l \geq 0}$, where $P_{d,l}(x) = f_\lambda^l(P_{d+l}(x)) = P_d(f_\lambda^l(x))$.

The position (d, l) is called critical if $P_{d,l}(x)$ contains a critical point in C_λ. If $P_{d,l}(x)$ contains a critical point $c \in C_\lambda$, the position (d, l) is called a c-position. We call $T(x)$ a critical tableau if $x \in C_\lambda$.

For any $x \in \overline{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, the tableau $T(x)$ satisfies the following three rules:

(T1) For each column $l \geq 0$, either the position (d, l) is critical for all $d \geq 0$ or there is a unique integer $d_0 \geq 0$ such that the position (d, l) is critical for all $d < d_0$ and not critical for $d \geq d_0$.

(T2) If $P_{d,l}(x) = P_d(y)$ for some $y \in \overline{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, then $P_{i,l+j}(x) = P_{i,j}(y)$ for $0 \leq i + j \leq d$.

(T3) Let $T(c)$ be a critical tableau. Assume

(a) $P_{d+1-l,l}(c) = P_{d+1-l}(c')$ for some critical point $c' \in C_\lambda$, $0 \leq l < d$, and $P_{d-i,i}(c)$ contains no critical points for $0 < i < l$,

(b) $P_{d,m}(x) = P_d(c)$ and $P_{d+1,m}(x) \neq P_{d+1}(c)$ for some $m > 0$,

Then $P_{d+1-l,m+l}(x) \neq P_{d+1-l}(c')$.

Remark 5.4.2. *The tableau rule (T3) is based on the fact that every puzzle piece of depth $d \geq 1$ contains at most one critical point in C_λ.*

Definition 5.4.2. *1. The tableau $T(x)$ is non-critical if there is an integer $d_0 \geq 0$ such that (d_0, j) is not critical for all $j > 0$. Otherwise, $T(x)$ is called critical.*

2. The tableau $T(x)$ is called pre-periodic if there exist two integers $l \geq 0$ and $p \geq 1$ such that $P_{d,l+p}(x) = P_{d,l}(x)$ for all $d \geq 0$. In this case, if $l = 0$, $T(x)$ is called periodic and the smallest integer $p \geq 1$ is called the period of $T(x)$.

3. Let $\mathrm{Row}_c(d)$ *be the d-th row of the critical tableau $T(c)$, we say* $\mathrm{Row}_c(d+l)$ *with $l > 0$ is a child of* $\mathrm{Row}_c(d)$ *if there is a critical point $c' \in C_\lambda$ such that $A_d(f_\lambda^l(c)) = A_d(c')$ and $f_\lambda^l : A_{d+l}(c) \to A_d(c')$ is a degree two covering map.*

4. (following 3) For $d \geq 1$, we say $\mathrm{Row}_c(d)$ *is excellent if $A_d(f_\lambda^l(c))$ is not semi-critical for all $l \geq 0$.*

Remark 5.4.3. *By Lemma 5.4.1, and the fact $f_\lambda^k(\omega z) = \pm f_\lambda^k(z)$ for $k \geq 1, \omega^{2n} = 1$, we have*

1. If (d,l) is a critical position for some critical tableau, then (d,l) is a critical position for every critical tableau.

2. If some critical tableau $T(c)$ is critical, non-critical or pre-periodic, then every critical tableau is critical, non-critical or pre-periodic, respectively.

3. If $\mathrm{Row}_c(d)$ *is excellent or has a child* $\mathrm{Row}_c(d+l)$ *for some critical point $c \in C_\lambda$, then for every $c' \in C_\lambda$,* $\mathrm{Row}_{c'}(d)$ *is excellent or has a child* $\mathrm{Row}_{c'}(d+l)$*, respectively.*

Lemma 5.4.7. *Suppose some critical tableau $T(c)$ is critical but not pre-periodic, then*

1. For every $d \geq 1$, $\mathrm{Row}_c(d)$ has at least one child.

2. If $\mathrm{Row}_c(d)$ is excellent, then $\mathrm{Row}_c(d)$ has at least two children.

3. If $\mathrm{Row}_c(d)$ is excellent and $\mathrm{Row}_c(d+l)$ is its child, then $\mathrm{Row}_c(d+l)$ is also excellent.

4. If $\mathrm{Row}_c(d)$ has only one child, say $\mathrm{Row}_c(d+l)$, then $\mathrm{Row}_c(d+l)$ is excellent.

Proof. 1. By hypothesis, for every $d \geq 1$, we can find a smallest integer $l > 0$, such that the annulus $A_d(f_\lambda^l(c))$ is c'-critical for some $c' \in C_\lambda$. The map $f_\lambda^l : A_{d+l}(c) \to A_d(c')$ is a degree two covering map. This implies that $\mathrm{Row}_c(d+l)$ is a child of $\mathrm{Row}_c(d)$.

2. (following 1) There exists $d' > d$ such that the annulus $A_{d'}(f_\lambda^l(c))$ is c'-semi-critical. Since $\mathrm{Row}_c(d)$ is excellent, by tableau rule (T3), $A_{d'-t}(f_\lambda^{l+t}(c))$ is either off-critical or semi-critical for $0 < t \leq d' - d$. In particular, $A_d(f_\lambda^{l+d'-d}(c))$ is off-critical. Hence, we can find a smallest integer $l' > l+d'-d$ such that the annulus $A_d(f_\lambda^{l'}(c))$ is critical, so $\mathrm{Row}_c(d+l')$ is another child of $\mathrm{Row}_c(d)$.

3. If $\mathrm{Row}_c(d+l)$ is not excellent, then there is a column $l' \geq l$ such that $A_{d+l}(f_\lambda^{l'}(c))$ is semi-critical. By tableau rule (T3), $A_d(f_\lambda^{l+l'}(c))$ is also semi-critical, which contradict the fact that $\mathrm{Row}_c(d)$ is excellent.

4. If $\mathrm{Row}_c(d+l)$ is not excellent, then as in (3), $A_d(f_\lambda^{l+l'}(c))$ is semi-critical for some $l' \geq l$. Suppose $l' \geq l$ is the smallest integer. We can find a smallest integer $t > l' + l$ such that $A_d(f_\lambda^t(c))$ is c'-critical for some $c' \in C_\lambda$. Then $\mathrm{Row}_c(d+t)$ is also a child of $\mathrm{Row}_c(d)$, which is a contradiction. □

Lemma 5.4.8. *Suppose the critical tableau $T(c)$ is critical and pre-periodic.*

1. If n is odd, then there exist exactly two critical points $\pm c' \in C_\lambda$ such that $T(c')$ and $T(-c')$ are periodic.

2. If n is even, then there is a unique critical point $\tilde{c} \in C_\lambda$ such that $T(\tilde{c})$ is periodic.

Proof. Since $T(c)$ is critical, there exist a smallest integer $p \geq 1$ and a unique critical point $c' \in C_\lambda$ such that (d, p) is a c'-position for all $d \geq 0$.

1. n is odd. There are two possibilities, either $f_\lambda(c) = f_\lambda(c')$ or $f_\lambda(c) + f_\lambda(c') = 0$.

If $f_\lambda(c) = f_\lambda(c')$, then the critical tableaux $T(c')$ and $T(-c')$ are periodic with period p. In this case, there is an integer $d_0 \geq 0$ such that for any $d \geq d_0, 0 < l < p$, the position (d, l) is not critical. It's easy to check that for any $\tilde{c} \in C_\lambda \setminus \{\pm c'\}$, the tableau $T(\tilde{c})$ is strictly pre-periodic. In particular, if $p = 1$, then $P_d(c') = P_d(f_\lambda(c'))$ for all $d \geq 0$. This means that for any

$d \geq 0$, c' and $f_\lambda(c')$ lie in the same puzzle piece of depth d. Thus we conclude $\{\pm c'\} = \{c_0, c_n\}$.

If $f_\lambda(c)+f_\lambda(c') = 0$, then the critical tableau $T(c')$ and $T(-c')$ are periodic with period $2p$. Consider the tableau $T(c')$, there is an integer $d_0 \geq 0$, such that for any $d \geq d_0, 0 < l < p$, the position (d,l) is not critical and for any $d \geq 0$ the position (d,p) is $(-c')$-critical. It's easy to check that for any $\tilde{c} \in C_\lambda \setminus \{\pm c'\}$, the tableau $T(\tilde{c})$ is strictly pre-periodic. In particular, if $p = 1$, then $P_d(-c') = P_d(f_\lambda(c'))$ for all $d \geq 0$. This means that for any $d \geq 0$, $-c'$ and $f_\lambda(c')$ lie in the same puzzle piece of depth d. Thus we conclude $\{\pm c'\} = \{c_1, c_{n+1}\}$.

2. n is even. In this case, by the fact that $f_\lambda^k(v_\lambda^+) = f_\lambda^k(v_\lambda^-)$ for all $k \geq 1$, we conclude the tableau $T(f_\lambda(c'))$ is periodic with period p and the tableau $T(-f_\lambda(c'))$ is strictly pre-periodic. Thus there is a unique critical point $\tilde{c} \in f_\lambda^{-1}(f_\lambda(c'))$ such that $T(\tilde{c})$ is periodic. For this tableau, there is an integer $d_0 \geq 0$ such that for any $d \geq d_0, 0 < l < p$, the position (d,l) is not critical. It's easy to check that for any $c'' \in C_\lambda \setminus \{\tilde{c}\}$, the tableau $T(c'')$ is strictly pre-periodic. In particular, if $p = 1$ and $T(v_\lambda^+)$ is periodic, then $\tilde{c} = c_0$; if $p = 1$ and $T(v_\lambda^-)$ is periodic, then $\tilde{c} = c_{n+1}$. □

5.5 Renormalizations

In this section, we discuss the renormalizations of McMullen maps from the viewpoint of puzzle piece.

Definition 5.5.1. *If there exist a critical point c of f_λ, an integer $p \geq 1$ and two disks U and V containing c such that*

$$\varepsilon f_\lambda^p : U \to V$$

is a quadratic like map whose Julia set is connected (here $\varepsilon \in \{\pm 1\}$ is a symbol), then we say f_λ is p-renormalizable at c if $\varepsilon = 1$ and f_λ is p-$*$-renormalizable at c if $\varepsilon = -1$. In the former case, the triple (f_λ^p, U, V) is called a p-renormalization of f_λ at c. In the latter case, the triple $(-f_\lambda^p, U, V)$ is called a p-$*$-renormalization of f_λ at c.

In the following, we use $K_c = \{z \in U; (\varepsilon f_\lambda^p)^k(z) \in U, \forall\ k \geq 0\} = \bigcap_{k \geq 0}(\varepsilon f_\lambda^p)^{-k}(U)$ to denote the small filled Julia set of the ($*$-)renormalization $(\varepsilon f_\lambda^p, U, V)$. By straightening theorem of Douady and Hubbard [DH3], if $(\varepsilon f_\lambda^p, U, V)$ is a p-($*$-)renormalization of f_λ, then εf_λ^p is conjugate by a quasi-conformal map σ to a unique quadratic polynomial $p_\mu(z) = z^2 + \mu$ in a neighborhood of filled Julia set K_c. Let β be the β-fixed point (i.e. landing point of the zero external ray) of p_μ, and β' be the other preimage of β. We call

$\beta_c = \sigma^{-1}(\beta)$ the β-fixed point of the renormalization $(\varepsilon f_\lambda^p, U, V)$. The other preimage of β_c under the renormalization is $\beta_c' = \sigma^{-1}(\beta')$.

In this section, we always assume that the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is admissible.

5.5.1 Periodic critical tableau implies renormalization

Lemma 5.5.1. *Suppose the critical tableau $T(c)$ is pre-periodic.*

1. *If $T(c)$ is non-critical, then f_λ is critically finite.*
2. *If $T(c)$ is critical, then f_λ is either renormalizable or $*$-renormalizable.*

Proof. Since $T(c)$ is pre-periodic, there exist two integers $l \geq 0$ and $p \geq 1$ such that $P_d(f_\lambda^{l+p}(c)) = P_{d,l+p}(c) = P_{d,l}(c) = P_d(f_\lambda^l(c))$ for all $d \geq 0$.

1. $T(c)$ is non-critical. In this case, the tableaux $T(f_\lambda^l(c))$ and $T(f_\lambda^{l+p}(c))$ are also non-critical. By Lemma 5.7.1 (Notice that the proof of Lemma 5.7.1 is independent of Lemma 5.5.1), $\{f_\lambda^{l+p}(c)\} = \bigcap_{d\geq 0} P_d(f_\lambda^{l+p}(c)) = \bigcap_{d\geq 0} P_d(f_\lambda^l(c)) = \{f_\lambda^l(c)\}$. Therefore, $f_\lambda^{l+p}(c) = f_\lambda^l(c)$ and f_λ is critically finite.

2. $T(c)$ is critical. If n is odd, then by Lemma 5.4.8, there are exactly two critical points $\pm c' \in C_\lambda$ such that $T(c')$ and $T(-c')$ are periodic. Suppose the period is p. Consider the tableau $T(c')$, there are two possibilities :

Case 1. There is an integer $d_0 \geq 0$ such that for any $d \geq d_0, 0 < l < p$, the position (d, l) is not critical. Then $f_\lambda^p : P_{d_0+p}(c') \to P_{d_0}(c')$ is a quadratic-like map and $\{f_\lambda^{kp}(c'); k \geq 0\} \subset P_{d_0+p}(c')$. Thus $(f_\lambda^p, P_{d_0+p}(c'), P_{d_0}(c'))$ is a p-renormalization of f_λ at c'. Since f_λ is an odd function, $(f_\lambda^p, P_{d_0+p}(-c'), P_{d_0}(-c'))$ is a p-renormalization of f_λ at $-c'$.

Case 2. p is even and there is an integer $d_0 \geq 0$ such that for any $d \geq d_0, 0 < l < p/2$, the position (d, l) is not critical and for any $d \geq 0$, the position $(d, p/2)$ is $(-c')$-critical. Then $-f_\lambda^{p/2} : P_{d_0+p/2}(c') \to P_{d_0}(c')$ is a quadratic-like map with $\{(-1)^k f_\lambda^{kp/2}(c'); k \geq 0\} \subset P_{d_0+p/2}(c')$. Thus $(-f_\lambda^{p/2}, P_{d_0+p/2}(c'), P_{d_0}(c'))$ is a $p/2$-$*$-renormalization of f_λ at c'. It turns out that $(-f_\lambda^{p/2}, P_{d_0+p/2}(-c'), P_{d_0}(-c'))$ is a $p/2$-$*$-renormalization of f_λ at $-c'$.

If n is even, then by Lemma 5.4.8, there is a unique critical point $\tilde{c} \in C_\lambda$ such that $T(\tilde{c})$ is periodic. Suppose the period is p. Then there is an integer $d_0 \geq 0$ such that for any $d \geq d_0, 0 < l < p$, the position (d, l) is not critical. Then $f_\lambda^p : P_{d_0+p}(\tilde{c}) \to P_{d_0}(\tilde{c})$ is a quadratic-like map and $\{f_\lambda^{kp}(\tilde{c}); k \geq 0\} \subset P_{d_0+p}(\tilde{c})$. Thus $(f_\lambda^p, P_{d_0+p}(\tilde{c}), P_{d_0}(\tilde{c}))$ is a p-renormalization of f_λ at $\tilde{c}$. Since f_λ is an even function, $(-f_\lambda^p, P_{d_0+p}(-\tilde{c}), P_{d_0}(-\tilde{c}))$ is a p-$*$-renormalization of f_λ at $-\tilde{c}$. □

Remark 5.5.1. *Lemma 5.5.1 also holds when the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is non-admissible. Indeed, in this case, we can use modified puzzle pieces to define renormalizations.*

Proposition 5.5.1. *Suppose f_λ has a non-repelling cycle in $\mathbb{C}$, then f_λ is either renormalizable or $*$-renormalizable. Under this situation, there are three possibilities:*

1. If f_λ is renormalizable and n is odd, then f_λ has exactly two non-repelling cycles in $\mathbb{C}$.

2. If f_λ is $*$-renormalizable and n is odd, then f_λ has exactly one non-repelling cycle in $\mathbb{C}$.

3. If f_λ is renormalizable and n is even, then f_λ has exactly one non-repelling cycle in $\mathbb{C}$.

Proof. Let $\mathcal{C} = \{z_0, f_\lambda(z_0), \cdots, f_\lambda^q(z_0) = z_0\}$ be the non-repelling cycle of f_λ in $\mathbb{C}$. By Proposition 5.4.1, we can find an admissible graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. By Proposition 5.3.4, the cycle $\mathcal{C}$ avoid the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. Thus for any $z \in \mathcal{C}$ and any integer $d \geq 0$, the puzzle piece $P_d(z)$ is well-defined.

We claim that there exist $z \in \mathcal{C}$ and a critical point $c \in C_\lambda$, such that $P_d(z) = P_d(c)$ for all $d \geq 0$. Otherwise, the tableau $T(z)$ is non-critical for any $z \in \mathcal{C}$. So there is an integer $d_0 \geq 0$ such that the map $f_\lambda^q : P_{d_0+q}(z_0) \to P_{d_0}(z_0)$ is conformal. By Schwarz lemma, $|(f_\lambda^q)'(z_0)| > 1$, which is a contradiction.

In this way, we can find a critical point $c \in C_\lambda$ whose tableau $T(c)$ is periodic. By Lemma 5.5.1, f_λ is either renormalizable or $*$-renormalizable.

To continue, suppose the period of $T(c)$ is p, which is necessarily a divisor of q. By Lemma 5.5.1, there are three possibilities:

(P1). n is odd and $(f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is a p-renormalization of f_λ at c. In this case, $(f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is quasiconformally conjugate to a polynomial $z \mapsto z^2 + \mu$. Since a quadratic polynomial has at most one non-repelling cycle (See [CG] or [Sh1]), it turns out that $\mathcal{C}$ is the only non-repelling cycle contained in $\bigcup_{0 \leq j < p} f_\lambda^j(K_c)$. On the other hand, $-\mathcal{C}$ is the only non-repelling cycle contained in $\bigcup_{0 \leq j < p} f_\lambda^j(-K_c)$. Since there are exactly two periodic critical tableaux in this case and $(\cup_{0 \leq j < p} f_\lambda^j(K_c)) \cap (\cup_{0 \leq j < p} f_\lambda^j(-K_c)) = \emptyset$, we conclude that f_λ has exactly two non-repelling cycles in $\mathbb{C}$.

(P2). n is odd and $(-f_\lambda^{p/2}, P_{d_0+p/2}(c), P_{d_0}(c))$ is a $p/2$-$*$- renormalization of f_λ at c. In this case, the cycle $\mathcal{C}$ meets both K_c and $-K_c$. Similar argument as above, one see that $\mathcal{C}$ is the only non-repelling cycle contained in $\bigcup_{0 \leq j < p} f_\lambda^j(K_c)$. Since the cycle $-\mathcal{C}$ is also contained in $\bigcup_{0 \leq j < p} f_\lambda^j(K_c)$, it turns out that $\mathcal{C} = -\mathcal{C}$.

(P3). n is even and $(f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is a p-renormalization of f_λ at c. In this case, $T(c)$ is the only periodic critical tableau. Similar argument as above, we see that $\mathcal{C}$ is the only non-repelling cycle in $\mathbb{C}$. $\square$

In the following, we discuss the case when f_λ has an indifferent cycle of multiplier $e^{2\pi i\theta}$. Douady [Dou1] conjectured that for any rational map, whenever it is linearizable (i.e. the map is conformally conjugate to an irrational rotation) near an indifferent fixed point of multiplier $e^{2\pi i\theta}$, then θ must be a Brjuno number. Here an irrational number θ of convergent p_k/q_k (rational approximations obtained by the continued fraction expansion) is a Brjuno number (denoted by $\mathcal{B}$) if

$$\sum_{k\geq 1}\frac{\log q_{k+1}}{q_k}<+\infty.$$

It follows from Cremer, Siegel and Brjuno that if $\theta\in\mathcal{B}$, then every germ $f(z)=e^{2\pi i\theta}z+\mathcal{O}(z^2)$ is linearizable. Yoccoz [Yo] shows that if the quadratic polynomial $z\mapsto e^{2\pi i\theta}z+z^2$ is linearizable, then $\theta\in\mathcal{B}$. For general case, Geyer [Gey1] shows that for any $d\geq 2$, if $z\mapsto z^d+c$ has an indifferent cycle of multiplier $e^{2\pi i\theta}$ near which the map is linearizable, then $\theta\in\mathcal{B}$. Based on these results and Proposition 5.5.1, we establish immediately:

Proposition 5.5.2. *Suppose f_λ has an indifferent cycle of multiplier $e^{2\pi i\theta}$, then f_λ is linearizable near the indifferent cycle if and only if $\theta\in\mathcal{B}$.*

5.5.2 Properties of renormalizations

In this section, we assume that the critical tableau $T(c)$ is periodic with period k. By Lemma 5.5.1, f_λ is either k-renormalizable at c or $k/2$-$*$-renormalizable at c. Let $(\varepsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ be the corresponding renormalization, where

$$(\varepsilon,p)=\begin{cases}(1,k), & \text{if } f_\lambda \text{ is } k\text{-renormalizable at } c,\\ (-1,k/2), & \text{if } f_\lambda \text{ is } k/2\text{-}*\text{-renormalizable at } c.\end{cases}$$

The small filled Julia set $K_c=\bigcap_{d\geq 0}\overline{P_d(c)}=\bigcap_{d\geq 0}P_d(c)$.

If $K_c\cap\partial B_\lambda\neq\emptyset$, we will show that there is a unique external ray in B_λ accumulating on K_c. Before the proof, we need a classic result for quadratic polynomials:

Lemma 5.5.2. *Let $p_\mu(z)=z^2+\mu$ be a quadratic polynomial with a connected filled Julia set K. If there is a curve $\delta\subset\mathbb{C}\setminus K$ converging to $x\in K$ and $p_\mu(\delta)\supset\delta$, then x is the β-fixed point of p_μ.*

Here, a curve $\delta\subset\mathbb{C}\setminus K$ converges to $x\in K$ means that δ can be parameterized as $\delta:[0,1)\to\mathbb{C}\setminus K$ such that $\lim_{t\to 1}\delta(t)$ exists and $\lim_{t\to 1}\delta(t)=x\in K$. See [McM1] for a proof of Lemma 5.5.2. The conclusion also holds for quadratic like maps.

Lemma 5.5.3. *Suppose the critical tableau $T(c)$ is k-periodic and $K_c \cap \partial B_\lambda \neq \emptyset$, then*

1. The small filled Julia sets $K_c, f_\lambda(K_c), \cdots, f_\lambda^{k-1}(K_c)$ are pairwise disjoint.

2. There is a unique external ray $R_\lambda(t)$ in B_λ accumulating on K_c. This external ray lands at $\beta_c \in K_c$ and the angle t is k-periodic.

Proof. 1. If $f_\lambda^i(K_c) \cap f_\lambda^j(K_c) \neq \emptyset$ for some $0 \leq i < j < k$, then $K_c \cap f_\lambda^{k+i-j}(K_c) \neq \emptyset$. Thus $P_{d,k+i-j}(c) = f_\lambda^{k+i-j}(P_{d+k+i-j}(c)) = P_d(c)$ for all $d \geq 0$. This implies that the critical tableau $T(c)$ is $(k+i-j)$-periodic, which is a contradiction.

2. First note that $f_\lambda^k(P_{d+k}(c)) = P_d(c)$ for $d \geq 0$. Since $K_c \cap \partial B_\lambda \neq \emptyset$, $P_{mk}(c) \cap B_\lambda$ is nonempty and bounded by two external rays, say $R_\lambda(\theta_m^-)$ and $R_\lambda(\theta_m^+)$ with $\theta_m^- < \theta_m^+$. Let $Q(\theta_m^-, \theta_m^+) = \overline{P_{mk}(c) \cap B_\lambda}$, $m \geq 1$. Since $f_\lambda^k(Q(\theta_{m+1}^-, \theta_{m+1}^+)) = Q(\theta_m^-, \theta_m^+)$, we have

$$\theta_m^- \leq \theta_{m+1}^- \leq \cdots \leq \theta_{m+1}^+ \leq \theta_m^+, \quad \theta_m^+ - \theta_m^- = n^k(\theta_{m+1}^+ - \theta_{m+1}^-).$$

Thus there is a common limit $t = \lim \theta_m^+ = \lim \theta_m^-$. Since $\theta_m^- \leq t \leq \theta_m^+$ for any m, we have $n^k t \equiv t \pmod{\mathbb{Z}}$. Thus t is a periodic angle and the external ray $R_\lambda(t)$ lands at a point $z \in K_c \cap \partial B_\lambda$ (This is because rational external rays always land). Since $R_\lambda(n^j t)$ lands at $f_\lambda^j(z) \in f_\lambda^j(K_c) \cap \partial B_\lambda$ for $0 \leq j < k$ and the small filled Julia sets $K_c, f_\lambda(K_c), \cdots, f_\lambda^{k-1}(K_c)$ are pairwise disjoint, we conclude that the angles $t, nt, \cdots, n^{k-1}t$ are different from each other. Thus t is k-periodic.

Suppose θ is another angle such that the external ray $R_\lambda(\theta)$ accumulating on K_c. Then $\theta_m^- \leq \theta \leq \theta_m^+$ for any m. Thus $\theta = \lim \theta_m^+ = \lim \theta_m^- = t$.

To finish, we show $z = \beta_c$. Since $T(c)$ is k-periodic, f_λ is either k-renormalizable or $k/2$-$*$-renormalizable. In the former case, $f_\lambda^k(R_\lambda(t)) = R_\lambda(t)$. Thus by Lemma 5.5.2, $z = \beta_c$. In the latter case, since $R_\lambda(t)$ is the unique external ray accumulating on K_c, we conclude $R_\lambda(t+1/2) = -R_\lambda(t)$ is the unique external ray accumulating on $-K_c$. On the other hand, $f_\lambda^{k/2}(R_\lambda(t))$ is also an external ray accumulating on $-K_c$, we have $f_\lambda^{k/2}(R_\lambda(t)) = R_\lambda(t+1/2) = -R_\lambda(t)$. In this case, $-f_\lambda^{k/2}(R_\lambda(t)) = R_\lambda(t)$. Again by Lemma 5.5.2, $z = \beta_c$. □

5.6 A Criterion of Local Connectivity

In this section, we present a criterion to characterize the local connectivity of the immediate basin of attraction. This criterion together with Yoccoz puzzle techniques can be applied to study the local connectivity and higher regularity of the boundary ∂B_λ.

In the following discussion, let f be a rational map of degree at least two, $C(f)$ be the critical set of f and $P(f) = \overline{\bigcup_{k\geq 1} f^k(C(f))}$ be the postcritical set. Suppose that f has an attracting periodic point z_0 and the immediate basin B of z_0 is simply connected. Let $B(z,\delta) = \{x \in \mathbb{C}; |x-z| < \delta\}$.

Definition 5.6.1. *We say f satisfies* **BD***(i.e. bounded degree) condition on ∂B if for any $u \in \partial B$, there is a number $\varepsilon_u > 0$ such that for any integer $m \geq 0$ and any component $U_m(u)$ of $f^{-m}(B(u,\varepsilon_u))$ intersecting with ∂B, $U_m(u)$ is simply connected and the degree $\deg(f^m : U_m(u) \to B(u,\varepsilon_u))$ is bounded by some constant D, which is independent of u, m and $U_m(u)$.*

For the definition, here is a remark: since $f^m : U_m(u) \to B(u,\varepsilon_u)$ is a proper map between two disks, we conclude by Maximum Principle that for any disk $W \subset B(u,\varepsilon_u)$ and any component V of $f^{-m}(W)$ that lies inside $U_m(u)$, V is also a disk.

The aim of this section is to prove the following:

Proposition 5.6.1. *If f satisfies* **BD** *condition on ∂B, then*

1. ∂B is locally connected.

2. If furthermore ∂B is a Jordan curve, then ∂B is a quasicircle.

The proof of Proposition 5.6.1 is based on Theorem 2.3.2.

Proof. By replacing f with f^k, we assume z_0 is a fixed point of f. By quasiconformal surgery, we assume z_0 is a superattracting fixed point with local degree $d = \deg(f : B \to B) \geq 2$. Thus B contains no critical points other that z_0. By Möbius conjugation, we assume $z_0 = \infty$.

Since f satisfies **BD** condition on ∂B, there exists a constant $\delta > 0$ such that for any $u \in \partial B$, any integer $m \geq 0$ and any component $U_m(u)$ of $f^{-m}(B(u,\delta))$ that intersects with ∂B, $U_m(u)$ is simply connected and $\deg(f^m : U_m(u) \to B(u,\delta)) \leq D$. In fact, we can choose δ as the Lebesgue number of the family $\mathcal{F} = \{B(u,\varepsilon_u); u \in \partial B\}$, which is an open covering of the boundary ∂B.

The proof consists of four steps.

Step 1. *Let $V_m(z)$ be the component of $f^{-m}(B(z,\delta/2))$ contained in $U_m(z)$ and intersecting with ∂B, then*

$$\lim_{m\to\infty} \sup_{z\in\partial B} \operatorname{diam}(V_m(z)) = 0.$$

For else, there is a constant $d_0 \geq 0$ and two sequences $\{z_k\} \subset \partial B$ and $\{\ell_k\}$ such that $\operatorname{diam}(V_{\ell_k}(z_k)) \geq d_0$. For every $k \geq 1$, choose a point $y_k \in$

$f^{-\ell_k}(z_k) \cap V_{\ell_k}(z_k)$. By passing to a subsequence, we assume $y_k \to y_\infty \in \partial B$ and $z_k \to z_\infty \in \partial B$. By Theorem 2.3.2, there is a constant $C(D)$ such that

$$\mathrm{Shape}(V_{\ell_k}(z_k), y_k) \le C(D)\mathrm{Shape}(B(z_k, \delta/2), z_k) = C(D).$$

Since $\mathrm{diam}(V_{\ell_k}(z_k)) \ge d_0$, $V_{\ell_k}(z_k)$ contains a round disk centered at y_k of definite size. So there is a constant $r_0 = r_0(d_0, D)$ such that $V_{\ell_k}(z_k) \supset B(y_\infty, r_0)$ for large k. This means $f^{\ell_k}(B(y_\infty, r_0)) \subset B(z_k, \delta/2) \subset B(z_\infty, \delta)$. But this contradicts the fact that $f^{\ell_k}(B(y_\infty, r_0)) \supset J(f)$ when k is large.

Step 2. *There are two constants $L > 0$ and $\nu \in (0,1)$, such that for any $z \in \partial B$ and any $k \ge 1$, $\mathrm{diam}(V_k(z)) \le L\nu^k$.*

By Step 1, there is an integer $s > 0$, such that $\mathrm{diam}(V_s(z)) < \delta/4$ for all $z \in \partial B$. For each $x \in \partial B$ and each integer $k > 0$, let $V_{ks}(x)$ be a component of $f^{-ks}(B(x, \delta/2))$ intersecting with ∂B and $x_{ks} \in V_{ks}(x) \cap f^{-ks}(x)$. For $0 \le j \le k$, let $x_{js} = f^{(k-j)s}(x_{ks})$ and U_j be the component of $f^{-js}(B(x_{(k-j)s}, \delta/2))$ containing x_{ks}. Then

$$x_{ks} \in V_{ks}(x) = U_k \subset \cdots \subset U_0 = B(x_{ks}, \delta/2).$$

For every $1 \le j < k$, $f^{js} : U_j \to B(x_{(k-j)s}, \delta/2)$ is a proper map of degree $\le D$. Since $f^{js}(U_{j+1})$ is contained in $B(x_{(k-j)s}, \delta/4)$,

$$\mathrm{mod}(U_j \setminus \overline{U_{j+1}}) \ge \frac{1}{D}\mathrm{mod}(B(x_{(k-j)s}, \delta/2) \setminus \overline{f^{js}(U_{j+1})}) \ge \frac{\log 2}{2\pi D},$$

$$\mathrm{mod}(B(x_{ks}, \delta/2) \setminus \overline{V_{ks}(x)}) \ge \sum_{0 \le j < k} \mathrm{mod}(U_j \setminus \overline{U_{j+1}}) \ge \frac{k \log 2}{2\pi D}.$$

So there are two constants $M > 0$ and $\mu \in (0,1)$ such that for any $x \in \partial B$, $\mathrm{diam}(V_{ks}(x)) \le M\mu^k$. This implies that there are two constants $L > 0$ and $\nu \in (0,1)$ such that $\mathrm{diam}(V_k(x)) \le L\nu^k$ for all $k \ge 1$.

Step 3. *There exists a sequence of Jordan curves $\{\gamma_k : \mathbb{S} \to B\}$ such that γ_k converges uniformly to a continuous and surjective map $\gamma_\infty : \mathbb{S} \to \partial B$, where $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ is the unit circle. Hence, ∂B is locally connected.*

Recall that the Böttcher map $\phi : B \to \overline{\mathbb{C}} \setminus \bar{\mathbb{D}}$ defined by $\phi(z) = \lim_{k\to\infty} (f_\lambda^k(z))^{d^{-k}}$ is a conformal isomorphism. It satisfies $\phi^{-1}(r^d e^{2\pi i dt}) = f(\phi^{-1}(re^{2\pi it}))$ for $(r,t) \in (1, +\infty) \times \mathbb{S}$. Let $\ell(R,t) = \phi^{-1}([\sqrt[d]{R}, R]e^{2\pi it})$ for $(R,t) \in (1,2) \times \mathbb{S}$. By the boundary behavior of Poincaré metric, there is a constant $C > 0$ such that for any$(R,t) \in (1,2) \times \mathbb{S}$,

$$\begin{aligned}\mathrm{Eucl.length}(\ell(R,t)) &\le C\mathrm{Hyper.length}(\ell(R,t)) \cdot \mathrm{H.dist}(\phi^{-1}(R\mathbb{S}), \partial B) \\ &\le C(\log d)\mathrm{H.dist}(\phi^{-1}(R\mathbb{S}), \partial B) \quad (\to 0 \; as \; R \to 1),\end{aligned}$$

where Hyper.length is the hyperbolic length in B and H.dist is the Hausdorff distance in the sphere $\bar{\mathbb{C}}$. Thus we can choose R sufficiently close to 1 such that for any $t \in \mathbb{S}$, $\ell(R,t) \subset B(z,\delta/2)$ for some $z \in \partial B$. For $k \geq 0$, define a curve $\gamma_k : \mathbb{S} \to B$ by $\gamma_k(t) = \phi^{-1}(R^{1/d^k} e^{2\pi i t})$. Since $f^k(\gamma_{k+q}(t)) = \gamma_q(d^k t)$ for $q \geq 0$ and $\gamma_0(d^k t), \gamma_1(d^k t) \in \ell(R, d^k t) \subset B(z,\delta/2)$ for some $z \in \partial B$, we conclude that $\gamma_k(t)$ and $\gamma_{k+1}(t)$ lie in the same component of $f^{-k}(B(z,\delta/2))$ intersecting with ∂B. By Step 2,

$$\max_{t\in\mathbb{S}} |\gamma_{k+1}(t) - \gamma_k(t)| = \mathcal{O}(\nu^k).$$

So $\{\gamma_k : \mathbb{S} \to B\}$ is a Cauchy sequence, hence converges to a continuous map $\gamma_\infty : \mathbb{S} \to \partial B$. By construction, γ_∞ is surjective.

Step 4. *If furthermore ∂B is a Jordan curve, then ∂B is a quasi-circle.*

Since ∂B is a Jordan curve, the Böttcher map $\phi : B \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ can be extended to a homeomorphism $\phi : \overline{B} \to \overline{\mathbb{C}} \setminus \mathbb{D}$. Define a map $\psi : \mathbb{S} \to \partial B$ by $\psi(\zeta) = \phi^{-1}(\zeta)$ for $\zeta \in \mathbb{S}$. Then $f(\psi(\zeta)) = \psi(\zeta^d)$. Let $\varphi = \phi|_{\partial B}$ be the inverse of ψ. Both ψ and φ are uniformly continuous. Thus for any sufficiently small positive number ε, there are two small constants $a(\varepsilon), b(\varepsilon)$ such that

$$\begin{aligned} \forall\ (\zeta_1,\zeta_2) \in \mathbb{S}\times\mathbb{S}, |\zeta_1 - \zeta_2| < a(\varepsilon) \ &\Rightarrow\ |\psi(\zeta_1) - \psi(\zeta_2)| < \varepsilon; \\ \forall\ (z_1,z_2) \in \partial B\times\partial B, |z_1 - z_2| < b(\varepsilon) \ &\Rightarrow\ |\varphi(z_1) - \varphi(z_2)| < a(\varepsilon). \end{aligned}$$

Given two points $z_1, z_2 \in \partial B$, $\partial B \setminus \{z_1, z_2\}$ consists of two components, say E_1 and E_2. Let $L(z_1,z_2) \in \{\overline{E}_1, \overline{E}_2\}$ be the section of ∂B such that $\mathrm{diam}(L(z_1,z_2)) = \min\{\mathrm{diam}(E_1), \mathrm{diam}(E_2)\}$. Thus for any positive number $\varepsilon \ll \mathrm{diam}(\partial B)$, by uniform continuity, we have

$$|z_1 - z_2| < b(\varepsilon) \Rightarrow \mathrm{diam}(L(z_1,z_2)) < \varepsilon. \tag{5.1}$$

By Ahlfors' characterization of quasicircle [Ahl], to prove that ∂B is a quasicircle, it suffices to show that there is a constant $C > 0$ such that for any $z_1, z_2 \in \partial B$ with $z_1 \neq z_2$, $\Delta(L(z_1,z_2); z_1, z_2) \leq C$. In fact, if $|z_1 - z_2| \geq \varepsilon$ for some positive constant ε, then $\Delta(L(z_1,z_2); z_1,z_2) \leq \mathrm{diam}(\partial B)/\varepsilon$. So we just need consider the case when $|z_1 - z_2|$ is small. In the following, we assume $\delta \ll \mathrm{diam}(\partial B)$ and $|z_1 - z_2| \leq b(\delta/2)$, it turns out that $\mathrm{diam}(L(z_1,z_2)) < \delta/2$.

Since f is expanding on ∂B, there is an integer $N > 0$ such that $f^k(L(z_1,z_2)) = \partial B$ for all $k \geq N$. So we can find a smallest integer $\ell \geq 0$, such that

$$\mathrm{diam}(f^\ell(L(z_1,z_2))) < \delta/2,\ \mathrm{diam}(f^{\ell+1}(L(z_1,z_2))) \geq \delta/2.$$

On the other hand, there exist two points $w_1, w_2 \in f^\ell(L(z_1, z_2))$ such that

$$\begin{aligned}\operatorname{diam}(f^{\ell+1}(L(z_1, z_2))) &= |f(w_1) - f(w_2)| \le \int_{[w_1,w_2]} |f'(z)||dz| \\ &\le M|w_1 - w_2| \le M\operatorname{diam}(f^\ell(L(z_1, z_2))),\end{aligned}$$

where $[w_1, w_2]$ is the straight segment connecting w_1 with w_2 and

$$M = \max\{|f'(z)|; \text{Eucl.dist}(z, \partial B) \le \delta/2\}.$$

Thus we have

$$\frac{\delta}{2M} \le \operatorname{diam}(f^\ell(L(z_1, z_2))) = \operatorname{diam}(L(f^\ell(z_1), f^\ell(z_2))) < \frac{\delta}{2}.$$

By (5.1), there is a constant $c(\delta, M) > 0$ such that $|f^\ell(z_1) - f^\ell(z_2)| \ge c(\delta, M)$.

Applying Theorem 2.3.2 to the following situation $(V_1, U_1) = (U_\ell(f^\ell(z_1)), V_\ell(f^\ell(z_1)))$, $(V_2, U_2) = (B(f^\ell(z_1), \delta), B(f^\ell(z_1), \delta/2))$ and $g = f^\ell$, we conclude that there is a constant $C(D) > 0$ such that

$$\Delta(L(z_1, z_2); z_1, z_2) \le C(D)\Delta(f^\ell(L(z_1, z_2)); f^\ell(z_1), f^\ell(z_2)) \le \frac{C(D)\delta}{2c(\delta, M)}.$$

Thus for any $x, y \in \partial B$ with $x \ne y$, the turning $\Delta(L(x, y); x, y)$ is bounded by

$$\max\left\{\frac{\operatorname{diam}(\partial B)}{b(\delta/2)}, \frac{C(D)\delta}{2c(\delta, M)}\right\}.$$

□

Remark 5.6.1. *Using the same argument as [CJY], one can show further that if f satisfies* **BD** *condition on ∂B, then ∂B is a John domain.*

The following proposition gives a criterion when f satisfies **BD** condition on ∂B.

Proposition 5.6.2. *If $\#(P(f) \cap \partial B) < \infty$ and all periodic points in $P(f) \cap \partial B$ are repelling, then f satisfies* **BD** *condition on ∂B.*

Proof. The proof is based on the following claim.

Claim: *For any $u \in \partial B$, there is a constant $\varepsilon_u > 0$ such that for any $m \ge 0$ and any component $U_m(u)$ of $f^{-m}(B(u, \varepsilon_u))$ that intersects with ∂B, $U_m(u)$ contains at most one critical point of f^m.*

The claim implies that $U_m(u)$ is simply connected by Riemann-Hurwitz Formula. Since the sequence $U_m(u) \to f(U_m(u)) \to \cdots \to f^{m-1}(U_m(u)) \to$

$B(u,\varepsilon_u)$ meets every critical point of f at most once, we conclude that $\deg(f^m : U_m(u) \to B(u,\varepsilon_u))$ is bounded by $D = \Pi_{c\in C(f)}\deg(f,c)$.

In the following,we prove the claim.

First note that every point in $P(f)\cap\partial B$ is pre-periodic, we can decompose ∂B into three disjoint sets: X, Y and Z, where $X = \partial B \setminus P(f)$, Z is the union of all repelling cycles in $P(f)\cap\partial B$ and $Y = (P(f)\cap\partial B)\setminus Z$.

For any $x \in X$, choose a small number $\varepsilon_x > 0$ such that $B(x,\varepsilon_x)\cap P(f) = \emptyset$. Then for any component $W_m(x)$ of $f^{-m}(B(x,\varepsilon_x))$ intersecting with ∂B, $f^m : W_m(x) \to B(x,\varepsilon_x)$ is a conformal map.

The set Y consists of all strictly pre-periodic points. Thus there is an integer $q \geq 1$ such that for any $y \in Y$, $f^{-q}(y)\cap P(f)\cap\partial B = \emptyset$. For a open set U in $\bar{\mathbb{C}}$ and a point $u \in U$, we use $\mathrm{Comp}_u(U)$ to denote the component of U that contains u. For every $y \in Y$, choose $\varepsilon_y > 0$ small enough such that for any $x \in f^{-q}(y)\cap\partial B \subset X$, $\mathrm{Comp}_x(f^{-q}(B(y,\varepsilon_y))) \subset B(x,\varepsilon_x)$ and $\mathrm{Comp}_x(f^{-q}(B(y,\varepsilon_y)))$ contains at most one critical point of f^q.

Finally, we deal with Z. For $z \in Z$, suppose z lies in a repelling cycle of period p. Choose $\varepsilon_z > 0$ such that

(1) $B(z,\varepsilon_z)$ is contained in the linearizable neighborhood of z and $\mathrm{Comp}_z(f^{-p}(B(z,\varepsilon_z)))$ is a subset of $B(z,\varepsilon_z)$,

(2) For every $u \in (f^{-p}(z)\cap\partial B)\setminus\{z\} \subset X\cup Y$, $\mathrm{Comp}_u(f^{-p}(B(z,\varepsilon_z)))$ contains at most one critical point of f^p and $\mathrm{Comp}_u(f^{-p}(B(z,\varepsilon_z))) \subset B(u,\varepsilon_u)$.

One can easily verify that the collection of neighborhoods $\{B(u,\varepsilon_u), u \in \partial B\}$ are just as required. □

Corollary 5.6.1. *If f is critically finite, then f satisfies* **BD** *condition on ∂B.*

Proof. Since f is critically finite, every periodic point of f is either repelling or superattracting. This implies that $\#(P(f)\cap\partial B) < \infty$ and all periodic points in $P(f)\cap\partial B$ are repelling. Thus by Proposition 5.6.2, f satisfies **BD** condition on ∂B. □

5.7 The boundary ∂B_λ is a Jordan curve

In this section, we will prove:

Theorem 5.7.1. *For any $n \geq 3$ and any complex parameter λ, if the Julia set $J(f_\lambda)$ is not a Cantor set, then ∂B_λ is a Jordan curve.*

An immediate corollary of the theorem is the following:

Corollary 5.7.1. *Suppose f_λ has no Siegel disk and the Julia set $J(f_\lambda)$ is connected, then every Fatou component is a Jordan domain.*

For the higher regularity of ∂B_λ, we show

Theorem 5.7.2. *Suppose the Julia set $J(f_\lambda)$ is not a Cantor set, then the boundary ∂B_λ is a quasicircle if it contains neither parabolic point nor recurrent critical point.*

The strategy of the proof is as follows:

First, consider the McMullen maps f_λ with parameter $\lambda \in \mathcal{H}$. If f_λ is critically finite, then the Julia set is locally connected. Else, by Proposition 5.4.1, we can find an admissible graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. With respect to the Yoccoz puzzle induced by this graph, there are two possibilities of the critical tableaux:

Case 1: There is no periodic critical tableau. This case is discussed in section 5.7.1 and the local connectivity of $J(f_\lambda)$ follows from Proposition 5.7.1. The idea of the proof is based on the combinatorial analysis for tableaux introduced by Branner and Hubbard (see [BH], [M2]), together with 'modified puzzle piece' techniques.

Case 2: There is a periodic critical tableau $T(c)$. In this case, the map f_λ is either renormalizable or $*$-renormalizable. This case is discussed in section 5.7.2. The local connectivity of ∂B_λ follows from Proposition 5.7.2. The idea of the proof of Proposition 5.7.2 is to construct a closed curve separating ∂B_λ from the small filled Julia set K_c.

In section 5.7.3, we deal with the real parameters $\lambda \in \mathbb{R}^+$.

In section 5.7.4, we improve the regularity of the boundary ∂B_λ. We first include a proof of R.Devaney which claims that the local connectivity of ∂B_λ implies that ∂B_λ is a Jordan curve. Then we show ∂B_λ is a quasicircle except two specific cases.

In section 5.7.5, we present some corollaries.

5.7.1 No periodic critical tableau case

Recall that J_0 is the set of all points on the Julia set $J(f_\lambda)$ whose orbits eventually meet the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$.

Lemma 5.7.1. *Let $z \in J(f_\lambda) \setminus J_0$. If $T(z)$ is non-critical, then* $\mathrm{End}(z) := \bigcap_{d\geq 0} \overline{P_d(z)} = \{z\}$.

Proof. It suffice to prove $\mathrm{End}(f_\lambda(z)) = \{f_\lambda(z)\}$. Since $T(z)$ is non-critical, there is an integer $d_0 \geq 1$ such that for any $j > 0$, the position (d_0, j) is not critical. Let $\{\widehat{P}^{(i)}_{d_0-1}; 1 \leq i \leq M\}$ be the collection of all modified puzzle pieces of depth $d_0 - 1$, numbered so that $\widehat{P}^{(1)}_{d_0-1} = \widehat{P}_{d_0-1}(v^+_\lambda)$, $\widehat{P}^{(2)}_{d_0-1} = \widehat{P}_{d_0-1}(v^-_\lambda)$, recall that we use $\widehat{P}_d(w)$ to denote the modified puzzle piece of $P_d(w)$. Every modified puzzle piece of depth$\geq d_0$ is contained in a unique modified puzzle piece $\widehat{P}^{(i)}_{d_0-1}$ of depth d_0-1. Let $\text{dist}_i(x,y)$ be the Poincaré metric of $\widehat{P}^{(i)}_{d_0-1}$. For $2 < i \leq M$, there are exactly $2n$ branches of f_λ^{-1} on $\widehat{P}^{(i)}_{d_0-1}$, say $g_1^i, g_2^i, \cdots, g_{2n}^i$, and each g_k^i on $\widehat{P}^{(i)}_{d_0-1}$ is univalent and carries $\widehat{P}^{(\alpha)}_{d_0} \subset\subset \widehat{P}^{(i)}_{d_0-1}$ onto a proper subset of some $\widehat{P}^{(j)}_{d_0-1}$. It follows that there is a uniform constant $0<\nu<1$, such that

$$\text{dist}_j(g_k^i(x), g_k^i(y)) \leq \nu \text{dist}_i(x,y)$$

for any $x, y \in \widehat{P}^{(\alpha)}_{d_0} \subset\subset \widehat{P}^{(i)}_{d_0-1}$ and any $2 < i \leq M, 1 \leq k \leq 2n$.

Let D be the maximum of Poincaré diameters of the modified puzzle pieces of depth d_0. For any integer $h>0$, since the sequence

$$\widehat{P}_{d_0+h}(f_\lambda(z)) \to \widehat{P}_{d_0+h-1}(f_\lambda^2(z)) \to \cdots \widehat{P}_{d_0+1}(f_\lambda^h(z)) \to \widehat{P}_{d_0}(f_\lambda^{h+1}(z))$$

contains no critical point, it follows that

$$\text{Hyper.diam}\big(\widehat{P}_{d_0+h}(f_\lambda(z))\big) \leq D\nu^h$$

with respect to the Poincaré metric of $\widehat{P}_{d_0-1}(f_\lambda(z))$. Thus we have $\bigcap_{d\geq 0} \widehat{P}_d(f_\lambda(z)) = \{f_\lambda(z)\}$. By the construction of modified puzzle piece, $\overline{P_d(f_\lambda(z))} \subset \widehat{P}_d(f_\lambda(z)) \cup A_\lambda$ for any $d \geq 0$, thus $\text{End}(f_\lambda(z)) \subset \{f_\lambda(z)\} \cup A_\lambda$. Since $\text{End}(f_\lambda(z))$ has no intersection with A_λ, $\text{End}(f_\lambda(z)) = \{f_\lambda(z)\}$. □

Proposition 5.7.1. *If $T(c)$ is not periodic for any $c \in C_\lambda$, then the Julia set $J(f_\lambda)$ is locally connected.*

Proof. Note that $T(c)$ is either critical or non-critical. First we prove $\text{End}(c) = \{c\}$ and $\text{End}(z) = \{z\}$ for any $z \in J(f_\lambda) \setminus J_0$. Then we deal with the points that lie in J_0.

Case 1: $T(c)$ **is critical.** Since the graph is admissible, we can find a non-degenerate annulus $A_{d_0}(c)$. Consider the descendants of $\text{Row}_c(d_0)$. It's obvious that if $\text{Row}_c(t)$ is a descendent in the k-th generation of $\text{Row}_c(d_0)$, the annulus $A_t(c)$ is non-degenerate with modulus $\text{mod}(A_{d_0}(c))/2^k$. If $\text{Row}_c(d_0)$ has at least 2^k descendants in the k-th generation for each $k \geq 1$, then each of these contributes exactly $\text{mod}(A_{d_0}(c))/2^k$ to the sum $\sum_d \text{mod}(A_d(c))$. Hence $\sum_d \text{mod}(A_d(c)) = \infty$, as required. On the other hand, if there are fewer descendants in some generation, then one of them, say $\text{Row}_c(m)$ must be an only child, hence excellent by Lemma 5.4.7. Again by Lemma 5.4.7, we see that $\sum_d \text{mod}(A_d(c)) = \infty$. Therefore in either case, $\text{End}(c) = \{c\}$.

Now, consider a point $z \in J(f_\lambda) \setminus (J_0 \cup C_\lambda)$. If $T(z)$ is non-critical, then by Lemma 5.7.1, $\text{End}(z) = \{z\}$. If $T(z)$ is critical, then for each $d \geq 1$, there is a smallest integer $l_d \geq 0$ such that both (d, l_d) and (d, l_d+1) are critical positions. It follows that $f_\lambda^{l_d} : A_{d+l_d}(z) \to A_d(c')$ is a conformal map for some $c' \in C_\lambda$. In this case $\sum_d \mathrm{mod}(A_d(z)) \geq \sum_d \mathrm{mod}(A_{d+l_d}(z)) = \sum_d \mathrm{mod}(A_d(c)) = \infty$, hence $\mathrm{End}(z) = \{z\}$.

Case 2: $T(c)$ **is non-critical.** It follows from Lemma 5.7.1 that $\mathrm{End}(c) = \{c\}$. For $z \in J(f_\lambda) \backslash (J_0 \cup C_\lambda)$, we assume $T(z)$ is critical, for else $\mathrm{End}(z) = \{z\}$ by Lemma 5.7.1. Suppose $A_{d_0}(c)$ is a non-degenerate annulus, and $(d_0 + 1, l_1), (d_0 + 1, l_2), \cdots$ are all critical positions in the $(d_0 + 1)$-th row of the tableau $T(z)$. Since all critical tableaux are non-critical, there is a constant D such that $\deg(f_\lambda^{l_k} : P_{d_0+l_k}(z) \to P_{d_0,l_k}(z)) \leq D$ for all $k \geq 1$. Thus

$$\mathrm{mod}(A_{d_0+l_k}(z)) \geq D^{-1}\mathrm{mod}(A_{d_0}(c))$$

for all $k \geq 1$. Hence $\sum_d \mathrm{mod}(A_d(z)) \geq \sum_k \mathrm{mod}(A_{d_0+l_k}(z)) = \infty$ and $\mathrm{End}(z) = \{z\}$.

Points that lie in J_0**.** For any $z \in J_0$, the orbit $z \mapsto f_\lambda(z) \mapsto f_\lambda^2(z) \mapsto \cdots$ eventually meets the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. So the Euclidean distance between the critical set C_λ and the orbit $\{f_\lambda^k(z)\}_{k\geq 0}$ is bounded below by some positive number $\varepsilon(z)$. In addition, for every d large enough, z lies in the common boundary of exactly two puzzle pieces of depth d. We denote these two puzzle pieces by $P'_d(z)$ and $P''_d(z)$. In the previous argument, we have already proved that $\mathrm{End}(c) = \{c\}$, this implies $\mathrm{Eucl.diam}(P_d(c)) \to 0$ as $d \to \infty$. Choose d_0 large enough such that

$$\mathrm{Eucl.diam}(P_{d_0}(c)) < \varepsilon(z) \leq \mathrm{Eucl.dist}(C_\lambda, \{f_\lambda^k(z)\}_{k\geq 0}).$$

Then the orbit $z \mapsto f_\lambda(z) \mapsto f_\lambda^2(z) \mapsto \cdots$ avoids all the critical puzzle pieces of depth d_0. Let $P_d^*(z) = \overline{P'_d(z) \cup P''_d(z)}$ for d large enough. Then the proof of Lemma 5.7.1 applies equally well to this situation and $\bigcap_d P_d^*(z) = \{z\}$ follows immediately.

Connectivity of neighborhoods. Let

$$P_d^*(z) = \begin{cases} \overline{P_d(z)}, & \text{if } z \in J(f_\lambda) \setminus J_0, \\ \overline{P'_d(z) \cup P''_d(z)}, & \text{if } z \in J_0 \text{ and } d \text{ is large.} \end{cases}$$

By Lemma 5.4.2, for every $z \in J(f_\lambda)$ and every large integer d, the intersection $P_d^*(z) \cap J(f_\lambda)$ is a connected and compact subset of $J(f_\lambda)$. Thus $\{P_d^*(z) \cap J(f_\lambda)\}$ forms a basis of connected neighborhoods of z. Since $\bigcap(P_d^*(z) \cap J(f_\lambda)) = \{z\}$, the Julia set is locally connected at z. Note that z is arbitrarily chosen, $J(f_\lambda)$ is locally connected. □

5.7.2 Periodic critical tableau case

Suppose the critical tableau $T(c)$ is k-periodic for some $k > 0$. By the proof of Lemma 5.5.1, f_λ is either k-renormalizable at c or $k/2$-$*$-renormalizable at c.

Let $(\varepsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ be the renormalization, where d_0 is a large integer and

$$(\varepsilon, p) = \begin{cases} (1,k), & \text{if } f_\lambda \text{ is } k\text{-renormalizable at } c, \\ (-1, k/2), & \text{if } f_\lambda \text{ is } k/2\text{-}*\text{-renormalizable at } c. \end{cases}$$

The small filled Julia set of the renormalization $(\varepsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is denoted by K_c. Recall that β_c is the β-fixed point of the renormalization and β_c' is the other preimage of β_c under the map $\varepsilon f_\lambda^p|_{P_{d_0+p}(c)}$.

Assume now $K_c \cap \partial B_\lambda \neq \emptyset$, then by Lemma 5.5.3, $\beta_c \in K_c \cap \partial B_\lambda$ and there is a unique external ray, say $R_\lambda(\theta)$, landing at β_c. The angle θ is of the form $\frac{m}{2^k-1}$. It follows that $\beta_c' \in K_c \cap \partial T_\lambda$ and there is a unique radical ray $R_{T_\lambda}(\alpha_\theta)$ in T_λ landing at β_c'. The radical ray $R_{T_\lambda}(\alpha_\theta)$ satisfies $\varepsilon f_\lambda^p(R_{T_\lambda}(\alpha_\theta)) = R_\lambda(\theta)$. Let

$$K = K_c \cup \overline{R_\lambda(\theta)} \cup \overline{R_{T_\lambda}(\alpha_\theta)} \cup (-K_c) \cup (-\overline{R_\lambda(\theta)}) \cup (-\overline{R_{T_\lambda}(\alpha_\theta)}).$$

The set K is a connected and compact subset of $\bar{\mathbb{C}}$. Note that $-R_{T_\lambda}(\alpha_\theta) = R_{T_\lambda}(\alpha_\theta + 1/2)$. Let Δ_1 be the component of $\bar{\mathbb{C}} \setminus (K \cup \overline{B_\lambda})$ that intersects with $Q_{T_\lambda}(\alpha_\theta, \alpha_\theta + 1/2)$ and Δ_2 be the component of $\bar{\mathbb{C}} \setminus (K \cup \overline{B_\lambda})$ that intersects with $Q_{T_\lambda}(\alpha_\theta + 1/2, \alpha_\theta)$, where we use $Q_{T_\lambda}(\theta_1, \theta_2)$ to denote the set $\overline{\{\phi_{T_\lambda}(re^{2\pi i t}); 0 < r < 1, \theta_1 \le t \le \theta_2\}}$. Since $K \cup \overline{B_\lambda}$ is connected and compact, both Δ_1 and Δ_2 are disks. Let Z_i be the component of $\bar{\mathbb{C}} \setminus K$ that contains Δ_i.

The aim of this section is to prove:

Proposition 5.7.2. *Assume that $K_c \cap \partial B_\lambda \neq \emptyset$, then for $i \in \{1,2\}$, there is a curve $\mathcal{L}_i \subset \Delta_i \cup \{0\}$ stemming from T_λ and converging to β_c. More precisely, $\mathcal{L}_i$ can be parameterized as $\mathcal{L}_i : [0, +\infty) \to \Delta_i \cup \{0\}$ such that $\mathcal{L}_i(0) = 0, \mathcal{L}_i((0,+\infty)) \subset \Delta_i$ and $\lim_{t\to+\infty} \mathcal{L}_i(t) = \beta_c$.*

Proof. Let $\Gamma = \bigcup_{j\ge 0}(\pm f_\lambda^j(K_c \cup \overline{R_\lambda(\theta)}))$. By Lemma 5.5.3, any two different elements in the set $\{\pm f_\lambda^j(K_c \cup \overline{R_\lambda(\theta)}); j \ge 0\}$ intersect only at the point ∞. This implies $U = \bar{\mathbb{C}} \setminus \Gamma$ is a disk.

Step 1. *There exists $G_i : U \to U \cap Z_i$, which is an inverse branch of εf_λ^p, such that the sequence $\{G_i^k; k \ge 0\}$ converges locally and uniformly in U to a constant $z_i \in K_c$.*

Since U has no intersection with the critical orbits, its preimage $f_\lambda^{-1}(U)$ has exactly $2n$ components, say $V_1, \cdots, V_{2n}$. These components are arranged symmetrically about the origin under the rotation $z \mapsto e^{\pi i/n} z$. For every $1 \le j \le 2n$, $f_\lambda : V_j \to U$ is a conformal map. Moreover, $f_\lambda^{-1}(U) \subset \bar{\mathbb{C}} \setminus K$.

For $1 \le j \le p-1$, let Ω_j be the component of $f_\lambda^{-1}(U)$ such that $\overline{\Omega}_j \cap f_\lambda^j(K_c) \neq \emptyset$ and the inverse of $f_\lambda : \Omega_j \to U$ is denoted by g_j. For $j = 0$, let Ω_0^i

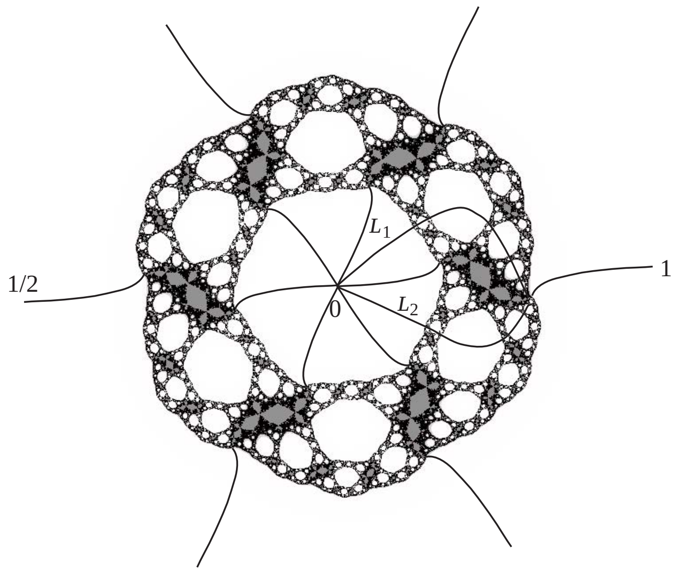

Figure 5.10: Constructing two curves L_1 and L_2 that converge to β_c, here $n = 3$ and f_λ is $1-$renormalizable at $c = c_0$.

be the component of $f_\lambda^{-1}(U)$ such that $\overline{\Omega_0^i} \cap K_c \neq \emptyset$ and $\Omega_0^i \subset Z_i$. The inverse of $f_\lambda : \Omega_0^i \to U$ is denoted by g_0^i for $i \in \{1, 2\}$.

Now, we define

$$G_i(z) = \begin{cases} g_0^i \circ g_1 \circ \cdots \circ g_{p-1}(\varepsilon z), & z \in U \quad \text{if } p \geq 2, \\ g_0^i(\varepsilon z), \quad z \in U & \text{if } p = 1. \end{cases}$$

Since $(\varepsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is a p-($*$-)renormalization of f_λ at c, we have $G_i(P_{d_0}(c) \cap U) \subset P_{d_0+p}(c) \cap Z_i$. The map $G_i : U \to U$ is not surjective, thus by Denjoy-Wolff theorem(See [M1]), the sequence $\{G_i^k; k \geq 0\}$ converges locally and uniformly in U to a constant z_i. It follows from $G_i(P_{d_0}(c) \cap U) \subset P_{d_0+p}(c) \cap Z_i$ that $z_i \in K_c$.

Step 2. *There exists a curve $C_i \subset U \cap (\Delta_i \cup \{0\})$ connecting 0 with $G_i(0)$ for $i \in \{1, 2\}$.*

Since the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$ is admissible, the filled Julia set K_c is disjoint from the boundary of any puzzle piece. Thus for any $\alpha \in \{\tau^s(\theta_j); 1 \leq j \leq N, s \geq 0\}$, Γ is disjoint from the cut ray Ω_λ^α outside ∞. For any angle $\alpha \in \{\tau^s(\theta_j); 1 \leq j \leq N, s \geq 0\}$ and any map $g \in \{g_0^1, g_0^2, g_1, \cdots, g_{p-1}\}$, by Proposition 5.4.2, only one curve of $g(\omega_\lambda^\alpha \setminus \{\infty\}), g(\omega_\lambda^{\alpha+1/2} \setminus \{\infty\})$ intersects with ∂B_λ and the other curve connects 0 with a preimage of 0.

Fix an angle $\alpha \in \{\tau^s(\theta_j); 1 \leq j \leq N, s \geq 0\}$, we define a curve family $\mathcal{F}$ by

$$\mathcal{F} = \{\varepsilon\omega_\lambda^\alpha \setminus \{\infty\};\ \varepsilon^{2n} = 1 \text{ and } \ \varepsilon\omega_\lambda^\alpha \subset \cup_{j \in \mathbb{I} \setminus \{0,n\}} S_j\}.$$

We construct the curve C_i by inductive procedure as following:

First, choose a curve $\zeta_{p-1} \in \mathcal{F}$ such that $g_{p-1}(\zeta_{p-1}) \cap \partial B_\lambda = \emptyset$, and let $\gamma_{p-1} = g_{p-1}(\zeta_{p-1})$. Suppose that for some $2 \leq j \leq p-1$, we have already constructed the curves $\gamma_{p-1}, \cdots, \gamma_j$. Then we choose $\zeta_{j-1} \in \mathcal{F}$ such that $g_{j-1}(\zeta_{j-1}) \cap \partial B_\lambda = \emptyset$ and $\zeta_{j-1} \cap \gamma_j = \emptyset$, and let $\gamma_{j-1} = g_{j-1}(\zeta_{j-1} \cup \gamma_j)$. In this way, we can construct a sequence of curves $\gamma_{p-1}, \gamma_{p-2}, \cdots, \gamma_2, \gamma_1$ step by step and each curve has no intersection with ∂B_λ, connecting 0 with some iterated preimage of 0. By construction,

$$\gamma_1 = \bigcup_{1 \leq j \leq p-1} g_1 \circ \cdots \circ g_j(\zeta_j).$$

Now, we choose $\zeta_0^i \in \mathcal{F}$ such that $g_0^i(\zeta_0^i) \cap \partial B_\lambda = \emptyset$ and $\zeta_0^i \cap \gamma_1 = \emptyset$, and let

$$C_i = \begin{cases} g_0^i(\zeta_0^i \cup \gamma_1) \cup \{0\}, & \text{if } p \geq 2, \\ g_0^i(\zeta_0^i) \cup \{0\}, & \text{if } p = 1. \end{cases}$$

The curve C_i connects 0 with $G_i(0)$ and $C_i \subset U \cap (\Delta_i \cup \{0\})$, as required.

Step 3. *The union $\mathcal{L}_i = \bigcup_{j \geq 0} G_i^j(C_i)$ is the curve contained in $\Delta_i \cup \{0\}$ and converging to β_c.*

By construction, $G_i(\mathcal{L}_i) \subset G_i(\mathcal{L}_i) \cup C_i = \mathcal{L}_i$ and $\mathcal{L}_i \setminus \{0\} \subset \Delta_i$.

To finish, we show $\mathcal{L}_i$ converges to β_c. By step 1, the sequence $\{G_i^k; k \geq 0\}$ converges uniformly on any compact subset of U to a constant $z_i \in K_c$. Since C_i is a compact subset of U, the curve $\mathcal{L}_i$ converges to $z_i \in K_c$ and $G_i(z_i) = z_i$. Since $\varepsilon f_\lambda^p(\mathcal{L}_i) \supset \mathcal{L}_i$, we conclude $z_i = \beta_c$ by Lemma 5.5.2. $\square$

Corollary 5.7.2. *If $T(c)$ is periodic for some $c \in C_\lambda$, then ∂B_λ is locally connected.*

Proof. We may assume that f_λ is not geometrically finite, otherwise the Julia set is locally connected (see [TY]). Thus f_λ has no parabolic point.

If $K_c \cap \partial B_\lambda = \emptyset$, then for all $j \geq 0$, $f_\lambda^j(K_c) \cap \partial B_\lambda = \emptyset$. Since $P(f_\lambda)$ is a subset of $(\bigcup_{j \geq 0} f_\lambda^j(\pm f_\lambda(K_c))) \bigcup \{\infty\}$, we conclude $P(f_\lambda) \cap \partial B_\lambda = \emptyset$. By Proposition 5.6.1 and Proposition 5.6.2, ∂B_λ is locally connected.

If $K_c \cap \partial B_\lambda \neq \emptyset$, then by Proposition 5.7.2, the closed curve $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{\beta_c\}$ separates $K_c \setminus \{\beta_c\}$ from $\partial B_\lambda \setminus \{\beta_c\}$. In this case, for all $j \geq 0$, $f_\lambda^j(K_c) \cap \partial B_\lambda = \{f_\lambda^j(\beta_c)\}$. Thus $\#(P(f_\lambda) \cap \partial B_\lambda) < \infty$ and all periodic points in $P(f_\lambda) \cap \partial B_\lambda$ are repelling. Again by Proposition 5.6.1 and Proposition 5.6.2, ∂B_λ is locally connected. $\square$

5.7.3 Real case

In this section, we will deal with real parameters. By the symmetry of the parameter plane, we just need consider $\lambda \in \mathbb{R}^+ = (0, +\infty)$. In this case, the

Julia set $J(f_\lambda)$ is symmetric about the real axis. If $C_\lambda \subset A_\lambda$, by 'The Escape Trichotomy' (Theorem 5.2.1), the Julia set $J(f_\lambda)$ is either a Cantor set, a Cantor set of circles or a Sierpinski curve. The local connectivity of ∂B_λ is already known in the latter two cases. In the following discussion, we assume $C_\lambda \cap A_\lambda = \emptyset$.

Lemma 5.7.2. *Suppose $\lambda \in \mathbb{R}^+$ and $C_\lambda \cap A_\lambda = \emptyset$, then f_λ is 1-renormalizable at $c_0 = \sqrt[2n]{\lambda}$.*

Proof. Let U be the interior of $(S_0 \cup S_{-(n-1)}) \setminus \{z \in B_\lambda \cup T_\lambda; G_\lambda(z) \geq 1\}$ and $V = \bar{\mathbb{C}} \setminus (\{z \in B_\lambda; G_\lambda(z) \geq n\} \cup [-\infty, v_\lambda^-])$. One can easily verify that $f_\lambda : U \to V$ is a quadratic-like map. Since $C_\lambda \cap A_\lambda = \emptyset$, the critical orbit $\{f_\lambda^k(c_0); k \geq 0\}$ is contained in $U \cap \mathbb{R}^+$. This implies that (f_λ, U, V) is a 1-renormalization of f_λ at c_0. □

Let $K_{c_0} = \bigcap_{k\geq 0} f_\lambda^{-k}(U)$ be the small filled Julia set of the renormalization (f_λ, U, V), β_{c_0} be the $\beta-$fixed point and β'_{c_0} be the preimage of β_{c_0}. It's easy to check that K_{c_0} is symmetric about the real axis and $K_{c_0} \cap \mathbb{R}^+$ is a connected and closed interval.

Proposition 5.7.3. $K_{c_0} \cap \partial B_\lambda = \{\beta_{c_0}\}$.

Proof. The idea of the proof is to construct a Jordan curve $\mathcal{C}$ that separates $K_{c_0} \setminus \{\beta_{c_0}\}$ from $\partial B_\lambda \setminus \{\beta_{c_0}\}$, similar as the proof of Proposition 5.7.2.

We first show that β_{c_0} is the landing point of the zero external ray $R_\lambda(0)$. Note that rational external rays (i.e. external rays with a rational angle) always land. Let z_0 be the landing point of $R_\lambda(0)$. Obviously, $R_\lambda(0) \subset \mathbb{R}^+$ and z_0 is a fixed point of f_λ. This implies $z_0 \in U \cap \mathbb{R}^+$ and the orbit of z_0 does not escape from U, so $z_0 \in K_{c_0}$. Since $R_\lambda(0)$ is an f_λ-invariant ray that lands at z_0, we conclude $z_0 = \beta_{c_0}$ by Lemma 5.5.2.

Let $K = K_{c_0} \cup [\beta_{c_0}, +\infty] \cup (-K_{c_0}) \cup [-\infty, -\beta_{c_0}]$. One can easily verify $f_\lambda^{-1}(K) = \bigcup_{\omega^{2n}=1} \omega(K_{c_0} \cup [0, +\infty])$. The set $Y = \bar{\mathbb{C}} \setminus K$ is a disk and its preimage $f_\lambda^{-1}(Y)$ consists of $2n$ components, which are symmetric about the origin under the rotation $z \mapsto e^{i\pi/n}z$. For each component X of $f_\lambda^{-1}(Y)$, $f_\lambda : X \to Y$ is a conformal map. Let X_0 be the component of $f_\lambda^{-1}(Y)$ that is contained in S_0 and g be the inverse map of $f_\lambda : X_0 \to Y$. By Denjoy-Wolff theorem, the sequence of maps $\{g^k; k \geq 0\}$ converges locally and uniformly in Y to a constant, say x. Since $g(Y \cap V) \subset X_0 \cap U$, we conclude $x \in K_{c_0}$.

Let Δ be the component of $\bar{\mathbb{C}} \setminus (\overline{B}_\lambda \cup K_{c_0} \cup (-K_{c_0}) \cup \mathbb{R})$ that intersects with T_λ and lies in the upper half plane.

Claim: *There is a path $\mathcal{L} \subset \Delta \cup \{0\}$ stemming from T_λ and converging to β_{c_0}. More precisely, $\mathcal{L}$ can be parameterized as $\mathcal{L} : [0, +\infty) \to \Delta \cup \{0\}$ such that $\mathcal{L}(0) = 0, \mathcal{L}((0, +\infty)) \subset \Delta$ and $\lim_{t\to+\infty} \mathcal{L}(t) = \beta_{c_0}$*

Let $p_0 = \sqrt[2n]{-\lambda}$ be the preimage of 0 that lies in S_0 and $\gamma_0 = [0, p_0]$ be the segment connecting 0 with p_0. Then $\gamma_0 \cap (K_{c_0} \cup \partial B_\lambda) = \emptyset$. Indeed, $\gamma_0 \cap K_{c_0} = \emptyset$ follows from the fact that $f_\lambda(\gamma_0) \cap K_{c_0} \subset i\mathbb{R} \cap K_{c_0} = \emptyset$. In the following, we show $\gamma_0 \cap \partial B_\lambda = \emptyset$. It suffices to show $B_\lambda \cap D = \emptyset$, where $D = \{z \in \mathbb{C}; |z| < \sqrt[2n]{\lambda}\}$. Otherwise, $B_\lambda \cap D \neq \emptyset$ implies $B_\lambda \cap \partial D \neq \emptyset$. Since $\varphi : z \mapsto \sqrt[n]{\lambda}/\bar{z}$ maps B_λ onto T_λ and the restriction $\varphi|_{\partial D}$ is the identity map, we have $B_\lambda \cap \partial D = \varphi(B_\lambda \cap \partial D) = T_\lambda \cap \partial D$. But this is a contradiction.

Note that g maps γ_0 outside D and $g(\gamma_0)$ connects p_0 with a preimage of p_0 that lies inside S_0. Let $\mathcal{L} = \bigcup_{k \geq 0} g^k(\gamma_0)$. By construction, $\mathcal{L} \cap (K_{c_0} \cup \partial B_\lambda) = \emptyset$ and $\mathcal{L}$ converges to $x \in K_{c_0}$. Since $f_\lambda(\mathcal{L}) = \mathcal{L} \cup f_\lambda(\gamma_0) \supset \mathcal{L}$, we conclude $x = \beta_{c_0}$ by Lemma 5.5.2.

Let $\mathcal{C} = \mathcal{L} \cup \mathcal{L}^* \cup \{\beta_{c_0}\}$, where $\mathcal{L}^* = \{\bar{z}; z \in \mathcal{L}\}$. $\mathcal{C}$ is a Jordan curve separating $K_{c_0} \setminus \{\beta_{c_0}\}$ from $\partial B_\lambda \setminus \{\beta_{c_0}\}$. The conclusion follows. $\square$

Remark 5.7.1. *From the proof of Proposition 5.7.3, we conclude*

$$\partial B_\lambda \cap \mathbb{R} = \{\pm\beta_{c_0}\},\ K_{c_0} \cap \mathbb{R} = [\beta'_{c_0}, \beta_{c_0}],\ \partial T_\lambda \cap \mathbb{R} = \{\pm\beta'_{c_0}\}.$$

Corollary 5.7.3. *Suppose $\lambda \in \mathbb{R}^+$ and $C_\lambda \cap A_\lambda = \emptyset$, then ∂B_λ is locally connected.*

Proof. By Proposition 5.7.3, if n is odd, then $P(f_\lambda) \cap \partial B_\lambda \subset (-K_{c_0} \cup K_{c_0}) \cap \partial B_\lambda \subset \{\pm\beta_{c_0}\}$; if n is even, then $P(f_\lambda) \cap \partial B_\lambda \subset K_{c_0} \cap \partial B_\lambda \subset \{\beta_{c_0}\}$. If β_{c_0} is a parabolic point, then f_λ is geometrically finite, the local connectivity of ∂B_λ follows from [TY]. Else, by Proposition 5.6.1 and Proposition 5.6.2, ∂B_λ is also locally connected. $\square$

5.7.4 Local connectivity implies higher regularity

Up to now, we have already proved that ∂B_λ is locally connected if the Julia set is not a Cantor set. By the arguments of Devaney [D1], we prove the following proposition which will lead to Theorem 5.1.1.

Proposition 5.7.4. *If ∂B_λ is locally connected, then ∂B_λ is a Jordan curve.*

Proof. Let W_0 be the component of $\overline{\mathbb{C}} - \overline{B}_\lambda$ containing 0. It's obvious that $\partial W_0 \subset \partial B_\lambda$, $T_\lambda \subset W_0$, $\partial T_\lambda \subset \overline{W}_0$. By Lemma 5.2.1, $e^{i\pi/n} W_0 = W_0$.

Recall that $H_\lambda(z) = \sqrt[n]{\lambda}/z$, so $H_\lambda(\partial W_0) \subset H_\lambda(\partial B_\lambda) = \partial T_\lambda \subset \overline{W}_0$. Since ∂B_λ is locally connected, ∂W_0 is locally connected. It follows that $\overline{\mathbb{C}} - \overline{W}_0$ is connected and $H_\lambda(\overline{\mathbb{C}} - \overline{W}_0) \subset W_0$.

Now we show $f_\lambda^{-1}(0) \subset W_0$. If not, $f_\lambda^{-1}(0) \cap (\overline{\mathbb{C}} - \overline{W}_0) \neq \emptyset$. By the symmetry of $f_\lambda^{-1}(0)$ and $\overline{\mathbb{C}} - \overline{W}_0$, we have $f_\lambda^{-1}(0) \subset \overline{\mathbb{C}} - \overline{W}_0$. This will contradict the fact that $f_\lambda^{-1}(0) = H_\lambda(f_\lambda^{-1}(0)) \subset H_\lambda(\overline{\mathbb{C}} - \overline{W}_0) \subset W_0$.

Since any point on ∂W_0 can not be mapped into W_0, we have $f_\lambda^{-1}(W_0) \subset W_0$ and $f_\lambda^{-1}(\overline{W}_0) \subset \overline{W}_0$. Take a point $z \in \partial W_0$, we have $\partial B_\lambda \subset J(f_\lambda) = \overline{\bigcup_{k\geq 0} f_\lambda^{-k}(z)} \subset \overline{W}_0$ and $\partial B_\lambda \subset \partial W_0$. Therefore $\partial W_0 = \partial B_\lambda$.

Now we show that ∂B_λ is a Jordan curve. If two different external rays, say $R_\lambda(t_1)$ and $R_\lambda(t_2)$, land at the same point $p \in \partial B_\lambda$, then $\overline{R_\lambda(t_1) \cup R_\lambda(t_2)}$ decomposes ∂B_λ into two parts. It turns out that $\partial W_0 \neq \partial B_\lambda$, which is a contradiction. □

The aim of this section is to prove that ∂B_λ is a quasicircle in almost all cases. Formally, we have the following

Theorem 5.7.3. *Suppose the Julia set $J(f_\lambda)$ is not a Cantor set, then the boundary ∂B_λ is a quasicircle if it contains neither parabolic point nor recurrent critical point.*

Proof. By Theorem 1 and Proposition 5.6.1, it suffices to show that f_λ satisfies **BD** condition on ∂B_λ. First we deal with three special cases:

Case 1. The critical orbit escapes to infinity.

Case 2. The parameter $\lambda \in \mathbb{R}^+$ and ∂B_λ contains no parabolic point.

Case 3. The map f_λ is critically finite.

In case 1, $P(f_\lambda) \cap \partial B_\lambda = \emptyset$. By Proposition 5.6.2, f_λ satisfies **BD** condition on ∂B_λ. For case 2, by Proposition 5.7.4, either $P(f_\lambda) \cap \partial B_\lambda = \emptyset$ or $P(f_\lambda) \cap \partial B_\lambda = \{\beta_c\}$ or $P(f_\lambda) \cap \partial B_\lambda = \{\pm\beta_c\}$. In either case, β_c is a repelling fixed point of f_λ. By Proposition 5.6.1, f_λ satisfies **BD** condition on ∂B_λ. For case 3, f_λ satisfies **BD** condition on ∂B_λ by Corollary 5.6.2.

In the remaining cases, we can use Yoccoz puzzle to study the higher regularity of ∂B_λ. There are two remaining cases:

Case 4. ∂B_λ contains no critical point.

Case 5. $C_\lambda \subset \partial B_\lambda$ and all critical points in C_λ are non-recurrent.

In either case, by Proposition 5.4.1, we can find an admissible graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. With respect to the Yoccoz puzzle induced by this graph, we consider the critical tableaux. For case 4, there are two possibilities:

Case 4.1. There is a periodic critical tableau $T(c)$.

Case 4.2. There is no periodic critical tableau.

For case 4.1, we conclude from Proposition 5.7.2 that $\#(P(f_\lambda) \cap \partial B_\lambda) < \infty$. Since ∂B_λ contains no parabolic point, all periodic points in $P(f_\lambda) \cap \partial B_\lambda$ are repelling. Thus by Proposition 5.6.2, f_λ satisfies **BD** condition on ∂B_λ.

For case 4.2, we have already shown that $\mathrm{End}(c) = \bigcap_{d\geq 0} \overline{P_d(c)} = \{c\}$ for $c \in C_\lambda$ in the proof of Proposition 5.7.1. Thus we can choose d_0 large enough such that

$$\mathrm{Eucl.diam}(P_{d_0}(c)) < \mathrm{Eucl.dist}(c, \partial B_\lambda).$$

For $d \geq d_0$, let U_d be the union of all puzzle pieces of depth d that intersect with ∂B_λ and V_d be the interior of $\overline{U_d}$. For every $u \in \partial B_\lambda$, there is a number $\varepsilon_u > 0$ such that $B(u, \varepsilon_u) \subset V_{d_0}$. For any $m \geq 0$ and any component $U_m(u)$ of $f_\lambda^{-m}(B(u, \varepsilon_u))$ intersecting with ∂B_λ, $U_m(u) \subset V_{d_0+m} \subset V_{d_0}$. By the choice of d_0, the sequence $U_m(u) \to \cdots \to f_\lambda^{m-1}(U_m(u)) \to B(u, \varepsilon_u)$ meets no critical point of f_λ, thus $f_\lambda^m : U_m(u) \to B(u, \varepsilon_u)$ is a conformal map. So in this case, f_λ satisfies **BD** condition on ∂B_λ.

In the following, we deal with case 5. Again by Proposition 5.7.1, $\mathrm{End}(c) = \{c\}$ for $c \in C_\lambda$. Thus in this case, one can verify that ∂B_λ contains no recurrent critical point if and only if all critical tableaux are non-critical. By Lemma 5.5.1, f_λ is critically finite. It follows from Corollary 5.6.1 that f_λ satisfies **BD** condition on ∂B_λ. □

5.7.5 Corollaries

In this section, we present some corollaries of Theorem 5.1.1.

Proposition 5.7.5. *If ∂B_λ contains a parabolic cycle, then the multiplier of the cycle is* 1 *and the Julia set $J(f_\lambda)$ contains a quasiconformal copy of quadratic Julia set of $z \mapsto z^2 + 1/4$.*

Proof. Suppose $\mathcal{C} = \{z_0, f_\lambda(z_0), \cdots, f_\lambda^q(z_0) = z_0\}$ is a parabolic cycle on ∂B_λ. We will first consider the case $\lambda \in \mathbb{R}^+$, then deal with the case $\lambda \in \mathcal{H}$.

First suppose $\lambda \in \mathbb{R}^+$. By Lemma 5.7.2 and Proposition 5.7.3, f_λ is $1-$renormalizable at c_0 and $P(f_\lambda) \cap \partial B_\lambda \subset (-K_{c_0} \cup K_{c_0}) \cap \partial B_\lambda = \{\pm\beta_{c_0}\}$. Since a parabolic point must attract a critical point, we conclude that β_{c_0} is a parabolic fixed point of f_λ. So (f_λ, U, V) is quasiconformally conjugate to a quadratic polynomial $z \mapsto z^2 + \mu$ whose $\beta-$fixed point is also a parabolic point, thus $\mu = 1/4$. The conclusion follows in this case.

In the following, we deal with the case $\lambda \in \mathcal{H}$. By Proposition 5.4.1, we can find an admissible graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. By Proposition 5.3.4, the parabolic cycle $\mathcal{C}$ avoid the graph $\mathbf{G}_\lambda(\theta_1, \cdots, \theta_N)$. With respect to the Yoccoz puzzle induced by this graph and by the similar argument as Corollary 5.5.1, we conclude that there is a critical point $c \in C_\lambda$ and a point $z \in \mathcal{C}$ such that $P_d(z) = P_d(c)$ for all $d \geq 0$. Thus the critical tableau $T(c)$ is periodic. Suppose the period of $T(c)$ is k. It is obvious that k is a divisor of q. By Lemma 5.5.1, when d_0 is large enough, the triple $(\varepsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is either a k-renormalization of f_λ at c (in this case, $(\varepsilon, p) = (1, k)$) or a $k/2$-$*$-renormalization of f_λ at c (in this case, $(\varepsilon, p) = (-1, k/2)$). Moreover, the small filled Julia set $K_c = \mathrm{End}(c) = \bigcap_{d \geq 0} \overline{P_d(c)}$ and $z \in K_c \cap \partial B_\lambda$.

On the other hand, by Lemma 5.5.3, there is a unique external ray $R_\lambda(t)$ landing at β_c, which is the β-fixed point of the renormalization

$(\varepsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$. Note that we have already proved that ∂B_λ is a Jordan curve, the intersection $\partial B_\lambda \cap \overline{P_d(c)}$ shrinks to a single point as $d \to \infty$. Thus we have $K_c \cap \partial B_\lambda = \{\beta_c\}$. By the previous argument, $\beta_c = z$.

By the straightening theorem of Douady and Hubbard, $(\varepsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is quasi-conformally conjugate to a quadratic polynomial $p_\mu(z) = z^2 + \mu$ in a neighborhood of the small filled Julia set K_c. For this quadratic polynomial, the β-fixed point is also a parabolic point, thus $\mu = 1/4$. This means that the Julia set $J(f_\lambda)$ contains a quasiconformal copy of quadratic Julia set of $z \mapsto z^2 + 1/4$. Since the multiplier of the parabolic point of $z \mapsto z^2 + 1/4$ if 1, it turns out that $(\varepsilon f_\lambda^p)'(z) = 1$, $(f_\lambda^k)'(z) = 1$ and $(f_\lambda^q)'(z) = 1$. □

Proposition 5.7.6. *Suppose f_λ has no Siegel disk and the Julia set $J(f_\lambda)$ is connected, then every Fatou component is a Jordan domain.*

Proof. By Proposition 5.7.4 and the fact $H_\lambda(B_\lambda) = T_\lambda$, we conclude that both T_λ and B_λ are Jordan domains.

If the critical orbit tends to ∞, then the Julia set is a Sierpinski curve which is locally connected, and all Fatou components are quasidisks (By Proposition 5.6.1).

If the critical orbit remains bounded, then for any $U \in \mathcal{P} \setminus \{T_\lambda, B_\lambda\}$, there is a smallest integer $k \geq 1$ such that $f_\lambda^k : U \to T_\lambda$ is a conformal map. Thus if two radial rays $R_U(\theta_1)$ and $R_U(\theta_2)$ land at the same point, then $R_{T_\lambda}(\theta_1) = f_\lambda^k(R_U(\theta_1))$ and $R_{T_\lambda}(\theta_2) = f_\lambda^k(R_U(\theta_2))$ also land at the same point. This implies that U is also a Jordan domain. If there are other Fatou components, then they are eventually mapped to a parabolic basin or an attracting basin. By Proposition 5.5.1, the map is either renormalizable or $*-$renormalizable. It is known that every bounded Fatou component of a quadratic polynomial(without Siegel disk) is a Jordan disk, it turns out that all Fatou components of f_λ are Jordan disks in this case. □

Proposition 5.7.7. *If f_λ has a Cremer point, then the Cremer point cannot lie on the boundary of any Fatou component. In other words, all Cremer points are buried on the Julia set.*

Proof. Suppose f_λ has a Cremer point z, then the Fatou set $F(f_\lambda) = \bigcup_{k\geq 0} f_\lambda^{-k}(B_\lambda)$. If z lies on the boundary of some Fatou component, then after integrations, one sees that $z \in \partial B_\lambda$. By Theorem 5.1.1, there is a periodic external ray $R_\lambda(t)$ landing at z. But this is a contradiction since every periodic external ray can only land at a parabolic point or a repelling point (By Snail Lemma, see [M1]). □

5.8 Local connectivity of the Julia set $J(f_\lambda)$

In this section, we study the local connectivity of the Julia set $J(f_\lambda)$. We will prove the following

Theorem 5.8.1. *Suppose f_λ has no Siegel disk and the Julia set $J(f_\lambda)$ is connected, then $J(f_\lambda)$ is locally connected in either of the following cases:*

1. *The critical orbit does not accumulate on the boundary ∂B_λ.*
2. *f_λ is neither renormalizable nor $*-$renormalizable.*
3. *The parameter λ is real and positive.*

The proof is based on the 'Characterization of Local Connectivity' (Proposition 5.8.1, See [Wh]) and 'Shrinking Lemma' (Proposition 5.8.2, See [TY] or [LM]), as follows

Proposition 5.8.1. *A connected and compact set $X \subset \overline{\mathbb{C}}$ is locally connected if and only if it satisfies the following conditions:*

1. *Every component of $\overline{\mathbb{C}} \setminus X$ is locally connected.*
2. *For any $\varepsilon > 0$, there are at most finitely many components of $\overline{\mathbb{C}} \setminus X$ with spherical diameter greater than ε.*

Proposition 5.8.2. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map and D be a topological disk whose closure $\overline{D}$ has no intersection with the post-critical set $P(f)$, then either $\overline{D}$ is contained in a Siegel disk or a Herman ring, or for any $\varepsilon > 0$, there are at most finitely many iterated preimages of D with spherical diameter greater than ε.*

Proof of Theorem 5.8.1.

1. If f_λ is geometrically finite, then $J(f_\lambda)$ is locally connected (See [TY]). Else, the Fatou set $F(f_\lambda) = \bigcup_{k\geq 0} f_\lambda^{-k}(B_\lambda)$. Since $\overline{B}_\lambda \cap P(f_\lambda) = \emptyset$, we conclude by Shrinking Lemma that for any $\varepsilon > 0$, there are at most finitely many iterated preimages of B_λ with spherical diameter greater than ε. By Proposition 5.8.1, $J(f_\lambda)$ is locally connected.

2. If f_λ is neither renormalizable nor $*-$renormalizable, then the parameter $\lambda \in \mathcal{H}$ by Lemma 5.7.2. We may assume f_λ is not critically finite, for else the Julia set is locally connected. Thus by Proposition 5.4.1, we can find an admissible graph. By Lemma 5.5.1, none of the critical tableaux is periodic. The local connectivity of $J(f_\lambda)$ follows from Proposition 5.7.1.

3. (The notations here are the same as in Section 5.7.3) We just need consider the case when f_λ is not geometrically finite. In this case, the Fatou set $F(f_\lambda) = \bigcup_{k\geq 0} f_\lambda^{-k}(B_\lambda)$. Note that for any $z > 0$, $f_\lambda(z) \geq 2\sqrt{z^n \cdot \frac{\lambda}{z^n}} = 2\sqrt{\lambda} = v_\lambda^+$. Thus $\{f_\lambda^k(v_\lambda^+); k \geq 0\} \subset [v_\lambda^+, \beta_{c_0}]$.

If $v_\lambda^+ = \beta'_{c_0}$, then one can easily verify that the triple (f_λ, U, V) is quasiconformally conjugate to the quadratic polynomial $z \mapsto z^2 - 2$, which is critically finite. So f_λ is also critically finite and the Julia set is locally connected.

If $v_\lambda^+ > \beta'_{c_0}$, then $\overline{T}_\lambda \cap [v_\lambda^+, \beta_{c_0}] = \emptyset$ by Remark 5.7.1. Since $P(f_\lambda) \subset [-\beta_{c_0}, v_\lambda^-] \cup [v_\lambda^+, \beta_{c_0}] \cup \{\infty\}$, we have $\overline{T}_\lambda \cap P(f_\lambda) = \emptyset$. By Proposition 5.8.2, for any $\varepsilon > 0$, there are at most finitely many iterated preimages of T_λ with spherical diameter greater than ε. By Proposition 5.8.1, the Julia set is locally connected. $\square$

Chapter 6

Decomposition Theorem and Thurston-type Theorems

6.1 Introduction

Let $f : S^2 \to S^2$ be an orientation preserving branched covering of degree at leat two. We denote by $\deg(f, x)$ the local degree of f at $x \in S^2$. The critical set Ω_f of f is defined by

$$\Omega_f = \{x \in S^2; \deg(f, x) > 1\},$$

and the postcritical set P_f of f is defined by

$$P_f = \overline{\bigcup_{n \geq 1} f^n(\Omega_f)}.$$

We say that f is postcritically finite (also called 'critically finite') if P_f is a finite set. Such a map is always called a Thurston map. For a Thurston map, we define a function $\nu_f : S^2 \to \mathbb{N} \cup \{\infty\}$ in the following way: For each $x \in S^2$, define $\nu_f(x)$ (may be ∞) as the least common multiple of the local degrees $\deg(f^n, y)$ for all $n > 0$ and all $y \in S^2$ such that $f^n(y) = x$. (Notice that $\nu_f(x) = 1$ if $x \notin P_f$). We call $\mathcal{O}_f = (S^2, \nu_f)$ the orbifold of f.

In 1980s, Thurston proved the following theorem:

Theorem 6.1.1. (Thurston) *Let $f : S^2 \to S^2$ be a critically finite branched covering. Suppose that $\mathcal{O}_f$ does not have signature $(2, 2, 2, 2)$. Then f is combinatorially equivalent to a rational function R if and only if for any f-stable multicurve Γ, we have $\lambda(\Gamma, f) < 1$. The rational function R is unique up to Möbius conjugation.*

Here, the definitions of 'multicurve' and 'combinatorially equivalent' will be presented below for a larger category of branched coverings, that covers the postcritically finite cases. A detailed proof of Thurston's theorem is given by Douady and Hubbard [DH1].

Thurston's theorem has connections with a number of related areas such as Teichmuller theory, quasiconformal surgery, dynamics of several complex variables, transversality, group theory, algorithm, etc.

There are many applications of Thurston's theorem. Here is an incomplete list: Geyer's sharp bounds on the number of harmonic polynomial roots [Gey2], Kiwi's characterization of polynomial laminations [Kiwi] (using previous work of Bielefield-Fisher-Hubbard [BFH] and Poirier [Poi]), Mikulich's classification of postcritically finite Newton maps, Milnor-Thurston's proof of monotonicity of entropy for unimodal maps [MT], McMullen's work on rational quotients [McM1], Pilgrim-Tan's cut-and-paste surgery along arcs ([PT1]), Rees' descriptions of parameter spaces [Rees2], Rees, Shishikura and Tan's studies on matings of polynomials ([Rees1],[ST], [Tan1], [Tan2]), ...

Over the years, there are several various attempts to generalize Thurston's theorem beyond postcritically finite rational maps. For example, David Brown [Bro], supported by the previous work of Hubbard and Schleicher [HS], has succeeded in extending the theory to the uni-critical polynomials with an infinite postcritical set (but always with a connected Julia set), and pushed it even further to the infinite degree case, namely the exponential maps. We would also like to mention a recent work of Hubbard-Schleicher-Shishikura [HSS]) extending Thurston's theorem to postcritically finite exponential maps. Cui-Tan[CT1] and Jiang-Zhang [JZ], independently, using different methods, extend Thurston's theorem to hyperbolic rational maps. Furthermore, Cui and Tan [CT2] extend Thurston's theorem to geometrically finite rational maps. Meanwhile, Zhang [Zh2] extends Thurston's theorem to a class of rational maps with Siegel disks.

In this work, we aim to extend Thurston's theorem to a large class of branched covering, namely 'non-parabolic' branched covering. Roughly speaking, a 'non-parabolic' branched covering is a proper branched covering for which each critical point either has finite orbit or is attracted to an attracting cycle, or is eventually mapped to the closure of some rotation domain. Before we are going on, we shall define these objects first.

We may identify S^2 with $\overline{\mathbb{C}}$.

Definition 6.1.1. (Rotation domain) *We say $\langle U_0, \cdots, U_{p-1}\rangle$ is a cycle of rotation domain of f if*

1. All U_i are disks or annuli, with disjoint closures and Jordan curve boundaries.

2. f should induce conformal isomorphisms

$$U_0 \xrightarrow{\cong} U_1 \xrightarrow{\cong} \cdots \xrightarrow{\cong} U_{p-1} \xrightarrow{\cong} U_p = U_0$$

and the return map $f^p : U_0 \to U_0$ is conformally conjugate to an irrational rotation.

3. Each boundary cycle of ∂U_j contains at least one critical point of f.

One may compare this definition with the definitions of Siegel disks and Herman rings for rational maps ([M1]). Let P'_f be the accumulation set of P_f.

Definition 6.1.2. (Attracting cycle) *We say $\langle z_0, \cdots, z_{p-1}\rangle$ is an attracting cycle of f if $\langle z_0, \cdots, z_{p-1}\rangle$ is contained in P'_f, and f is holomorphic in a neighborhood of this cycle with multiplier $|(f^p)'(z_0)| < 1$.*

We remark that: A periodic cycle near which the map f is holomorphic and attracting (i.e. the multiplier λ satisfies $|\lambda| < 1$) is not necessarily contained in P'_f. This kind of 'attracting' cycle may be artificial. This is one of the differences between branched coverings and rational maps. Another important and essential difference is, for an attracting cycle $\langle z_0, \cdots, z_{p-1}\rangle \subset P'_f$, the immediate attracting basin $\mathcal{A}_0 = \cup_{0\le i<p}\mathcal{A}_0(z_i)$ of $\langle z_0, \cdots, z_{p-1}\rangle$ does not necessarily contain a critical point, where $\mathcal{A}_0(z_i)$ is the component of $\{z \in \overline{\mathbb{C}}; f^{pk}(z) \to z_i \text{ as } k \to \infty\}$ that contains z_i. This case usually implies the existence of Thurston obstructions.

Definition 6.1.3. (Non-parabolic map) *We say that f is a non-parabolic map if each critical point of f either has finite orbit or is attracted to an attracting cycle, or is eventually mapped to the closure of some rotation domain.*

Given a non-parabolic map f, let $n_{RD}(f)$ be the number of rotation disk cycles, $n_{RA}(f)$ be the number of rotation annulus cycles and $n_A(f)$ be the number of attracting cycles. By definition, we see that

$$n_{RD}(f) + 2n_{RA}(f) + n_A(f) \le 2\deg(f) - 2.$$

Let f be a non-parabolic map,

$$\text{we call } f \begin{cases} \text{a Herman map, if } n_{RA}(f) > 0, n_{RD}(f) \ge 0 \text{ and } n_A(f) = 0, \\ \text{a Siegel map, if } n_{RA}(f) = 0, n_{RD}(f) > 0 \text{ and } n_A(f) = 0. \end{cases}$$

It's obvious that a Thurston map is a non-parabolic map with $n_{RA}(f) = n_{RD}(f) = n_A(f) = 0$.

Definition 6.1.4. (Marked set) *Let f be a non-parabolic map and R_f be the union of all rotation domains of f. A marked set P is a compact set that satisfies the following:*

1. *$f(P) \subset P$.*
2. *$P \supset P_f \cup \overline{R_f}$ and $P - (P_f \cup \overline{R_f})$ is a finite set.*

In the chapter, we always use a pair (f, P), a branched covering together with a marked set, to denote a non-parabolic map.

Definition 6.1.5. (C-equivalence) *Two non-parabolic maps (f,P) and (g,Q) are called combinatorially equivalent or 'c-equivalent' for short (resp. q.c-equivalent), if there is a pair (ϕ,ψ) of homeomorphisms (resp. quasiconformal maps) of $\overline{\mathbb{C}}$ such that*

1. $\phi\circ f = g\circ\psi$ and $\phi(P)=Q$.

2. ϕ and ψ are holomorphic in $R_f\cup N$, where R_f is the union of all rotation domains of f (if any) and N is a neighborhood of all attracting cycles (if any). If P'_f contains no attracting cycle, we set $N=\emptyset$.

3. ϕ and ψ are isotopic rel $P\cup N$. That is, there is a continuous map $H: [0,1]\times\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ such that for any $t\in[0,1]$, $H(t,\cdot):\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is a homeomorphism (resp. quasiconformal map), $H(0,\cdot)=\phi, H(1,\cdot)=\psi$ and $H(t,z)=\phi(z)$ for any $t\in[0,1]$ and any $z\in P\cup N$.

In this case, we say (f,P) is c-equivalent (resp. q.c-equivalent) to (g,Q) via (ϕ,ψ). Notice that a necessary condition for q.c-equivalence is that f is a qusiregular map.

Multicurve and Thurston obstruction

Let (f,P) be a non-parabolic map. A Jordan curve γ in $\overline{\mathbb{C}}\setminus P$ is called *null-homotopic* (resp. *peripheral*) in $\overline{\mathbb{C}}\setminus P$ if one of its complementary components contains no (resp. one) point of P, and called *non-peripheral* in $\overline{\mathbb{C}}\setminus P$ if each component of $\overline{\mathbb{C}}\setminus\gamma$ contains at least two points of P.

We say that $\Gamma=\{\gamma_1,\cdots,\gamma_n\}$ is a *multicurve* in $\overline{\mathbb{C}}\setminus P$ if each γ_i is a non-peripheral Jordan curve in $\overline{\mathbb{C}}\setminus P$, and they are mutually disjoint and no two homotopic in $\overline{\mathbb{C}}\setminus P$. Its (f,P)-transition matrix $W_\Gamma=(a_{ij})$ is defined by

$$a_{ij}=\sum_{\alpha\sim\gamma_i}\frac{1}{\deg(f:\alpha\to\gamma_j)},$$

where the summation is taken over all the components α of $f^{-1}(\gamma_j)$ which are homotopic to γ_i in $\overline{\mathbb{C}}\setminus P$.

A multicurve Γ in $\overline{\mathbb{C}}\setminus P$ is called (f,P)-*stable* if every component of $f^{-1}(\gamma)$ for $\gamma\in\Gamma$ is either null-homotopic, or peripheral, or homotopic in $\overline{\mathbb{C}}\setminus P$ to a curve $\delta\in\Gamma$.

We say that a multicurve Γ is a *Thurston obstruction* of (f,P) if Γ is (f,P)-stable and the leading eigenvalue $\lambda(\Gamma,f)$ of its transition matrix W_Γ satisfies $\lambda(\Gamma,f)\geq 1$.

For convention, an empty set $\Gamma=\emptyset$ is always considered as a (f,P)-stable multicurve with $\lambda(\Gamma,f)=0$.

The main theorem of this chapter is:

Theorem 6.1.2. (Decomposition Theorem) *Let (f,P) be a non-parabolic map, then there exist a (f,P)-stable multicurve Γ and a collection of Siegel* *maps or Thurston maps, say* $\{(h_k, P_k), k \in \Lambda\}$*, where* Λ *is a finite index set, such that*

1. (Combinatorial part) (f, P) *has no Thurston obstructions if and only if* $\lambda(\Gamma, f) < 1$ *and for each* $k \in \Lambda$*,* (h_k, P_k) *has no Thurston obstructions.*

2. (Surgery part) (f, P) *is q.c-equivalent to a rational map if and only if* $\lambda(\Gamma, f) < 1$ *and for each* $k \in \Lambda$*,* (h_k, P_k) *is q.c-equivalent to a rational map.*

3. (Analytic part) (f, P) *is q.c-equivalent to a unique rational map up to Möbius conjugation if and only if* $\lambda(\Gamma, f) < 1$ *and for each* $k \in \Lambda$*,* (h_k, P_k) *q.c-equivalent to a unique rational map up to Möbius conjugation.*

From the viewpoint of 'decomposition', this theorem means that every non-parabolic branched covering can be decomposed along a stable multicurve into finitely many Siegel maps or Thurston maps, such that the combinatorics and rational realizations of these resulting maps essentially dominate the original one. These resulting maps can be viewed as the renormalizations of the original map.

From the viewpoint of 'reduction', the theorem implies that Thurston-type Theorems for every non-parabolic branched covering can be reduced to Thurston-type Theorems for finitely many Siegel type branched coverings. In particular, Thurston-type Theorems for rational maps with Herman rings can be reduced to Thurston-type Theorems for rational maps with Siegel disks.

Remark 6.1.1. *1. The number of Siegel maps is bounded by* $n_{RD}(f) + 2n_{RA}(f)$*.*

2. In the surgery part of Theorem 6.1.2, we require that (f, P) *is a quasiregular branched covering, and the pair of c-equivalences are quasiconformal. This is simply because we want to apply Measurable Riemann Mapping Theorem in our proof. In fact, this part can be restated as '*(f, P) *is c-equivalent to a rational map if and only if* $\lambda(\Gamma, f) < 1$ *and for each* $k \in \Lambda$*,* (h_k, P_k) *is c-equivalent to a rational map' and the proof goes through without any difficulty. The only modification of the proof is to replace the Measurable Riemann Mapping Theorem by the Uniformization Theorem.*

3. The condition $\lambda(\Gamma, f) < 1$ *implies that if* (h_k, P_k) *is a Thurston map for some* $k \in \Lambda$*, then the signature of the orbifold of* (h_k, P_k) *is not* $(2, 2, 2, 2)$*. Thus by Thurston's Theorem,* (h_k, P_k) *has no Thurston obstructions if and only if* (h_k, P_k) *is c-equivalent to a rational map.*

4. Theorem 6.1.2 consists of four cases:

1). $\Gamma = \emptyset$*,* $\Lambda = \emptyset$*. In this case,* (f, P) *is c-equivalent to a unique rational map up to Möbius conjugation. Hyperbolic polynomials with Cantor Julia sets provide such examples.*

2). $\Gamma \neq \emptyset$*,* $\Lambda = \emptyset$*. In this case,* (f, P) *has no Thurston obstructions (or 'is c-equivalent to a rational map') if and only if* $\lambda(\Gamma, f) < 1$*. Such* *examples of rational maps can be found in the family of McMullen maps: $f_{n,\lambda}(z) = z^n + \lambda/z^n$ with $n \geq 3$ and λ suitably chosen such that $J(f_{n,\lambda})$ is a Cantor set of circles.*

3). $\Gamma = \emptyset$, $\Lambda \neq \emptyset$. In this case, (f, P) is a Herman map. See Example 6.2.1.

4). $\Gamma \neq \emptyset$, $\Lambda \neq \emptyset$. This is the general case.

The 'Decomposition Theorem' provides a mechanism to produce Thurston type Theorems for non-parabolic maps. Thus it has many applications. For example, it can reduce the Thurston-type Theorem for hyperbolic maps to the so-called 'Marked Thurston Theorem' (this is the idea of Cui-Tan's work [CT1]), which is slightly stronger than Thurston's original theorem, as follows:

Theorem 6.1.3. (Marked Thurston Theorem) *Let (f, P) be a Thurston map. Suppose that $\mathcal{O}_f$ does not have signature $(2, 2, 2, 2)$. Then (f, P) is c-equivalent to a rational function (R, Q) if and only if for any (f, P)-stable multicurve Γ, we have $\lambda(\Gamma, f) < 1$. The rational function (R, Q) is unique up to Möbius conjugation.*

The detailed proof the 'Marked Thurston Theorem' can be found in [BCT], using the same idea as Douady-Hubbard's original one.

As another application, the 'Decomposition Theorem' enables us to give a characterization of a class of rational maps with Herman rings based on Zhang's work [Zh2] and the 'Marked Thurston Theorem', as follows:

Theorem 6.1.4. (Characterization of rational maps with Herman rings) *Let (f, P) be a non-parabolic map, with only one rotation annulus cycle which is of period one and has rotation number of bounded type, and without rotation disk. Then (f, P) is c-equivalent to a rational map (R, Q) if and only if (f, P) has no Thurston obstructions. Moreover, the Lebesgue measure of the Julia set $J(R)$ is zero, and (R, Q) is unique up to Möbius conjugation.*

There is no reason to believe that the absence of Thurston obstruction is always equivalent to rational realization for postcritically infinite branched covering, even if the equivalence is true for hyperbolic case ([CT1], [JZ]), some Siegel cases [Zh2] and Herman cases (Theorem 6.1.4). The mating of two quadratic Siegel polynomials $f_\theta(z) = z^2 + c_\theta$ and $f_{-\theta}(z) = z^2 + c_{-\theta}$, where $c_\alpha = \frac{e^{2\pi i\alpha}}{2}(1 - \frac{e^{2\pi i\alpha}}{2})$, provides a non-parabolic map $g = f_\theta \sqcup f_{-\theta}$ for which the equivalence is false. As a supplement to the Decomposition Theorem, following the same idea as Shishikura's construction [Sh1] of rational maps with prescribed numbers of non-repelling cycles and Herman rings, we can construct many such examples by surgery.

Theorem 6.1.5. *Given nonnegative integers n_A, n_{RD}, n_{RA}, d satisfying*

$$n_A + n_{RD} + 2n_{RA} \leq 2d - 2,\ 1 \leq n_{RA} \leq d - 2,\ n_{RD} + n_{RA} \geq 2.$$

There exists a non-parabolic map (f, P) of degree d, such that

1. $n_A(f) = n_A, n_{RD}(f) = n_{RD}, n_{RA}(f) = n_{RA}$, and the rotation number of each rotation cycle is of bounded type.

2. (f, P) has no Thurston obstructions.

3. (f, P) is not c-equivalent to a rational map.

This chapter is organized as follows:

From Section 6.2 to Section 6.4, we prove the 'combinatorial part' and 'surgery part' of Theorem 6.1.2 for Herman maps. More precisely, in Section 6.2, we will decompose a Herman map into finitely many Siegel maps and Thurston maps based on Shishikura's 'Herman ring-Siegel disk' surgery. In Section 6.3, we show the equivalence of absence of Thurston obstructions between the original map and the resulting maps. In Section 6.4, we show the equivalence of rational realizations between the original map and the resulting maps.

From Section 6.5 to Section 6.7, we prove that a non-parabolic map with $n_A(f) > 0$ can be decomposed along a stable multicurve into finitely many Herman maps, Siegel maps and Thurston maps whose combinatorics and rational realizations essentially dominate the original one. The proof is based on Cui-Tan's repelling system theory. The decomposition procedure, 'combinatorial part' and 'surgery part' are discussed in Section 6.5, Section 6.6 and Section 6.7, respectively.

In Section 6.8, we prove the 'combinatorial part' and 'surgery part' of Theorem 6.1.2.

In Section 6.9, we discuss the renormalizations of rational maps and prove the 'analytic part' of Theorem 6.1.2.

In Section 6.10, we give many applications of Theorem 6.1.2. These include characterizations of hyperbolic rational maps and a class of rational maps with Siegel disks. As another application, we prove a Thurston-type theorem for a class of rational maps with Herman rings.

In Section 6.11, as a supplement to the Decomposition Theorem, we show that for postcritically infinite non-parabolic maps, no Thurston obstruction does not always imply rational realization. We construct many such examples by surgery.

Definitions and Notations:

1. Given a collection of Jordan curves $\mathcal{C}$ (not necessarily a multicurve) in $\overline{\mathbb{C}} - P$. For any integer $k \geq 0$, we denote by $f^{-k}(\mathcal{C})$ the collection of all components δ of $f^{-k}(\gamma)$ for $\gamma \in \mathcal{C}$. Set $\cup\mathcal{C} := \cup_{\gamma\in\mathcal{C}}\gamma$.

2. Let $A = (a_{ij})$ be a $n \times n$ real matrix. The Banach norm $\|A\|$ of A is defined to be either $\sum |a_{ij}|$ or $(\sum |a_{ij}|^2)^{1/2}$ according to different situations. The *spectral radius* $\mathrm{sp}(A)$ of A is defined by $\mathrm{sp}(A) := \lim \sqrt[n]{\|A^n\|}$.

3. Given two multicurves Σ_1 and Σ_2 in $\overline{\mathbb{C}} - P$. We say that Σ_1 is *homotopically contained* in Σ_2, denoted by $\Sigma_1 \prec \Sigma_2$, if each curve $\alpha \in \Sigma_1$ is homotopic in $\overline{\mathbb{C}} - P$ to some curve $\beta \in \Sigma_2$. We say that Σ_1 is identical to Σ_2 up to homotopy, if $\Sigma_1 \prec \Sigma_2$ and $\Sigma_2 \prec \Sigma_1$.

4. Let D and Ω be two planar domains and $f : D \to \Omega$ be a quasiregular map, the Beltrami coefficient μ_f of f is defined by $\mu_f = \frac{\partial f}{\partial \bar{z}} / \frac{\partial f}{\partial z}$.

5. We use $\#E$ to denote the cardinality of the set E. The characteristic function $\chi_E : E \to \{0, 1\}$ is defined by

$$\chi_E(z) = \begin{cases} 1, & \text{if } z \in E, \\ 0, & \text{if } z \notin E. \end{cases}$$

6.2 Decompositions of Herman maps

In the following three sections we will prove the following theorem:

Theorem 6.2.1. (Herman=multicurve+Siegel+Thurston) *Let (f, P) be a Herman map, then there exist a (f, P)-stable multicurve Γ and a collection of Siegel maps or Thurston maps, say $\{(h_k, P_k), k \in \Lambda\}$, where Λ is a finite index set, such that*

1. (f, P) has no Thurston obstructions if and only if $\lambda(\Gamma, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) has no Thurston obstructions.

2. (f, P) is q.c-equivalent to a rational map if and only if $\lambda(\Gamma, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) is q.c-equivalent to a rational map.

In this section, we will decompose a Herman map into finitely many Siegel maps and Thurston maps along a collection of f-periodic Jordan curves and their suitably chosen preimages. The method we use here is called 'Herman-Siegel' surgery which is pioneered by Shishikura [Sh1].

Let (f, P) be a Herman map, $\mathcal{A}$ be the collection of all rotation annuli of f and $\cup\mathcal{A} := \bigcup_{A \in \mathcal{A}} A$ be the union of all these annuli. For each $A \in \mathcal{A}$, we choose an analytic curve $\gamma_A \subset A$ such that $\gamma_A \cap f(P - \cup\mathcal{A}) = \emptyset$ (This implies that γ_A avoids the postcritical points and the images of other marked points) and $f(\gamma_A) = \gamma_{f(A)}$. It's obvious that if $f^p(A) = A$, then $f^p(\gamma_A) = \gamma_A$.

Let $\Gamma_0 = \{\gamma_A; A \in \mathcal{A}\}$, we first show that Γ_0 can generate a unique (f, P)-stable multicurve up to homotopy.

Lemma 6.2.1. *Given a choice of Γ_0, there is a (f, P)-stable multicurve Γ such that:*

• *(Invariant) For any $\gamma \in \Gamma$, we have $f(\gamma) \in \Gamma \cup \Gamma_0$.*

• *(Maximal) Γ represents all homotopy classes of non-peripheral curves of $\cup_{k\geq 1} f^{-k}(\Gamma_0) - \Gamma_0$ in $\overline{\mathbb{C}} - P$.*

Moreover, the multicurve Γ is unique up to homotopy.

Proof. First, there is a multicurve Γ_1 in $\overline{\mathbb{C}} - P$ such that $\Gamma_1 \subset f^{-1}(\Gamma_0) - \Gamma_0$ and Γ_1 represents all homotopy classes of non-peripheral curves of $f^{-1}(\Gamma_0) - \Gamma_0$.

Such Γ_1 is not uniquely chosen. But any two such multicurves are identical up to homotopy, thus they have the same number of curves.

For $n \geq 2$, we define Γ_n inductively in the following way:

• $\Gamma_n \subset f^{-1}(\Gamma_{n-1})$.

• $\Gamma_1 \cup \cdots \cup \Gamma_n$ is a multicurve in $\overline{\mathbb{C}} - P$.

• $\Gamma_1 \cup \cdots \cup \Gamma_n$ represents all homotopy classes of non-peripheral curves of $f^{-n}(\Gamma_0) - \Gamma_0$.

Since any two different curves in $\cup_{k\geq 1} f^{-k}(\Gamma_0) - \Gamma_0$ are disjoint and P has finitely many components, we conclude that $\cup_{k\geq 1} f^{-k}(\Gamma_0) - \Gamma_0$ has finitely many homotopy classes of non-peripheral curves in $\overline{\mathbb{C}} - P$. It turns out that $\#(\Gamma_1 \cup \cdots \cup \Gamma_n)$ is uniformly bounded above by some constant $C(P)$. Thus there is an integer $N \geq 0$ such that $\Gamma_N \neq \emptyset$ and $\Gamma_{N+1} = \Gamma_{N+2} = \cdots = \emptyset$. (It can happen that $N = 0$, see Example 6.2.1.)

We set $\Gamma = \emptyset$ if $N = 0$ and $\Gamma = \cup_{1\leq j\leq N} \Gamma_j$ if $N \geq 1$. By the choice of N, Γ is a (f, P)-stable multicurve. By construction, for any $\gamma \in \Gamma$, we have $f(\gamma) \in \Gamma \cup \Gamma_0$. The homotopy classes of Γ is uniquely determined by those of non-peripheral curves of $\cup_{k\geq 1} f^{-k}(\Gamma_0) - \Gamma_0$. So Γ is unique up to homotopy. □

Here we give an example to show that Γ can be an empty set.

Example 6.2.1. **($\Gamma = \emptyset$)** *The example is borrowed from Shishikura's paper [Sh1]. Let*

$$f(z) = \frac{e^{i\alpha}}{z}\Big(\frac{z-r}{1-rz}\Big)^2,$$

where $\alpha \in \mathbb{R}$ and $0 < r < 1/5$. We may assume that α is properly chosen such that f has a fixed Herman ring H containing the unit circle $\mathbb{S}$, with bounded type rotation number (Remark: in this case, each boundary component of H is a quasicircle containing a critical point of f). There are two other critical points: r and $1/r$, which are eventually mapped to a repelling cycle of period two, and $f(r) = f^3(r) = 0, f^2(r) = f(1/r) = \infty$. We choose $\Gamma_0 = \{\mathbb{S}\}$. Let $P = \overline{H} \cup P_f = \overline{H} \cup \{0, \infty\}$. Since each component of $\overline{\mathbb{C}} - \overline{H}$ is a disk containing exactly one point in the marked set P, the set Γ is necessarily empty.

Let $\Sigma = \Gamma_0 \cup \Gamma$. In the following, we will use Σ to decompose the complex sphere $\overline{\mathbb{C}}$ into finitely many pieces. We define

$$\mathcal{S} = \{\overline{U};\ U \text{ is a connected component of } \overline{\mathbb{C}} - \cup\Sigma\},$$
$$\mathcal{E} = \{\overline{V};\ V \text{ is a connected component of } \overline{\mathbb{C}} - \cup f^{-1}(\Sigma)\}.$$

Each element of $\mathcal{S}$ (resp. $\mathcal{E}$) is called an $\mathcal{S}$-piece (resp. $\mathcal{E}$-piece). Given an $\mathcal{S}$-piece S (resp. $\mathcal{E}$-piece E), let $\partial(S)$ (resp. $\partial(E)$) be the collection of all boundary curves of S (resp. E). One should notice that ∂S and $\partial(S)$ are different notations, they satisfy $\partial S = \cup\partial(S)$.

The following facts are easy to verify:

- *Every $\mathcal{E}$-piece E is contained in a unique $\mathcal{S}$-piece and $f(E) \in \mathcal{S}$.*
- *For every $\mathcal{S}$-piece S, we have $\#(S \cap P) + \#\partial(S) \geq 3$.*
- *For each curve $\gamma \in \Sigma$, there exist exactly two $\mathcal{S}$-pieces, say S_γ^+ and S_γ^-, that share γ as a common boundary component.*

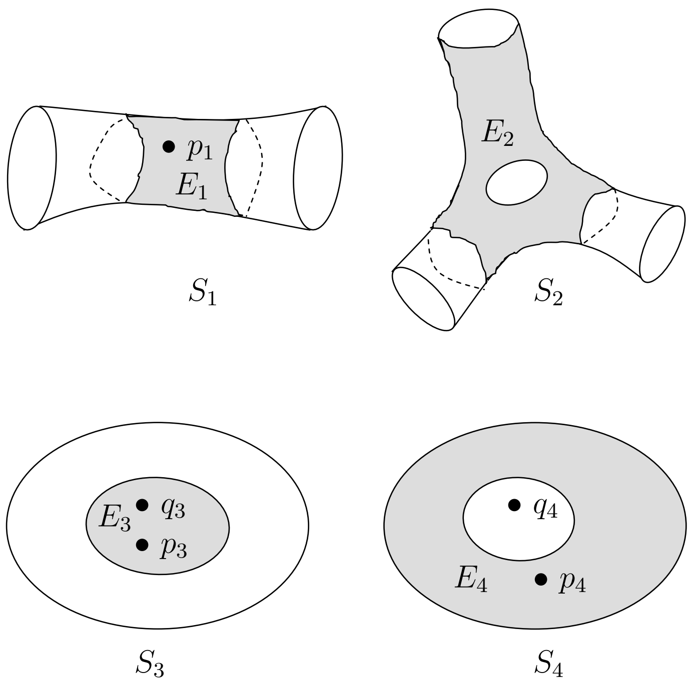

Figure 6.1: Four examples: E_i (shadow region) is parallel to S_i. p_i, q_i are marked points in P. Here, S_1 is an annulus with one marked point, S_2 has three boundary curves and contains no marked point, both S_3 and S_4 are disks with two marked points.

Let S be an $\mathcal{S}$-pieces, T is a connected and closed subset of S, we say T is *parallel* to S if $\partial T \cap P = \emptyset$ and each component of $S \setminus T$ is either

• an annulus contained in $S - P$, or
• a disk that contains at most one point of P.

Notice that if T is parallel to S and A is an annular component of $S \backslash T$, then one boundary curve of A is on S and $\#(T \cap P) + \#\partial(T) \geq \#(S \cap P) + \#\partial(S)$.

Here is an important property of the $\mathcal{S}$-pieces:

Lemma 6.2.2. *For every $\mathcal{S}$-piece S, there is a unique $\mathcal{E}$-piece, say E_S, parallel to S.*

Proof. The proof is based on the 'maximal' property of the (f, P)-stable multicurve Γ. We omit the details. □

We define a map

$$f_* : \begin{cases} \mathcal{S} \to \mathcal{S}, \\ S \mapsto f(E_S). \end{cases}$$

Since there are finitely many $\mathcal{S}$-pieces, every $\mathcal{S}$-piece is eventually periodic under the map f_*.

For each curve $\gamma \in \partial(S)$, there is a unique boundary curve $\beta_\gamma \in \partial(E_S)$ such that either $\beta_\gamma = \gamma$, or β_γ and γ bound an annulus in $S - P$. We define three sets $\partial_0(S), \partial_1(S), \partial_2(S)$ as follows:

$$\begin{array}{rcl} \partial_0(S) & = & \{\gamma \in \partial(S); \gamma \in \Gamma_0\}, \\ \partial_1(S) & = & \{\gamma \in \partial(S); \gamma \neq \beta_\gamma\}, \\ \partial_2(S) & = & \{\gamma \in \partial(S); \gamma = \beta_\gamma\} - \Gamma_0. \end{array}$$

Lemma 6.2.3. *If $\partial_0(S) \neq \emptyset$, then we have:*

1. *For any $\gamma \in \partial_0(S)$, $\gamma = \beta_\gamma$.*
2. *S is f_*-periodic.*
3. *$\#\partial_0(S) = \#\partial_0(f_*(S))$.*

Proof. 1. Notice that every component of $S - E_S$ is either a disk containing at most one point in P, or an annulus in $\overline{\mathbb{C}} - P$. It follows that if $\gamma \in \partial_0(S)$, then $\gamma \subset P$ and $\gamma = \beta_\gamma$.

2. Take $\gamma \in \partial_0(S)$. Then there is a rotation annulus $A_\gamma \in \mathcal{A}$ containing γ. Then from 1 we see that $S \cap A_\gamma = E_S \cap A_\gamma$. This implies $f(S \cap A_\gamma) = f_*(S) \cap f(A_\gamma)$. Let $k \geq 1$ be the period of A_γ. Then we have $S \cap A_\gamma = f^k(S \cap A_\gamma) = f_*^k(S) \cap f^k(A_\gamma) = f_*^k(S) \cap A_\gamma$. Thus $f_*^k(S) = S$. So S is f_*-periodic, and the period of S is a divisor of k.

3. It follows from 1 that if $\gamma \in \partial_0(S)$, then $f(\gamma) \in \partial_0(f_*(S))$. So $\#\partial_0(S) \leq \#\partial_0(f_*(S)) \leq \cdots$. Since S is f_*-periodic (by 2), we have $\#\partial_0(S) = \#\partial_0(f_*(S))$. □

It follows from Lemma 6.2.3 that $\partial_i(S), i \in \{0,1,2\}$ are mutually disjoint and $\partial(S) = \partial_0(S) \sqcup \partial_1(S) \sqcup \partial_2(S)$.

Remark 6.2.1. *Suppose $\partial_0(S) \neq \emptyset$. For each $\gamma \in \partial_0(S)$, let $\mathrm{per}(\gamma)$ be the period of γ. From Lemma 6.2.3 we see that the f_*-period of S is a devisor of $\gcd\{\mathrm{per}(\gamma); \gamma \in \partial_0(S)\}$. In particular, if $\gcd\{\mathrm{per}(\gamma); \gamma \in \partial_0(S)\} = 1$, then $f_*(S) = S$ and for every $\gamma \in \partial_0(S)$ and every integer $k \geq 0$, we have $f^k(\gamma) \in \partial_0(S)$.*

For example, suppose that (f, P) has two cycles of rotation annuli whose periods are different prime numbers, say p and q. If $\partial_0(S) \neq \emptyset$, then $\#\partial_0(S)$ takes only four possible values: 1, p, q and $p+q$.

6.2.1 Marked disk extension

For each $\mathcal{S}$-piece S, we denote by $\overline{\mathbb{C}}(S)$ the Riemann sphere containing S. We always consider that different $\mathcal{S}$-pieces are embedded into different copies of Riemann spheres.

In the following, we will extend $f|_{E_S}$ to a quasiregular branched covering $H_S : \overline{\mathbb{C}}(S) \to \overline{\mathbb{C}}(f_*(S))$ such that $\deg(H_S) = \deg(f|_{E_S})$. To do this, we need to define the map $H_S : \overline{\mathbb{C}}(S) - E_S \to \overline{\mathbb{C}}(f_*(S)) - f_*(S)$ such that $H_S|_{\partial E_S} = f|_{\partial E_S}$. We will define H_S component by componet.

Notice that each component of $\overline{\mathbb{C}}(S) - E_S$ is a disk. Let U be such a component with boundary curve γ.

We first deal with the case when $\gamma \in \partial_0(S)$. In this case, there is a rotation annulus A_γ containing γ. Let $k \geq 1$ be the period of A_γ. Let $\phi_0 : S \cap A_\gamma \to \mathbb{A}_R := \{z \in \mathbb{C}; 1 < |z| < R\}$ be the conformal map such that $\phi_0 f^k \phi_0^{-1}(z) = e^{2\pi i\theta} z$ for $z \in \mathbb{A}_R$. For $1 \leq j \leq k-1$, we define a conformal map from $f^j(S \cap A_\gamma)$ onto $\mathbb{A}_R$ by $\phi_j = \phi_0 f^{k-j}|_{f^j(S \cap A_\gamma)}$. Then we have the following commutative diagram

$$\begin{tikzcd}
S \cap A_\gamma \arrow[r, "f"] \arrow[d, "\phi_0"'] & f(S \cap A_\gamma) \arrow[r, "f"] \arrow[d, "\phi_1"'] & \cdots \arrow[r] & f^{k-1}(S \cap A_\gamma) \arrow[r, "f"] \arrow[d, "\phi_{k-1}"'] & S \cap A_\gamma \arrow[d, "\phi_0"] \\
\mathbb{A}_R \arrow[r, "z \mapsto e^{2\pi i\theta} z"'] & \mathbb{A}_R \arrow[r, "id"'] & \cdots \arrow[r] & \mathbb{A}_R \arrow[r, "id"'] & \mathbb{A}_R
\end{tikzcd}$$

Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$. For $0 \leq j < k$, we consider the disk Δ_j obtained by gluing $f^j(S \cap A_\gamma)$ and $\mathbb{D}_R$ via the map ϕ_j. The disk Δ_j inherits a natural complex structure from $\mathbb{D}_R$ since ϕ_j is holomorphic.

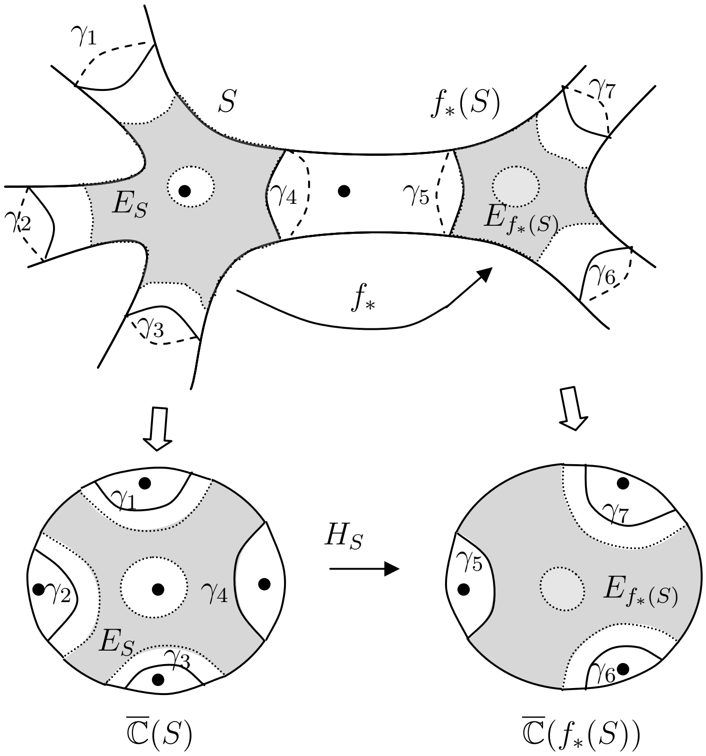

Figure 6.2: Marked disk extension. Here $\partial S = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, $\partial f_*(S) = \gamma_5 \cup \gamma_6 \cup \gamma_7$. Marked points are labeled by '$\bullet$'.

The map $H_{f_*^j(S)} : \Delta_j \to \Delta_{j+1}$ defined by

$$H_{f_*^j(S)}(z) = \begin{cases} f(z), & z \in f^j(S \cap A_\gamma),\ 0 \le j < k, \\ e^{2\pi i\theta} z, & z \in \mathbb{D},\ j = 0, \\ z, & z \in \mathbb{D},\ 1 \le j < k. \end{cases}$$

is a holomorphic extension of $f|_{E_{f_*^j(S)}}$ along the boundary curve $f^j(\gamma) \in \partial_0(f_*^j(S))$. We call $(\Delta_j, 0)$ a holomorphic marked disk of $H_{f_*^j(S)}$. This construction allows us to define the extensions of $f|_{E_S}, \cdots, f|_{E_{f_*^{l-1}(S)}}$ (where l is the f_*-period of S) along the curves in $\partial_0(S) \cup \cdots \cup \partial_0(f_*^{l-1}(S))$ at the same time.

Now, we consider the case when $\gamma = \partial U \notin \partial_0(S)$. Notice that either U is a disk in S containing at most one point of P, or it contains a unique component V of $\overline{\mathbb{C}}(S) - S$. In the former case, if U contains a marked point $p \in P$, we get a marked disk (U, p); if $U \cap P = \emptyset$, we don't mark any point in U. In the latter case, we mark a point $p \in V$ and get two marked disks (U, p) and (V, p).

Now we extend $f|_{E_S}$ to U in the following fashion:

We require that H_S maps U onto (W, q) with $\deg(H_S|_U) = \deg(f|_{\partial U})$, where (W, q) is the unique marked disk of $\overline{\mathbb{C}}(f_*(S)) - f_*(S)$ whose boundary curve is $f(\partial U)$. If U contains a marked point p, we require further $H(p) = q$ and the local degree of H_S at p is equal to $\deg(f|_{\partial U})$. Else, we require that q is the only possible critical value (This implies that U contains at most one ramification point of H_S).

In this way, for each $\mathcal{S}$-piece S, we can get an extension $H_S : \overline{\mathbb{C}}(S) \to \overline{\mathbb{C}}(f_*(S))$ of $f|_{E_S}$. Let $D(S)$ be the union of all holomorphic marked disks of H_S. Notice that if $\partial_0(S) = \emptyset$, then $D(S) = \emptyset$. Set

$$P(S) = (P \cap S) \cup \{\text{all marked points in } \overline{\mathbb{C}}(S) - S\} \cup \overline{D(S)}.$$

We call $(\overline{\mathbb{C}}(S), P(S))$ a marked sphere of $\overline{\mathbb{C}}(S)$. By the construction of H_S, we see that $H_S(P(S)) \subset P(f_*(S))$.

Notice that every $\mathcal{S}$-piece is eventually periodic under the map f_*. Let n be the number of all f_*-cycles of $\mathcal{S}$-pieces. These cycles are listed as follows:

$$S_\nu \mapsto f_*(S_\nu) \mapsto \cdots \mapsto f_*^{p_\nu - 1}(S_\nu) \mapsto f_*^{p_\nu}(S_\nu) = S_\nu, \quad 1 \le \nu \le n,$$

where S_ν is a representative of the ν-th cycle and p_ν is the period of S_ν.

Set

$$h_\nu = H_{f_*^{p_\nu - 1}(S_\nu)} \circ \cdots \circ H_{f_*(S_\nu)} \circ H_{S_\nu}, \; P_\nu = P(S_\nu), \quad 1 \le \nu \le n.$$

Then $h_\nu : \overline{\mathbb{C}}(S_\nu) \to \overline{\mathbb{C}}(S_\nu)$ is a branched covering with $h_\nu(P_\nu) \subset P_\nu$.

These resulting maps $(h_1, P_1), \cdots, (h_n, P_n)$ can be considered as the renormalizations of the original map (f, P). There are three types of them:

• $\partial_0(S_\nu) \neq \emptyset$ or S_ν contains at least one rotation disk of (f, P). In this case, (h_ν, P_ν) has at least one cycle of rotation disks, so (h_ν, P_ν) is a Siegel map. Moreover, a curve $\gamma \in \partial_0(S_\nu)$ contained in a rotation annulus of f with period p and rotation number θ becomes a periodic curve contained in a rotation disk of h_ν, with period p/p_ν and rotation number θ. One may verify that the number of these resulting Siegel maps is at least two, and bounded above by $2n_{RA}(f) + n_{RD}(f)$.

• $\partial_0(S_\nu) = \emptyset$, S_ν contains no rotation disk of (f, P) and $\deg(h_\nu) > 1$. In this case, P_ν is a finite set and (h_ν, P_ν) is a Thurston map.

• $\partial_0(S_\nu) = \emptyset$, S_ν contains no rotation disk of (f, P) and $\deg(h_\nu) = 1$. In this case, (h_ν, P_ν) is a homeomorphism of $\overline{\mathbb{C}}(S_\nu)$ and $h_\nu(P_\nu) = P_\nu$. So every point of P_ν is periodic. Moreover, for any $S \in \{S_\nu, f_*(S_\nu), \cdots, f_*^{p_\nu - 1}(S_\nu)\}$, we have $\#\partial(E_S) = \#\partial(S)$.

Let Λ be the index set consisting of all $\nu \in \{1, \cdots, n\}$ such that $\deg(h_\nu) > 1$. That is, for each $\nu \in \Lambda$, (h_ν, P_ν) is either a Siegel map or a Thurston map. Let $\Lambda^* = \{1, \cdots, n\} - \Lambda$.

We use the following notation to record the decomposition procedure:

$$\mathrm{Dec}(f,P)=\left(\bigoplus_{\nu\in\Lambda\cup\Lambda^*}(h_\nu,P_\nu)\right)_\Gamma.$$

Lemma 6.2.4. *If $\lambda(\Gamma,f)<1$, then*

1. For any $1\le\nu\le n$, every point in $(\overline{\mathbb{C}}(S_\nu)-S_\nu)\cap P_\nu$ is eventually mapped to either the center of some rotation disk or a periodic critical point of (h_ν,P_ν).

2. $\Lambda^*=\emptyset$.

3. If (h_ν,P_ν) is a Thurston map, then the signature of the orbifold of (h_ν,P_ν) is not $(2,2,2,2)$.

Proof. Since $h_\nu((\overline{\mathbb{C}}(S_\nu)-S_\nu)\cap P_\nu)\subset(\overline{\mathbb{C}}(S_\nu)-S_\nu)\cap P_\nu$ and $(\overline{\mathbb{C}}(S_\nu)-S_\nu)\cap P_\nu$ is a finite set, every point in $(\overline{\mathbb{C}}(S_\nu)-S_\nu)\cap P_\nu$ is eventually periodic under the map (h_ν,P_ν). Let z_0 be a periodic point in $(\overline{\mathbb{C}}(S_\nu)-S_\nu)\cap P_\nu$ with period k. Suppose that z_0 is not the center of rotation disk, and let β be the boundary curve of S_ν that encloses z_0. Then there is a unique component of $h_\nu^{-k}(\beta)$, denoted by α, such that $\alpha\subset S_\nu$ and α is homotopic to β in $\overline{\mathbb{C}}(S_\nu)-P_\nu$. Thus

$$\deg(h_\nu^k,z_0)=\deg(h_\nu^k:\alpha\to\beta)=\deg(f^{kp_\nu}:\alpha\to\beta)\ge\lambda(\Gamma,f)^{-kp_\nu}>1.$$

This implies that z_0 lies in a critical cycle and $\deg(h_\nu)>1$. It follows that $\Lambda^*=\emptyset$ and there is no $(2,2,2,2)$-type Thurston map among $(h_\nu,P_\nu),\nu\in\Lambda$. □

A multicurve $\Gamma=\{\gamma_1,\cdots,\gamma_k\}$ is called a *Levy cycle* of (f,P), if for each $1\le i\le k$, $f^{-1}(\gamma_i)$ has a component α_{i-1} homotopic to γ_{i-1} in $\overline{\mathbb{C}}-P$ (set $\gamma_0=\gamma_k$) and $\deg(f:\alpha_{i-1}\to\gamma_i)=1$.

Corollary 6.2.1. *If $\Lambda^*\ne\emptyset$, then $\lambda(\Gamma,f)\ge1$ and Γ contains a Levy cycle of (f,P).*

Proof. If $\Lambda^*\ne\emptyset$, then $\lambda(\Gamma,f)\ge1$ follows from Lemma 6.2.4. Take some $\nu\in\Lambda^*$, the boundary multicurve $\partial(S_\nu)$ contains a submulticurve $\{\gamma_1,\cdots,\gamma_k\}$ labeled in the way that for any $i\in[1,k]$, $h_\nu^{-1}(\gamma_i)$ is homotopic to γ_{i-1} ($\gamma_0:=\gamma_k$) in $\overline{\mathbb{C}}-P$. In particular, $\alpha:=h_\nu^{-k}(\gamma_k)$ is homotopic to γ_k in $\overline{\mathbb{C}}-P$. So there exist two integers $0\le m<n\le kp_\nu$ such that $f^m(\alpha)$ is homotopic to $f^n(\alpha)$ in $\overline{\mathbb{C}}-P$ and $\{f^k(\alpha);m<k\le n\}$ is a multicurve in $\overline{\mathbb{C}}-P$. Since Γ is (f,P)-stable, there is a submulticurve $\Gamma_*\subset\Gamma$ identical to $\{f^k(\alpha);m<k\le n\}$ up to homotopy. One may verify that Γ_* is a Levy cycle of (f,P). □

To end this section, we give a concrete example to illustrate how the decomposition procedure works:

Example 6.2.2. *We still consider the rational map f defined as in Example 6.2.1. Recall that f has a fixed Herman ring containing the unit circle $\mathbb{S}$. Suppose $f^{-1}(\mathbb{S}) = \mathbb{S} \cup \gamma_1 \cup \gamma_2$, where γ_1 (resp. γ_2) is a Jordan curve in the connected component D_0 (resp. D_∞) of $\overline{\mathbb{C}} - \overline{H}$ containing 0 (resp. ∞). One may deduce that γ_1 encloses 0 and γ_2 encloses ∞. Let $D_0^{\gamma_1}$ (resp. $D_\infty^{\gamma_2}$) be the disk neighborhood of 0 (resp. ∞) whose boundary curve is γ_1 (resp. γ_2). Notice that $r \in D_0 - D_0^{\gamma_1}$ and $1/r \in D_\infty - D_\infty^{\gamma_2}$.*

One may check both $f^{-1}(\gamma_1)$ and $f^{-1}(\gamma_2)$ consist of two components. We denote $f^{-1}(\gamma_1) = \gamma_4 \cup \gamma_5$, where $\gamma_4 \subset D_\infty^{\gamma_2}$ and γ_4 encloses ∞, $\gamma_5 \subset D_0 - D_0^{\gamma_1}$ and γ_5 encloses the critical point r. Correspondingly, We denote $f^{-1}(\gamma_2) = \gamma_3 \cup \gamma_6$, where $\gamma_3 \subset D_0^{\gamma_1}$ and γ_3 encloses 0, $\gamma_6 \subset D_\infty - D_\infty^{\gamma_2}$ and γ_6 encloses the critical point $1/r$.

Let $D_0^{\gamma_3}$ (resp. $D_\infty^{\gamma_4}$) be the disk neighborhood of 0 (resp. ∞) whose boundary curve is γ_3 (resp. γ_4). We choose two points p and q, with $p \in D_0^{\gamma_3}$ and $q \in D_\infty^{\gamma_4}$. We modify the mappings $f|_{D_0^{\gamma_3}} : D_0^{\gamma_3} \to D_\infty^{\gamma_2}$ and $f|_{D_\infty^{\gamma_4}} : D_\infty^{\gamma_4} \to D_0^{\gamma_1}$, by $\varphi : D_0^{\gamma_3} \to D_\infty^{\gamma_2}$ and $\psi : D_\infty^{\gamma_4} \to D_0^{\gamma_1}$, respectively, such that $\varphi|_{\gamma_3} = f|_{\gamma_3}, \psi|_{\gamma_4} = f|_{\gamma_4}, \varphi(0) = \infty, \varphi(p) = q, \psi(\infty) = p, \psi(q) = 0$. Then we define a new branched covering

$$h(z) = \begin{cases} \varphi(z), & z \in D_0^{\gamma_3}, \\ \psi(z), & z \in D_\infty^{\gamma_4}, \\ f(z), & z \in \overline{\mathbb{C}} - D_0^{\gamma_3} \cup D_\infty^{\gamma_4}. \end{cases}$$

Then the postcritical set P_h of h is $\partial H \cup \{0, p, q, \infty\}$ and (h, P) is a Herman map, where $P = P_h \cup H$. The curves $\gamma_i, i \in [1, 4]$ are all non-peripheral in $\overline{\mathbb{C}} - P$.

According to the decomposition procedure, $\Gamma = \{\gamma_1, \gamma_2\}$ and $\Sigma = \{\mathbb{S}, \gamma_1, \gamma_2\}$. There are four $\mathcal{S}$-pieces: S_1, S_2, S_3, S_4, and $h_*(S_1) = S_1, h_*(S_2) = S_2, h_*(S_3) = S_4, h_*(S_4) = S_3$. So there are three h_*-periodic cycles, we denote the resulting map associating with $S_i, i = 1, 2, 3$ by (h_i, P_i).

One may check the resulting maps (h_1, P_1) and (h_2, P_2) are both degree two Siegel maps, c-equivalent to $z \mapsto e^{2\pi i\theta}(z-1)^2/z$. Moreover, (h_3, P_3) is a homeomorphism, satisfying $h_3(0) = p, h_3(p) = 0$.

The (h, P)-transition matrix of Γ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus Γ is a Levy cycle. It follows that (h, P) is not c-equivalent to a rational map (To see this, one may use 'modulus argument' or apply Theorem 6.4.2).

6.3 Combinatorial part

The aim of this section is to prove the following:

Theorem 6.3.1. *Let (f,P) be a Herman map, and*

$$\mathrm{Dec}(f,P)=\left(\bigoplus_{\nu\in\Lambda\cup\Lambda^*}(h_\nu,P_\nu)\right)_\Gamma.$$

Then (f,P) has no Thurston obstructions if and only if $\lambda(\Gamma,f)<1$ and for each $\nu\in\Lambda$, (h_ν,P_ν) has no Thurston obstructions.

Notice that if (f,P) has no Thurston obstructions or $\lambda(\Gamma,f)<1$, then $\Lambda^*=\emptyset$ (See Lemma 6.2.4).

The proof of the 'sufficiency' of Theorem 6.3.1 is based on the decomposition of (f,P)-stable multicurves. We will show that every (f,P)-stable multicurve contains an 'essential' submulticurve (see Lemma 6.3.1), and every such essential submulticurve can be decomposed into a 'Γ-part' multicurve together with a (h_ν,P_ν)-stable multicurve for each $\nu\in[1,n]$. Moreover, the leading eigenvalues of their transition matrices satisfy the so-called 'reduction identity' (Theorem 6.3.2).

To prove the 'necessity' of Theorem 6.3.1, we will show that every (h_ν,P_ν)-stable multicurve Σ can generate a (f,P)-stable multicurve $\mathcal{C}$ with $\lambda(\Sigma,h_\nu)\le\lambda(\mathcal{C},f)^{p_\nu}$.

Lemma 6.3.1. (Essential submulticurve) *Let Σ_0 be a (f,P)-stable multicurve, then there is a (f,P)-stable multicurve Σ, such that*

1. *Σ is homotopically contained in Σ_0.*
2. *Each curve of Σ is contained in the interior of some $\mathcal{S}$-piece.*
3. *$\lambda(\Sigma,f)=\lambda(\Sigma_0,f)$.*

Proof. For $n\ge1$, we define a multicurve Σ_n inductively in the following way: $\Sigma_n\subset f^{-1}(\Sigma_{n-1})$ and Σ_n represents all homotopy classes of non-peripheral curves of $f^{-1}(\Sigma_{n-1})$. Since Σ_0 is a (f,P)-stable multicurve, all Σ_n are (f,P)-stable, and Σ_n is homotopically contained in Σ_{n-1}. Let W_n be the (f,P)-transition matrix of Σ_n for $n\ge0$, then

$$W_n=\begin{pmatrix} W_{n+1} & * \\ O & O\end{pmatrix}.$$

Thus $\lambda(\Sigma_0,f)=\lambda(\Sigma_1,f)=\lambda(\Sigma_2,f)=\cdots$. By the construction of Γ, there is an integer $N\ge0$ such that $\Gamma\subset f^{-N}(\Gamma_0)$, where Γ_0 is the choice of a collection of f-periodic curves in the rotation annuli (see the previous section). Since $\cup\Gamma_0$ has no intersection with $\cup\Sigma_0$, we conclude that $f^{-n}(\cup\Gamma_0)$

has no intersection with $f^{-n}(\cup\Sigma_0)$ for all $n \geq 1$. Thus when $n \geq N$, we have $\cup\Gamma \subset f^{-n}(\cup\Gamma_0)$ and each curve of Σ_n is contained in the interior of some $\mathcal{S}$-piece. The proof is completed if we set $\Sigma = \Sigma_n$ for some $n \geq N$. $\square$

Theorem 6.3.2. (Decomposition of stable multicurve) *Let $\mathcal{C}$ be a (f, P)-stable multicurve. Suppose that each curve of $\mathcal{C}$ is contained in the interior of some $\mathcal{S}$-piece. Let*

$$\mathcal{C}_\Gamma = \{\gamma \in \mathcal{C}; \gamma \text{ is homotopic to a curve of } \Gamma\},$$
$$\Sigma_\nu = \{\gamma \in \mathcal{C} - \mathcal{C}_\Gamma; \gamma \text{ is contained in } S_\nu\},\ \nu \in \Lambda \cup \Lambda^* = [1, n].$$

Then $\mathcal{C}_\Gamma$ is a (f, P)-stable multicurve, Σ_ν is a (h_ν, P_ν)-stable multicurve for each $\nu \in [1, n]$, and we have the following reduction identity:

$$\lambda(\mathcal{C}, f) = \max\left\{\lambda(\mathcal{C}_\Gamma, f),\ \sqrt[p_1]{\lambda(\Sigma_1, h_1)}, \cdots,\ \sqrt[p_n]{\lambda(\Sigma_n, h_n)}\right\}.$$

Remark 6.3.1. *In Theorem 6.3.2, the multicurve Σ_ν can be viewed as a multicurve of (h_ν, P_ν), this is because under the inclusion map $\iota_\nu : S_\nu \hookrightarrow \overline{\mathbb{C}}(S_\nu)$, the set $\iota_\nu(\Sigma_\nu) := \{\iota_\nu(\gamma); \gamma \in \Sigma_\nu\}$ is a multicurve in $\overline{\mathbb{C}}(S_\nu) - P_\nu$. We still use Σ_ν to denote the multicurve $\iota_\nu(\Sigma_\nu)$ if there is no confusion.*

One may show directly that if $\Lambda^* \neq \emptyset$, then for any $\nu \in \Lambda^*$,

$$\lambda(\Sigma_\nu, h_\nu) = \begin{cases} 1, & \text{if } \Sigma_\nu \neq \emptyset, \\ 0, & \text{if } \Sigma_\nu = \emptyset. \end{cases}$$

This observation can simplify the reduction identity.

Proof. The fact that $\mathcal{C}_\Gamma$ is (f, P)-stable is easy to verify since both Γ and $\mathcal{C}$ are (f, P)-stable. Let $\Sigma_\nu^k = \{\gamma \in \mathcal{C} - \mathcal{C}_\Gamma; \gamma \text{ is contained in } f_*^k(S_\nu)\}$ for $0 \leq k \leq p_\nu$. It's obvious that $\Sigma_\nu^0 = \Sigma_\nu^{p_\nu} = \Sigma_\nu$. Since $\mathcal{C}$ is (f, P)-stable, each non-peripheral component of $f^{-1}(\gamma)$ for $\gamma \in \Sigma_\nu^{k+1} (0 \leq k < p_\nu)$ is homotopic in $\overline{\mathbb{C}} - P$ to either a curve $\alpha \in \mathcal{C}_\Gamma$, or a curve $\beta \in \Sigma_\nu^k$, or a curve δ contained in a non-periodic $\mathcal{S}$-piece.

By the definition of the marked set $P(f_*^k(S_\nu))$, one can verify that the set Σ_ν^k is a multicurve in $\overline{\mathbb{C}}(f_*^k(S_\nu)) - P(f_*^k(S_\nu))$. Moreover, each curve $\gamma \in \mathcal{C}_\Gamma$ contained in $f_*^k(S_\nu)$ and homotopic (in $\overline{\mathbb{C}} - P$) to a boundary curve of $f_*^k(S_\nu)$ is peripheral in $\overline{\mathbb{C}}(f_*^k(S_\nu)) - P(f_*^k(S_\nu))$. Thus for $0 \leq k < p_\nu$, each non-peripheral component of $H^{-1}_{f_*^k(S_\nu)}(\gamma)$ for $\gamma \in \Sigma_\nu^{k+1}$ is homotopic to a curve $\delta \in \Sigma_\nu^k$ in $\overline{\mathbb{C}}(f_*^k(S_\nu)) - P(f_*^k(S_\nu))$. It follows that each non-peripheral component of $h_\nu^{-1}(\gamma)$ for $\gamma \in \Sigma_\nu$ is homotopic to a curve $\delta \in \Sigma_\nu$ in $\overline{\mathbb{C}}(S_\nu) - P_\nu$. This means Σ_ν is a (h_ν, P_ν)-stable multicurve.

In the following, we will prove the 'reduction identity'. Let $W_{\mathcal{C}_\Gamma}$ be the (f, P)-transition matrix of $\mathcal{C}_\Gamma$. We define $\mathcal{C}_s := \{\gamma \in \mathcal{C} -$

$\mathcal{C}_\Gamma$; γ is contained in a strictly preperiodic $\mathcal{S}$-piece$\}$ with (f,P)-transition matrix W_s. Let $\mathcal{C}_\nu = \Sigma_\nu^0 \cup \cdots \cup \Sigma_\nu^{p_\nu - 1}$ with (f,P)-transition matrix W_ν. Then the (f,P)-transition matrix $W_{\mathcal{C}}$ of $\mathcal{C}$ has the following block decomposition:

$$W_{\mathcal{C}} = \begin{pmatrix} W_{\mathcal{C}_\Gamma} & * & * & \cdots & * \\ O & W_s & * & \cdots & * \\ O & O & W_1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & W_n \end{pmatrix}.$$

It follows that $\lambda(\mathcal{C}, f) = \max\left\{\lambda(\mathcal{C}_\Gamma, f), \lambda(\mathcal{C}_s, f), \lambda(\mathcal{C}_1, f), \cdots, \lambda(\mathcal{C}_n, f)\right\}$. By the definition of $\mathcal{C}_s$, there is an integer $M > 0$ such that for any $\gamma \in \mathcal{C}_s$ and any component α of $f^{-M}(\gamma)$, α is either null-homotopic, or peripheral, or homotopic to a curve $\delta \in \mathcal{C}_\Gamma$ in $\overline{\mathbb{C}} - P$. This implies $W_s^M = 0$ and $\lambda(\mathcal{C}_s, f) = 0$. So we have

$$\lambda(\mathcal{C}, f) = \max\left\{\lambda(\mathcal{C}_\Gamma, f), \lambda(\mathcal{C}_1, f), \cdots, \lambda(\mathcal{C}_n, f)\right\}.$$

Notice that the (f,P)-transition matrix W_ν of $\mathcal{C}_\nu$ has the form

$$W_\nu = \begin{pmatrix} O & B_0 & O & \cdots & O \\ O & O & B_1 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & B_{p_\nu - 2} \\ B_{p_\nu - 1} & O & O & \cdots & O \end{pmatrix},$$

where B_j is a $n_j \times n_{j+1}$ matrix, n_j is equal to the number of curves in Σ_ν^j for $0 \le j \le p_\nu - 1$. By a direct calculation,

$$W_\nu^{p_\nu} = \begin{pmatrix} B_0 B_1 \cdots B_{p_\nu - 1} & O & \cdots & O \\ O & B_1 B_2 \cdots B_0 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_{p_\nu - 1} B_0 \cdots B_{p_\nu - 2} \end{pmatrix}.$$

For a square matrix $A = (a_{ij})$, we use the norm $\|A\| = \sqrt{\sum |a_{ij}|^2}$. Then for any $k \ge 1$, we have

$$\|(W_\nu^{p_\nu})^k\|^2 = \|(B_0 B_1 \cdots B_{p_\nu - 1})^k\|^2 + \cdots + \|(B_{p_\nu - 1} B_0 \cdots B_{p_\nu - 2})^k\|^2.$$

It follows from Lemma 6.3.2 that

$$\mathrm{sp}(W_\nu)^{p_\nu} = \mathrm{sp}(B_0 B_1 \cdots B_{p_\nu - 1}) = \cdots = \mathrm{sp}(B_{p_\nu - 1} B_0 \cdots B_{p_\nu - 2}).$$

On the other hand, one can verify that the (h_ν, P_ν)-transition matrix of Σ_ν is $B_0 B_1 \cdots B_{p_\nu - 1}$. It follows from Perron-Frobenius Theorem that

$$\lambda(\Sigma_\nu, h_\nu) = \mathrm{sp}(B_0 B_1 \cdots B_{p_\nu-1}) = \mathrm{sp}(W_\nu)^{p_\nu} = \lambda(\mathcal{C}_\nu, f)^{p_\nu}.$$

Finally we have

$$\lambda(\mathcal{C}, f) \quad = \quad \max\Big\{\lambda(\mathcal{C}_\Gamma, f),\ \sqrt[p_1]{\lambda(\Sigma_1, h_1)}, \cdots,\ \sqrt[p_n]{\lambda(\Sigma_n, h_n)}\Big\}.$$

□

Lemma 6.3.2. *Let B_ν be a $n_\nu \times n_{\nu+1}$ real matrix for $1 \le \nu \le k$, $n_{k+1} = n_1$, then*

$$\mathrm{sp}(B_1 B_2 \cdots B_k) = \mathrm{sp}(B_2 B_3 \cdots B_1) \cdots = \mathrm{sp}(B_k B_1 \cdots B_{k-1}).$$

Proof. For a square matrix $A = (a_{ij})$, we define a norm $\|A\| := \sqrt{\sum |a_{ij}|^2}$. The basic property of this norm is $\|AB\| \le \|A\|\|B\|$. First we assume $n_1 = \cdots = n_k$, then for any $1 \le \nu \le k$,

$$\begin{aligned}
\mathrm{sp}(B_1 B_2 \cdots B_k) &= \lim_{n\to\infty} \sqrt[n]{\|(B_1 B_2 \cdots B_k)^n\|} \\
&= \lim_{n\to\infty} \sqrt[n]{\|(B_1 \cdots B_{\nu-1})(B_\nu B_{\nu+1} \cdots B_{\nu-1})^{n-1}(B_\nu \cdots B_k)\|} \\
&\le \lim_{n\to\infty} \sqrt[n]{\|B_1 \cdots B_{\nu-1}\|\|(B_\nu B_{\nu+1} \cdots B_{\nu-1})^{n-1}\|\|B_\nu \cdots B_k\|} \\
&= \mathrm{sp}(B_\nu B_{\nu+1} \cdots B_{\nu-1}).
\end{aligned}$$

The same argument leads to the other direction of the inequality. In the following, we deal with the general case. Choose $n \ge \max\{n_1, \cdots, n_k\}$, for any $1 \le \nu \le k$, we define a $n \times n$ matrix $\widehat{B}_\nu$ by

$$\widehat{B}_\nu = \begin{pmatrix} B_\nu & O_{n_\nu \times (n - n_{\nu+1})} \\ O_{(n-n_\nu)\times n_{\nu+1}} & O_{(n-n_\nu)\times(n-n_{\nu+1})} \end{pmatrix},$$

where we use $O_{p\times q}$ to denote the $p \times q$ zero matrix. Then by the above argument, $\mathrm{sp}(\widehat{B}_1 \widehat{B}_2 \cdots \widehat{B}_k) = \mathrm{sp}(\widehat{B}_\nu \widehat{B}_{\nu+1} \cdots \widehat{B}_{\nu-1})$. On the other hand,

$$\widehat{B}_\nu \widehat{B}_{\nu+1} \cdots \widehat{B}_{\nu-1} = \begin{pmatrix} B_\nu B_{\nu+1} \cdots B_{\nu-1} & O_{n_\nu \times (n-n_\nu)} \\ O_{(n-n_\nu)\times n_\nu} & O_{(n-n_\nu)\times(n-n_\nu)} \end{pmatrix}.$$

This implies that $\|(B_1 B_2 \cdots B_k)^n\| = \|(\widehat{B}_1 \widehat{B}_2 \cdots \widehat{B}_k)^n\|$ for all $n \ge 1$. So

$$\mathrm{sp}(B_1 B_2 \cdots B_k) = \mathrm{sp}(\widehat{B}_1 \widehat{B}_2 \cdots \widehat{B}_k) = \mathrm{sp}(\widehat{B}_\nu \widehat{B}_{\nu+1} \cdots \widehat{B}_{\nu-1}) = \mathrm{sp}(B_\nu B_{\nu+1} \cdots B_{\nu-1}).$$

□

Proof of Theorem 6.3.1 .

Sufficiency. Let $\mathcal{C}$ be a (f, P)-stable multicurve in $\overline{\mathbb{C}} - P$. The multicurves $\mathcal{C}_\Gamma, \Sigma_1, \cdots, \Sigma_n$ are the subsets of $\mathcal{C}$ defined as in Theorem 6.3.2. We may assume that each curve $\gamma \in \mathcal{C}$ is contained in the interior of some $\mathcal{S}$-piece by Lemma 6.3.1. If $\lambda(\Gamma, f) < 1$ (notice that this implies $\Lambda^* = \emptyset$ by Lemma 6.2.4) and (h_ν, P_ν) has no Thurston obstructions for each $\nu \in \Lambda$, then by Theorem 6.3.2, we have

$$\begin{aligned}\lambda(\mathcal{C}, f) &= \max\left\{\lambda(\mathcal{C}_\Gamma, f),\ \sqrt[p_1]{\lambda(\Sigma_1, h_1)}, \cdots,\ \sqrt[p_n]{\lambda(\Sigma_n, h_n)}\right\} \\ &\leq \max\left\{\lambda(\Gamma, f),\ \sqrt[p_1]{\lambda(\Sigma_1, h_1)}, \cdots,\ \sqrt[p_n]{\lambda(\Sigma_n, h_n)}\right\} < 1.\end{aligned}$$

This means (f, P) has no Thurston obstructions.

Necessity. Suppose that (f, P) has no Thurston obstructions. Then $\lambda(\Gamma, f) < 1$ and $\Lambda^* = \emptyset$. Let Σ be a (h_ν, P_ν)-stable multicurve in $\overline{\mathbb{C}}(S_\nu) - P_\nu$. Up to homotopy, we may assume that each curve $\gamma \in \Sigma$ is contained in the interior of S_ν, so Σ can be considered as a multicurve in $\overline{\mathbb{C}} - P$. In the following, we will use Σ to generate a (f, P)-stable multicurve $\mathcal{C}$.

For $k \geq 0$, let $\Lambda_k \subset f^{-k}(\Sigma)$ be a multicurve in $\overline{\mathbb{C}} - P$, representing all homotopy classes of non-peripheral curves in $f^{-k}(\Sigma)$. We claim that

For any $\alpha \in \Lambda_i, \beta \in \Lambda_j$ with $0 \leq i < j$, if α is not homotopic to β in $\overline{\mathbb{C}} - P$, then α and β are homotopically disjoint.

In fact, the claim is obviously true in either of the following cases:

1. The curves α and β are contained in two different $\mathcal{S}$-pieces.
2. Either α or β is homotopic a curve in Γ.

So in the following discussion, we assume that α and β are contained in the same $\mathcal{S}$-piece S, and neither is homotopic to a boundary curve of S. We assume further that they intersect homotopically. In this case, one may check that both $f^i(\alpha)$ and $f^i(\beta)$ are contained in $f^i_*(S) = S_\nu$, but neither of $f^i(\alpha)$ and $f^i(\beta)$ is homotopic to a boundary curve of S_ν. So $f^i(\beta)$ is contained in the unique component of $f^{i-j}(S_\nu)$ that is parallel to S_ν. This implies $i \equiv j \mod p_\nu$. Since $f^j(\beta) \in \Sigma$ and Σ is (h_ν, P_ν)-stable, we have that $f^i(\beta)$ is homotopic in $\overline{\mathbb{C}} - P$ to either a curve of Σ or a curve of Γ. But neither is possible due to our assumption. This ends the proof of the claim.

For $k \geq 0$, we define a collection of Jordan curves $\mathcal{C}_k$ such that $\Sigma \subset \mathcal{C}_k \subset \Lambda_0 \cup \cdots \cup \Lambda_k$ and $\mathcal{C}_k$ represents all homotopy classes of non-peripheral curves in $\Lambda_0 \cup \cdots \cup \Lambda_k$. It follows from the above claim that we can consider $\mathcal{C}_k$ as a multicurve in $\overline{\mathbb{C}} - P$ up to homotopy. Notice that $\mathcal{C}_k$ is homotopically contained in $\mathcal{C}_{k+1}$, we have $\#\mathcal{C}_k \leq \#\mathcal{C}_{k+1}$. Since P has finitely many components, $\#\mathcal{C}_k$ is uniformly bounded above for all k. So there is an integer $N \geq 0$, such that $\#\mathcal{C}_n = \#\mathcal{C}_N$ for all $n \geq N$.

Let $\mathcal{C} = \mathcal{C}_N$, then $\mathcal{C}$ is a (f, P)-stable multicurve by the choice of N. Let $\mathcal{C}_\Gamma = \{\gamma \in \mathcal{C}; \gamma$ is homotopic to a curve in $\Gamma\}$, one may verify that $\Sigma = \{\gamma \in \mathcal{C} - \mathcal{C}_\Gamma; \gamma$ is contained in $S_\nu\}$. By Theorem 6.3.2,

$$\lambda(\Sigma, h_\nu) \leq \lambda(\mathcal{C}, f)^{p_\nu} < 1.$$

Thus (h_ν, P_ν) has no Thurston obstructions. $\square$

6.4 Surgery part: Gluing holomorphic models

The aim of this section is to prove the following:

Theorem 6.4.1. *Let (f, P) be a Herman map, and*

$$\mathrm{Dec}(f, P) = \left(\bigoplus_{\nu \in \Lambda \cup \Lambda^*} (h_\nu, P_\nu) \right)_\Gamma .$$

Then (f, P) is q.c-equivalent to a rational map if and only if $\lambda(\Gamma, f) < 1$ and for each $\nu \in \Lambda$, (h_ν, P_ν) is q.c-equivalent to a rational map.

The proof of Theorem 6.4.1 is based on the quasiconformal surgery.

In Section 6.4.1, we prove the necessity of Theorem 6.4.1. The idea is as follows: we use the rational realization of (f, P), say (R, Q), to generate the partial holomorphic models of $(h_\nu, P_\nu), \nu \in \Lambda$. The partial holomorphic model of (h_ν, P_ν) takes the form $R^{p_\nu}|_{E_\nu}$, where E_ν is a multi-connected domain in the Riemann sphere $\overline{\mathbb{C}}$. The holomorphic map $R^{p_\nu}|_{E_\nu}$ can be extended to a Siegel map or Thurston map, say (g_ν, Q_ν), q.c-equivalent to (h_ν, P_ν). The map (g_ν, Q_ν) can be made holomorphic outside a neighborhood of the boundary ∂E_ν. In the final step, we apply quasiconformal surgery to make the map (g_ν, Q_ν) globally holomorphic and get a rational realization of (h_ν, P_ν).

In Section 6.4.2, we prove the sufficiency of Theorem 6.4.1 assuming $\Gamma = \emptyset$. This part is the inverse procedure of Section 6.4.1. We use the rational realizations of $(h_\nu, P_\nu), \nu \in \Lambda$ to generate the partial holomorphic models for (f, P). These partial holomorphic models can be glued along $\Sigma = \Gamma_0$ in a suitable fashion into a branched covering (g, Q), holomorphic in most part of $\overline{\mathbb{C}}$ and q.c-equivalent to (f, P). Finally, we apply quasiconformal surgery to make the map (g, Q) globally holomorphic.

In Section 6.4.3, we prove the sufficiency of Theorem 6.4.1 in the more general case $\Gamma \neq \emptyset$. The idea of reconstruction of the rational realization of (f, P) via gluing the rational realizations of $(h_\nu, P_\nu), \nu \in \Lambda$ is essentially the same as that in Section 6.4.2. But this setion provides very interesting and technical flavor because of the algebraic condition $\lambda(\Gamma, f) < 1$. In most part of this section, we deal with this condition and shows that this algebraic condition is equivalent to the Grötzsch inequality in the homorphic setting. Thus it enables us to glue the partial homorphic models of (f,P) along Σ in a suitable fashion into a branched covering (g,Q), holomorphic in most part of $\overline{\mathbb{C}}$ and q.c-equivalent to (f,P). The last step is similar to the previous sections, it is a quasiconformal surgery procedure.

6.4.1 Rational realizations can descend

Theorem 6.4.2. (Marked McMullen Theorem) *Let R be a rational map, M be a closed set containing the postcritical set P_R and $R(M) \subset M$. Let Γ be a multicurve in $\overline{\mathbb{C}} - M$. Then $\lambda(\Gamma, R) \leq 1$. If $\lambda(\Gamma, R) = 1$, then either R is postcritically finite whose orbifold has signature (2, 2, 2, 2); or R is postcritically infinite, and Γ includes a curve contained in a periodic Siegel disk or Herman ring.*

We remark that the definition of the multicurve in $\overline{\mathbb{C}} - M$ is similar to the definition of the multicurve in $\overline{\mathbb{C}} - P$. The 'Marked McMullen Theorem' is slightly stronger than McMullen's original result [McM1], but the proof goes through without any problem.

Proof of the necessity of Theorem 6.4.1 Suppose that (f,P) is q.c-equivalent to a rational map (R,Q) via a pair of quasiconformal maps (ϕ_0,ϕ_1). Then the (f,P)-stable multicurve Γ in $\overline{\mathbb{C}} - P$ induces a (R,Q)-stable multicurve $\phi_0(\Gamma) := \{\phi_0(\gamma); \gamma \in \Gamma\}$ in $\overline{\mathbb{C}} - Q$. Since the marked set Q contains all possible Siegel disks and Herman rings of R, it follows from Theorem 6.4.2 that $\lambda(\Gamma, f) = \lambda(\phi_0(\Gamma), R) < 1$.

Notice that $\lambda(\Gamma, f) < 1$ implies $\Lambda^* = \emptyset$ by Lemma 6.2.4. In the following, we will show that for each $\nu \in \Lambda$, (h_ν, P_ν) is q.c-equivalent to a rational map.

Let $H_0 : [0,1] \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an isotopy between ϕ_0 and ϕ_1 rel P. That is, $H_0 : [0,1] \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a continuous map such that $H_0(0,\cdot) = \phi_0, H_0(1,\cdot) = \phi_1$ and $H_0(t,z) = \phi_0(z)$ for all $(t,z) \in [0,1] \times P$. Moreover, for any $t \in [0,1]$, $H_0(t,\cdot) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a quasiconformal map. Then there is a unique lift of H_0, say H_1, such that $H_0(t, f(z)) = R(H_1(t,z))$ for all $(t,z) \in [0,1] \times \overline{\mathbb{C}}$, with basepoint $H_1(0,\cdot) = \phi_1$. Set $\phi_2 = H_1(1,\cdot)$. Inductively, for any $k \geq 1$, let H_{k+1} be the unique lift of H_k such that $H_k(t, f(z)) = R(H_{k+1}(t,z))$ for all $(t,z) \in [0,1] \times \overline{\mathbb{C}}$, and $H_{k+1}(0,\cdot) = \phi_{k+1}$. Set $\phi_{k+2} = H_{k+1}(1,\cdot)$. In this way, we can get a sequence of quasiconformal maps $\phi_0, \phi_1, \phi_2, \cdots$, such that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r, "f"] & (\overline{\mathbb{C}}, P) \arrow[r, "f"] \arrow[d, "\phi_3"'] & (\overline{\mathbb{C}}, P) \arrow[r, "f"] \arrow[d, "\phi_2"'] & (\overline{\mathbb{C}}, P) \arrow[r, "f"] \arrow[d, "\phi_1"'] & (\overline{\mathbb{C}}, P) \arrow[d, "\phi_0"'] \\
\cdots \arrow[r, "R"'] & (\overline{\mathbb{C}}, Q) \arrow[r, "R"'] & (\overline{\mathbb{C}}, Q) \arrow[r, "R"'] & (\overline{\mathbb{C}}, Q) \arrow[r, "R"'] & (\overline{\mathbb{C}}, Q)
\end{tikzcd}$$

One can verify that for any $k \geq 0$, ϕ_{k+1} is isotopic to ϕ_k rel $f^{-k}(P)$.

Fix some $\nu \in \Lambda$, let D_ν be the union of all rotation disks of (h_ν, P_ν) intersecting ∂S_ν. We set $D_\nu = \emptyset$ if $\partial_0(S_\nu) = \emptyset$.

Choose a large integer $\ell > 0$ such that $\cup\Gamma \subset f^{-\ell+p_\nu}(P)$. Then we extend $\phi_\ell|_{S_\nu}$ to a quasiconformal map $\Phi : \overline{\mathbb{C}}(S_\nu) \to \overline{\mathbb{C}}$. We require that Φ is holomorphic in D_ν if $D_\nu \neq \emptyset$.

Notice that there is a unique component E_ν of $f^{-p_\nu}(S_\nu)$ parallel to S_ν. By the choice of ℓ, $\partial E_\nu \subset f^{-p_\nu}(\cup\Gamma) \subset f^{-\ell}(P)$, so $\phi_{\ell+p_\nu}$ and ϕ_ℓ are isotopic rel $f^{-\ell}(P)$. In particular, $\phi_{\ell+p_\nu}|_{\partial E_\nu} = \phi_\ell|_{\partial E_\nu} = \Phi|_{\partial E_\nu}$.

Denote the components of $\overline{\mathbb{C}}(S_\nu) - (E_\nu \cup D_\nu)$ by $\{U_j; j \in I\}$, where I is a finite index set. Each U_j is a disk, containing at most one point in P_ν. For any $j \in I$, let $V_j \Subset U_j$ be a disk such that $V_j \cap P_\nu = U_j \cap P_\nu$ and $U_j \setminus V_j \subset f^{-\ell}(P) \setminus P$. By Measurable Riemann Mapping Theorem, there is a quasiconformal homeomorphism $\Psi_j : V_j \to \Phi(V_j)$ whose Beltrami coefficient satisfies $\mu_{\Psi_j}(z) = \mu_{\Phi\circ h_\nu}(z)$ for $z \in V_j$. If U_j contains a point $p \in P_\nu$, we further require that $\Psi_j(p) = \Phi(p)$.

We can construct a quasiconformal map $\Psi : \overline{\mathbb{C}}(S_\nu) \to \overline{\mathbb{C}}$ by

$$\Psi(z) = \begin{cases} \Phi(z), & z \in D_\nu, \\ \phi_{\ell+p_\nu}(z), & z \in E_\nu, \\ \Psi_j(z), & z \in V_j, j \in I, \\ \textit{q.c interpolation}, & z \in U_j \setminus V_j, j \in I. \end{cases}$$

One may verify that Φ is homotopic to Ψ rel P_ν. Thus (h_ν, P_ν) is c-equivalent to $(g_\nu, Q_\nu) := (\Phi \circ h_\nu \circ \Psi^{-1}, \Phi(P_\nu))$ via (Φ, Ψ). Moreover, (g_ν, Q_ν) is holomorphic outside $\Psi(\cup_{j\in I}(U_j \setminus \overline{V_j}))$.

In the following, we will construct a (g_ν, Q_ν)-invariant complex structure. For each $j \in I$, we may assume that the annulus $U_j \setminus \overline{V_j}$ is thin enough such that for $k > 1$ large enough, $g_\nu^k(\Psi(U_j \setminus \overline{V_j}))$ is contained either in a rotation disk of g_ν, or in a neighborhood of a periodic critical point near which g_ν is holomorphic. Let $k_j \geq 1$ be the first integer such that g_ν is holomorphic in $g_\nu^{k_j}(\Psi(U_j \setminus \overline{V_j}))$. Define a complex structure in $\Psi(U_j \setminus \overline{V_j})$ by pulling back the standard complex structure in $g_\nu^{k_j}(\Psi(U_j \setminus \overline{V_j}))$ via $g_\nu^{k_j}$. Then we define a complex structure in $g_\nu^{-k}(\Psi(\cup_{j\in I}(U_j \setminus \overline{V_j})))$ by pulling back the complex structure in $\Psi(\cup_{j\in I}(U_j \setminus \overline{V_j}))$ via g_ν^k for all $k \geq 0$ and define the standard complex structure elsewhere. In this way, we get a (g_ν, Q_ν)-invariant complex structure σ. The Beltrami coefficient μ of σ satisfies $\|\mu\|_\infty < 1$ since (g_ν, Q_ν) is holomorphic outside $\Psi(\cup_{j\in I}(U_j \setminus \overline{V_j}))$.

By Measurable Riemann Mapping Theorem, there is a quasiconformal map $\zeta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ whose Beltrami coefficient is μ. Let $f_\nu = \zeta \circ g_\nu \circ \zeta^{-1}$, then f_ν is a rational map and (h_ν, P_ν) is q.c-equivalent to $(f_\nu, \zeta \circ \Phi(P_\nu))$ via $(\zeta \circ \Phi, \zeta \circ \Psi)$. See the following commutative diagram.

$$\begin{tikzcd}
\overline{\mathbb{C}}(S_\nu) \arrow[r, "\Psi"] \arrow[d, "h_\nu"'] & \overline{\mathbb{C}} \arrow[r, "\zeta"] \arrow[d, "g_\nu"'] & \overline{\mathbb{C}} \arrow[d, "f_\nu"'] \\
\overline{\mathbb{C}}(S_\nu) \arrow[r, "\Phi"'] & \overline{\mathbb{C}} \arrow[r, "\zeta"'] & \overline{\mathbb{C}}
\end{tikzcd}$$

6.4.2 Promotion of rational realizations when $\Gamma = \emptyset$

Proof of the sufficiency of Theorem 6.4.1, assuming $\Gamma = \emptyset$

Since $\Gamma = \emptyset$, for each $\mathcal{S}$-piece S, we have $\partial(S) = \partial_0(S) \subset \Gamma_0$, where Γ_0 is the collection of (f, P)-periodic curves defined in Section 6.2. It follows from Lemma 6.2.3 that S is f_*-periodic. So each $\mathcal{S}$-piece is f_*-periodic, and $\mathcal{S}$ can be written as $\{f_*^j(S_\nu); 0 \leq j < p_\nu, \nu \in \Lambda\}$. Moreover, any two $\mathcal{S}$-pieces contained in the same f_*-cycle have the same number of boundary curves.

Suppose that (h_ν, P_ν) is c-equivalent to a rational map (R_ν, Q_ν) via a pair of quasiconformal maps (Φ_ν, Ψ_ν) for $\nu \in \Lambda = [1, n]$.

Step 1: Getting partial holomorphic models. *For each $\mathcal{S}$-piece S, there exist a pair of quasiconformal maps $(\Phi_S, \Psi_S) : \overline{\mathbb{C}}(S) \to \overline{\mathbb{C}}$ and a rational map R_S such that the following diagram commutes:*

$$\begin{tikzcd}
\overline{\mathbb{C}}(S) \arrow[r, "H_S"] \arrow[d, "\Psi_S"'] & \overline{\mathbb{C}}(f_*(S)) \arrow[d, "\Phi_{f_*(S)}"] \\
\overline{\mathbb{C}} \arrow[r, "R_S"'] & \overline{\mathbb{C}}
\end{tikzcd}$$

It suffices to show that for each f_*-cycle $\langle S_\nu, \cdots, f_*^{p_\nu - 1}(S_\nu)\rangle$, there exist a sequence of quasiconformal maps $\Psi_{S_\nu},\ \Phi_{f_*^k(S_\nu)}, 0 \leq k < p_\nu$ and a sequence of rational maps $R_{f_*^k(S_\nu)}, 0 \le k < p_\nu$ such that the following diagram commutes

$$\begin{tikzcd}
\overline{\mathbb{C}}(S_\nu) \arrow[r, "H_{S_\nu}"] \arrow[d, "\Psi_{S_\nu}"'] & \overline{\mathbb{C}}(f_*(S_\nu)) \arrow[r, "H_{f_*(S_\nu)}"] \arrow[d, "\Phi_{f_*(S_\nu)}"'] & \overline{\mathbb{C}}(f_*^2(S_\nu)) \arrow[r] \arrow[d, "\Phi_{f_*^2(S_\nu)}"'] & \cdots \arrow[r] & \overline{\mathbb{C}}(f_*^{p_\nu-1}(S_\nu)) \arrow[r, "H_{f_*^{p_\nu-1}(S_\nu)}"] \arrow[d, "\Phi_{f_*^{p_\nu-1}(S_\nu)}"'] & \overline{\mathbb{C}}(S_\nu) \arrow[d, "\Phi_{S_\nu}"'] \\
\overline{\mathbb{C}} \arrow[r, "R_{S_\nu}"'] & \overline{\mathbb{C}} \arrow[r, "R_{f_*(S_\nu)}"'] & \overline{\mathbb{C}} \arrow[r] & \cdots \arrow[r] & \overline{\mathbb{C}} \arrow[r, "R_{f_*^{p_\nu-1}(S_\nu)}"'] & \overline{\mathbb{C}}
\end{tikzcd}$$

The constructions of the two sequences of maps are as follows: First, we set $\Phi_{S_\nu} = \Phi_\nu$ and $\Psi_{S_\nu} = \Psi_\nu$. By Measurable Riemann Mapping Theorem, there is a quasiconformal map $\Phi_{f_*^{p_\nu-1}(S_\nu)} : \overline{\mathbb{C}}(f_*^{p_\nu-1}(S_\nu)) \to \overline{\mathbb{C}}$ such that $\Phi^*_{f_*^{p_\nu-1}(S_\nu)}(\sigma_0) = (\Phi_{S_\nu} \circ H_{f_*^{p_\nu-1}(S_\nu)})^*(\sigma_0)$, where σ_0 is the standard complex structure. Then $R_{f_*^{p_\nu-1}(S_\nu)} = \Phi_{S_\nu} \circ H_{f_*^{p_\nu-1}(S_\nu)} \circ \Phi^{-1}_{f_*^{p_\nu-1}(S_\nu)}$ is a rational map.

Inductively, for $i = p_\nu - 2, \cdots, 1$, we can get a quasiconformal map $\Phi_{f_*^i(S_\nu)} : \overline{\mathbb{C}}(f_*^i(S_\nu)) \to \overline{\mathbb{C}}$ such that $R_{f_*^i(S_\nu)} = \Phi_{f_*^{i+1}(S_\nu)} \circ H_{f_*^i(S_\nu)} \circ \Phi^{-1}_{f_*^i(S_\nu)}$ is a rational map.

Finally, we set $R_{S_\nu} = \Phi_{f_*(S_\nu)} \circ H_{S_\nu} \circ \Psi^{-1}_{S_\nu}$. Then $R_\nu = R_{f_*^{p_\nu-1}(S_\nu)} \circ \cdots \circ R_{f_*(S_\nu)} \circ R_{S_\nu}$. Hence R_{S_ν} is also a rational map.

Set $\Psi_{f_*^i(S_\nu)} = \Phi_{f_*^i(S_\nu)}$ for $1 \le i < p_\nu$. Then the pair of quasiconformal maps $(\Phi_{f_*^i(S_\nu)}, \Psi_{f_*^i(S_\nu)})$ and the rational map $R_{f_*^i(S_\nu)} (0 \le i < p_\nu)$ are as required.

Step 2: Gluing holomorphic models. For each $\mathcal{S}$-piece S, recall that E_S is the unique $\mathcal{E}$-piece parallel to S. Since $\Gamma = \emptyset$, each boundary curve of S is also a boundary curve of E_S. So each component of $S - E_S$ is a disk, containing at most one point in P. Let $\{U_k; k \in I_S\}$ be the collection of all components of $S \setminus E_S$, where I_S is the finite index set induced by S. For any $k \in I_S$, let $V_k \Subset U_k$ be a disk such that $V_k \cap P = U_k \cap P$ and $U_k \setminus V_k \subset f^{-1}(P) \setminus P$. By the Measurable Riemann Mapping Theorem, there is a quasiconformal homeomorphism $\phi_k : V_k \to \Psi_S(V_k)$ whose Beltrami coefficient satisfies

$$\mu_{\phi_k}(z) = \sum_{\mathcal{E} \ni E \subset U_k} \chi_E(z) \mu_{\Phi_{f(E)} \circ f}(z), \ z \in V_k.$$

Here the summation is taken over all the $\mathcal{E}$-pieces contained in U_k. If V_k contains a point $p \in P$, we further require that $\phi_k(p) = \Phi_S(p)$.

Now we define a quasiconformal homeomorphism $\psi_S : S \to \Phi_S(S)$ by

$$\psi_S(z) = \begin{cases} \Psi_S(z), & z \in E_S, \\ \phi_k(z), & z \in V_k, k \in I_S, \\ q.c\ interpolation, & z \in U_k \setminus V_k, k \in I_S. \end{cases}$$

Define a quasiconformal map $\Theta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by $\Theta|_S = \psi_S^{-1} \circ \Phi_S$ for all $S \in \mathcal{S}$. The map Θ is isotopic to the identity map rel P. Let $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasiconformal map such that

$$\mu_\Phi(z) = \sum_{S\in\mathcal{S}} \chi_S(z)\mu_{\Phi_S}(z),\ z \in \overline{\mathbb{C}}.$$

Let $\Psi = \Phi \circ \Theta^{-1}$. The pair of quasiconformal maps (Φ, Ψ) can be considered as the gluing of $(\Phi_S|_S, \Psi_S|_S)_{S\in\mathcal{S}}$. In this way, (f, P) is q.c-equivalent to the Herman map $(g, Q) := (\Phi \circ f \circ \Psi^{-1}, \Phi(P))$ via (Φ, Ψ).

Step 3: Applying quasiconformal surgery. We first show that the Herman map (g, Q) is holomorphic in most parts of $\overline{\mathbb{C}}$. In fact, it is holomorphic outside $X := \Psi(\cup_{S\in\mathcal{S}} \cup_{k\in I_S} (U_k \setminus V_k))$. To see this, we fix some $\mathcal{S}$-piece S. The restriction $g|_{\Psi(E_S)}$ can be decomposed into

$$g|_{\Psi(E_S)} = (\Phi \circ \Phi^{-1}_{f(E_S)}) \circ (\Phi_{f(E_S)} \circ f \circ \Psi_S^{-1})|_{\Psi_S(E_S)} \circ (\Phi_S \circ \Phi^{-1})|_{\Psi(E_S)}.$$

For any $k \in I_S$, any $\mathcal{E}$-piece $E \subset U_k$, the restriction $g|_{\Psi(V_k\cap E)}$ can be decomposed into

$$g|_{\Psi(V_k\cap E)} = (\Phi \circ \Phi^{-1}_{f(E)}) \circ (\Phi_{f(E)} \circ f \circ \phi_k^{-1})|_{\phi_k(V_k\cap E)} \circ (\Phi_S \circ \Phi^{-1})|_{\Psi(V_k\cap E)}.$$

In either case, each factor of the decompositions of g is holomorphic in its domain of definition. So $g|_S$ is holomorphic outside $\Psi(\cup_{k\in I_S}(U_k \setminus V_k))$. It follows that (g, Q) is holomorphic outside X.

Let R_A be the union of all rotation annuli of g. Then one can check that $X \subset g^{-1}(R_A) \setminus R_A$. Let σ_0 be the standard complex structure in $\overline{\mathbb{C}}$. Define a g-invariant complex structure σ by

$$\sigma = \begin{cases} (g^k)^*(\sigma_0), & \text{in } g^{-k}(R_A) \setminus g^{-k+1}(R_A),\ k \geq 1, \\ \sigma_0, & \text{in } \overline{\mathbb{C}} - \cup_{k\geq 1}(g^{-k}(R_A) \setminus g^{-k+1}(R_A)). \end{cases}$$

Since g is holomorphic outside X, the Beltrami coefficient μ of σ satisfies $\|\mu\|_\infty < 1$. By Measurable Riemann Mapping Theorem, there is a quasiconformal map $\zeta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\zeta^*(\sigma_0) = \sigma$. Let $R = \zeta \circ g \circ \zeta^{-1}$, then R is a rational map and (f, P) is c-equivalent to $(R, \zeta \circ \Phi(P))$ via $(\zeta \circ \Phi, \zeta \circ \Psi)$. $\square$

6.4.3 Promotion of rational realizations when $\Gamma \neq \emptyset$

This is the technical part. We assume in this section that $\Gamma \neq \emptyset$, $\lambda(\Gamma, f) < 1$ and for each $\nu \in \Lambda = [1, n]$, the map (h_ν, P_ν) is q.c-equivalent to a rational map, we will show that (f, P) is q.c-equivalent to a rational map.

To begin with, we recall a result on non-negative matrix. Let W be a non-negative square matrix (i.e. each entry is a nonnegative real number).

It's known from Perron-Frobenius Theorem that the spectral radius of W is an eigenvalue of W, named the *leading eigenvalue*. Let $v = (v_1, \cdots, v_n)^t \in \mathbb{R}^n$ be a vector, we say $v > 0$ if for each i, $v_i > 0$. The following Lemma can be found in [CT1], Lemma A.1.

Lemma 6.4.1. *Let W be a non-negative square matrix with leading eigenvalue λ. Then $\lambda < 1$ iff there is a vector $v > 0$ such that $Wv < v$.*

With the help of Lemma 6.4.1, we turn to our discussion. First, $\lambda(\Gamma, f) < 1$ implies $Wv < v$, where W is the (f, P)-transition matrix of Γ and $v \in \mathbb{R}^\Gamma$ is a positive vector. That is, there is a positive function $v : \Gamma \to \mathbb{R}^+$ such that for any $\gamma \in \Gamma$,

$$(Wv)(\gamma) = \sum_{\beta \in \Gamma} \sum_{\alpha \sim \gamma} \frac{v(\beta)}{\deg(f : \alpha \to \beta)} < v(\gamma),$$

where the second summation is taken over all components α of $f^{-1}(\beta)$ homotopic to γ in $\overline{\mathbb{C}} - P$.

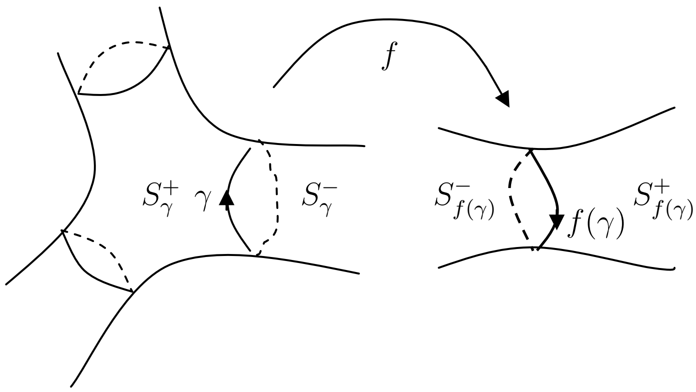

Figure 6.3: Orientation and labeling

Recall that for each curve $\gamma \in \Sigma$, there exist exactly two $\mathcal{S}$-pieces, say S_γ^+ and S_γ^-, such that $S_\gamma^+ \cap S_\gamma^- = \gamma$. For each curve $\gamma \in \Sigma$, we can associate an orientation such that f preserves the orientation. We may assume that the notations S_γ^+ and S_γ^- are chosen such that S_γ^+ lies on the left side of γ and S_γ^- lies on the right side of γ.

Here, we borrow some notations from Lemma 6.2.1. Recall that Γ_0 is the collection of the (f, P)-periodic curves that generates Γ, and $\Gamma_n = \{\gamma \in \Gamma; n$ is the first integer such that $f^n(\gamma) \in \Gamma_0\}$ for $n \geq 1$.

One may verify that if $\delta \in f^{-1}(\Gamma)$ is homotopic to a curve $\gamma \in \Gamma$ in $\overline{\mathbb{C}} - P$, then δ is necessarily contained in $S_\gamma^+ \cup S_\gamma^-$. Moreover, if $\gamma \in \Gamma_1$, then $\delta \neq \gamma$; if $\gamma \in \Gamma_k$ for some $k \geq 2$, it can happen that $\delta = \gamma$.

In the following, for each curve $\gamma \in \Gamma = \bigcup_{n\geq 1} \Gamma_n$, we will associate two positive numbers $\rho(S_\gamma^+, \gamma)$ and $\rho(S_\gamma^-, \gamma)$ inductively.

For $\gamma \in \Gamma_1$, we choose two positive numbers $\rho(S_\gamma^+, \gamma)$ and $\rho(S_\gamma^-, \gamma)$ such that

$$\rho(S_\gamma^+, \gamma) + \rho(S_\gamma^-, \gamma) = 1,$$

$$\sum_{\beta\in\Gamma} \sum_{\alpha\sim\gamma, \alpha\subset S_\gamma^\omega} \frac{v(\beta)}{\deg(f:\alpha\to\beta)} < v(\gamma)\rho(S_\gamma^\omega, \gamma),\ \omega \in \{\pm\}.$$

Suppose that for each curve $\alpha \in \Gamma_1 \cup \cdots \cup \Gamma_k$, we have already chosen two numbers $\rho(S_\alpha^+, \alpha)$ and $\rho(S_\alpha^-, \alpha)$. Then for $\gamma \in \Gamma_{k+1}$ (notice that $f(\gamma) \in \Gamma_k$), we can find two positive numbers $\rho(S_\gamma^+, \gamma)$ and $\rho(S_\gamma^-, \gamma)$ such that:

$$\rho(S_\gamma^+, \gamma) + \rho(S_\gamma^-, \gamma) = 1,$$

$$\frac{v(f(\gamma))}{\deg(f|_\gamma)}\rho(S_{f(\gamma)}^\omega, f(\gamma)) + \sum_{\beta\in\Gamma} \sum_{\alpha\sim\gamma, \alpha\subset S_\gamma^\omega\setminus\gamma} \frac{v(\beta)}{\deg(f:\alpha\to\beta)} < v(\gamma)\rho(S_\gamma^\omega, \gamma),\ \omega \in \{\pm\}.$$

In fact, we can take

$$\rho(S_\gamma^\omega, \gamma) = \frac{\dfrac{v(f(\gamma))}{\deg(f|_\gamma)}\rho(S_{f(\gamma)}^\omega, f(\gamma)) + \displaystyle\sum_{\beta\in\Gamma} \sum_{\alpha\sim\gamma, \alpha\subset S_\gamma^\omega\setminus\gamma} \frac{v(\beta)}{\deg(f:\alpha\to\beta)}}{\displaystyle\sum_{\beta\in\Gamma}\sum_{\alpha\sim\gamma} \frac{v(\beta)}{\deg(f:\alpha\to\beta)}},\ \omega \in \{\pm\}.$$

Potentials

Let $\mathbb{D}$ be the unit disk. A marked disk is a pair (Δ, a) with Δ an open hyperbolic disk in $\overline{\mathbb{C}}$ and $a \in \Delta$ a marked point. An equipotential γ of the marked disk (Δ, a) is a Jordan curve that is mapped to a round circle with center zero under a conformal isomorphism $\phi : \Delta \to \mathbb{D}$ with $\phi(a) = 0$. The potential $\varpi(\gamma)$ of γ is defined to be $\mathrm{mod}(A(\partial\Delta, \gamma))$, the modulus of the annulus between $\partial\Delta$ and γ. One may check that these definitions are independent of the choice of the Riemann mapping ϕ.

Suppose that (f, P) is either a Thurston rational map or a Siegel rational map, with a non-empty Fatou set. Recall that P is a marked set containing the poscritical set P_f. Then each periodic Fatou component is either a superattracting domain or a Siegel disk. If f has a superattracting Fatou component D, then every Fatou component Δ which is eventually mapped onto D can be marked by the unique eventually periodic point $a \in \Delta$. We call (Δ, a) a I-type marked disk of f. Notice that every equipotential in a superattracting Fatou component corresponds to a round circle in Böttcher coordinates. If f has a Siegel disk D, then it is known that the boundary ∂D is contained in the postcritical set P_f. Let z_0 be the center of the Siegel disk D, the intersection $P\cap(D-\{z_0\})$ is either empty or consists of finitely many (f, P)-periodic Jordan curves. Let $D_0 \subset D$ be the component of $\overline{\mathbb{C}}-(P\setminus\{z_0\})$ containing z_0. For any $k \geq 0$ and any component Δ of $f^{-k}(D_0)$, one can verify that Δ is a disk and there is a unique point $a \in \Delta \cap f^{-k}(z_0)$. We call (Δ, a) a II-type marked disk of (f, P).

A positive function

For each curve $\gamma \in \Sigma$, we associate a very thin annular neighborhood A^γ of γ. The annulas A^γ is chosen as follows: If $\gamma \in \Gamma_0$, then A^γ is a proper subset of the rotation annulus containing γ such that $f(A^\gamma) = A^{f(\gamma)}$ and $\overline{A^\gamma} \cap f(P - \cup\mathcal{A}) = \emptyset$. If $\gamma \in \Gamma_k$ for some $k \geq 1$, then A^γ is the component of $f^{-k}(A^{f^k(\gamma)})$ containing γ.

We define

$$\mathcal{S}^\star = \{U;\ U \text{ is a connected component of } \overline{\mathbb{C}} - \cup_{\gamma\in\Sigma} A^\gamma\},$$
$$\mathcal{E}^\star = \{V;\ V \text{ is a connected component of } \overline{\mathbb{C}} - f^{-1}(\cup_{\gamma\in\Sigma} A^\gamma)\}.$$

Each element of $S^\star$ (resp. $E^\star$) is called an $\mathcal{S}^\star$-piece (resp. $\mathcal{E}^\star$-piece). We will use $S^\star$ (resp. $E^\star$) to denote an $\mathcal{S}^\star$-piece (resp. $\mathcal{E}^\star$-piece). Notice that for each $\mathcal{S}$-piece S (resp. $\mathcal{E}$-piece E), there is a unique $\mathcal{S}^\star$-piece (resp. $\mathcal{E}^\star$-piece) contained in S (resp. E), we denote this piece by $S^\star$ (resp. $E^\star$). Similarly as in Section 6.2, we can define $E_{S^\star}$, the unique $\mathcal{E}^\star$-piece contained in $S^\star$ and parallel to $S^\star$. The map $f_* : \mathcal{S}^\star \to \mathcal{S}^\star$ is defined by $f_*(S^\star) = f(E_{S^\star})$. Moreover, the notations $\partial(E^\star), \partial(S^\star), \partial_0(S^\star), \partial_1(S^\star), \partial_2(S^\star)$, $\overline{\mathbb{C}}(S^\star)$ and the marked disk extension $H_{S^\star} : \overline{\mathbb{C}}(S^\star) \to \overline{\mathbb{C}}(f_*(S^\star))$ are defined similarly. Let

$$h^\star_\nu = H_{f_*^{p_\nu-1}(S^\star_\nu)} \circ \cdots \circ H_{f_*(S^\star_\nu)} \circ H_{S^\star_\nu},\ P^\star_\nu = P(S^\star_\nu),\ 1 \leq \nu \leq n.$$

One can see that this modification doesn't change the combinatorics and rational realizations of the maps $(h_\nu, P_\nu),\ 1 \leq \nu \leq n$. That is to say

- (h_ν, P_ν) has no Thurston obstructions if and only if $(h^\star_\nu, P^\star_\nu)$ has no Thurston obstructions.
- (h_ν, P_ν) is c-equivalent to a rational map if and only if $(h^\star_\nu, P^\star_\nu)$ is c-equivalent to a rational map.

The virtue of this modification is that we can construct deformations in a neighborhood of each curve $\gamma \in \Sigma$. This will be seen in the last step of the proof of Theorem 6.4.1 when we apply the quasiconformal surgery to glue all holomorphic models together to obtain a rational realization of (f, P).

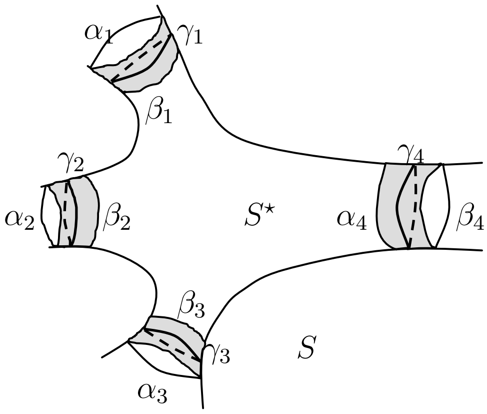

Figure 6.4: A $\mathcal{S}$-piece S with boundary $\partial S = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$. S contains a $\mathcal{S}^\star$-piece $S^\star$, whose boundary curves are $\beta_1, \beta_2, \beta_3$ and α_4.

For each curve $\gamma \in \Sigma$, let $\alpha_\gamma, \beta_\gamma$ be the two boundary curves of A^γ. Define

$$\Sigma^\star = \{\alpha_\gamma, \beta_\gamma; \gamma \in \Sigma\}, \quad \Gamma^\star = \{\alpha_\gamma, \beta_\gamma; \gamma \in \Gamma\}, \quad \Gamma_k^\star = \{\alpha_\gamma, \beta_\gamma; \gamma \in \Gamma_k\}, k \geq 0.$$

We define a map $\pi : \Sigma^\star \to \Sigma$ by $\pi(\alpha) = \gamma$ if α is a boundary curve of A^γ. It's obvious that for each curve $\gamma \in \Sigma$, $\pi^{-1}(\gamma) = \{\alpha_\gamma, \beta_\gamma\}$. For each curve $\gamma \in \Sigma^\star$, let S_γ (resp. $S_\gamma^\star$) be the unique $\mathcal{S}$-piece (resp. $\mathcal{S}^\star$-piece) that contains γ.

Now we define a positive function $\sigma_t : \Sigma^\star \to \mathbb{R}^+$, where t is a positive parameter, as follows:

First suppose $\gamma \in \Gamma^\star$. In this case, $\gamma \in \partial_1(S_\gamma^\star) \cup \partial_2(S_\gamma^\star)$. If $\gamma \in \partial_1(S_\gamma^\star)$, we define

$$\sigma_t(\gamma) = t \cdot \rho(S_\gamma, \pi(\gamma)) \cdot v(\pi(\gamma)).$$

If $\gamma \in \partial_2(S_\gamma^\star)$, we define

$$\sigma_t(\gamma) = \begin{cases} \dfrac{\sigma_t(f(\gamma))}{\deg(f|_\gamma)}, & \text{if } S_\gamma^\star \in \mathcal{S}^\star - \{S_1^\star, \cdots, S_n^\star\}, \\ t \cdot \rho(S_\gamma, \pi(\gamma)) \cdot v(\pi(\gamma)), & \text{if } S_\gamma^\star \in \{S_1^\star, \cdots, S_n^\star\}. \end{cases}$$

Now we consider $\gamma \in \Gamma_0^\star$. In this case, there is an integer $p \geq 0$ such that $f^p(\gamma) \in \partial_0(S_k^\star)$ for some $k \in \{1, \cdots, n\}$. Then $f^p(\gamma)$ is contained in a rotation disk of $(h_k^\star, P_k^\star)$, say D, with the center a. Notice that there is an annulus $A \subset \Delta$ such that:

• the inner boundary of A is $f^p(\gamma)$,

• the outer boundary of A is a $(h_k^\star, P_k^\star)$-periodic curve in the marked set $P_k^\star$,

- $A \cap P_k^\star = \emptyset$.

We define $\sigma_t(\gamma)$ to be the modulus of A. By definition, $\sigma_t(\gamma) = \sigma_t(f(\gamma)) = \cdots$. In this way, for all curves $\gamma \in \Sigma^\star$, the quantity $\sigma_t(\gamma)$ is well defined.

Lemma 6.4.2. *When t is large enough, the function $\sigma_t : \Sigma^\star \to \mathbb{R}^+$ satisfies:*

1. For any $\gamma \in \Gamma^\star$, $\sigma_t(\gamma) \le t \cdot \rho(S_\gamma, \pi(\gamma)) \cdot v(\pi(\gamma))$.

2. For every $\gamma \in \Sigma$, suppose that $\pi^{-1}(\gamma) = \{\alpha_\gamma, \beta_\gamma\}$. Then

$$\sigma_t(\alpha_\gamma) + \sigma_t(\beta_\gamma) \le \begin{cases} tv(\gamma), & \text{if } \gamma \in \Gamma, \\ \mathrm{mod}(A_\gamma), & \text{if } \gamma \in \Gamma_0, \end{cases}$$

where A_γ is the rotation annulus of (f, P) that contains γ if $\gamma \in \Gamma_0$.

3. For every $\gamma \in \Gamma^\star$, if $\gamma \in \partial_1(S_\gamma^\star)$, then we have the following inequality:

$$\sum_{\beta \in \Gamma^\star} \sum_{\alpha \sim \gamma, \alpha \subset S_\gamma^\star} \frac{\sigma_t(\beta)}{\deg(f : \alpha \to \beta)} < \sigma_t(\gamma)$$

where the second summation is taken over all components of $f^{-1}(\beta)$ contained in $S_\gamma^\star$ and homotopic to γ in $\overline{\mathbb{C}} - P$.

Proof. 1. Notice that if $\gamma \in \Gamma^\star$, then $\gamma \in \partial_1(S_\gamma^\star) \cup \partial_2(S_\gamma^\star)$. If $\gamma \in \partial_1(S_\gamma^\star)$ or $S_\gamma^\star \in \{S_1^\star, \cdots, S_n^\star\}$, then by definition, $\sigma_t(\gamma) = t\rho(S_\gamma, \pi(\gamma))v(\pi(\gamma))$. Now suppose $\gamma \in \partial_2(S_\gamma^\star)$ and $S_\gamma^\star \in \mathcal{S}^\star - \{S_1^\star, \cdots, S_n^\star\}$. Let $p \ge 1$ be the first integer such that $f_*^p(S_\gamma^\star) \in \{S_1^\star, \cdots, S_n^\star\}$. There is a largest number $k \in \{0, \cdots, p\}$ such that $f^j(\gamma) \in \partial_2(f_*^j(S_\gamma^\star))$ for $0 \le j < k$. Thus we have

$$\sigma_t(\gamma) = \frac{\sigma_t(f(\gamma))}{\deg(f|_\gamma)} = \cdots = \frac{\sigma_t(f^k(\gamma))}{\deg(f^k|_\gamma)}.$$

If $f^k(\gamma) \in \partial_0(f_*^k(S_\gamma^\star))$, then $\sigma_t(f^k(\gamma))$ is a constant independent of t, thus $\sigma_t(\gamma) \le t\rho(S_\gamma, \pi(\gamma))v(\pi(\gamma))$ when t is large.

If $f^k(\gamma) \in \partial_1(f_*^k(S_\gamma^\star))$, then

$$\sigma_t(\gamma) = \frac{t \cdot \rho(S_{f^k(\gamma)}, \pi(f^k(\gamma))) \cdot v(\pi(f^k(\gamma)))}{\deg(f^k|_\gamma)}.$$

By the choice of the numbers $\{\rho(S_\gamma^+, \gamma), \rho(S_\gamma^-, \gamma); \gamma \in \Gamma\}$, we see that for any curve $\beta \in \Gamma - \Gamma_1 = \cup_{n \ge 2} \Gamma_n$,

$$\frac{v(f(\beta))\rho(S^\omega_{f(\beta)}, f(\beta))}{\deg(f|_\beta)} < v(\beta)\rho(S^\omega_\beta, \beta), \ \omega \in \{\pm\}.$$

Since for each $\gamma \in \Gamma^\star$, $\deg(f|_\gamma) = \deg(f|_{\pi(\gamma)})$, we have that

$$\begin{aligned} \sigma_t(\gamma) &< \frac{t\rho(S_{f^{k-1}(\gamma)}, \pi(f^{k-1}(\gamma)))v(\pi(f^{k-1}(\gamma)))}{\deg(f^{k-1}|_\gamma)} < \cdots \\ &< \frac{t\rho(S_{f(\gamma)}, \pi(f(\gamma)))v(\pi(f(\gamma)))}{\deg(f|_\gamma)} < t\rho(S_\gamma, \pi(\gamma))v(\pi(\gamma)). \end{aligned}$$

If $f^k(\gamma) \in \partial_2(f_*^k(S_\gamma^\star))$, in this case, we have $k = p$ by the choice of k and

$$\sigma_t(\gamma) = \frac{t\rho(S_{f^p(\gamma)}, \pi(f^p(\gamma)))v(\pi(f^p(\gamma)))}{\deg(f^p|_\gamma)}.$$

With the same argument as above, we have $\sigma_t(\gamma) < t\rho(S_\gamma, \pi(\gamma))v(\pi(\gamma))$.

2. It follows from 1 and the definition of σ_t.

3.

$$\begin{aligned} &\sum_{\beta\in\Gamma^\star} \sum_{\alpha\sim\gamma, \alpha\subset S_\gamma^\star} \frac{\sigma_t(\beta)}{\deg(f:\alpha\to\beta)} \\ = &\sum_{\beta\in\Gamma^\star} \sum_{\alpha\sim\gamma, \alpha\subset S_\gamma^\star\setminus\gamma} \frac{\sigma_t(\beta)}{\deg(f:\alpha\to\beta)} + \frac{\sigma_t(f(\gamma))}{\deg(f|_\gamma)} \\ \le &\sum_{\beta\in\Gamma^\star} \sum_{\alpha\sim\gamma, \alpha\subset S_\gamma^\star\setminus\gamma} \frac{\sigma_t(\beta)}{\deg(f:\alpha\to\beta)} + \frac{t\rho(S_{f(\gamma)}, \pi(f(\gamma)))v(\pi(f(\gamma)))}{\deg(f|_\gamma)} \quad (By\ 1) \\ = &\sum_{\delta\in\Gamma} \sum_{\zeta\in\pi^{-1}(\delta)} \sum_{\alpha\sim\gamma, \alpha\subset S_\gamma^\star\setminus\gamma} \frac{\sigma_t(\zeta)}{\deg(f:\alpha\to\zeta)} + \frac{t\rho(S_{f(\gamma)}, \pi(f(\gamma)))v(\pi(f(\gamma)))}{\deg(f|_\gamma)} \\ = &\sum_{\delta\in\Gamma} \sum_{\alpha\sim\pi(\gamma), \alpha\subset S_\gamma\setminus\pi(\gamma)} \frac{\sum_{\zeta\in\pi^{-1}(\delta)} \sigma_t(\zeta)}{\deg(f:\alpha\to\delta)} + \frac{t\rho(S_{f(\gamma)}, \pi(f(\gamma)))v(\pi(f(\gamma)))}{\deg(f|_\gamma)} \\ \le &\sum_{\delta\in\Gamma} \sum_{\alpha\sim\pi(\gamma), \alpha\subset S_\gamma\setminus\pi(\gamma)} \frac{tv(\delta)}{\deg(f:\alpha\to\delta)} + \frac{t\rho(S_{f(\gamma)}, \pi(f(\gamma)))v(\pi(f(\gamma)))}{\deg(f|_\gamma)} \quad (By\ 2) \\ < \ &t\rho(S_\gamma, \pi(\gamma))v(\pi(\gamma)) = \sigma_t(\gamma). \quad (By\ the\ choice\ of\ the\ number\ \rho) \end{aligned}$$

□

Holomorphic Models

We first decompose $\mathcal{S}^\star$ into $\mathcal{S}_0^\star \sqcup \mathcal{S}_1^\star \sqcup \cdots$, where

$$\mathcal{S}_0^\star = \{f_*^j(S_\nu^\star); 0 \le j < p_\nu, 1 \le \nu \le n\},$$
$$\mathcal{S}_k^\star = \{S^\star \in \mathcal{S}^\star; k \text{ is the first integer such that } f_*^k(S^\star) \in \mathcal{S}_0^\star\}, k \ge 1.$$

It's obvious that $\mathcal{S}_0^\star$ consists of all f_*-periodic $\mathcal{S}^\star$-pieces.

Lemma 6.4.3. (Pre holomorphic models) *Suppose that $(h^\star_\nu, P^\star_\nu)$ is q.c-equivalent to a rational map (R_ν, Q_ν) via a pair of quasiconformal maps (Φ_ν, Ψ_ν) for $1 \le \nu \le n$. Then for each $\mathcal{S}^\star$-piece $S^\star$, there exist a pair of quasiconformal maps $(\Phi_{S^\star}, \Psi_{S^\star}) : \overline{\mathbb{C}}(S^\star) \to \overline{\mathbb{C}}$ and a rational map $R_{S^\star}$ such that $\Phi_{S^\star}$ is isotopic to $\Psi_{S^\star}$ rel $P(S^\star)$ and the following diagram commutes:*

$$\begin{tikzcd}
\overline{\mathbb{C}}(S^\star) \arrow[r, "H_{S^\star}"] \arrow[d, "\Psi_{S^\star}"'] & \overline{\mathbb{C}}(f_*(S^\star)) \arrow[d, "\Phi_{f_*(S^\star)}"] \\
\overline{\mathbb{C}} \arrow[r, "R_{S^\star}"'] & \overline{\mathbb{C}}
\end{tikzcd}$$

Proof. Using the same argument as the proof of the sufficiency of Theorem 6.4.1 (see Section 6.4.2, step 1), one can show that for any $1 \le \nu \le n$ and any $0 \le k < p_\nu$, there exist a quasiconformal map $\Phi_{f_*^k(S^\star_\nu)}$ and a rational map $R_{f_*^k(S^\star_\nu)}$ such that the following diagram commutes

$$\begin{tikzcd}
\overline{\mathbb{C}}(S^\star_\nu) \arrow[r, "H_{S^\star_\nu}"] \arrow[d, "\Psi_{S^\star_\nu}=\Psi_\nu"'] & \overline{\mathbb{C}}(f_*(S^\star_\nu)) \arrow[r, "H_{f_*(S^\star_\nu)}"] \arrow[d, "\Phi_{f_*(S^\star_\nu)}"'] & \cdots \arrow[r] & \overline{\mathbb{C}}(f_*^{p_\nu-1}(S^\star_\nu)) \arrow[r, "H_{f_*^{p_\nu-1}(S^\star_\nu)}"] \arrow[d, "\Phi_{f_*^{p_\nu-1}(S^\star_\nu)}"'] & \overline{\mathbb{C}}(S^\star_\nu) \arrow[d, "\Phi_{S^\star_\nu}=\Phi_\nu"] \\
\overline{\mathbb{C}} \arrow[r, "R_{S^\star_\nu}"'] & \overline{\mathbb{C}} \arrow[r, "R_{f_*(S^\star_\nu)}"'] & \cdots \arrow[r] & \overline{\mathbb{C}} \arrow[r, "R_{f_*^{p_\nu-1}(S^\star_\nu)}"'] & \overline{\mathbb{C}}
\end{tikzcd}$$

We set $\Psi_{f_*^k(S^\star_\nu)} = \Phi_{f_*^k(S^\star_\nu)}$ for $0 < k < p_\nu$.

For each $S^\star \in \mathcal{S}^\star_1$, notice that $f_*(S^\star) \in \mathcal{S}^\star_0$, we pull back the standard complex structure of $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}(S^\star)$ via $\Phi_{f_*(S^\star)} \circ H_{S^\star}$ and integrate it to get a quasiconformal map $\Phi_{S^\star} : \overline{\mathbb{C}}(S^\star) \to \overline{\mathbb{C}}$. Then $R_{S^\star} := \Phi_{f_*(S^\star)} \circ H_{S^\star} \circ \Phi^{-1}_{S^\star}$ is a rational map. We set $\Psi_{S^\star} = \Phi_{S^\star}$.

By the inductive procedure, for each $\mathcal{S}^\star_n$-piece $(n = 2, 3, \cdots)$, we can get a pair of quasiconformal maps $(\Phi_{S^\star}, \Psi_{S^\star})$ and a rational map $R_{S^\star}$, as required. □

Lemma 6.4.4. (Holomorphic model for periodic pieces) *Fix a periodic piece $S^\star \in \mathcal{S}^\star_0$. Let p be the period of $S^\star$. Then for any large parameter $t > 0$, there exist a pair of quasiconformal maps $(\Phi^t_{S^\star}, \Psi^t_{S^\star}) : \overline{\mathbb{C}}(S^\star) \to \overline{\mathbb{C}}$ such that*

1. $\Psi^t_{S^\star}$ is isotopic to $\Phi^t_{S^\star}$ rel $P(S^\star)$.

2. $\Phi^t_{f_*(S^\star)} \circ f \circ (\Psi^t_{S^\star})^{-1}|_{\Psi^t_{S^\star}(E_{S^\star})} = R_{S^\star}|_{\Psi^t_{S^\star}(E_{S^\star})}$, where $R_{S^\star}$ is defined in Lemma 6.4.3.

3. The return map $f_i := R_{f_*^{i-1}(S^\star)} \circ \cdots \circ R_{S^\star} \circ R_{f_*^{p-1}(S^\star)} \circ \cdots \circ R_{f_*^i(S^\star)}$ is either a Siegel map or a Thurston map.

4. For each $i \ge 0$ and each curve $\gamma \in \partial(f_*^i(S^\star))$, let β_γ be the unique boundary curve of $E_{f_*^i(S^\star)}$ such that either $\gamma = \beta_\gamma$, or γ and β_γ bound an annulus in $S^\star - P$. Then both $\Phi^t_{f_*^i(S^\star)}(\gamma)$ and $\Psi^t_{f_*^i(S^\star)}(\beta_\gamma)$ are equipotentials in the same marked disk of f_i, with potentials

$$\varpi(\Phi^t_{f_*^i(S^\star)}(\gamma)) = \sigma_t(\gamma),\ \varpi(\Psi^t_{f_*^i(S^\star)}(\beta_\gamma)) = \frac{\sigma_t(f(\beta_\gamma))}{\deg(f|_{\beta_\gamma})}.$$

Proof. For each $\nu \in [1,n]$ and each $i \geq 0$, the critical values of $H_{f_*^i(S_\nu^\star)}$ are contained in $P(f_*^{i+1}(S_\nu^\star))$ and $H_{f_*^i(S_\nu^\star)}(P(f_*^i(S_\nu^\star))) \subset P(f_*^{i+1}(S_\nu^\star))$. Let $(\Phi_{f_*^i(S_\nu^\star)}, \Psi_{f_*^i(S_\nu^\star)}) : \overline{\mathbb{C}}(f_*^i(S_\nu^\star)) \to \overline{\mathbb{C}}$ be the quasiconformal maps constructed in Lemma 6.4.3. Since $\Phi_{S_\nu^\star}$ is isotopic to $\Psi_{S_\nu^\star}$ rel $P_\nu^\star = P(S_\nu^\star)$, there is a quasiconformal map $\phi_{f_*^{p_\nu-1}(S_\nu^\star)} : \overline{\mathbb{C}}(f_*^{p_\nu-1}(S_\nu^\star)) \to \overline{\mathbb{C}}$ isotopic to $\Phi_{f_*^{p_\nu-1}(S_\nu^\star)}$ rel $P(f_*^{p_\nu-1}(S_\nu^\star))$ and $\Psi_{S_\nu^\star} \circ H_{f_*^{p_\nu-1}(S_\nu^\star)} = R_{f_*^{p_\nu-1}(S_\nu^\star)} \circ \phi_{f_*^{p_\nu-1}(S_\nu^\star)}$. Inductively, there is a sequence of quasiconformal maps $\phi_{f_*^i(S_\nu^\star)} : \overline{\mathbb{C}}(f_*^i(S_\nu^\star)) \to \overline{\mathbb{C}}$ for $i = p_\nu - 2, \cdots, 1$, such that $\phi_{f_*^i(S_\nu^\star)}$ is isotopic to $\Phi_{f_*^i(S_\nu^\star)}$ rel $P(f_*^i(S_\nu^\star))$ and the following diagram commutes:

$$\begin{tikzcd}
\overline{\mathbb{C}}(f_*(S_\nu^\star)) \arrow[r, "H_{f_*(S_\nu^\star)}"] \arrow[d, "\phi_{f_*(S_\nu^\star)}"'] & \overline{\mathbb{C}}(f_*^2(S_\nu^\star)) \arrow[r, "H_{f_*^2(S_\nu^\star)}"] \arrow[d, "\phi_{f_*^2(S_\nu^\star)}"'] & \cdots \arrow[r] & \overline{\mathbb{C}}(f_*^{p_\nu-1}(S_\nu^\star)) \arrow[r, "H_{f_*^{p_\nu-1}(S_\nu^\star)}"] \arrow[d, "\phi_{f_*^{p_\nu-1}(S_\nu^\star)}"'] & \overline{\mathbb{C}}(S_\nu^\star) \arrow[d, "\Psi_{S_\nu^\star}"'] \\
\overline{\mathbb{C}} \arrow[r, "R_{f_*(S_\nu^\star)}"'] & \overline{\mathbb{C}} \arrow[r, "R_{f_*^2(S_\nu^\star)}"'] & \cdots \arrow[r] & \overline{\mathbb{C}} \arrow[r, "R_{f_*^{p_\nu-1}(S_\nu^\star)}"'] & \overline{\mathbb{C}}
\end{tikzcd}$$

This diagram together with the diagram in Lemma 6.4.3 implies that for any $1 \leq i < p_\nu$, the map $H_{f_*^{i-1}(S_\nu^\star)} \circ \cdots \circ H_{S_\nu^\star} \circ H_{f_*^{p_\nu-1}(S_\nu^\star)} \circ \cdots \circ H_{f_*^i(S_\nu^\star)}$ is q.c-equivalent to $f_i = R_{f_*^{i-1}(S_\nu^\star)} \circ \cdots \circ R_{S_\nu^\star} \circ R_{f_*^{p_\nu-1}(S_\nu^\star)} \circ \cdots \circ R_{f_*^i(S_\nu^\star)}$ via $(\Phi_{f_*^i(S_\nu^\star)}, \phi_{f_*^i(S_\nu^\star)})$. Notice that $f_i(\phi_{f_*^i(S_\nu^\star)}(P(f_*^i(S_\nu^\star)))) \subset \phi_{f_*^i(S_\nu^\star)}(P(f_*^i(S_\nu^\star)))$, so f_i is either a Siegel map or a Thurston map.

The relation $f_{i+1} \circ R_{f_*^i(S_\nu^\star)} = R_{f_*^i(S_\nu^\star)} \circ f_i$ with $f_{p_\nu} = R_\nu$ (here, R_ν is the rational map defined in Lemma 6.4.3) means that $R_{f_*^i(S_\nu^\star)}$ is a semi-conjugacy between f_{i+1} and f_i, so their Julia sets satisfy $J(f_i) = R^{-1}_{f_*^i(S_\nu^\star)}(J(f_{i+1}))$. One can check that $R_{f_*^i(S_\nu^\star)}$ maps the marked disks of f_i onto the marked disks of f_{i+1}, and maps the equipotentials of f_i to the equipotentials of f_{i+1}.

In the following, we will construct a pair of quasiconformal maps $(\Phi^t_{S^\star}, \Psi^t_{S^\star}) : \overline{\mathbb{C}}(S^\star) \to \overline{\mathbb{C}}$ that satisfy the required properties.

Step 1: Construction of $\Phi^t_{S_\nu^\star}$ and $\Phi^t_{f_*^{p_\nu-1}(S_\nu^\star)}$. We first modify $\Phi_{S_\nu^\star}$ to a new quasiconformal map $\Phi^t_{S_\nu^\star} : \overline{\mathbb{C}}(S_\nu^\star) \to \overline{\mathbb{C}}$ such that $\Phi^t_{S_\nu^\star}$ is isotopic to $\Phi_{S_\nu^\star}$ rel $P(S_\nu^\star)$, and for each curve $\gamma \in \partial(S_\nu^\star)$, the curve $\Phi^t_{S_\nu^\star}(\gamma)$ is the equipotential in a marked disk of $f_{p_\nu} = R_\nu$ with potential $\varpi(\Phi^t_{S_\nu^\star}(\gamma)) = \sigma_t(\gamma)$.

Then, we lift $\Phi^t_{S_\nu^\star}$ via $R_{f_*^{p_\nu-1}(S_\nu^\star)}$ and $H_{f_*^{p_\nu-1}(S_\nu^\star)}$ and get a quasiconformal map $\widehat{\Phi}^t_{f_*^{p_\nu-1}(S_\nu^\star)}$ isotopic to $\Phi_{f_*^{p_\nu-1}(S_\nu^\star)}$ rel $P(f_*^{p_\nu-1}(S_\nu^\star))$. See the following commutative diagram:

$$\begin{tikzcd} \overline{\mathbb{C}}(f_*^{p_\nu-1}(S_\nu^\star)) \arrow[r, "H_{f_*^{p_\nu-1}(S_\nu^\star)}"] \arrow[d, "\widehat{\Phi}^t_{f_*^{p_\nu-1}(S_\nu^\star)}(\sim\Phi_{f_*^{p_\nu-1}(S_\nu^\star)})"'] & \overline{\mathbb{C}}(S_\nu^\star) \arrow[d, "\Phi^t_{S_\nu^\star}(\sim\Phi_{S_\nu^\star})"] \\ \overline{\mathbb{C}} \arrow[r, "R_{f_*^{p_\nu-1}(S_\nu^\star)}"'] & \overline{\mathbb{C}} \end{tikzcd}$$

Now, we modify $H_{f_*^{p_\nu-1}(S_\nu^\star)}$ to another marked disk extension of $f|_{E_{f_*^{p_\nu-1}(S_\nu^\star)}}$, say $\widehat{H}_{f_*^{p_\nu-1}(S_\nu^\star)}$, such that for each curve $\gamma \in \partial_1(f_*^{p_\nu-1}(S_\nu^\star))$, the curve $\Phi^t_{S_\nu^\star}(\widehat{H}_{f_*^{p_\nu-1}(S_\nu^\star)}(\gamma))$ is an equipotential in some marked disk of $f_{p_\nu} = R_\nu$. Since $\gamma \in \partial_1(f_*^{p_\nu-1}(S_\nu^\star))$, the potential of $\Phi^t_{S_\nu^\star}(\widehat{H}_{f_*^{p_\nu-1}(S_\nu^\star)}(\gamma))$ should be larger than $\varpi(\Phi^t_{S_\nu^\star}(f(\beta_\gamma))) = \sigma_t(f(\beta_\gamma))$. It follows from Lemma 6.4.2 that $\deg(f|_{\beta_\gamma})\sigma_t(\gamma) > \sigma_t(f(\beta_\gamma))$ when t is large. So we designate $\varpi(\Phi^t_{S_\nu^\star}(\widehat{H}_{f_*^{p_\nu-1}(S_\nu^\star)}(\gamma)))$ to be $\deg(f|_{\beta_\gamma}) \cdot \sigma_t(\gamma)$.

Since both $H_{f_*^{p_\nu-1}(S_\nu^\star)}$ and $\widehat{H}_{f_*^{p_\nu-1}(S_\nu^\star)}$ are marked disk extensions of $f|_{E_{f_*^{p_\nu-1}(S_\nu^\star)}}$, there is a quasiconformal map $\xi_{p_\nu-1} : \overline{\mathbb{C}}(f_*^{p_\nu-1}(S_\nu^\star)) \to \overline{\mathbb{C}}(f_*^{p_\nu-1}(S_\nu^\star))$ isotopic to the identity map rel $E_{f_*^{p_\nu-1}(S_\nu^\star)} \cup P(f_*^{p_\nu-1}(S_\nu^\star))$ such that $\widehat{H}_{f_*^{p_\nu-1}(S_\nu^\star)} = H_{f_*^{p_\nu-1}(S_\nu^\star)} \circ \xi_{p_\nu-1}$.

We set $\Phi^t_{f_*^{p_\nu-1}(S_\nu^\star)} = \widehat{\Phi}^t_{f_*^{p_\nu-1}(S_\nu^\star)} \circ \xi_{p_\nu-1}$. It's obvious that $\Phi^t_{S_\nu^\star} \circ \widehat{H}_{f_*^{p_\nu-1}(S_\nu^\star)} = R_{f_*^{p_\nu-1}(S_\nu^\star)} \circ \Phi^t_{f_*^{p_\nu-1}(S_\nu^\star)}$.

Step 2: Construction of $\Phi^t_{f_*^i(S_\nu^\star)}$ for $i = p_\nu - 2, \cdots, 1$ and $\Psi^t_{S_\nu^\star}$. By the same argument as in Step 1, we can lift $\Phi^t_{f_*^{p_\nu-1}(S_\nu^\star)}$ via $R_{f_*^{p_\nu-2}(S_\nu^\star)}$ and $H_{f_*^{p_\nu-2}(S_\nu^\star)}$ and get a map $\widehat{\Phi}^t_{f_*^{p_\nu-2}(S_\nu^\star)}$ isotopic to $\Phi_{f_*^{p_\nu-2}(S_\nu^\star)}$ rel $P(f_*^{p_\nu-2}(S_\nu^\star))$. Then we modify $H_{f_*^{p_\nu-2}(S_\nu^\star)}$ to another marked disk extension of $f|_{E_{f_*^{p_\nu-2}(S_\nu^\star)}}$, say $\widehat{H}_{f_*^{p_\nu-2}(S_\nu^\star)} = H_{f_*^{p_\nu-2}(S_\nu^\star)} \circ \xi_{p_\nu-2}$, where $\xi_{p_\nu-2} : \overline{\mathbb{C}}(f_*^{p_\nu-2}(S_\nu^\star)) \to \overline{\mathbb{C}}(f_*^{p_\nu-2}(S_\nu^\star))$ is a quasiconformal map isotopic to the identity map rel $E_{f_*^{p_\nu-2}(S_\nu^\star)} \cup P(f_*^{p_\nu-2}(S_\nu^\star))$, such that for each $\gamma \in \partial_1(f_*^{p_\nu-2}(S_\nu^\star))$, the curve $\Phi^t_{f_*^{p_\nu-1}(S_\nu^\star)}(\widehat{H}_{f_*^{p_\nu-2}(S_\nu^\star)}(\gamma))$ is an equipotential of $f_{p_\nu-1}$ with potential

$$\varpi(\Phi^t_{f_*^{p_\nu-1}(S_\nu^\star)}(\widehat{H}_{f_*^{p_\nu-2}(S_\nu^\star)}(\gamma))) = \deg(f|_{\beta_\gamma})\sigma_t(\gamma).$$

We set $\Phi^t_{f_*^{p_\nu-2}(S_\nu^\star)} = \widehat{\Phi}^t_{f_*^{p_\nu-2}(S_\nu^\star)} \circ \xi_{p_\nu-2}$.

Inductively, we can get a sequence of new marked disk extensions $\widehat{H}_{f_*^i(S_\nu^\star)}, i = p_\nu - 1, \cdots, 0$, and a sequence of quasiconformal maps $\Phi^t_{f_*^i(S_\nu^\star)}, i = p_\nu - 1, \cdots, 1$, $\Psi^t_{S_\nu^\star}$ such that the following diagram commutes

$$
\begin{tikzcd}
\overline{\mathbb{C}}(S_\nu^\star) \arrow[r, "\widehat{H}_{S_\nu^\star}"] \arrow[d, "\Psi^t_{S_\nu^\star}"'] & \overline{\mathbb{C}}(f_*(S_\nu^\star)) \arrow[r, "\widehat{H}_{f_*(S_\nu^\star)}"] \arrow[d, "\Phi^t_{f_*(S_\nu^\star)}"'] & \cdots \arrow[r] & \overline{\mathbb{C}}(f_*^{p_\nu-1}(S_\nu^\star)) \arrow[r, "\widehat{H}_{f_*^{p_\nu-1}(S_\nu^\star)}"] \arrow[d, "\Phi^t_{f_*^{p_\nu-1}(S_\nu^\star)}"'] & \overline{\mathbb{C}}(S_\nu^\star) \arrow[d, "\Phi^t_{S_\nu^\star}"] \\
\overline{\mathbb{C}} \arrow[r, "R_{S_\nu^\star}"'] & \overline{\mathbb{C}} \arrow[r, "R_{f_*(S_\nu^\star)}"'] & \cdots \arrow[r] & \overline{\mathbb{C}} \arrow[r, "R_{f_*^{p_\nu-1}(S_\nu^\star)}"'] & \overline{\mathbb{C}}
\end{tikzcd}
$$

Moreover, for each $i \in [0, p_\nu - 1]$ and each curve $\gamma \in \partial_1(f_*^i(S_\nu^\star))$, we require

$$\varpi(\Phi^t_{f_*^{i+1}(S_\nu^\star)}(\widehat{H}_{f_*^i(S_\nu^\star)}(\gamma))) = \deg(f|_{\beta_\gamma})\sigma_t(\gamma).$$

Finally, we set $\Psi^t_{f_*^i(S_\nu^\star)} = \Phi^t_{f_*^i(S_\nu^\star)}$ for $1 \le i \le p_\nu - 1$.

Step 3: Prescribed potentials. In this step, we will show that for each $0 \le i \le p_\nu - 1$ and each curve $\gamma \in \partial(f_*^i(S_\nu^\star))$,

$$\varpi(\Phi^t_{f_*^i(S_\nu^\star)}(\gamma)) = \sigma_t(\gamma),\ \varpi(\Psi^t_{f_*^i(S_\nu^\star)}(\beta_\gamma)) = \frac{\sigma_t(f(\beta_\gamma))}{\deg(f|_{\beta_\gamma})}. \tag{6.1}$$

Notice that for each curve $\gamma \in \partial(S_\nu^\star) \cup \cup_{0<i<p_\nu} \partial_0(f_*^i(S_\nu^\star))$, the first equation of (6.1) holds by the evaluation of ϖ.

If $\gamma \in \partial_1(f_*^i(S_\nu^\star))$ for some $0 < i < p_\nu$, then by construction, $\Phi^t_{f_*^{i+1}(S_\nu^\star)}(\widehat{H}_{f_*^i(S_\nu^\star)}(\gamma))$ is an equipotential in a marked disk (Δ_{i+1}, a) of f_{i+1}. Since $\Phi^t_{f_*^{i+1}(S_\nu^\star)} \circ \widehat{H}_{f_*^i(S_\nu^\star)}(\gamma) = R_{f_*^i(S_\nu^\star)} \circ \Phi^t_{f_*^i(S_\nu^\star)}(\gamma)$, we conclude that $\Phi^t_{f_*^i(S_\nu^\star)}(\gamma)$ is also an equipotential of some marked disk of f_i, denoted by (Δ_i, b). Then $R_{f_*^i(S_\nu^\star)} : \Delta_i - \{b\} \to \Delta_{i+1} - \{a\}$ is a covering map of degree $\deg(f|_{\beta_\gamma})$. The potential of $\Phi^t_{f_*^i(S_\nu^\star)}(\gamma)$ satisfies (Here, we use $A(\alpha, \beta)$ to denote the annulus bounded by α and β)

$$
\begin{aligned}
\varpi(\Phi^t_{f_*^i(S_\nu^\star)}(\gamma)) &= \mathrm{mod}\Big(A(\partial\Delta_i, \Phi^t_{f_*^i(S_\nu^\star)}(\gamma))\Big) \\
&= \mathrm{mod}\Big(A(\partial\Delta_{i+1}, \Phi^t_{f_*^{i+1}(S_\nu^\star)}(\widehat{H}_{f_*^i(S_\nu^\star)}(\gamma)))\Big)/\deg(f|_{\beta_\gamma}) \\
&= \varpi(\Phi^t_{f_*^{i+1}(S_\nu^\star)}(\widehat{H}_{f_*^i(S_\nu^\star)}(\gamma)))/\deg(f|_{\beta_\gamma}) \\
&= \sigma_t(\gamma).
\end{aligned}
$$

Now we consider $\gamma \in \partial_2(f_*^i(S_\nu^\star))$ for some $0 < i < p_\nu$. In this case, by the same argument as above, we can see that

$$\varpi(\Phi^t_{f_*^i(S_\nu^\star)}(\gamma)) = \frac{\varpi(\Phi^t_{f_*^{i+1}(S_\nu^\star)}(f(\gamma)))}{\deg(f|_\gamma)}.$$

By the definition of σ_t, for $\gamma \in \partial_2(f_*^i(S_\nu^\star))$, we have

$$\sigma_t(\gamma) = \frac{\sigma_t(f(\gamma))}{\deg(f|_\gamma)}.$$

Based on this observation, we conclude by induction that $\varpi(\Phi^t_{f_*^i(S_\nu^\star)}(\gamma)) = \sigma_t(\gamma)$.

To finish, we show that the second equation of (6.1) holds. Since for each $i \in [0, p_\nu - 1]$ and each curve $\gamma \in \partial(f_*^i(S_\nu^\star))$, the curve $\Phi^t_{f_*^{i+1}(S_\nu^\star)}(f(\beta_\gamma))$ is an equipotential, it follows from the relation

$$\Phi^t_{f_*^{i+1}(S_\nu^\star)} \circ f \circ (\Psi^t_{f_*^i(S_\nu^\star)})^{-1}|_{\Psi^t_{f_*^i(S_\nu^\star)}(E_{f_*^i(S_\nu^\star)})} = R_{f_*^i(S_\nu^\star)}|_{\Psi^t_{f_*^i(S_\nu^\star)}(E_{f_*^i(S_\nu^\star)})}$$

that $\Psi^t_{f_*^i(S_\nu^\star)}(\beta_\gamma)$ is also an equipotential. Using a similar argument as above, we obtain

$$\varpi(\Psi^t_{f_*^i(S_\nu^\star)}(\beta_\gamma)) = \frac{\varpi(\Phi^t_{f_*^{i+1}(S_\nu^\star)}(f(\beta_\gamma)))}{\deg(f|_{\beta_\gamma})} = \frac{\sigma_t(f(\beta_\gamma))}{\deg(f|_{\beta_\gamma})}.$$

The proof is completed. $\square$

Now, we deal with the strictly pre-periodic $\mathcal{S}^\star$-pieces. Let $S^\star \in \mathcal{S}_k^\star$ for some $k \geq 1$. Then $f_*^k(S^\star)$ is a f_*-periodic $\mathcal{S}^\star$-piece. Notice that for $0 \leq i < k$, $H_{f_*^i(S^\star)}(P(f_*^i(S^\star))) \subset P(f_*^{i+1}(S^\star))$ and each critical value of $H_{f_*^i(S^\star)}$ is contained in $P(f_*^{i+1}(S^\star))$, we have that $R_{f_*^i(S^\star)} \circ \Phi_{f_*^i(S^\star)}(P(f_*^i(S^\star))) \subset \Phi_{f_*^{i+1}(S^\star)}(P(f_*^{i+1}(S^\star)))$ and every critical value of $R_{f_*^{k-1}(S^\star)} \circ \cdots \circ R_{S^\star}$ is contained in $\Phi_{f_*^k(S^\star)}(P(f_*^k(S^\star))) = \Phi^t_{f_*^k(S^\star)}(P(f_*^k(S^\star)))$, here $R_{f_*^i(S^\star)}$ and $\Phi_{f_*^i(S^\star)}$ are defined in Lemma 6.4.3. For any marked point $a \in P(S^\star) \cap (\overline{\mathbb{C}}(S^\star) - S^\star)$, the point $R_{f_*^{k-1}(S^\star)} \circ \cdots \circ R_{S^\star}(\Phi_{S^\star}(a))$ is the center of some marked disk (Δ, q) of some f_j, where f_j is a return map defined in Lemma 6.4.4. The component $\Delta_{\Phi_{S^\star}(a)}$ of $(R_{f_*^{k-1}(S^\star)} \circ \cdots \circ R_{S^\star})^{-1}(\Delta)$ that contains $\Phi_{S^\star}(a)$ is also a disk. We call $(\Delta_{\Phi_{S^\star}(a)}, \Phi_{S^\star}(a))$ a marked disk of $R_{f_*^{k-1}(S^\star)} \circ \cdots \circ R_{S^\star}$.

By the same argument as in the proof of Lemma 6.4.4, we can show that

Lemma 6.4.5. *For any $k \geq 1$, any $S^\star \in \mathcal{S}_k^\star$ and any large parameter $t > 0$, there exist a pair of quasiconformal maps $\Phi^t_{S^\star} = \Psi^t_{S^\star} : \overline{\mathbb{C}}(S^\star) \to \overline{\mathbb{C}}$ such that*

1. $\Phi^t_{f_*(S^\star)} \circ f \circ (\Psi^t_{S^\star})^{-1}|_{\Psi^t_{S^\star}(E_{S^\star})} = R_{S^\star}|_{\Psi^t_{S^\star}(E_{S^\star})}$, where $R_{S^\star}$ is defined in Lemma 6.4.3.

2. For each curve $\gamma \in \partial(S^\star)$, let β_γ be the unique curve in $\partial(E_{S^\star})$ homotopic to γ in $\overline{\mathbb{C}} - P$. Then both $\Phi^t_{S^\star}(\gamma)$ and $\Phi^t_{S^\star}(\beta_\gamma)$ are equipotentials in the same marked disk of $R_{f_*^{k-1}(S^\star)} \circ \cdots \circ R_{S^\star}$, with potentials

$$\varpi(\Phi^t_{S^\star}(\gamma)) = \sigma_t(\gamma),\ \varpi(\Phi^t_{S^\star}(\beta_\gamma)) = \frac{\sigma_t(f(\beta_\gamma))}{\deg(f|_{\beta_\gamma})}.$$

We decompose $\mathcal{E}^\star$ into $\mathcal{E}^\star_{ess} \sqcup \mathcal{E}^\star_A \sqcup \mathcal{E}^\star_D$, where

• $\mathcal{E}^\star_{ess} = \{E_{S^\star}; S^\star \in \mathcal{S}^\star\}$, it consists of all $\mathcal{E}^\star$-pieces parallel to some $\mathcal{S}^\star$-piece;

• $\mathcal{E}^\star_A$ is the collection of all $\mathcal{E}^\star$-pieces $E^\star$ contained essentially in an annular component of $S^\star - E_{S^\star}$ for some $\mathcal{S}^\star$-piece $S^\star$ (Here 'essential' means at least one boundary curve of $E^\star$ is non-peripheral in $\overline{\mathbb{C}} - P$);

• $\mathcal{E}^\star_D = \mathcal{E}^\star - (\mathcal{E}^\star_{ess} \sqcup \mathcal{E}^\star_A)$. One may verify that each $\mathcal{E}^\star_D$-piece is contained in a disk component of $S^\star - E_{S^\star}$ for some $\mathcal{S}^\star$-piece $S^\star$.

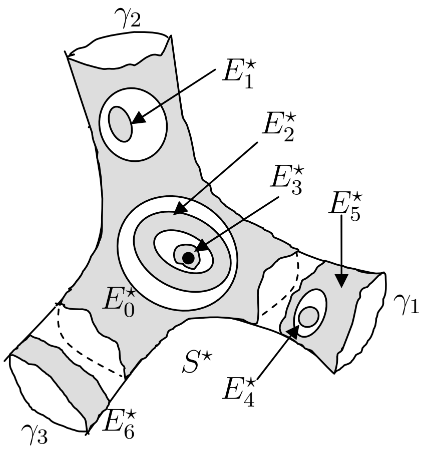

Figure 6.5: Different types of $\mathcal{E}^\star$-pieces: Here, $S^\star$ is a $\mathcal{S}^\star$-piece with boundary $\partial S^\star = \gamma_1 \cup \gamma_2 \cup \gamma_3$. $E_0^\star$ is a $\mathcal{E}^\star_{ess}$-piece. $E_5^\star$ and $E_6^\star$ are $\mathcal{E}^\star_A$-pieces. $E_1^\star, E_2^\star$, $E_3^\star$ and $E_4^\star$ are $\mathcal{E}^\star_D$-pieces.

In the following, for every $\mathcal{E}^\star_A$-piece $E^\star$, we will construct a holomorphic model for $f|_{E^\star}$. Given an $\mathcal{E}^\star_A$-piece $E^\star$, first notice that $E^\star$ has no intersection with the marked set P. As we did before, we also associate a Riemann sphere $\overline{\mathbb{C}}(E^\star)$ for $E^\star$. We mark a point in each component of $\overline{\mathbb{C}}(E^\star) - E^\star$, and let $P(E^\star)$ be the collection of all these marked points. We can get a marked disk extension of $f|_{E^\star}$, say $H_{E^\star} : \overline{\mathbb{C}}(E^\star) \to \overline{\mathbb{C}}(f(E^\star))$, such that $H_{E^\star}|_{E^\star} = f|_{E^\star}$, $H_{E^\star}(P(E^\star)) \subset P(f(E^\star))$ and all critical values (if any) of $H_{E^\star}$ are contained in $P(f(E^\star))$. Let $\Phi^t_{E^\star} : \overline{\mathbb{C}}(E^\star) \to \overline{\mathbb{C}}$ be a quasiconformal map such that $R_{E^\star} := \Phi^t_{f(E^\star)} \circ H_{E^\star} \circ (\Phi^t_{E^\star})^{-1}$ is holomorphic. We remark that if we change $\Phi^t_{f(E^\star)}$ to another quasiconformal map $\Phi^{t_1}_{f(E^\star)}$ isotopic to $\Phi^t_{f(E^\star)}$ rel $P(f(E^\star))$, then we can modify $\Phi^t_{E^\star}$ to a new map $\Phi^{t_1}_{E^\star}$, isotopic to $\Phi^t_{E^\star}$ rel $P(E^\star)$, such that $R_{E^\star} = \Phi^{t_1}_{f(E^\star)} \circ H_{E^\star} \circ (\Phi^{t_1}_{E^\star})^{-1}$. This means that once we get the holomorphic map $R_{E^\star}$, we can always assume that it is independent of the parameter t. We set $\Psi^t_{E^\star} = \Phi^t_{E^\star}$.

Notice that the $\mathcal{E}^\star_A$-piece $E^\star$ has exactly two boundary curves α and

β which are non-peripheral and homotopic to each other in $\overline{\mathbb{C}} - P$. By the choice of $\Phi^t_{S^\star}$ for $S^\star \in \mathcal{S}^\star$, both $\Phi^t_{f(E^\star)}(f(\alpha))$ and $\Phi^t_{f(E^\star)}(f(\beta))$ are equipotentials in the marked disks of some f_j (defined in Lemma 6.4.4) or some $R_{f_*^{k-1}(S^\star)} \circ \cdots \circ R_{S^\star}$. We denote the marked disk that contains $\Phi^t_{f(E^\star)}(f(\alpha))$ (resp. $\Phi^t_{f(E^\star)}(f(\beta))$) by (Δ_a, a) (resp. (Δ_b, b)). It can happen that $(\Delta_a, a) = (\Delta_b, b)$. Let Δ_α (resp. Δ_β) be the component of $R^{-1}_{E^\star}(\Delta_a)$ (resp. $R^{-1}_{E^\star}(\Delta_b)$) that contains $\Phi^t_{E^\star}(\alpha)$ (resp. $\Phi^t_{E^\star}(\beta)$). Then Δ_α (resp. Δ_β) contains a marked point in $P(E^\star)$, say z_α (resp. z_β). The marked disks $(\Delta_\alpha, z_\alpha)$ and (Δ_β, z_β) are called the marked disks of $R_{E^\star}$. They are independent of the choice of t. Clearly, $\Phi^t_{E^\star}(\alpha)$ is an equipotential in the marked disk $(\Delta_\alpha, z_\alpha)$ and $\Phi^t_{E^\star}(\beta)$ is an equipotential in the marked disk (Δ_β, z_β), with potentials

$$\begin{aligned}
\varpi(\Phi^t_{E^\star}(\alpha)) &= \frac{\varpi(\Phi^t_{f(E^\star)}(f(\alpha)))}{\deg(f|_\alpha)} = \frac{\sigma_t(f(\alpha))}{\deg(f|_\alpha)}, \\
\varpi(\Phi^t_{E^\star}(\beta)) &= \frac{\varpi(\Phi^t_{f(E^\star)}(f(\beta)))}{\deg(f|_\beta)} = \frac{\sigma_t(f(\beta))}{\deg(f|_\beta)}.
\end{aligned}$$

We denote by $A(E^\star) \subset \overline{\mathbb{C}}(E^\star)$ the annulus bounded by α and β. By Reversed Grötzsch Inequality (See Theorem 2.2.3, also Lemma B.1 in [CT1]), there is a constant $C(E^\star)$, independent of the parameter t, such that

$$\operatorname{mod}(\Phi^t_{E^\star}(A(E^\star))) \le \frac{\sigma_t(f(\alpha))}{\deg(f|_\alpha)} + \frac{\sigma_t(f(\beta))}{\deg(f|_\beta)} + C(E^\star).$$

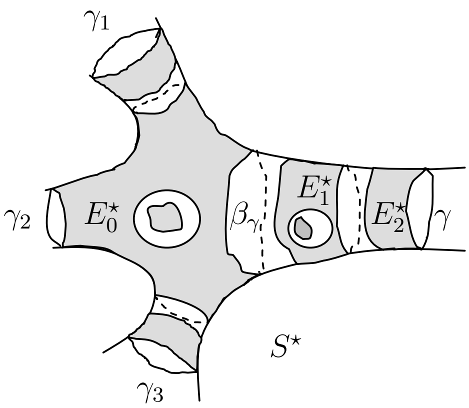

Figure 6.6: A $\mathcal{S}^\star$-piece $S^\star$ with boundary $\partial S^\star = \gamma \cup \gamma_1 \cup \gamma_2 \cup \gamma_3$. Here, $E_0^\star$ is the $\mathcal{E}^\star_{ess}$-piece parallel to $S^\star$, $E_1^\star$ and $E_2^\star$ are two $\mathcal{E}^\star_A$-pieces between γ and β_γ.

For any $\mathcal{S}^\star$-piece $S^\star$ and any $\gamma \in \partial_1(S^\star)$, let $A^\gamma_{S^\star}$ be the annulus bounded by γ and β_γ. By the construction of $\Phi^t_{S^\star}, \Psi^t_{S^\star} : \overline{\mathbb{C}}(S^\star) \to \overline{\mathbb{C}}$, both $\Phi^t_{S^\star}(\gamma)$

and $\Psi^t_{S^\star}(\beta_\gamma)$ are equipotentials. We denote the annulus between $\Phi^t_{S^\star}(\gamma)$ and $\Psi^t_{S^\star}(\beta_\gamma)$ by $A^t(S^\star, \gamma)$. It's obvious that

$$\mathrm{mod}(A^t(S^\star,\gamma)) = \varpi(\Phi^t_{S^\star}(\gamma)) - \varpi(\Psi^t_{S^\star}(\beta_\gamma)) = \sigma_t(\gamma) - \frac{\sigma_t(f(\beta_\gamma))}{\deg(f|_{\beta_\gamma})}.$$

Then we have the following

Lemma 6.4.6. (Large parameter implies Grötzsch inequality) *When t is large enough, for any $\mathcal{S}^\star$-piece $S^\star$ and any $\gamma \in \partial_1(S^\star)$, we have*

$$\sum_{\mathcal{E}^\star_A \ni E^\star \subset \overline{A^\gamma_{S^\star}}} \mathrm{mod}(\Psi^t_{E^\star}(A(E^\star))) < \mathrm{mod}(A^t(S^\star,\gamma)),$$

where the summation is taken over all the $\mathcal{E}^\star_A$-pieces contained in $\overline{A^\gamma_{S^\star}}$.

Proof. It suffices to show that when t is large enough,

$$\sum_{\mathcal{E}^\star_A \ni \widehat{E} \subset \overline{A^\gamma_{S^\star}}} \left(\frac{\sigma_t(f(\alpha_{E^\star}))}{\deg(f|_{\alpha_{E^\star}})} + \frac{\sigma_t(f(\beta_{E^\star}))}{\deg(f|_{\beta_{E^\star}})} + C(E^\star) \right) + \frac{\sigma_t(f(\beta_\gamma))}{\deg(f|_{\beta_\gamma})} < t \cdot \rho(S_\gamma, \pi(\gamma)) \cdot v(\pi(\gamma)),$$

where $\alpha_{E^\star}$ and $\beta_{E^\star}$ are the boundary curves of $E^\star$, homotopic to γ in $\overline{\mathbb{C}} - P$.

One can verify that

$$\sum_{\mathcal{E}^\star_A \ni E^\star \subset \overline{A^\gamma_{S^\star}}} \left(\frac{\sigma_t(f(\alpha_{E^\star}))}{\deg(f|_{\alpha_{E^\star}})} + \frac{\sigma_t(f(\beta_{E^\star}))}{\deg(f|_{\beta_{E^\star}})} \right) + \frac{\sigma_t(f(\beta_\gamma))}{\deg(f|_{\beta_\gamma})} = \sum_{\beta \in \Sigma^\star} \sum_{\alpha \sim \gamma, \alpha \subset S^\star_\gamma} \frac{\sigma_t(\beta)}{\deg(f : \alpha \to \beta)}.$$

Since $\Sigma^\star = \Gamma^\star_0 \cup \Gamma^\star$, we can decompose the summation into two parts:

$$I = \sum_{\beta \in \Gamma^\star} \sum_{\alpha \sim \gamma, \alpha \subset S^\star_\gamma} \frac{\sigma_t(\beta)}{\deg(f : \alpha \to \beta)}, \quad II = \sum_{\beta \in \Gamma^\star_0} \sum_{\alpha \sim \gamma, \alpha \subset S^\star_\gamma} \frac{\sigma_t(\beta)}{\deg(f : \alpha \to \beta)}.$$

It follows from the proof of Lemma 6.4.2 that $I \le t\omega(\gamma)$, where

$$\omega(\gamma) := \frac{\rho(S_{f(\gamma)}, \pi(f(\gamma)))v(\pi(f(\gamma)))}{\deg(f|_\gamma)} + \sum_{\delta \in \Gamma} \sum_{\pi(\gamma) \sim \alpha \subset S_\gamma \setminus \pi(\gamma)} \frac{v(\delta)}{\deg(f : \alpha \to \delta)},$$

if $f(\gamma) \in \Gamma^\star$ (or equivalently $\gamma \in \Gamma^\star_2 \cup \Gamma^\star_3 \cup \cdots$); and

$$\omega(\gamma) := \sum_{\delta \in \Gamma} \sum_{\pi(\gamma) \sim \alpha \subset S_\gamma \setminus \pi(\gamma)} \frac{v(\delta)}{\deg(f : \alpha \to \delta)},$$

if $f(\gamma) \in \Gamma_0^\star$ (or equivalently $\gamma \in \Gamma_1^\star$).

For the second term, we have

$$II \leq \sum_{A \in \mathcal{A}} \sum_{\alpha \in f^{-1}(\Gamma_0) \setminus \Gamma_0} \frac{\text{mod}(A)}{\deg(f|_\alpha)},$$

where $\mathcal{A}$ is the collection of all rotation annuli of (f, P), Γ_0 is the collection of (f, P)-invariant curves defined in Section 6.2.

So if we choose t large enough such that for any $\gamma \in \cup_{S^\star \in \mathcal{S}^\star} \partial_1(S^\star)$,

$$\sum_{E^\star \in \mathcal{E}_A^\star} C(E^\star) + \sum_{A \in \mathcal{A}} \sum_{\alpha \in f^{-1}(\Gamma_0) \setminus \Gamma_0} \frac{\text{mod}(A)}{\deg(f|_\alpha)} < t\Big(\rho(S_\gamma, \pi(\gamma)) \cdot v(\pi(\gamma)) - \omega(\gamma)\Big),$$

then the conclusion follows (notice that by the choice of the number ρ, we have $\rho(S_\gamma, \pi(\gamma)) \cdot v(\pi(\gamma)) - \omega(\gamma) > 0$ for all $\gamma \in \Gamma^\star$). $\square$

Now, we are ready to complete the proof of Theorem 6.4.1. Here is a fact used in the proof, which is equivalent to the Grötzsch inequality, and we will state it as follows. Let $A, B \subset \overline{\mathbb{C}}$ be two annuli. We say that B can be embedded into A essentially and holomorphically if there is a holomorphic injection $\phi : B \to A$ such that $\phi(B)$ separates the two boundary components of A.

Fact *Let $A, A_1, \cdots, A_n \subset \overline{\mathbb{C}}$ be annuli, then $A_1, \cdots, A_n$ can be embedded into A essentially and holomorphically such that the closures of the images of A_i's are mutually disjoint if and only if*

$$\sum_{i=1}^{n} \text{mod}(A_i) < \text{mod}(A).$$

Proof of the sufficiency of Theorem 6.4.1, assuming $\Gamma \neq \emptyset$

The idea of the proof is to glue the holomorphic models together along the stable multicurve Γ. Here is the detailed proof:

Recall that for each $\mathcal{S}^\star$-piece $S^\star$, we use S to denote the $\mathcal{S}$-piece that contains $S^\star$. For each curve $\gamma \in \Sigma$, A^γ is the annular neighborhood of γ such that A^γ avoids $P \cup f(P - \cup\mathcal{A})$ and $f(A^\gamma) = A^{f(\gamma)}$, where $\mathcal{A}$ is the collection of all rotation annuli of (f, P).

For each $\mathcal{S}^\star$-piece $S^\star$, we extend $\Phi_{S^\star}^t : S^\star \to \Phi_{S^\star}^t(S^\star)$ to a quasiconformal homeomorphism $\Phi_S : S \to \Phi_S(S)$ such that Φ_S is holomorphic in $(S - S^\star) \cap (\cup\mathcal{A})$.

We first choose t large enough such that Lemma 6.4.6 holds. In this case, one can embedded $\Psi_{E^\star}^t(E^\star)$ holomorphically into the interior of $\Phi_S(S)$ for each $\mathcal{E}_A^\star$-piece $E^\star$ contained in S according to the original order of their non-peripheral boundary curves so that the embedded images are mutually disjoint. In other words, there is a quasiconformal homeomorphism $\psi_S : S \to \Phi_S(S)$ such that

• $\psi_S|_{\partial S} = \Phi_S|_{\partial S}$ and ψ_S is isotopic to Φ_S rel $\partial S \cup (S \cap P)$. Moreover $\psi_S|_{S\cap(\cup\mathcal{A})} = \Phi_S|_{S\cap(\cup\mathcal{A})}$.

• $\psi_S|_{E_{S^\star}} = \Psi^t_{S^\star}|_{E_{S^\star}}$, where $E_{S^\star}$ is the unique $\mathcal{E}^\star$-piece parallel to $S^\star$.

• For each curve $\gamma \in \partial_1(S)$, $\Phi_S(S \cap A^\gamma) = \psi_S(S \cap A^\gamma)$.

• For every $\mathcal{E}^\star_A$-piece $E^\star$ with $E^\star \subset S$, the map $\Psi^t_{E^\star} \circ \psi_S^{-1}$ is holomorphic in $\psi_S(E^\star)$.

We define a subset $\mathcal{E}_A$ of $\mathcal{E}$ by $\mathcal{E}_A = \{E; E^\star \in \mathcal{E}^\star_A\}$. Let $\mathcal{D}(S)$ be the collection of all disk components of $S - E_S \cup (\cup_{\mathcal{E}_A \ni E \subset S} E)$, here E_S is the unique $\mathcal{E}$-piece parallel to S. For each $D \in \mathcal{D}(S)$, we construct a quasiconformal homeomorphism $\zeta_D : D \to \psi_S(D)$, whose Beltrami coefficient satisfies

$$\mu_{\zeta_D}(z) = \sum_{\mathcal{E} \ni E \subset D} \chi_E(z)\mu_{\Phi_{f(E)}\circ f}(z),$$

here the summation is taken over all $\mathcal{E}$-pieces contained in D. We further require $\zeta_D(p) = \psi_S(p)$ if D contains a marked point $p \in P$.

Let Γ_S be the collection of all boundary curves of $\cup_{D\in\mathcal{D}(S)} D$. For each $\gamma \in \Gamma_S$, notice that $f(\gamma) \in \Sigma$. Let A^γ be the component of $f^{-1}(A^{f(\gamma)})$ that contains γ. It's obvious that A^γ is an annular neighborhood of γ. We define a quasiconformal homeomorphism $\Psi_S : S \to \Phi_S(S)$ by

$$\Psi_S(z) = \begin{cases} \zeta_D(z), & z \in D, D \in \mathcal{D}(S), \\ \psi_S(z), & z \in S - (\cup_{D\in\mathcal{D}(S)} D) \cup (\cup_{\gamma\in\Gamma_S} A^\gamma), \\ q.c\ interpolation, & z \in \cup_{\gamma\in\Gamma_S} A^\gamma - \cup_{D\in\mathcal{D}(S)} D. \end{cases}$$

The map Ψ_S satisfies:

• $\Psi_S|_{\partial S} = \Phi_S|_{\partial S}$ and Ψ_S is isotopic to Φ_S rel $\partial S \cup (S \cap P)$. Moreover $\Psi_S|_{S\cap(\cup\mathcal{A})} = \Phi_S|_{S\cap(\cup\mathcal{A})}$.

• For every $\mathcal{E}^\star_{ess} \cup \mathcal{E}^\star_A$-piece $E^\star \subset S$, the map $\Phi_{f(E)} \circ f \circ \Psi_S^{-1}$ is holomorphic in $\Psi_S(E^\star)$.

• For every $\mathcal{E}$-piece $E \subset \cup_{D\in\mathcal{D}(S)} D$, the map $\Phi_{f(E)} \circ f \circ \Psi_S^{-1}$ is holomorphic in $\Psi_S(E)$.

Now, we define a quasiconformal map $\Theta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by $\Theta|_S = \Psi_S^{-1} \circ \Phi_S$ for all $S \in \mathcal{S}$. It's obvious that Θ is isotopic to the identity map rel P. Moreover, for each curve $\gamma \in \Gamma$, we have $\Theta(\gamma) = \gamma$ and $A^\gamma \subset \Theta^{-1}(A^\gamma)$. Let $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasiconformal map whose Beltrami coefficient satisfies

$$\mu_\Phi(z) = \sum_{S\in\mathcal{S}} \chi_S(z)\mu_{\Phi_S}(z),\ z \in \overline{\mathbb{C}}.$$

Set $\Psi = \Phi \circ \Theta^{-1}$. Then (f, P) is q.c-equivalent to the Herman map $(g, Q) := (\Phi \circ f \circ \Psi^{-1}, \Phi(P))$ via (Φ, Ψ).

One can verify that g is holomorphic outside $X := \Psi(\cup_{\gamma\in\Gamma\cup(\cup_{S\in\mathcal{S}}\Gamma_S)} A^\gamma)$. To see this, notice that if $E^\star \in \mathcal{E}^\star_{ess} \cup \mathcal{E}^\star_A$ and $E^\star$ is contained in some $\mathcal{S}$-piece S, then the decomposition

$$g|_{\Psi(E^\star)} = (\Phi \circ \Phi^{-1}_{f(E)}) \circ (\Phi_{f(E)} \circ f \circ \Psi_S^{-1}) \circ (\Phi_S \circ \Phi^{-1})|_{\Psi(E^\star)}$$

implies that g is holomorphic in $\Psi(E^\star)$ since each factor is holomorphic. If $E \in \mathcal{E}$ and $E \subset D \in \mathcal{D}(S)$, then

$$g|_{\Psi(E)} = (\Phi \circ \Phi^{-1}_{f(E)}) \circ (\Phi_{f(E)} \circ f \circ \zeta_D^{-1}) \circ (\Phi_S \circ \Phi^{-1})|_{\Psi(E)},$$

so g is holomorphic in $\Psi(E)$.

The last step is to apply quasiconformal surgery. For each curve $\gamma \in \Gamma$, let $\iota(\gamma)$ be the first integer $p \geq 1$ such that $f^p(\gamma) \in \Gamma_0$ and $L = \max_{\gamma\in\Gamma} \iota(\gamma)$. One may verify by induction that for any $j \geq 1$,

$$g^{-j}(\Psi(\cup\mathcal{A})) = \Psi((\Theta \circ f)^{-j}(\cup\mathcal{A})) \supset \Psi(\cup_{\gamma\in\Gamma,\iota(\gamma)\leq j} A^\gamma).$$

In particular, $g^{-L-1}(\Psi(\cup\mathcal{A})) \supset X$. Let σ_0 be the standard complex structure in $\overline{\mathbb{C}}$. Define a (g, Q)-invariant complex structure σ by

$$\sigma = \begin{cases} (g^k)^*(\sigma_0), & \text{in } g^{-k}(\Psi(\cup\mathcal{A})) \setminus g^{-k+1}(\Psi(\cup\mathcal{A})),\ k \geq 1, \\ \sigma_0, & \text{in } \overline{\mathbb{C}} - \cup_{k\geq 1}(g^{-k}(\Psi(\cup\mathcal{A})) \setminus g^{-k+1}(\Psi(\cup\mathcal{A}))). \end{cases}$$

Since (g, Q) is holomorphic outside X, the Beltrami coefficient μ of σ satisfies $\|\mu\|_\infty < 1$. By Measurable Riemann Mapping Theorem, there is a quasiconformal map $\zeta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\zeta^*(\sigma_0) = \sigma$. Let $R = \zeta \circ g \circ \zeta^{-1}$, then R is a rational map and (f, P) is q.c-equivalent to $(R, \zeta \circ \Phi(P))$ via $(\zeta \circ \Phi, \zeta \circ \Psi)$. □

6.5 Decomposition part II

In the following three sections, we will prove the following:

Theorem 6.5.1. *Let (f, P) be a non-parabolic map with $n_A(f) > 0$, then there exist a (f, P)-stable multicurve $\mathcal{Y}$ and a finite collection of Herman maps, or Siegel maps, or Thurston maps, say $\{(h_k, P_k), k \in \Lambda\}$, such that*

1. (f, P) has no Thurston obstructions if and only if $\lambda(\mathcal{Y}, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) has no Thurston obstructions.

2. (f, P) is q.c-equivalent to a rational map if and only if $\lambda(\mathcal{Y}, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) is q.c-equivalent to a rational map.

This theorem can be read as
Non-parabolic=multicurve+Herman+Siegel+Thurston.

The proof of Theorem 6.5.1 is based on Cui-Tan's repelling system theory [CT1].

6.5.1 The hole-filling operator

Let (f, P) be a non-parabolic map with $n_A(f) > 0$.

A *bordered surface* $S \subsetneq \overline{\mathbb{C}}$ is the Riemann sphere $\overline{\mathbb{C}}$ minus at most finitely many open quasidisks whose closures are mutually disjoint. The set of all boundary curves of S is denoted by $\partial(S)$.

Let S be a bordered surface with $\partial S \cap P = \emptyset$. The *topological complexity* of S with respect to the marked set P is defined by

$$\mathcal{T}_P(S) = \#\partial(S) + \#(S \cap P).$$

We say S is of *disk type* if S is a closed disk containing at most one point in P; of *annular type* if S is a closed annulus disjoint from P. Otherwise, we say S is of *complex type.* One may verify that S is of complex type if and only if $\mathcal{T}_P(S) \geq 3$.

A *surface puzzle* $\mathbf{S} = S_1 \sqcup \cdots \sqcup S_k$ is a finite union of disjoint bordered surfaces. Each S_i is called an $\mathbf{S}$-piece.

Let S be a bordered surface. We define $\mathcal{D}(S)$ by

$$\mathcal{D}(S) = \{U; U \text{ is a component of } \overline{\mathbb{C}} - S \text{ with } \#(U \cap P) \leq 1\}.$$

The hole-filling of S, denoted by $\mathcal{H}(S)$, is defined by

$$\mathcal{H}(S) = S \cup (\cup_{V \in \mathcal{D}(S)} V).$$

The hole-filling of the surface puzzle $\mathbf{S} = S_1 \sqcup \cdots \sqcup S_k$, denoted by $\mathcal{H}(\mathbf{S})$, is defined by

$$\mathcal{H}(\mathbf{S}) = \bigcup_{1 \leq j \leq k} \mathcal{H}(S_j).$$

One can verify that the hole-filling operator satisfies the following properties:

1. Let S_1 and S_2 be two bordered surfaces. If $S_1 \subset S_2$, then $\mathcal{H}(S_1) \subset \mathcal{H}(S_2)$. If $S_1 \Subset S_2$, then $\mathcal{H}(S_1) \Subset \mathcal{H}(S_2)$.

2. Let S_1 and S_2 be two disjoint bordered surfaces. Then either $\mathcal{H}(S_1) \cap \mathcal{H}(S_2) = \emptyset$, or $\mathcal{H}(S_1) \Subset \mathcal{H}(S_2)$, or $\mathcal{H}(S_2) \Subset \mathcal{H}(S_1)$.

3. For any bordered surface S, we have $f^{-1} \circ \mathcal{H}(S) \subset \mathcal{H} \circ f^{-1}(S)$ and $\mathcal{H} \circ \mathcal{H}(S) = \mathcal{H}(S)$.

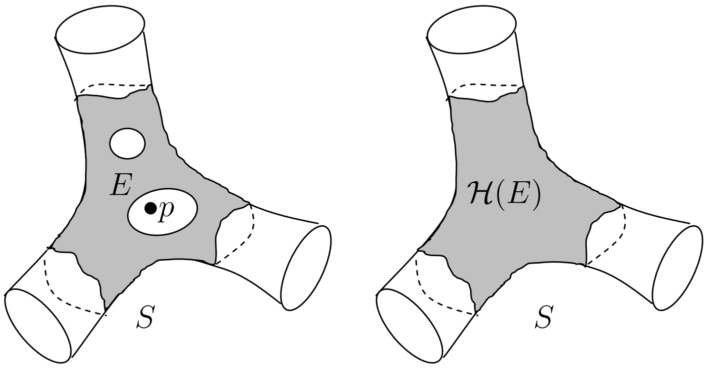

Figure 6.7: The hole-filling of E (Suppose that each boundary curve of S is non-peripheral in $\overline{\mathbb{C}} - P$).

6.5.2 Surface puzzle of constant complexity

Now, we denote by $\mathcal{A}(f)$ the union of all attracting cycles of f in P'_f. For each point $z \in \mathcal{A}(f)$, there is a small disk neighborhood U_z of z such that

1. ∂U_z is a quasicircle, and $\partial U_z \cap P = \emptyset$.
2. $f^{-1}(\overline{\mathbb{C}} - \cup_{z\in\mathcal{A}(f)} U_z)$ is contained in the interior of $\overline{\mathbb{C}} - \cup_{z\in\mathcal{A}(f)} U_z$.
3. f is holomorphic in $\cup_{z\in\mathcal{A}(f)} U_z$.

Let $\mathbf{S}_0 = \overline{\mathbb{C}} - \cup_{z\in\mathcal{A}(f)} U_z$. For $n \geq 1$, we define the surface puzzle $\mathbf{S}_n$ inductively in the following way:

$$\mathbf{S}_n = \mathcal{H} \circ f^{-1}(\mathbf{S}_{n-1}) = \cdots = (\mathcal{H} \circ f^{-1})^n(\mathbf{S}_0).$$

One can verify by induction that $\mathbf{S}_{n+1} \Subset \mathbf{S}_n$ for all $n \geq 0$.

Definition 6.5.1. (Parallel) *Let S, E be two bordered surfaces, with $E \Subset S$ and S is of complex type. We say that E is parallel to S if each component of $S - E$ is either a disk containing at most one point in P or an annulus containing no point in P (notice that in the latter case, one boundary curve of the annulus is on ∂S and the other is on ∂E).*

Notice that if S is of complex type and E is parallel to S, then $\mathcal{T}_P(E) \geq \mathcal{T}_P(S) \geq 3$. This means E is also of complex type.

Definition 6.5.2. (Constant complexity) *Let $\mathbf{S} = S_1 \sqcup \cdots \sqcup S_k$ be a surface puzzle with $f^{-1}(\mathbf{S}) \Subset \mathbf{S}$ and $\partial \mathbf{S} \cap P = \emptyset$. We say $\mathbf{S}$ is of constant complexity, if either there is no complex type $\mathbf{S}$-piece, or every complex type $\mathbf{S}$-piece S contains a $f^{-1}(\mathbf{S})$-piece E which is parallel to S.*

One may compare with [CT1] for the definitions of 'hole-filling', 'parallel' and 'constant complexity'. However, our definitions on these objects are slightly different from Cui-Tan's original ones, in order to make Theorem 6.6.2 sharp.

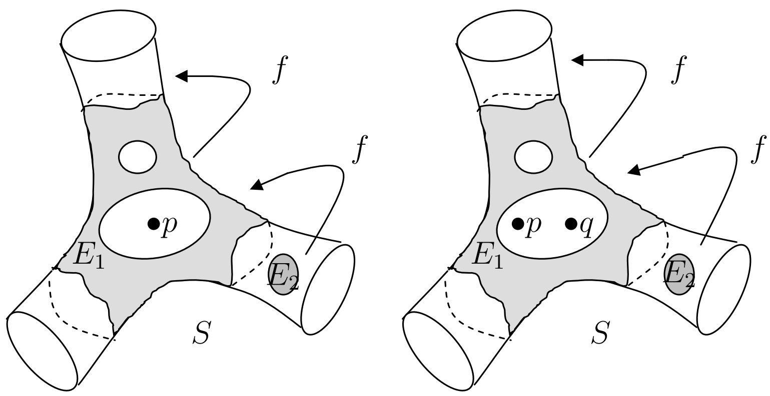

Figure 6.8: $f^{-1}(S) = E_1 \cup E_2$ (two shadow regions). In the left, E_1 is parallel to S and S is of constant complexity. In the right, neither E_1 nor E_2 is parallel to S, so S is not of constant complexity.

Proposition 6.5.1. *When n is large enough, $\mathbf{S}_n$ is of constant complexity.*

Proof. 1. We claim that: for every $n \geq 0$, and every annular type or complex type $\mathbf{S}_n$-piece S, all boundary curves of S are non-peripheral in $\overline{\mathbb{C}} - P$.

It's obvious for $n = 0$. So we just consider the case when $n \geq 1$. Let S be a $\mathbf{S}_n$-piece, either of annular type or of complex type. Since the hole-filling of two disjoint bordered surfaces either are disjoint or one contains the other, we conclude that there is a unique $f^{-1}(\mathbf{S}_{n-1})$-piece T such that $S = \mathcal{H}(T)$. Notice that the hole-filling operator satisfies $\mathcal{H} \circ \mathcal{H} = \mathcal{H}$, we have $S = \mathcal{H}(S)$. This implies all boundary curves of S are non-peripheral in $\overline{\mathbb{C}} - P$.

2. For every $n \geq 0$, one can verify that $\mathbf{S}_n \cap P$ has finitely many connected components, and $\mathbf{S}_{n+1} \cap P \subset \mathbf{S}_n \cap P$. We can choose n_0 large enough such that for all $n \geq n_0$, the number of connected components of $\mathbf{S}_n \cap P$ remains constant.

3. For any $n \geq 0$, let k_n be the number of homotopy classes of non-peripheral curves of $\cup_{0 \leq j \leq n} \partial \mathbf{S}_j$ in $\overline{\mathbb{C}} - P$. It's obvious that $k_n \leq k_{n+1}$. Since $\partial \mathbf{S}_0 \cup (\mathbf{S}_0 \cap P)$ has finitely many components, there exists $n_1 \geq n_0$, such that for all $n \geq n_1$, $k_n = k_{n_1}$.

4. By the choice of n_1, for any $n > n_1$ and any non-peripheral curve γ of $\partial \mathbf{S}_{n+1}$, there is a curve $\alpha \subset \cup_{0 \leq j \leq n_1} \partial \mathbf{S}_j$ homotopic to γ in $\overline{\mathbb{C}} - P$. Let $A(\gamma, \alpha)$

be the annulus bounded by γ and α. First notice that there is a unique $\mathbf{S}_{n+1}$-piece, say S_{n+1}, containing γ as a boundary curve. The piece S_{n+1} is either of annular type or of complex type. Since $\mathbf{S}_{n+1} \Subset \mathbf{S}_n$, there is a unique $\mathbf{S}_n$-piece S_n containing S_{n+1} as a proper subset. The piece S_n is either of annular type or of complex type. By 1, each boundary curve of S_n is non-peripheral in $\overline{\mathbb{C}} - P$. It follows that there is a unique boundary curve β of S_n contained in $A(\gamma, \alpha)$. Moreover, β is homotopic to γ in $\overline{\mathbb{C}} - P$, and the annulus $A(\gamma, \beta)$ bounded by γ and β is contained in S_n.

Claim: Every complex type $\mathbf{S}_n$-piece S_n contains at most one complex type $\mathbf{S}_{n+1}$-piece, say S_{n+1}. Each component of $S_n - S_{n+1}$ is an annulus.

To see this, let S_{n+1} and S'_{n+1} be two $\mathbf{S}_{n+1}$-pieces contained in S_n with S_{n+1} of complex type. By the previous argument, each component of $S_n - S_{n+1}$ is an annulus in $\overline{\mathbb{C}} - P$. So there exist $\gamma_n \subset \partial S_n$ and $\gamma_{n+1} \subset \partial S_{n+1}$ such that $S'_{n+1} \subset A(\gamma_n, \gamma_{n+1}) \subset \overline{\mathbb{C}} - P$, where $A(\gamma_n, \gamma_{n+1})$ is the annulus bounded by γ_n and γ_{n+1}. This implies that S'_{n+1} is either of disk type or of annular type.

Let c_n be the number of complex type $\mathbf{S}_n$-pieces for $n \geq 0$. Notice that every complex type $\mathbf{S}_{n+1}$-piece is contained in a unique complex type $\mathbf{S}_n$-piece, we have that for $n > n_1$, $c_{n+1} \leq c_n$. So there is $n_2 \geq n_1$ such that for all $n \geq n_2$, $c_n = c_{n_2}$.

To finish, we show that for any $n \geq n_2$, $\mathbf{S}_n$ is of constant complexity. Let S_n be a complex type $\mathbf{S}_n$-piece. Then there is a unique $\mathbf{S}_{n+1}$-piece S_{n+1} that is parallel to S_n. Since $c_n = c_{n+1}$, there is a unique $f^{-1}(\mathbf{S}_n)$-piece, say T, such that $S_{n+1} = \mathcal{H}(T)$. We have

$$S_n - T = (S_n - S_{n+1}) \cup (S_{n+1} - T) = (S_n - S_{n+1}) \cup (\cup_{V \in \mathcal{D}(T)} V).$$

This means that T is parallel to S_n. $\square$

From the proof of Proposition 6.5.1, we see that when $n \geq n_2$, $\mathbf{S}_n$ is of constant complexity. Fix some $n \geq n_2$, the surface puzzle $\mathbf{S} = \mathbf{S}_n$ satisfies:

- *For each annular type or complex type* $\mathbf{S}$*-piece* S*, each boundary curve* γ *of* S *is non-peripheral in* $\overline{\mathbb{C}} - P$.
- *For every complex type* $\mathbf{S}$*-piece* S*, there is a unique* $f^{-1}(\mathbf{S})$*-piece* E_S *parallel to* S*. And* $f(E_S)$ *is also a complex type* $\mathbf{S}$*-piece.*

Let $\mathbf{S}_D$ be the union of all disk type $\mathbf{S}$-pieces, $\mathbf{S}_A$ the union of all annular type $\mathbf{S}$-pieces and $\mathbf{S}_C$ the union of all complex type $\mathbf{S}$-pieces. If $\mathbf{S}_C \neq \emptyset$, then for every $\mathbf{S}_C$-piece S, there is a unique $f^{-1}(\mathbf{S})$-piece E_S parallel to S, and $f(E_S)$ is also a complex type $\mathbf{S}$-piece. We define a map f_* from all $\mathbf{S}_C$-pieces to themselves by

$$f_*(S) = f(E_S),$$

where E_S is the unique $f^{-1}(\mathbf{S})$-piece that is parallel to S. Since $\mathbf{S}_C$ has finitely many components, every $\mathbf{S}_C$-piece is eventually periodic under f_*.

Let $\mathcal{Y} \subset \{$all boundary curves of $\mathbf{S}_A \cup \mathbf{S}_C\}$ be a multicurve that represents all homotopy classes of boundary curves of $\mathbf{S}_A \cup \mathbf{S}_C$ in $\overline{\mathbb{C}} - P$. Such multicurve $\mathcal{Y}$ is unique up to homotopy.

The constant complexity of $\mathbf{S}$ implies:

Lemma 6.5.1. *The multicurve $\mathcal{Y}$ is (f, P)-stable.*

6.5.3 Marked disk extension

We assume in this section that $\mathbf{S}_C \neq \emptyset$. This section is an analogue of Section 6.2. For each $\mathbf{S}_C$-piece S, we denote by $\overline{\mathbb{C}}(S)$ the Riemann sphere containing S. We always assume that different $\mathbf{S}_C$-pieces are embedded into different copies of Riemann spheres.

Since each component U_i of $\overline{\mathbb{C}}(S) - E_S$ either is contained in S or contains a unique component V_j of $\overline{\mathbb{C}}(S) - S$. In the former case, if U_i contains a marked point $p \in P$, then we get a marked disk (U_i, p); else, we don't mark any point in U_i. In the latter case, we mark a point $p \in V_j \subset U_i$, and get two marked disks (V_j, p) and (U_i, p).

We define

$$P(S) = (P \cap S) \cup \{\text{all marked points in } \overline{\mathbb{C}}(S) - S\}.$$

We call $(\overline{\mathbb{C}}(S), P(S))$ a marked sphere of S.

Now we extend $f|_{E_S}$ to a branched covering $H_S : \overline{\mathbb{C}}(S) \to \overline{\mathbb{C}}(f_*(S))$. For each component U_i of $\overline{\mathbb{C}}(S) - E_S$, we define $H_S|_{U_i}$ in the following way:

a). If U_i contains a marked point $p \in P(S)$, then H_S maps the marked disk (U_i, p) to the marked disk (W_j, q). Here W_j is the component of $\overline{\mathbb{C}}(f_*(S)) - f_*(S)$ whose boundary curve is $f(\partial U_i)$. We require further that $H_S(p) = q$ and p is the only possible critical point, with local degree $\deg(f|_{\partial U_i})$.

b). If U_i contains no marked point in $P(S)$, then H_S maps U_i to the marked disk (W_j, q), such that $\deg(H_S|_{U_i}) = \deg(f|_{\partial U_i})$ and q is the only possible critical value. Here, W_j is the component of $\overline{\mathbb{C}}(f_*(S)) - f_*(S)$ whose boundary curve is $f(\partial U_i)$.

In this way, for each $\mathbf{S}_C$-piece S, we can get an extension H_S of $f|_{E_S}$. It satisfies $H_S(P(S)) \subset P(f_*(S))$.

Notice that every $\mathbf{S}_C$-piece is eventually periodic under the map f_*. We list all periodic $\mathbf{S}_C$-pieces in the following:

$$S_\nu \mapsto f_*(S_\nu) \mapsto \cdots \mapsto f_*^{p_\nu - 1}(S_\nu) \mapsto f_*^{p_\nu}(S_\nu) = S_\nu, \quad 1 \leq \nu \leq n,$$

where n is the number of f_*-periodic cycles, S_ν is a representative in the ν-th cycle and p_ν is the period of the piece S_ν.

Set

$$h_\nu = H_{f_*^{p_\nu-1}(S)} \circ \cdots \circ H_{f_*(S)} \circ H_S,\ P_\nu = P(S_\nu),\ 1 \le \nu \le n.$$

Then $h_\nu : \overline{\mathbb{C}}(S_\nu) \to \overline{\mathbb{C}}(S_\nu)$ is a branched covering with $h_\nu(P_\nu) \subset P_\nu$. There are four types of the resulting map (h_ν, P_ν):

• S_ν contains at least one rotation annulus of (f, P). In this case, (h_ν, P_ν) has at least one cycle of rotation annulus, and each boundary cycle of which contains at least one critical point of h_ν. So (h_ν, P_ν) is a Herman map.

• S_ν contains no rotation annulus of (f, P) but contains at least one rotation disk of (f, P). In this case, (h_ν, P_ν) has at least one cycle of rotation disk whose boundary cycle contains at least one critical point of h_ν. So (h_ν, P_ν) is a Siegel map.

• S_ν contains neither rotation annulus nor rotation disk of (f, P) and $\deg(h_\nu) > 1$. In this case, (h_ν, P_ν) is a Thurston map.

• S_ν contains neither rotation annulus nor rotation disk of (f, P) and $\deg(h_\nu) = 1$. In this case, (h_ν, P_ν) is a homeomorphism and all points in P_ν are periodic.

Let Λ be the index set consisting of all $\nu \in \{1, \cdots, n\}$ such that $\deg(h_\nu) > 1$. For each $\nu \in \Lambda$, (h_ν, P_ν) is either a Herman map, or a Siegel map, or a Thurston map. Let Λ^* be the set of all indices $\nu \in \{1, \cdots, n\}$ such that $\deg(h_\nu) = 1$.

We use the following notation to record the decomposition procedure:

$$\mathrm{Dec}_0(f, P) = \left(\bigoplus_{\nu \in \Lambda \cup \Lambda^*} (h_\nu, P_\nu) \right)_{\mathcal{Y}}.$$

By the same argument as Lemma 6.2.4, one may show that if $\lambda(\mathcal{Y}, f) < 1$, then the following holds:

1. For any $1 \le \nu \le n$, every point in $(\overline{\mathbb{C}}(S_\nu) - S_\nu) \cap P_\nu$ is eventually mapped to a periodic critical point of (h_ν, P_ν).

2. $\Lambda^* = \emptyset$.

3. If (h_ν, P_ν) is a Thurston map, then the signature of the orbifold of (h_ν, P_ν) is not $(2, 2, 2, 2)$.

6.6 Combinatorial part II

In this section, we will prove

Theorem 6.6.1. *Let (f, P) be a non-parabolic map with $n_A(f) > 0$. and*

$$\mathrm{Dec}_0(f, P) = \left(\bigoplus_{\nu \in \Lambda \cup \Lambda^*} (h_\nu, P_\nu) \right)_{\mathcal{Y}}.$$

Then (f, P) has no Thurston obstructions if and only if $\lambda(\mathcal{Y}, f) < 1$ and for each $\nu \in \Lambda$, (h_ν, P_ν) has no Thurston obstructions.

Here is the idea of the proof:

We first show that any (f, P)-stable multicurve induces a (F, Q)-stable multicurve (Here, (F, Q) is a suitable restriction of (f, P), which will be called a 'repelling system' of (f, P), see Section 6.6.1), and vice versa. These two multicurves have the same leading eigenvalues of their respective transition matrices. This implies that (f, P) has no Thurston obstructions if and only if (F, Q) has no Thurston obstructions (see Theorems 6.6.2 and 6.6.3).

We then show that every (F, Q)-stable multicurve can be decomposed into a '$\mathcal{Y}$-part' and a (h_ν, P_ν)-stable multicurve for each $\nu \in \Lambda \cup \Lambda^*$. Conversely, every (h_ν, P_ν)-stable multicurve can generate a (F, Q)-stable multicurve. This enables us to prove that (F, Q) has no Thurston obstructions if and only if $\lambda(\mathcal{Y}, f) < 1$ and for each $\nu \in \Lambda$, the map (h_ν, P_ν) has no Thurston obstructions (Theorems 6.6.4 and 6.6.5).

Then Theorem 6.6.1 follows by combining the above two steps.

6.6.1 Multicurves for repelling system

Let $\mathbf{B} = f^{-1}(\mathbf{S}) \Subset \mathbf{S}$, $F = f|_{\mathbf{B}}$, $Q = \mathbf{S} \cap P$. We call (F, Q) a *repelling system* of (f, P).

A Jordan curve $\gamma \subset \mathbf{S} \setminus Q$ is called *null-homotopic* (resp. *peripheral*, *non-peripheral*) in $\mathbf{S} \setminus Q$ if it is null-homotopic (resp. peripheral, non-peripheral) in $\overline{\mathbb{C}} \setminus P$.

We say that $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$ is a *multicurve* in $\mathbf{S} \setminus Q$ if each γ_i is a non-peripheral Jordan curve in $\mathbf{S} \setminus Q$, and they are mutually disjoint and no two homotopic in $\mathbf{S} \setminus Q$. Its (F, Q)-transition matrix $W_\Gamma = (a_{ij})$ is defined by

$$a_{ij} = \sum_{\alpha \sim \gamma_i} \frac{1}{\deg(F : \alpha \to \gamma_j)},$$

where the summation is taken over all components α of $F^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbf{S} \setminus Q$.

Notice that if two Jordan curves are homotopic in $\mathbf{S} \setminus Q$, then they are necessarily homotopic in $\overline{\mathbb{C}} \setminus P$. But the converse is not true.

A multicurve Γ in $\mathbf{S} \setminus Q$ is called (F, Q)-*stable* if each component of $F^{-1}(\gamma)$ for $\gamma \in \Gamma$ is either null-homotopic or peripheral, or homotopic to a curve of Γ in $\mathbf{S} \setminus Q$.

We say a multicurve Γ in $\mathbf{S} \setminus Q$ is a *Thurston obstruction* of (F, Q) if Γ is (F, Q)-stable and the leading eigenvalue $\lambda(\Gamma, F)$ of its transition matrix W_Γ satisfies $\lambda(\Gamma, F) \geq 1$.

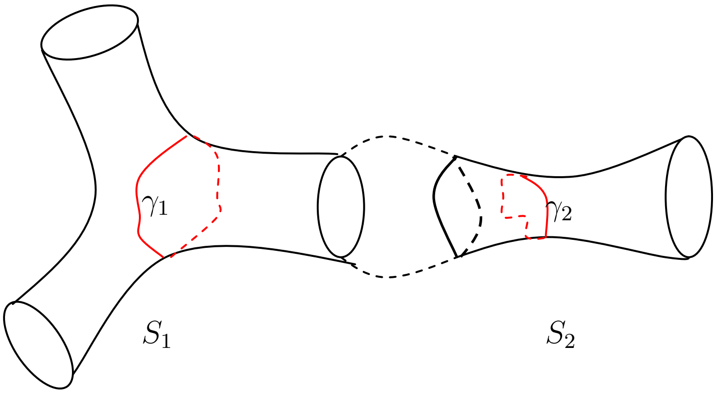

Figure 6.9: Suppose that the annulus $A(\gamma_1, \gamma_2)$ bounded by γ_1 and γ_2 contains no point in P, then γ_1 and γ_2 are homotopic in $\overline{\mathbb{C}} - P$. But they are not homotopic in $\mathbf{S} \setminus Q$.

For convention, an empty set $\Gamma = \emptyset$ is also considered as a (F, Q)-stable multicurve with $\lambda(\Gamma, F) = 0$.

Lemma 6.6.1. *Given two different* $\mathbf{S}$*-pieces* S_1 *and* S_2*, each is either of annular type or of complex type. If there are two non-peripheral curves* $\gamma_i \subset S_i, i = 1, 2,$ *homotopic to each other in* $\overline{\mathbb{C}} - P$*, then* γ_i *is homotopic in* S_i *to a boundary curve of* $S_i, i = 1, 2$.

Proof. We consider the annulus $A(\gamma_1, \gamma_2)$ bounded by γ_1 and γ_2. Since γ_1 is homotopic to γ_2 in $\overline{\mathbb{C}} - P$, we have that $A(\gamma_1, \gamma_2) \cap P = \emptyset$. Since S_i is either of annular type or of complex type, each boundary curve of S_i is non-peripheral in $\overline{\mathbb{C}} - P$. This implies that the closure of $A(\gamma_1, \gamma_2)$ necessarily contains a unique boundary curve of $S_i, i = 1, 2$. The conclusion follows immediately. □

Given a (f, P)-stable multicurve Γ, suppose that each curve $\gamma \in \Gamma$ is contained in $\mathbf{S}_A \cup \mathbf{S}_C$. Obviously, Γ is not necessarily a (F, Q)-stable multicurve in $\mathbf{S} - Q$. But we can use Γ to generate a (F, Q)-stable multicurve as follows:

Take a non-peripheral curve $\gamma \in f^{-1}(\Gamma)$ that is not homotopic to any curve of Γ in $\mathbf{S} - Q$ (if any), then γ is contained in some $\mathbf{S}_A \cup \mathbf{S}_C$-piece, say S. Since Γ is (f, P)-stable, there is a curve $\delta \in \Gamma$, homotopic to γ in $\overline{\mathbb{C}} - P$. We conclude by Lemma 6.6.1 that there is a unique boundary curve $\beta(\gamma)$ of S such that γ is homotopic to $\beta(\gamma)$ in $S - Q$. We define

$$\begin{aligned}\Gamma_1 = \Gamma \cup \{&\beta(\gamma); \gamma \in f^{-1}(\Gamma) \text{ is non-peripheral} \\ &\text{and not homotopic to any curve of } \Gamma \text{ in } \mathbf{S} - Q\}.\end{aligned}$$

For $n > 1$, we define Γ_n inductively in the following way: Let Σ_n be the set of all non-peripheral curves in $f^{-n}(\Gamma)$ that are not homotopic to any curve of Γ_{n-1} in $\mathbf{S} - Q$. For each $\gamma \in \Sigma_n$, let $\beta(\gamma)$ be the unique boundary curve of $\mathbf{S}$ homotopic to γ in $\mathbf{S} - Q$. Notice that given two different curves $\gamma_1, \gamma_2 \in \Sigma_n$, it may happen $\beta(\gamma_1) = \beta(\gamma_2)$. We define

$$\Gamma_n = \Gamma_{n-1} \cup \{\beta(\gamma); \gamma \in \Sigma_n\}.$$

One may verify that for each $n \geq 1$, the set Γ_n is a multicurve in $\mathbf{S} - Q$. Since $\partial \mathbf{S}_A \cup \partial \mathbf{S}_C \cup (\mathbf{S}_C \cap P)$ has finitely many components, there exists a positive integer N such that $\#\Gamma_n = \#\Gamma_N$ for all $n \geq N$. We define $\mathbf{G}(\Gamma) = \Gamma_N$. By the choice of N, one can verify that $\mathbf{G}(\Gamma)$ is a (F, Q)-stable multicurve. We call $\mathbf{G}(\Gamma)$ a (F, Q)-stable multicurve generated by Γ.

Now, let Σ be a (F, Q)-stable multicurve. We define an equivalent relation for the curves in Σ: $\alpha \sim \beta$ if α and β are homotopic in $\overline{\mathbb{C}} - P$. In this way, we can decompose Σ into finitely many equivalent classes $\Sigma_1 \sqcup \cdots \sqcup \Sigma_k$. For each equivalent class Σ_j, we choose a representative $\gamma_j \in \Sigma_j$. We define a multicurve $\pi(\Sigma)$ as follows:

$$\pi(\Sigma) = \{\gamma_1, \cdots, \gamma_k\}.$$

It's easy to check that $\pi(\Sigma)$ is a (f, P)-stable multicurve.

Theorem 6.6.2. *For any (f, P)-stable multicurve Γ, suppose that each curve $\gamma \in \Gamma$ is contained in $\mathbf{S}$, then $\lambda(\Gamma, f) = \lambda(\mathbf{G}(\Gamma), F)$. Conversely, for any (F, Q)-stable multicurve Σ, we have $\lambda(\Sigma, F) = \lambda(\pi(\Sigma), f)$.*

Proof. For any (f, P)-stable multicurve Γ, we decompose $\mathbf{G}(\Gamma)$ as follows: $\mathbf{G}(\Gamma) = \cup_{\gamma \in \Gamma} \Gamma_\gamma$, where $\Gamma_\gamma = \{\delta \in \mathbf{G}(\Gamma); \delta \text{ is homotopic to } \gamma \text{ in } \overline{\mathbb{C}} - P\}$. Let $W_\Gamma = (a_{\alpha\beta})$ be the (f, P)-transition matrix of Γ and $W_{\mathbf{G}(\Gamma)} = (b_{\xi\eta})$ be the (F, Q)-transition matrix of $\mathbf{G}(\Gamma)$. Then

$$a_{\alpha\beta} = \sum_{\delta \in \Lambda_{\alpha\beta}} \frac{1}{\deg(f : \delta \to \beta)}, \quad b_{\xi\eta} = \sum_{\zeta \in \Omega_{\xi\eta}} \frac{1}{\deg(F : \zeta \to \eta)},$$

where $\Lambda_{\alpha\beta}$ is the collection of components of $f^{-1}(\beta)$ which are homotopic to α in $\overline{\mathbb{C}} - P$ and $\Omega_{\xi\eta}$ is the collection of components of $F^{-1}(\eta)$ which are homotopic to ξ in $\mathbf{S} - Q$.

We claim that for any $\alpha, \beta \in \Gamma$ and any $\eta \in \Gamma_\beta$,

$$a_{\alpha\beta} = \sum_{\xi \in \Gamma_\alpha} b_{\xi\eta}.$$

To prove this, we first assume $\eta = \beta$. One may check that

$$\begin{aligned}\cup_{\xi\in\Gamma_\alpha}\Omega_{\xi\beta} &= \{\delta \subset F^{-1}(\beta) \text{ is homotopic to a curve of } \Gamma_\alpha \text{ in } \mathbf{S}-Q\}\\ &\subset \{\delta \subset F^{-1}(\beta) \text{ is homotopic to } \alpha \text{ in } \overline{\mathbb{C}}-P\} = \Lambda_{\alpha\beta}.\end{aligned}$$

Conversely, for any curve $\delta \subset F^{-1}(\beta)$ which is homotopic to α in $\overline{\mathbb{C}}-P$, since $\mathbf{G}(\Gamma)$ is (F,Q)-stable, there is a curve $\xi \in \mathbf{G}(\Gamma)$, homotopic to δ in $\mathbf{S}-Q$. This implies $\Lambda_{\alpha\beta} \subset \cup_{\xi\in\Gamma_\alpha}\Omega_{\xi\beta}$. So we have $\cup_{\xi\in\Gamma_\alpha}\Omega_{\xi\beta} = \Lambda_{\alpha\beta}$ and

$$a_{\alpha\beta} = \sum_{\delta\in\cup_{\xi\in\Gamma_\alpha}\Omega_{\xi\beta}} \frac{1}{\deg(f:\delta\to\beta)} = \sum_{\xi\in\Gamma_\alpha}\sum_{\delta\in\Omega_{\xi\beta}} \frac{1}{\deg(F:\delta\to\beta)} = \sum_{\xi\in\Gamma_\alpha} b_{\xi\eta}.$$

If $\eta \neq \beta$, we can replace η by β in Γ. This replacement doesn't change the transition matrix W_Γ. So the claim holds.

Now for any $p\times q$ real matrix $A = (a_{ij})$, we define $\|A\| = \sum_{ij}|a_{ij}|$. It's obvious that if $p = q$, then $\|A\|$ is the Banach norm of A. In the following, we will show that:

There exist two constants $C_2 \geq C_1 > 0$ such that for all $n \geq 1$,

$$C_1\|W_\Gamma^n\| \leq \|W_{\mathbf{G}(\Gamma)}^n\| \leq C_2\|W_\Gamma^n\|.$$

To prove this relation, we make a block decomposition of $W_{\mathbf{G}(\Gamma)}$: $W_{\mathbf{G}(\Gamma)} = (W_{\alpha\beta})_{\alpha,\beta\in\Gamma}$, where $W_{\alpha\beta}$ is the (α,β)-th block. Moreover, $W_{\alpha\beta}$ is a $\#\Gamma_\alpha \times \#\Gamma_\beta$ matrix, whose entries are $b_{\xi\eta}, \xi\in\Gamma_\alpha, \eta\in\Gamma_\beta$. One can verify that

$$\|W_{\alpha\beta}\| = \sum_{\xi\in\Gamma_\alpha}\sum_{\eta\in\Gamma_\beta} b_{\xi\eta} = \#\Gamma_\beta \cdot a_{\alpha\beta}.$$

We set $C_1 = \min\{\#\Gamma_\gamma; \gamma\in\Gamma\}, C_2 = \max\{\#\Gamma_\gamma; \gamma\in\Gamma\}$. Then we have

$$\|W_{\mathbf{G}(\Gamma)}\| = \sum_{\alpha,\beta\in\Gamma}\|W_{\alpha\beta}\| = \sum_{\alpha,\beta\in\Gamma}\#\Gamma_\beta a_{\alpha\beta}\begin{cases}\geq C_1\|W_\Gamma\|,\\ \leq C_2\|W_\Gamma\|.\end{cases}$$

For any $n \geq 1$, the (α,β)-th block of $W_{\mathbf{G}(\Gamma)}^n$ is $\displaystyle\sum_{\alpha_1,\cdots,\alpha_{n-1}} W_{\alpha\alpha_1}W_{\alpha_1\alpha_2}\cdots W_{\alpha_{n-1}\beta}$.

One can prove by induction that

$$\Big\|\sum_{\alpha_1,\cdots,\alpha_{n-1}} W_{\alpha\alpha_1}W_{\alpha_1\alpha_2}\cdots W_{\alpha_{n-1}\beta}\Big\| = \#\Gamma_\beta\cdot\sum_{\alpha_1,\cdots,\alpha_{n-1}} a_{\alpha\alpha_1}a_{\alpha_1\alpha_2}\cdots a_{\alpha_{n-1}\beta}.$$

It follows that

$$\|W_{\mathbf{G}(\Gamma)}^n\| = \sum_{\alpha,\beta\in\Gamma}\#\Gamma_\beta\cdot\sum_{\alpha_1,\cdots,\alpha_{n-1}} a_{\alpha\alpha_1}a_{\alpha_1\alpha_2}\cdots a_{\alpha_{n-1}\beta}\begin{cases}\geq C_1\|W_\Gamma^n\|,\\ \leq C_2\|W_\Gamma^n\|.\end{cases}$$

The proof of the claim is completed. This implies

$$\mathrm{sp}(W_\Gamma) = \lim_{n\to\infty} \sqrt[n]{\|W_\Gamma^n\|} = \lim_{n\to\infty} \sqrt[n]{\|W_{\mathbf{G}(\Gamma)}^n\|} = \mathrm{sp}(W_{\mathbf{G}(\Gamma)}).$$

By Perron-Frobenius Theorem, $\lambda(\Gamma, f) = \lambda(\mathbf{G}(\Gamma), F)$. The second statement follows from the same argument. □

6.6.2 Reduction of no Thurston obstructions

We first prove the following

Theorem 6.6.3. *(f, P) has no Thurston obstructions if and only if (F, Q) has no Thurston obstructions.*

Proof. Let Γ be a (f, P)-stable multicurve. Using the same argument as in Lemma 6.3.1, we can show that there is a (f, P)-stable multicurve Γ_*, such that

- Γ_* is homotopically contained in Γ.
- Each curve of Γ_* is contained in a $\mathbf{S}$-piece.
- $\lambda(\Gamma_*, f) = \lambda(\Gamma, f)$.

The multicurve Γ_* can generate a (F, Q)-stable multicurve $\mathbf{G}(\Gamma_*)$. By Theorem 6.6.2, $\lambda(\mathbf{G}(\Gamma_*), F) = \lambda(\Gamma_*, f) = \lambda(\Gamma, f)$. Thus that (F, Q) has no Thurston obstructions implies that (f, P) has no Thurston obstructions.

On the other hand, given a (F, Q)-stable multicurve Σ, the set $\pi(\Sigma)$ is a (f, P)-stable multicurve, and $\lambda(\Sigma, F) = \lambda(\pi(\Sigma), f)$. Thus if (f, P) has no Thurston obstructions, then (F, Q) has no Thurston obstructions either. □

Definition 6.6.1.

$$\text{We say that } (F,Q) \text{ is } \begin{cases} \text{a disk-covering, if } \mathbf{S}_C = \mathbf{S}_A = \emptyset, \\ \text{an annular-covering, if } \mathbf{S}_C = \emptyset \text{ but } \mathbf{S}_A \neq \emptyset, \\ \text{a complex-covering, if } \mathbf{S}_C \neq \emptyset. \end{cases}$$

Notice that if $F : (\mathbf{E}, Q) \to (\mathbf{S}, Q)$ is a disk-covering or an annular-covering, then (f, P) has no rotation domains.

Let $\mathcal{B}$ be a multicurve of (F, Q), consisting of all homotopy classes the boundary curves of $\mathbf{S}_A \cup \mathbf{S}_C$ in $\mathbf{S} - Q$. In fact, $\mathcal{B}$ can be written as

$$\mathcal{B} = \{\gamma \in \partial(S); S \text{ is an } \mathbf{S}_C\text{-piece}\} \cup \{\gamma_S; S \text{ is an } \mathbf{S}_A\text{-piece}\},$$

where γ_S is a boundary curve of S if S is an $\mathbf{S}_A$-piece (notice that for each annular piece S, $\mathcal{B}$ contains only one boundary curve of S).

Lemma 6.6.2. *The multicurve $\mathcal{B}$ is (F, Q)-stable and $\lambda(\mathcal{B}, F) = \lambda(\mathcal{Y}, f)$.*

Proof. The constant complexity of $\mathbf{S}$ implies that $\mathcal{B}$ is (F,Q)-stable. Notice that $\mathcal{Y}$ is identical to $\pi(\mathcal{B})$ up to homotopy. The equality $\lambda(\mathcal{B},F)=\lambda(\mathcal{Y},f)$ follows from Theorem 6.6.2. □

Theorem 6.6.4. (Decomposition of stable multicurve) *Suppose that (F,Q) is a complex-covering and let Υ be a (F,Q)-stable multicurve. We define*

$$\Upsilon_{\mathcal{B}}=\{\gamma\in\Upsilon;\gamma \text{ is homotopic to a curve } \delta\in\mathcal{B} \text{ in } \mathbf{S}-Q\},$$
$$\Sigma_\nu=\{\gamma\in\Upsilon-\Upsilon_{\mathcal{B}};\gamma \text{ is contained in } S_\nu\}, \nu\in\Lambda\cup\Lambda^*=[1,n].$$

Then $\Upsilon_{\mathcal{B}}$ is a (F,Q)-stable multicurve, Σ_ν is a (h_ν,P_ν)-stable multicurve for $1\le\nu\le n$, and we have the following identity:

$$\lambda(\Upsilon,F)=\max\left\{\lambda(\Upsilon_{\mathcal{B}},F),\ \sqrt[p_1]{\lambda(\Sigma_1,h_1)},\cdots,\ \sqrt[p_n]{\lambda(\Sigma_n,h_n)}\right\}.$$

The proof of Theorem 6.6.4 is the same as that of Theorem 6.3.2, we omit the details.

Suppose that (F,Q) is a complex-covering. For each $1\le\nu\le n$, let Σ_ν be a (h_ν,P_ν)-stable multicurve. Up to homotopy, we may assume that each curve of Σ_ν is contained in S_ν.

In the following, we will use $\Sigma_1,\cdots,\Sigma_n$ to generate a (F,Q)-stable multicurve.

Let $\Gamma_k\subset F^{-k}(\Sigma_1\cup\cdots\cup\Sigma_n)$ be a multicurve in $\mathbf{S}-Q$, such that Γ_k represents all homotopy classes of non-peripheral curves of $F^{-k}(\Sigma_1\cup\cdots\cup\Sigma_n)$.

Lemma 6.6.3. *For any pair of curves $\gamma_i\in\Gamma_i$, $\gamma_j\in\Gamma_j$ with $0\le i<j$, if γ_i is not homotopic to γ_j in $\mathbf{S}-Q$, then γ_i and γ_j are homotopically disjoint.*

Proof. Since γ_i is not homotopic to γ_j in $\mathbf{S}-Q$, either γ_i and γ_j are contained in two different $\mathbf{S}_A\cup\mathbf{S}_C$-pieces, or γ_i and γ_j are contained in the same $\mathbf{S}_C$-piece, say S. If we are in the former case, then the proof is done. So we just consider the latter case. In this case, if one of the curves γ_i,γ_j is homotopic to a boundary curve of S, then they are obviously homotopically disjoint.

We assume that neither of γ_i,γ_j is homotopic to a boundary curve of S. We further assume by contradiction that γ_i and γ_j intersect homotopically. There is $\nu\in[1,n]$ such that $f^i(\gamma_i)$ and $f^i(\gamma_j)$ are contained in S_ν. Moreover, neither of $f^i(\gamma_i),f^i(\gamma_j)$ is homotopic to a boundary curve of S_ν. It follows that $f^i(\gamma_j)$ is contained in the unique $F^{i-j}(S_1\cup\cdots\cup S_n)$-piece that is parallel to S_ν. This implies $i\equiv j \bmod p_\nu$, where p_ν is the f_*-period of S_ν. Since Σ_ν is (h_ν,P_ν)-stable, $f^i(\gamma_j)$ is homotopic to either a curve $\delta\in\Sigma_\nu$, or a boundary curve of S_ν. But neither is possible since $f^i(\gamma_j)$ intersects $f^i(\gamma_i)\in\Sigma_\nu$ homotopically. □

It follows from Lemma 6.6.3 that for any $k \geq 1$, there is a multicurve Δ_k in $\mathbf{S}-Q$ such that $\Sigma_1 \cup \cdots \cup \Sigma_n \subset \Delta_k$ and Δ_k represents all homotopy classes of non-peripheral curves in $\Gamma_0 \cup \cdots \cup \Gamma_k$. Obviously, Δ_k is homotopically contained in Δ_{k+1}. Since $\partial\mathbf{S}_A \cup \partial\mathbf{S}_C \cup (\mathbf{S}_C \cap P)$ has finitely many components, there exists a positive integer N such that $\#\Delta_k = \#\Delta_N$ for all $k \geq N$.

By the choice of N, the set Δ_N is a (F, Q)-stable multicurve. We denote Δ_N by $\mathbf{G}_F(\Sigma_1, \cdots, \Sigma_n)$ since it is generated by $\Sigma_1, \cdots, \Sigma_n$. We define

$$\mathcal{B}_F(\Sigma_1, \cdots, \Sigma_n) = \{\gamma \in \mathbf{G}_F(\Sigma_1, \cdots, \Sigma_n); \gamma \text{ is homotopic to a curve of } \mathcal{B} \text{ in } \mathbf{S}-Q\}.$$

Obviously $\mathcal{B}_F(\Sigma_1, \cdots, \Sigma_n)$ is a (F, Q)-stable multicurve. One may check that for all $\nu \in [1, n]$,

$$\Sigma_\nu = \{\gamma \in \mathbf{G}_F(\Sigma_1, \cdots, \Sigma_n) - \mathcal{B}_F(\Sigma_1, \cdots, \Sigma_n); \gamma \subset S_\nu\}.$$

Theorem 6.6.1 is essentially equivalent to the following:

Theorem 6.6.5. *We have that*

1. If (F, Q) is a disk-covering, then (f, P) has no Thurston obstructions.

2. If (F, Q) is an annular-covering, then (f, P) has no Thurston obstructions if and only if $\lambda(\mathcal{Y}, f) < 1$.

3. If (F, Q) is a complex-covering, then (f, P) has no Thurston obstructions if and only if $\lambda(\mathcal{Y}, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) has no Thurston obstructions.

Proof. 1. Since (F, Q) is a disk-covering, any (F, Q)-stable multicurve $\mathcal{X}$ is an empty set. So we have $\lambda(\mathcal{X}, F) = 0$ and (F, Q) has no Thurston obstructions. It follows from Theorem 6.6.3 that (f, P) has no Thurston obstructions either.

2. For any (F, Q)-stable multicurve $\mathcal{X}$, $\mathcal{X}$ is homotopically contained in $\mathcal{B}$. So $\lambda(\mathcal{X}, F) \leq \lambda(\mathcal{B}, F)$. This implies that (F, Q) has no Thurston obstructions if and only if $\lambda(\mathcal{B}, F) < 1$. It follows from Lemma 6.6.2 and Theorem 6.6.3 that (f, P) has no Thurston obstructions if and only if $\lambda(\mathcal{Y}, f) < 1$.

3. The 'sufficiency' follows from Theorem 6.6.3 and 6.6.4. We just prove the 'necessity'. We may assume that (F, Q) has no Thurston obstructions by Theorem 6.6.3. It follows immediately $\lambda(\mathcal{B}, F) < 1$ and $\Lambda^* = \emptyset$. Let Σ_ν be a (h_ν, P_ν)-stable multicurve for $\nu \in \Lambda = [1, n]$. By homotopic deformations, we may assume that each curve of Σ_ν is contained in the piece S_ν. Let $\mathbf{G}_F(\Sigma_1, \cdots, \Sigma_n)$ be the (F, Q)-stable multicurve generated by $\Sigma_1, \cdots, \Sigma_n$. Again by Theorem 6.6.4, for each $\nu \in \Lambda$,

$$\lambda(\Sigma_\nu, h_\nu) \leq \lambda(\mathbf{G}_F(\Sigma_1, \cdots, \Sigma_n), F)^{p_\nu} < 1.$$

So (h_ν, P_ν) has no Thurston obstructions for each $\nu \in \Lambda$. □

6.7 Surgery part II

The section is a sketch of Cui-Tan's work [CT1], with slight modifications.

In the pervious section, we call (F, Q) a repelling system of (f, P). In this section, to begin with, we give a more general definition of repelling systems:

Definition 6.7.1. *We say that a map $H : \mathbf{U} \to \mathbf{V}$ is a repelling system, if*

- *Both $\mathbf{U}$ and $\mathbf{V}$ are surface puzzles, with $\mathbf{U} \Subset \mathbf{V}$;*
- *The map H maps every $\mathbf{U}$-piece U properly onto a $\mathbf{V}$-piece V as a quasiregular map;*
- *The orbit $\{H^n(c); n \geq 0\}$ of every critical point c of H is disjoint from the boundary of $\mathbf{U}$.*

Notice that in the definition, we don't require that $H(\mathbf{U}) = \mathbf{V}$.

For example, a polynomial-like map is a repelling system.

Let $C(H)$ be the critical set of the repelling system $H : \mathbf{U} \to \mathbf{V}$, the postcritical set $P(H)$ is defined as the closure of $\{H^n(c) \in \mathbf{V}; c \in C(H), n \geq 1\}$. A marked set of the repelling system $H : \mathbf{U} \to \mathbf{V}$ is a compact set $M \subset \mathbf{V}$ such that $H(M \cap \mathbf{U}) \subset M$ and $M - P(H)$ is a finite set. We will use $H : (\mathbf{U}, M) \to (\mathbf{V}, M)$ to denote a marked repelling system. If there is no confusion, the marked repelling system $H : (\mathbf{U}, M) \to (\mathbf{V}, M)$ is also denoted by (H, M).

Definition 6.7.2. *We say that two marked repelling systems $H_1 : (\mathbf{U}_1, M_1) \to (\mathbf{V}_1, M_1)$ and $H_2 : (\mathbf{U}_2, M_2) \to (\mathbf{V}_2, M_2)$ are q.c-equivalent if there is a pair of quasiconformal homeomorphisms $\Phi, \Psi : \mathbf{V}_1 \to \mathbf{V}_2$, such that*

- *$\Psi(\mathbf{U}_1) = \mathbf{U}_2$, $\Psi(M_1) = M_2$;*
- *Ψ is isotopic to Φ rel $\partial \mathbf{V}_1 \cup M_1$;*
- *$\Phi \circ H_1 = H_2 \circ \Psi$.*

In this case, we say that (H_1, M_1) is q.c-equivalent to (H_2, M_2) via (Φ, Ψ).

Let $H : (\mathbf{U}, M) \to (\mathbf{V}, M)$ be a marked repelling system. We say that a $\mathbf{U}$-piece U is of *disk type*, if one boundary curve of U bounds a disk Δ_U such that $U \subset \Delta_U \subset \mathbf{V}$ and Δ_U contains at most one point in M.

Let $\mathbf{U}_0$ the union of all disk type $\mathbf{U}$-pieces. A repelling system $H_1 : (\mathbf{U}_1, M_1) \to (\mathbf{V}_1, M_1)$ is called a *sub-marked repelling system* of (H, M), if $\mathbf{U}_1$ is the union of some $\mathbf{U}$-pieces, $\mathbf{V}_1$ is the union of some $\mathbf{V}$-pieces, and $M_1 = \mathbf{V}_1 \cap M$.

Lemma 6.7.1. *Let $H : (\mathbf{U}, M) \to (\mathbf{V}, M)$ be a marked repelling system and $H_1 : (\mathbf{U}_1, M_1) \to (\mathbf{V}_1, M_1)$ be a sub-marked repelling system of (H, M), with $\mathbf{U} - \mathbf{U}_1 \subset \mathbf{U}_0$. Then (H, M) is q.c-equivalent to a holomorphic marked repelling system if and only if (H_1, M_1) is q.c-equivalent to a holomorphic marked repelling system.*

Proof. Suppose that (H, M) is q.c-equivalent to a holomorphic marked repelling system via (Φ, Ψ), then (H_1, M_1) is q.c-equivalent to a holomorphic marked repelling system via $(\Phi|_{\mathbf{V}_1}, \Psi|_{\mathbf{V}_1})$.

On the other hand, suppose that (H_1, M_1) is q.c-equivalent to a holomorphic marked repelling system via (Φ_1, Ψ_1). We first extend Φ_1 to a quasiconformal map $\Phi : \mathbf{V} \to \Phi(\mathbf{V}) \subset \overline{\mathbb{C}}$. Then we define a qusiconformal homeomorphism $\Psi : \mathbf{V} \to \Phi(\mathbf{V})$ isotopic to Φ rel $\partial\mathbf{V} \cup M$, as follows:

Given a $\mathbf{V}$-piece V, if V is a disk containing at most one point in M, then we choose a closed disk V' such that $V \cap \mathbf{U} \Subset V' \Subset V$ and $V' \cap M = V \cap M$. By Measurable Riemann Mapping Theorem, there is a q.c homeomorphism $\varphi : V' \to \Phi(V')$ with Beltrami coefficient:

$$\mu_\varphi(z) = \sum_{U \subset V \cap \mathbf{U}} \chi_U(z)\mu_{\Phi\circ F}(z),$$

where the summation is taken over all $\mathbf{U}$-pieces contained in V. We further require that $\varphi(p) = \Phi(p)$ if V contains a marked point $p \in M$.

We define a quasiconformal map $\Psi : V \to \Phi(V)$ by

$$\Psi(z) = \begin{cases} \Phi(z), & z \in \partial V, \\ \varphi(z), & z \in V', \\ q.c\ interpolation, & z \in V - V'. \end{cases}$$

Now, suppose that V is either an annulus or $\#\partial(V) + \#(V \cap M) \geq 3$. Let $\mathcal{U}(V)$ be the collection of all $(\mathbf{U} - \mathbf{U}_1)$-pieces that are contained in V. Notice that for $U_1, U_2 \in \mathcal{U}(V)$, either $\Delta_{U_1} \cap \Delta_{U_2} = \emptyset$ or one of Δ_{U_i} contains another. We denote all components of $\cup_{U \in \mathcal{U}(V)} \Delta_U$ by $\{D_i\}_{i \in \Lambda_V}$, where Λ_V is an index set induced by V. We thicken each D_i a little bit along the boundary ∂D_i to get a larger disk U_i. Take a quasiconformal homeomorphism $\phi_i : D_i \to \psi(D_i)$ (here, we set $\psi = \Psi_1$ if $V \subset \mathbf{V}_1$ and $\psi = \Phi$ if $V \subset \mathbf{V} - \mathbf{V}_1$) such that

$$\mu_{\phi_i}(z) = \sum_{\mathcal{U}(V) \ni U \subset D_i} \chi_E(z)\mu_{\Phi\circ F}(z).$$

We further require that $\phi_i(p) = \Phi(p)$ if D_i contains a marked point $p \in M$. Now, we define

$$\Psi(z) = \begin{cases} \psi(z), & z \in V - \cup_{i \in \Lambda_V} U_i, \\ \phi_i(z), & z \in D_i, i \in \Lambda_V, \\ q.c\ interpolation, & z \in \cup_{i \in \Lambda_V}(U_i - D_i). \end{cases}$$

In this way, we can construct a quasiconformal map $\Psi : \mathbf{V} \to \Phi(\mathbf{V})$, isotopic to Φ rel $\partial\mathbf{V} \cup M$. One may verify that (H, M) is q.c-equivalent to a holomorphic marked repelling system via (Φ, Ψ). $\square$

Let (f, P) be a non-parabolic map with $n_A(f) > 0$, and (F, Q) the marked repelling system of (f, P) defined in Section 6.6.1.

Theorem 6.7.1. *(f, P) is q.c-equivalent to a rational map if and only if (F, Q) is q.c-equivalent to a holomorphic marked repelling system.*

Proof. Suppose that (f, P) is q.c-equivalent to a rational map (R, M) via (ϕ_0, ϕ_1), then we can construct a sequence of quasiconformal maps $\phi_n, n \geq 0$ such that $R\phi_{n+1} = \phi_n f$ and ϕ_n is isotopic to ϕ_{n+1} rel $f^{-n}(P \cup N)$, where N is a neighborhood of all attracting cycles of (f, P) in P'_f. By the construction of the repelling system $F : (\mathbf{E}, Q) \to (\mathbf{S}, Q)$, when n is large enough, $\overline{\mathbb{C}} - \mathbf{S} \subset f^{-n}(N)$, thus ϕ_{n+1} and ϕ_n are identical on the boundary $\partial \mathbf{S}$. We set $\Phi = \phi_n|_{\mathbf{S}}, \Psi = \phi_{n+1}|_{\mathbf{S}}$ for such large n. Then (F, Q) is q.c-equivalent to the holomorphic marked repelling system $R|_{\Psi(\mathbf{E})} : (\Psi(\mathbf{E}), \Psi(Q)) \to (\Psi(\mathbf{S}), \Psi(Q))$ via (Φ, Ψ).

Sufficiency. It follows from [CT1], Proposition 2.4. □

Theorem 6.7.2. *Suppose that (f, P) is a quasiregular map, then*

1. If (F, Q) is a disk-covering, then (f, P) is q.c-equivalent to a rational map. This rational map is unique up to Möbius conjugation.

2. If (F, Q) is an annular-covering, then (f, P) is q.c-equivalent to a rational map (R, M) if and only if $\lambda(\mathcal{Y}, f) < 1$. The rational realization (R, M) is unique up to Möbius conjugation.

3. If (F, Q) is a complex-covering, then (f, P) is q.c-equivalent to a rational map if and only if $\lambda(\mathcal{Y}, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) is q.c-equivalent to a rational map.

Proof. 1. If (F, Q) is a disk-covering, then all $\mathbf{E}$-pieces are of disk type. It follows from Lemma 6.7.1 that (F, Q) is q.c-equivalent to a holomorphic marked repelling system. By Theorem 6.7.1, (f, P) is q.c-equivalent to a rational map, say (R, M). The uniqueness of the rational realization follows from Lemma 6.9.2 below and the fact that $J(R)$ has zero Lebesgue measure.

2. Let $\mathbf{E}_A$ be the union of all $\mathbf{E}$-pieces E which are contained essentially in the $\mathbf{S}_A$-pieces (here, 'essentially' means that E separates the two boundary curves of some $\mathbf{S}_A$-piece). One may verify that $\mathbf{E}_A \Subset \mathbf{S}_A$ and each $\mathbf{E}_A$-piece is mapped properly onto some $\mathbf{S}_A$-piece. Let $G = F|_{\mathbf{E}_A}$, then $G : \mathbf{E}_A \to \mathbf{S}_A$ is a repelling system.

One may check that $\mathcal{B}$ is also a stable multicurve for $G : \mathbf{E}_A \to \mathbf{S}_A$, with $\lambda(\mathcal{B}, G) = \lambda(\mathcal{B}, F)$. Since $\lambda(\mathcal{B}, F) < 1$, it follows from Lemma 6.2 in [CT1] that $G : \mathbf{E}_A \to \mathbf{S}_A$ is q.c-equivalent to a holomorphic repelling system. Notice that each $(\mathbf{E} - \mathbf{E}_A)$-piece is of disk type, we deduce by Lemma 6.7.1 that (F, Q) is q.c-equivalent to a holomorphic marked repelling system.

By Theorem 6.7.1, (f, P) is q.c-equivalent to a rational map, say (R, M).

The uniqueness of the rational realization follows from Lemma 6.9.2 and the fact that $J(R)$ has zero Lebesgue measure.

3. The 'necessity' part is essentially the same as the proof of 'necessity' of Theorem 6.4.1 (See Section 6.4.1). The 'sufficiency' part is essentially due to Cui Guizhen and Tan Lei [CT1]. Even if they deal only with maps without rotation domains, their proof applies equally well to our situations. For more details, one may refer Cui-Tan's paper ([CT1], Section 7 'Proof of Theorem 5.4 for a cycle of complex pieces' and Section 8 'Proof of Theorem 5.4') □

6.8 Proof of Theorem 6.1.2, the first two parts

Let (f, P) be a non-parabolic map. Notice that if (f, P) itself is a Siegel map or a Thurston map, then Theorem 6.1.2 follows immediately (We may take $\Gamma = \emptyset$, and the resulting map as (f, P) itself). If (f, P) is a Herman map or a non-parabolic map without rotation annulus, then Theorem 6.1.2 follows from Theorem 6.2.1 and Theorem 6.5.1.

So in the following, we need only consider the case when $n_A(f) > 0, n_{RD}(f) \geq 0$ and $n_{RA}(f) > 0$. By Theorem 6.5.1, there is a (f, P)-stable multicurve Γ_0, and finitely many branched coverings $(h_k, P_k), k \in \Lambda$, each is either a Herman map, or Siegel map, or Thurston map, such that

• (f, P) has no Thurston obstructions if and only if $\lambda(\Gamma_0, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) has no Thurston obstructions.

• (f, P) is q.c-equivalent to a rational map if and only if $\lambda(\Gamma_0, f) < 1$ and for each $k \in \Lambda$, (h_k, P_k) is q.c-equivalent to a rational map.

Suppose that there are exactly n Herman maps in the resulting maps. We may relabel them such that $(h_1, P_1), \cdots, (h_n, P_n)$ are Herman maps. By Theorem 6.2.1, for each $k \in [1, n]$, there is a decomposition

$$\mathrm{Dec}(h_k, P_k) = \left(\bigoplus_{j \in \Lambda_k \cup \Lambda_k^*} (h_{k,j}, P_{k,j}) \right)_{\Sigma_k},$$

such that: (h_k, P_k) has no Thurston obstructions if and only if $\lambda(\Sigma_k, h_k) < 1$ and for each $j \in \Lambda_k$, $(h_{k,j}, P_{k,j})$ has no Thurston obstructions; (h_k, P_k) is q.c-equivalent to a rational map if and only if $\lambda(\Sigma_k, h_k) < 1$ and for each $j \in \Lambda_k$, $(h_{k,j}, P_{k,j})$ is q.c-equivalent to a rational map.

Now let (F, Q) be the repelling system of (f, P) defined as in Section 6.6.1. We may assume that for each $k \in [1, n]$, each curve of Σ_k is contained in the **S**-piece S_k. In this way, Σ_k can be considered as a multicurve of (F, Q). We use $\Sigma_1, \cdots, \Sigma_n$ to generate a (f, P)-stable multicurve

$G_F(\Sigma_1, \cdots, \Sigma_n)$ (see Section 6.6.2). Let $\Gamma_1 = \pi \circ G_F(\Sigma_1, \cdots, \Sigma_n)$, then Γ_1 is a (f, P)-stable multicurve. We decompose Γ_1 into two submulticurves $\Gamma_{1,0} := \{\gamma \in \Gamma_1; \gamma \text{ is homotopic to a curve in } \Gamma_0\}$ and $\Gamma_{1,1} = \Gamma_1 - \Gamma_{1,0}$. One may check that $\Gamma_{1,0}$ is (f, P)-stable and the (f, P)-transition matrix W_1 (resp. $W_{1,0}$, $W_{1,1}$) of Γ_1 (resp. $\Gamma_{1,0}$, $\Gamma_{1,1}$) satisfies:

$$W_1 = \begin{pmatrix} W_{1,0} & * \\ O & W_{1,1} \end{pmatrix}.$$

Thus we have $\lambda(\Gamma_1, f) = \max\{\lambda(\Gamma_{1,0}, f), \lambda(\Gamma_{1,1}, f)\}$.

Define $\Gamma = \Gamma_0 \cup \Gamma_{1,1}$. Then Γ is a (f, P)-stable multicurve and

$$\lambda(\Gamma, f) = \max\{\lambda(\Gamma_0, f), \lambda(\Gamma_{1,1}, f)\} = \max\{\lambda(\Gamma_0, f), \lambda(\Gamma_1, f)\}.$$

By Theorem 6.6.2 ,

$$\lambda(G_F(\Sigma_1, \cdots, \Sigma_n), F) = \lambda(\Gamma_1, f),$$
$$\lambda(\mathcal{B}_F(\Sigma_1, \cdots, \Sigma_n), F) = \lambda(\pi \circ \mathcal{B}_F(\Sigma_1, \cdots, \Sigma_n), f).$$

Since $\pi \circ \mathcal{B}_F(\Sigma_1, \cdots, \Sigma_n)$ is homotopically contained in Γ_0, we have $\lambda(\pi \circ \mathcal{B}_F(\Sigma_1, \cdots, \Sigma_n), f) \le \lambda(\Gamma_0, f)$. It follows from Theorem 6.6.4 that

$$\lambda(\Gamma, f) = \max\left\{\lambda(\Gamma_0, f),\ \sqrt[p_1]{\lambda(\Sigma_1, h_1)}, \cdots,\ \sqrt[p_n]{\lambda(\Sigma_n, h_n)}\right\}.$$

This implies $\lambda(\Gamma, f) < 1$ if and only if $\lambda(\Gamma_0, f) < 1$ and for each $k \in [1, n]$, $\lambda(\Sigma_k, h_k) < 1$.

The proof is completed if we take the (f, P)-stable multicurve as Γ and the resulting maps as $(h_{k,j}, P_{k,j}), k \in [1, n], j \in \Lambda_k$ and $(h_k, P_k), k \in \Lambda - [1, n]$.

6.9 Analytic part

In this section, we will discuss the rational-like maps, renormalizations of rational maps and prove the analytic part of Theorem 6.1.2.

6.9.1 Rational-like maps

A *rational-like map* $g : U \to V$ is a proper and holomorphic map between two multi-connected domains such that $\overline{U} \subset V \subset \overline{\mathbb{C}}$ and the complementary set $\overline{\mathbb{C}} - X$ of $X \in \{U, V\}$ consists of finitely many topological disks. In our discussion, we always assume $V \neq \overline{\mathbb{C}}$ and the degree of g is at least two. The filled Julia set is defined by $K(g) = \bigcap_{n \ge 1} g^{-n}(V)$, the Julia set is defined by $J(g) = \partial K(g)$. The filled Julia set $K(g)$ is not necessarily a full set. This implies that $J(g)$ is not necessarily connected even if $K(g)$ is connected.

Two rational-like maps g_1 and g_2 are hybrid equivalent if there is a quasiconformal conjugacy ϕ between g_1 and g_2, defined in a neighborhood of their respective filled Julia sets, such that $\overline{\partial}\phi = 0$ on $K(g_1)$. We call ϕ a hybrid conjugacy between g_1 and g_2. These definitions are simply the generalizations of Douady-Hubbard's definitions of polynomial-like maps.

The following is an analogue of Douady-Hubbard's straightening theorem for polynomial-like maps.

Theorem 6.9.1. (Straightening Theorem) *Let $g : U \to V$ be a rational-like map of degree $d \geq 2$, then*

1. The map g is hybrid equivalent to a rational map R of degree d.

2. If $K(g)$ is connected, then g is hybrid equivalent to a rational map R of degree d, which is postcritically finite outside $\phi(K(g))$. Here ϕ is the hybrid conjugacy. Such R is unique up to Möbius conjugation.

Remark 6.9.1. *1. A rational-like map $g : U \to V$ can be hybrid equivalent to a rational map of degree greater than d.*

2. Even if $K(g)$ is connected, the rational-like map g can be hybrid equivalent to a rational map of degree greater than d, which is postcritically finite outside $\phi(K(g))$. Such example can be found in the family of rational maps: $f_\lambda(z) = z^n + \lambda/z^n, n \geq 3$, where λ is a complex parameter. We denote by B_λ the immediate attracting basin of ∞. We assume that each critical point of the form $\sqrt[2n]{\lambda}$ has an orbit meeting an attracting cycle other than ∞. In this case, the Julia set is connected since the map is postcritically finite, and f_λ is strictly expanding on ∂B_λ. There is an annular neighborhood A of ∂B_λ such that $f_\lambda|_A : A \to f_\lambda(A)$ is a proper map of degree n. (Such annulus can be chosen as the union of puzzles pieces that intersect with ∂B_λ, see the previous chapter). The rational-like map $f_\lambda|_A$ can be hybrid equivalent to the power map $z \mapsto z^n$, whose degree is lower than that of f_λ.

3. If $K(g)$ is connected and $\overline{\mathbb{C}} - K(g)$ consists of two components, then there are two annuli U', V' such that $K(g) \subset U' \subset V' \subset V$ and the restriction $g|_{U'} : U' \to V'$ is a rational-like map. One may show that $K(g)$ is a quasicircle by quasiconformal surgery.

Proof. 1. The proof is a standard surgery procedure. By shrinking V a little bit, we may assume that each boundary curve of U and V is a quasicircle. We then extend $g : U \to V$ to a quasiregular branched covering $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that G is holomorphic in $\overline{\mathbb{C}} - \overline{V}$ and G maps each component U_k of $\overline{\mathbb{C}} - U$ onto a connected component V_j of $\overline{\mathbb{C}} - V$, with degree equal to $\deg(g|_{\partial U_k})$. Such extension keeps the degree. By pulling back the standard complex structure σ_0 on $\overline{\mathbb{C}} - V$ via G, we get a G-invariant complex structure

$$\sigma = \begin{cases} (G^k)^*(\sigma_0), & \text{in } G^{-k}(\overline{\mathbb{C}} - V),\ k \geq 1, \\ \sigma_0, & \text{in } K(g). \end{cases}$$

The Beltrami coefficient μ of σ satisfies $\mu|_{K(g)} = 0$ and $\|\mu\|_\infty < 1$. Let ϕ solve the Beltrami equation $\overline{\partial}\phi = \mu\partial\phi$. Then $R = \phi \circ G \circ \phi^{-1}$ is a rational map and ϕ is a hybrid conjugacy between g and R.

2. By a hole-filling process, we can find a suitable restriction $g|_{U'} : U' \to V'$ of g with $K(g) \Subset U' \Subset V' \Subset V$ such that

a). All postcritical points of $g|_{U'}$ in V' are contained in $K(g)$.

b). Each connected component of $V' - \overline{U'}$ is either an annulus or a disk.

Notice that such V' can be chosen arbitrarily close to the filled Julia set $K(g)$. (To see this, one may replace V' by $g^{-k}(V')$ for some large k, and a), b) still holds.)

In this way, each component U_i of $\overline{\mathbb{C}} - \overline{U'}$ either is contained in V' or contains a unique component V_j of $\overline{\mathbb{C}} - \overline{V'}$. In the former case, we mark a point $p \in U_i$ and get a marked disk (U_i, p); in the latter case, we mark a point $p \in V_j$, and get two marked disks (V_j, p) and (U_i, p). We extend $g|_{U'}$ to a quasiregular branched covering $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

a). For each component U_i of $\overline{\mathbb{C}} - \overline{U'}$, G maps the marked disk (U_i, p) to the marked disk (V_k, q), where V_k is the component of $\overline{\mathbb{C}} - \overline{V'}$ whose boundary is $g(\partial U_i)$. We require that $G(p) = q$ and p is the only possible critical point, with local degree equal to $\deg(g|_{\partial U_i})$.

b). We further require that G is holomorphic in $\overline{\mathbb{C}} - \overline{V'}$.

By pulling back the standard complex structure on $\overline{\mathbb{C}} - V'$, we can get a G-invariant complex structure whose Beltrami coefficient μ satisfies $\mu|_{K(g)} = 0$ and $\|\mu\|_\infty < 1$. Let ϕ solve the Beltrami equation $\overline{\partial}\phi = \mu\partial\phi$. Then $f = \phi \circ G \circ \phi^{-1}$ is a rational map, postcritically finite outside $\phi(K(g))$, as required.

To prove the uniqueness, we need investigate some mapping properties of R, a rational map of degree d, to which $g|_{U'}$ is hybrid equivalent via ϕ, and postcritically finite outside $\phi(K(g))$. We assume V' is sufficiently close to $K(g)$ such that ϕ is defined on V'. Then $g|_{U'}$ induces a suitable restriction $R|_{\phi(U')}$. Let $\mathcal{X}_1$ be the collection of all components of $\overline{\mathbb{C}} - \phi(K(g))$ which intersect with the boundary curves of $\phi(V')$ and $\mathcal{X}_2$ be the collection of all components of $\overline{\mathbb{C}} - \phi(K(g))$ which intersect with the boundary curves of $\phi(U')$. It's obvious that $\mathcal{X}_1 \subset \mathcal{X}_2$. Since the degree of R is equal to d (This is very important), we have that

$$\{U \text{ is a component of } R^{-1}(X); X \in \mathcal{X}_1\} = \mathcal{X}_2.$$

Thus for each $X \in \mathcal{X}_2$, $R(X) \in \mathcal{X}_2$. This implies that each $X \in \mathcal{X}_2$ is eventually periodic under the map R. Let $X \in \mathcal{X}_2$ be a periodic element, with period p. Since R is poscritically finite outside $\phi(K(g))$, $R^p|_X : X \to X$ is proper and each critical point in X has finite orbit. Thus $R^p|_X$ is conformally conjugate to $z \mapsto z^d$, where $d = \deg(R^p|_X) \geq 2$ (For a proof of this fact, see [DH2] Lemma 4.1). It follows that for all $X \in \mathcal{X}_2$, the proper map $R|_X : X \to R(X)$ has only one possible critical point, which is eventually mapped to a superattracting cycle. Base on these observations, we are now ready to prove the uniqueness part of the theorem.

Suppose that R_1 and R_2 are two rational maps of degree d, both are hybrid equivalent to $g|_{U'}$ and poscritically finite outside $\phi_1(K(g))$ and $\phi_2(K(g))$, respectively. Here, ϕ_i is a hybrid conjugacy between $g|_{U'}$ and R_i, $i = 1, 2$. We assume that V' is sufficiently close to $K(g)$ such that ϕ_i is defined on U'. Then $g|_{U'}$ induces two restrictions $R_i|_{\phi_i(U')}, i = 1, 2$ and a hybrid conjugacy $\phi = \phi_2 \circ \phi_1^{-1}$ between them. One can construct a pair of quasiconformal maps $\varphi_0, \varphi_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

a). $\varphi_0 \circ R_1 = R_2 \circ \varphi_1$ on $\overline{\mathbb{C}}$.

b). φ_0, φ_1 are isotopic rel $\phi_1(K(g)) \cup P_{R_1}$ and $\varphi_0|_{\phi_1(U')} = \varphi_1|_{\phi_1(U')} = \phi|_{\phi_1(U')}$.

c). φ_0, φ_1 are holomorphic and identical in a neighborhood N of all superattracting cycles of R_1 in $\overline{\mathbb{C}} - \phi_1(K(g))$.

By Thurston algorithm, there is a sequence of quasiconformal maps $\{\varphi_n, n \geq 0\}$ such that $\varphi_n \circ R_1 = R_2 \circ \varphi_{n+1}$ and φ_n is isotopic to φ_{n+1} rel $R_1^{-n}(\phi_1(U') \cup P_{R_1} \cup N)$. The quasiconformal map φ_n satisfies $\overline{\partial}\varphi_n = 0$ on $\phi_1(K(g)) \cup R_1^{-n}(N)$. The sequence $\{\varphi_n\}$ has a limit quasiconformal map $\varphi = \lim \varphi_n$. Since the Lebesgue measure of $\overline{\mathbb{C}} - \phi_1(K(g)) \cup R_1^{-n}(N)$ tends to zero as $n \to \infty$, the map φ satisfies $\overline{\partial}\varphi = 0$ outside a zero measure set. It is in fact a holomorphic conjugacy between R_1 and R_2. □

6.9.2 Renormalizations of rational maps

Let f be a rational map. Its Julia set, critical set and postcritical set are denoted by $J(f)$, Ω_f and P_f, respectively. Let P'_f be the accumulation set of P_f. A Julia component is a connected component of $J(f)$.

We say f^p is *renormalizable* if there exist two multi-connected domains U, V such that $f^p : U \to V$ is a rational-like map of degree at least two, and the filled Julia set $K(f^p|_U)$ is connected. The triple (f^p, U, V) is called a *renormalization* of f. A renormalization (f^p, U, V) is of *annular type* if $\overline{\mathbb{C}} - K(f^p|_U)$ consists of two components (In this case, $K(f^p|_U)$ is necessarily a quasicircle and we may assume that U, V are annuli, see Remark 6.9.1). By Theorem 6.9.1, the rational-like map $f^p : U \to V$ is hybrid equivalent to a unique rational map R via some quasiconformal map ϕ, such that $\deg(R) = \deg(f^p|_U)$ and R is postcritically finite outside $\phi(K(f^p|_U))$. Such R is called the *canonical straightening map* of $f^p : U \to V$.

In this section, we will prove:

Theorem 6.9.2. (Renormalization) *Let f be a rational map of degree at least two. Suppose that $J(f) \cap P_f$ is contained in finitely many Julia components and P'_f contains attracting cycles. Then f admits finitely many non-annular type renormalizations $(f^{p_i}, U_i, V_i), i \in \Lambda$ (Λ is a finite index set) which satisfy*

1. For every $i \in \Lambda$, V_i contains no attracting cycles of f in P'_f.

2. The Julia set $J(f)$ has zero Lebesgue measure (resp. carries no invariant line fields) if and only if for each $i \in \Lambda$, $J(f^{p_i}|_{U_i})$ has zero Lebesgue measure (resp. carries no invariant line fields).

Remark 6.9.2. *Theorem 6.9.2 also holds when f is a rational-like map.*

The proof is based on the following ([McM1]):

Theorem 6.9.3. (Ergodic or attracting) *Let f be a rational map of degree at least two, then either*

- *$J(f) = \overline{\mathbb{C}}$ and the action of f on $\overline{\mathbb{C}}$ is ergodic, or*
- *the spherical distance $d(f^n(z), P_f) \to 0$ for almost every $z \in J(f)$ as $n \to \infty$.*

Proof of Theorem 6.9.2. Let $\mathcal{A}(f)$ be the union of all attracting cycles of f in P'_f. For each point $z \in \mathcal{A}(f)$, there is a disk neighborhood U_z of z such that

1. ∂U_z is a quasicircle, and $\partial U_z \cap P_f = \emptyset$.
2. $f^{-1}(\overline{\mathbb{C}} - \cup_{z\in\mathcal{A}(f)} U_z)$ is contained in the interior of $\overline{\mathbb{C}} - \cup_{z\in\mathcal{A}(f)} U_z$.

Let $\mathbf{S}_0 = \overline{\mathbb{C}} - \cup_{z\in\mathcal{A}(f)} U_z$, $P = P_f$. For $n \geq 1$, we define the surface puzzle $\mathbf{S}_n$ inductively in the following way:

$$\mathbf{S}_n = \mathcal{H} \circ f^{-1}(\mathbf{S}_{n-1}) = \cdots = (\mathcal{H} \circ f^{-1})^n(\mathbf{S}_0).$$

Since $J(f) \cap P_f$ is contained in finitely many Julia components, the same proof as Lemma 6.5.1 yields that when n is large enough, $\mathbf{S}_n$ is of constant complexity. We set $\mathbf{S} = \mathbf{S}_n$ for such large n and $\mathbf{E} = f^{-1}(\mathbf{S})$. Let $\mathbf{S}_D, \mathbf{S}_A, \mathbf{S}_C$ be the union of all disk pieces, annular pieces and complex pieces of $\mathbf{S}$, respectively.

If $\mathbf{S}_C = \emptyset$, then each $\mathbf{S}$-piece is either of disk type or of annular type. In this case, $P_f \cap J(f)$ is a finite set. This implies that the map f has neither indifferent cycles nor rotation domains. So the orbit of every point $z \in P_f \cap J(f)$ meets a repelling cycle. It follows that the Julia set $J(f)$ has Lebesgue measure zero. The conclusion follows if we set $\Lambda = \emptyset$.

If $\mathbf{S}_C \neq \emptyset$, then each $\mathbf{S}_C$-piece is eventually periodic under the map f_*. We list all f_*-periodic cycles of $\mathbf{S}_C$-pieces in the following:

$$S_\nu \mapsto f_*(S_\nu) \mapsto \cdots \mapsto f_*^{p_\nu - 1}(S_\nu) \mapsto f_*^{p_\nu}(S_\nu) = S_\nu, \quad 1 \leq \nu \leq n,$$

where n is the number of f_*-periodic cycles, S_ν is a representative in the ν-th cycle and p_ν is the period of the piece S_ν. For $i \in [1, n]$, let $V_i = S_i$, and U_i be the unique component of $f^{-p_i}(S_i)$ that is parallel to V_i (Recall that 'parallel' means each component of $V_i - U_i$ is either a disk containing at most one point in P, or an annulus in $\overline{\mathbb{C}} - P$ containing a boundary curve of V_i). Then $(f^{p_i}, U_i, V_i), i \in [1, n]$ are the renormalizations of f. (To see this, one should prove that $\deg(f^{p_i}|_{U_i}) \geq 2$, this follows from the fact $\lambda(\partial(V_i), f^{p_i}|_{U_i}) < 1$, with the same argument as Lemma 6.2.4.)

We claim that

The Lebesgue measure of $J(f)$ is zero if and only if for each $i \in [1, n]$, the Lebesgue measure of $J(f^{p_i}|_{U_i})$ is zero.

Let $\mathcal{E}_C$ be the collection of all $\mathbf{E}$-pieces that are parallel to the $\mathbf{S}_C$-pieces. $\mathcal{E}_{AD} = \{D; D$ is an $\mathbf{E}$-piece contained in $\mathbf{S}_A \cup \mathbf{S}_D \cup (\mathbf{S}_C - \cup_{E \in \mathcal{E}_C} E)\}$. Each element $E \in \mathcal{E}_{AD}$ contains at most one point in the postcritical set P_f. Let $\mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_{AD}$. For each $E \in \mathcal{E}$, the boundary of E is contained in the Fatou set $F(f)$.

Notice that $J(f) \subset \bigcap_{k \geq 0} f^{-k}(\cup_{E \in \mathcal{E}} E)$, we can define an itinerary map by:

$$iter : \begin{cases} J(f) \to \mathcal{E}^{\mathbb{N}}, \\ z \mapsto (E_0(z), E_1(z), E_2(z), \cdots). \end{cases}$$

where $E_k(z)$ is the unique element in $\mathcal{E}$ that contains $f^k(z)$.

For simplicity, we denote $J_i = J(f^{p_i}|_{U_i})$ for $i \in [1, n]$. Given a point $z \in J(f)$ with itinerary $iter(z) = (E_0(z), E_1(z), E_2(z), \cdots)$, one can verify that $z \in \bigcup_{k \geq 0} f^{-k}(J_1 \cup \cdots \cup J_n)$ if and only if there is an integer N (depending on z) such that for all $k \geq N$, $E_k(z) \in \mathcal{E}_C$. Moreover, $\bigcup_{k \geq 0} f^{-k}(J_1 \cup \cdots \cup J_n)$ contains all possible parabolic cycles, Cremer cycles and the boundaries of rotation domains, together with their preimages.

This implies that if $z \in J(f) - \bigcup_{k \geq 0} f^{-k}(J_1 \cup \cdots \cup J_n)$, then there exists a sequence of integers $\{n_j; j \geq 1\}$ such that $E_{n_j} \in \mathcal{E}_{AD}$ for all $j \geq 0$. We consider the sequence $\{F_j(z); j \geq 1\}$, where $F_j(z) = E_{n_j}(z)$. It contains a subsequence $\{F_{j_i}(z); i \geq 1\}$ that satisfies either of the following three cases:

1. $F_{j_i}(z) \cap P_f = \emptyset$ for all $i \geq 1$.

2. For all $i \geq 1$, $F_{j_i}(z) \cap P_f \neq \emptyset$ and $F_{j_i}(z)$ contains a point in P_f which is contained either in the Fatou set or in the grand orbit of a repelling cycle.

3. For all $i \geq 1$, $F_{j_i}(z) \cap P_f \neq \emptyset$ and $F_{j_i}(z)$ contains a point in P_f whose orbit accumulates at $P_f' \cap J(f)$.

In the first two cases, one may easily check that $\limsup d(f^n(z), P_f) > 0$.

In the last case, the set $\{F_{j_i}(z); i \geq 1\}$ can be rewritten as $\{E_1, \cdots, E_m\}$, which is a finite subset of $\mathcal{E}_{AD}$. Since each E_k is contained either in a disk component of $\mathbf{S}_C - \cup_{E \in \mathcal{E}_C} E$ or in a $\mathbf{S}_D$-piece, there is an integer $M > 0$ such that $f^{-M}(E_1 \cup \cdots \cup E_m) \cap P_f = \emptyset$. If $\limsup d(f^n(z), P_f) = 0$, then there exists a sequence of integers $\{\ell_j\}$ such that $d(f^{\ell_j}(z), (E_1 \cup \cdots \cup E_m) \cap P_f) \to 0$ as $j \to \infty$. It follows that $f^{\ell_j - M}(z) \in f^{-M}(E_1 \cup \cdots \cup E_m)$ for all large j. Since the boundary of each component of $f^{-M}(E_1 \cup \cdots \cup E_m)$ is contained in the Fatou set $F(f)$, there is an integer $\varepsilon(z) > 0$ such that $d(f^{\ell_j - M}(z), P_f) \geq \varepsilon(z)$ for all large j, which is contradiction. So in this case, we also have $\limsup d(f^n(z), P_f) > 0$.

Thus, for any $z \in J(f) - \cup_{k \geq 0} f^{-k}(J_1 \cup \cdots \cup J_n)$, we have

$$\limsup d(f^n(z), P_f) > 0.$$

It follows from Theorem 6.9.3 that the Lebesgue measure of $J(f) - \cup_{k \geq 0} f^{-k}(J_1 \cup \cdots \cup J_n)$ is zero. This means $Leb(J(f)) = 0$ if and only if for each $k \in \Lambda$, $Leb(J_k) = 0$.

Now we set Λ as the indices $i \in \{1, ..., n\}$ such that the renormalization (f^{p_i}, U_i, V_i) is not of annular type.

1. Notice that the Julia set of an annular type renormalization is a quasicircle, whose Lebesgue measure is zero. It follows that $Leb(J(f)) = 0$ if and only if for each $k \in \Lambda$, $Leb(J_k) = 0$.

2. Suppose that $J(f)$ carries an invariant line field. That is, there is a measurable Beltrami differential μ supported on a positive measure subset E of $J(f)$ such that $f^*\mu = \mu$ a.e, and $|\mu| = 1$ on E. Let $\mu_k = \mu|_{J_k}$ for $k \in \Lambda$. It follows from 1 that there exists $\ell \in \Lambda$ such that $Leb(J_\ell \cap E) > 0$, then μ_ℓ is an invariant line field for $f^{p_\ell}|_{U_\ell}$ since $(f^{p_\ell}|_{U_\ell})^*\mu_\ell = \mu_\ell$. On the other hand, suppose that μ_ℓ is an invariant line field for $f^{p_\ell}|_{U_\ell}$, then the Beltrami differential defined by $\mu = (f^k)^*\mu_\ell$ on $f^{-k}(J_\ell), k \geq 1$ is an invariant line field for f. □

Remark 6.9.3. (Hyperbolic rational maps) *It follows from Theorem 6.9.1 and Theorem 6.9.2 that: For every hyperbolic rational map f, either*

- *each Julia component is a single point or a quasicircle, or*
- *It admits finitely many rational maps $f_1, \cdots, f_n$ as renormalizations, and for each $1 \leq i \leq n$, $3 \leq \#P_{f_i} < \infty$.*

Let f be a rational map of degree d that satisfies the condition of Theorem 6.9.2. Here is a question concerning the number $\#\Lambda$ of non annular type renormalizations of f, posed by Cui Guizhen and Tan Lei:

Question 6.9.1. (Cui-Tan) *Is there a constant $C = C(d)$ depending on d, such that $\#\Lambda \leq C(d)$?*

The answer is yes if f is a polynomial. In fact, we can take $C(d) = d - 1$ in the polynomial case. However, for general rational maps, we don't know much.

6.9.3 Herman-Siegel renormalization

In [Sh1], Shishikura developed a surgery which transfers a rational map with Herman rings into finitely many rational maps with Siegel disks. These resulting maps can be considered as the renormalizations of the original map. However, we will see in the following that this kind of renormalization does not fit our definition in Section 6.9.2.

To begin with, we restate the 'Herman-Siegel surgery' following Shishikura. Let f be a rational map with Herman rings. For our purpose, we assume that $P_f \cap J(f)$ is contained in finitely many Julia components (This assumption enables us to obtain a stable multicurve).

Let $\mathcal{A}$ be the collection of all Herman rings of f. For each $A \in \mathcal{A}$, we choose an analytic curve $\gamma_A \subset A$ such that $\gamma_A \cap P_f = \emptyset$ and $f(\gamma_A) = \gamma_{f(A)}$. Let $\Gamma_0 = \{\gamma_A; A \in \mathcal{A}\}$, $P = P_f \cup \cup_{A\in\mathcal{A}}\overline{A}$. By the same argument as in Lemma 6.2.1, we can use Γ_0 to generate a (f, P)-stable multicurve Γ such that:

1. For any $\gamma \in \Gamma$, $f(\gamma) \in \Gamma \cup \Gamma_0$.

2. Γ represents all homotopy classes of non-peripheral curves of $\cup_{k\geq 1} f^{-k}(\Gamma_0) - \Gamma_0$ in $\overline{\mathbb{C}} - P$.

Here, we follow the notations in Section 6.2. Recall that $\mathcal{S}$ is the set of all closures of connected components of $\overline{\mathbb{C}} - \cup(\Gamma \cup \Gamma_0)$, $\mathcal{E}$ is the set of all closures of connected components of $\overline{\mathbb{C}} - \cup f^{-1}(\Gamma \cup \Gamma_0)$. For each $S \in \mathcal{S}$, let $E_S \in \mathcal{E}$ be the unique $\mathcal{E}$-piece that is parallel to S. We define a map $f_* : \mathcal{S} \to \mathcal{S}$ by $f_*(S) = f(E_S)$. Since there are finitely many $\mathcal{S}$-pieces, every $\mathcal{S}$-piece is eventually periodic under the map f_*.

We list all periodic cycles of $\mathcal{S}$-pieces in the following:

$$S_\nu \mapsto f_*(S_\nu) \mapsto \cdots \mapsto f_*^{p_\nu - 1}(S_\nu) \mapsto f_*^{p_\nu}(S_\nu) = S_\nu, \quad 1 \leq \nu \leq n,$$

where S_ν is a representative of the ν-th cycle and p_ν is the period of S_ν.

For $i \in [1, n]$, let $V_i = S_i$ and U_i be the unique component of $f^{-p_i}(S_i)$ that is contained in S_i and parallel to S_i. The triple (f^{p_i}, U_i, V_i) can be considered as a renormalization of f. In general, U_i is not contained in the interior of V_i (For example, if there is a boundary curve $\gamma \in \partial(V_i)$ such that $\gamma \in \Gamma_0$, then γ is necessarily a boundary curve of U_i). For this reason, we call (f^{p_i}, U_i, V_i) a Herman-Siegel (HS for short) renormalization of f.

We should show that $\deg(f^{p_i}|_{U_i}) \geq 2$ for all $i \in [1, n]$. If $\deg(f^{p_i}|_{U_i}) = 1$ for some $i \in [1, n]$, then $\#\partial(U_i) = \#\partial(V_i)$. There are two possibilities:

a). $\partial(V_i)-\Gamma_0 \neq \emptyset$. In this case, for each curve $\gamma \in \partial(V_i)-\Gamma_0$, $(f^{p_i}|_{U_i})^{-1}(\gamma)$ is a curve in $\partial(U_i)-\Gamma_0$. Conversely, each curve $\alpha \in \partial(U_i)-\Gamma_0$ is homotopic to a curve in $\partial(V_i)-\Gamma_0$. Thus $\partial(V_i)-\Gamma_0$ contains a Levy cycle of f^{p_i}. But this contradicts Theorem 6.4.2.

b). $\partial(V_i)-\Gamma_0 = \emptyset$. In this case, $U_i = V_i$ and $f(U_i) = U_i$. This implies that U_i is contained in the Fatou set of f, which is again a contradiction.

So in either case, $\deg(f^{p_i}|_{U_i}) \geq 2$.

The filled Julia set of the HS renormalization (f^{p_i}, U_i, V_i) is defined by $K(f^{p_i}|_{U_i}) = \bigcap_{k\geq 0}(f^{p_i}|_{U_i})^{-k}(U_i)$, and the Julia set is defined by $J(f^{p_i}|_{U_i}) = K(f^{p_i}|_{U_i}) \cap J(f)$ (Notice that $\partial K(f^{p_i}|_{U_i})$ is not a reasonable definition of the Julia set because $\partial K(f^{p_i}|_{U_i})$ may contain a curve in the Herman ring of f). One may check that $K(f^{p_i}|_{U_i})$ is connected. Moreover, if $\partial(V_i)\cap\Gamma_0 = \emptyset$, then $K(f^{p_i}|_{U_i})$ is contained in the interior of V_i and $J(f^{p_i}|_{U_i}) = \partial K(f^{p_i}|_{U_i})$.

We say (f^{p_i}, U_i, V_i) is hybrid equivalent to a rational map R, if there is a qusiconformal map ϕ defined in a neighborhood N of $K(f^{p_i}|_{U_i})$ such that $N \subset U_i$, $\overline{\partial}\phi = 0$ on $K(f^{p_i}|_{U_i})$ and $\phi \circ f^{p_i}|_N = R \circ \phi$. Notice that ∂N may intersect with $\partial K(f^{p_i}|_{U_i})$.

Theorem 6.9.4. (HS Renormalization) *Let f be a rational map with Herman rings, assume that $P_f \cap J(f)$ is contained in finitely many Julia components. Let $(f^{p_i}, U_i, V_i), i \in [1, n]$ be all the HS renormalizations defined as above. Then*

1. For each $i \in [1, n]$, the HS renormalization (f^{p_i}, U_i, V_i) is hybrid equivalent to a rational map R_i of degree $\deg(f^{p_i}|_{U_i})$ which is postcritically finite outside $\phi(K(f^{p_i}|_{U_i}))$. Here ϕ is the hybrid conjugacy. Such R_i is unique up to Möbius conjugation.

2. The Julia set $J(f)$ has zero Lebesgue measure (resp. carries no invariant line fields) if and only if for each $i \in [1, n]$, the Julia set $J(f^{p_i}|_{U_i})$ has zero Lebesgue measure(resp. carries no invariant line fields).

The proof of the first statement follows from the same line as Theorem 6.9.1, the proof of the second statement is essentially the same as the proof of Theorem 6.9.2. We omit the details here.

6.9.4 Q.c-equivalence vs Möbius conjugation

Lemma 6.9.1. (Q.c-equivalence implies q.c-conjugacy) *Let (f, P) and (g, Q) be two non-parabolic rational maps, and $J(f) \neq \overline{\mathbb{C}}$. If (f, P) and (g, Q) are q.c-equivalent via a pair of q.c maps (ϕ_0, ϕ_1), then they are q.c conjugate. That is, there is a quasiconformal map ϕ, holomorphic in the Fatou set $F(f)$, such that $\phi f = g\phi$.*

Proof. By the definition of q.c-equivalence, ϕ_0 and ϕ_1 are holomorphic and identical in the union of all rotation domains R of f (if any) and a neighborhood N_A of all attracting cycles in P'_f (if any).

If f has a superattracting cycle $\langle z_0, z_1, \cdots, z_{p-1}\rangle \subset P_f - P'_f$, then we can modify ϕ_0 and ϕ_1 such that they are holomorphic and identical near these superattracting cycles. The modification is as follows:

First, notice that for any $\zeta \in \langle z_0, z_1, \cdots, z_{p-1}\rangle$, $\phi_0(\zeta)(= \phi_1(\zeta))$ is a superattracting point of g. We can choose a neighborhood U_ζ of ζ (resp. $V_{\phi_0(\zeta)}$ of $\phi_0(\zeta)$), a Böttcher coordinate $B^f_\zeta : U_\zeta \to \mathbb{D}$ (resp. $B^g_{\phi_0(\zeta)} : V_{\phi_0(\zeta)} \to \mathbb{D}$), such that the following diagram commutes:

$$\begin{tikzcd}
U_\zeta \arrow[r, "B^f_\zeta"] \arrow[d, "f"'] & \mathbb{D} \arrow[d, "z\mapsto z^{d_\zeta}"] & V_{\phi_0(\zeta)} \arrow[l, "B^g_{\phi_0(\zeta)}"'] \arrow[d, "g"] \\
U_{f(\zeta)} \arrow[r, "B^f_{f(\zeta)}"'] & \mathbb{D} & V_{\phi_0(f(\zeta))} \arrow[l, "B^g_{\phi_0(f(\zeta))}"]
\end{tikzcd}$$

where d_ζ is the local degree of f at ζ. By a suitable choice of the neighborhoods U_ζ and a suitable choice of the Böttcher coordinates, we can modify ϕ_0, ϕ_1 such that $\phi_0|_{U_\zeta} = \phi_1|_{U_\zeta} = (B^g_{\phi_0(\zeta)})^{-1} \circ B^f_\zeta$. A suitable modification elsewhere guarantees $\phi_0 f = g\phi_1$.

In this way, ϕ_0 and ϕ_1 can be made holomorphic in a neighborhood N_{SA} of all superattracting cycles in $P_f - P'_f$ (if any). Then we construct a sequence of q.c maps $\{\phi_n; n \geq 0\}$ by $\phi_n f = g\phi_{n+1}$ so that ϕ_n is isotopic to ϕ_{n+1} rel $f^{-n}(P \cup N_A \cup N_{SA})$. The sequence ϕ_n has a unique limit ϕ, which is holomorphic in $\cup_{n\geq 0} f^{-n}(R \cup N_A \cup N_{SA}) = F(f)$, as required. □

Lemma 6.9.2. *Suppose that the non-parabolic map (f, P) is q.c-equivalent to a rational map (R, Q) with $J(R) \neq \overline{\mathbb{C}}$. Then the rational realization (R, Q) is unique up to Möbius conjugation if and only if $J(R)$ carries no invariant line field.*

Proof. It's obvious that if $J(R)$ carries an invariant line field, then the rational realization (R, Q) is not unique up to Möbius conjugation. Conversely, let (R_1, Q_1) be another rational realization of (f, P). Then it follows from Lemma 6.9.1 that (R, Q) and (R_1, Q_1) are q.c conjugate via some q.c map ϕ, which is holomorphic in $F(R)$. This implies $R^*\mu_\phi = \mu_\phi$ on $J(R)$. Since $J(R)$ carries no invariant line field, $\mu_\phi = 0$ almost everywhere on $\overline{\mathbb{C}}$. This implies that ϕ is a Möbius transformation. □

Proof of the analytic part of Theorem 6.1.2. It follows from Theorem 6.9.2 Theorem 6.9.4 and Lemma 6.9.2. □

Q.c-equivalence is a special case of c-equivalence. It follows that the rigidity of c-equivalence always implies the rigidity of q.c-equivalence.On the other hand, c-equivalences between two non-parabolic rational maps without rotation domains can always be promoted to q.c-equivalences, this is because the pair of c-equivalences are holomorphic in a neighborhood of all attracting cycles. For two non-parabolic rational maps with Siegel disks, to the author's knowledge, whether the promotion works depends on the boundary regularity of the Siegel disks.

It's known from Gaofei Zhang [Zh1] that the boundary of every bounded type Siegel disk of a rational map must be a quasicircle containing at least one critical point.

In [Zh2], Zhang showed that given a rational map R with a fixed Siegel disk and postcritically finite outside this Siegel disk, then the Lebesgue measure of the Julia set $J(R)$ is zero. He told the author that his method also works for more general case:

Given a rational map R with Siegel disks, all with bounded type rotation numbers and postcritically finite outside these Siegel disks, then the Julia set $J(R)$ has zero Lebesgue measure.

Based on Zhang's Theorems and Theorems 6.9.4, we have

Theorem 6.9.5. (Rigidity) *Let (f, P) be a non-parabolic rational map with rotation domains(i.e. Siegel disk or Herman ring), and the rotation number of each rotation domain is of bounded type, then the Lebesgue measure of the Julia set $J(f)$ is zero. Thus if two such rational maps are c-equivalent, then they are conformally conjugate.*

6.10 Applications

Besides of the independent interest, Theorem 6.1.2 enables us to extend Thurston's Theorem beyond postcritically finite cases, and give characterizations of hyperbolic rational maps, and rational maps with rotation domains (Siegel disks and Herman rings).

6.10.1 Characterization of hyperbolic rational maps

Theorem 6.10.1. (Cui-Tan [CT1], Jiang-Zhang [JZ]) *Let (f, P) be a non-parabolic map without rotation domains. Then (f, P) is c-equivalent to a rational map (R, Q) if and only if (f, P) has no Thurston obstructions. The rational map (R, Q) is unique up to Möbius conjugation.*

Remark 6.10.1. *In fact, the family of non-parabolic rational maps without rotation domains is slightly larger than the family of hyperbolic rational maps. It is conjectured to be dense in the parameter space.*

Proof. The proof here in fact follows from Cui-Tan's original one [CT1]. If (f, P) is a disk covering or annular covering, then the proof is done by Theorem 6.6.5 and Theorem 6.7.2. Else, by the Decomposition Theorem, (f, P) admits finitely many decompositions $(h_k, P_k), k \in [1, n]$ along some stable multicurve Γ. Since (f, P) has no rotation domains, all of these resulting maps are Thurston maps. If (f, P) has no Thurston obstructions, then each resulting map has no Thurston obstructions and the signature of its orbifold is not $(2, 2, 2, 2)$ (Lemma 6.2.4). By Marked Thurston Theorem, all (h_k, P_k) have rational realizations, so does (f, P). The uniqueness of the rational realization of (f, P) follows from Lemma 6.9.2 the fact that any such rational realization has a Julia set of zero Lebesgue measure. □

6.10.2 Characterization of rational maps with Siegel disks

For rational maps with Siegel disks, Zhang [Zh2] proved the following:

Theorem 6.10.2. (Zhang) *Let (f, P) be a non-parabolic map, with only one rotation disk cycle which is of period one and has rotation number of bounded type, and without rotation annulus. Then (f, P) is c-equivalent to a rational map (R, Q) if and only if (f, P) has no Thurston obstructions. Moreover, the Lebesgue measure of the Julia set $J(R)$ is zero, and (R, Q) is unique up to Möbius conjugation.*

Zhang's Theorem requires that the non-parabolic map has only one rotation disk, and it is postcritically finite outside the rotation disk. It's possible to generalize Zhang's Theorem to a more general setting without the assumptions of

- the postcritical finiteness outside the rotation disks, and
- the number of rotation disk cycles.

But for compensation, we usually need a separate condition for these rotation disks.

For a non-parabolic map (f, P) with $n_A(f) > 0$, let $\mathcal{A}$ be the union of all attracting cycles. The filled Julia set K_f of f is defined by

$$K_f = \{z \in \overline{\mathbb{C}}; \limsup d(f^n(z), \mathcal{A}) > 0\},$$

where $d(\cdot, \cdot)$ is the spherical distance. K_f is a compact subset of $\overline{\mathbb{C}}$.

Theorem 6.10.3. *Let (f, P) be a non-parabolic map, with $n_A(f) > 0, n_{RD}(f) > 0$ and $n_{RA}(f) = 0$. Suppose that the rotation numbers of all rotation disk cycles are of bounded type, and all rotation disks are contained in different components of the filled Julia set K_f. Then (f, P) is c-equivalent to a rational map (R, Q) if and only if (f, P) has no Thurston obstructions. Moreover, the Lebesgue measure of the Julia set $J(R)$ is zero, and (R, Q) is unique up to Möbius conjugation.*

Proof. By the Decomposition Theorem, (f, P) admits finitely many decompositions (h_k, P_k), $k \in [1, n]$. Since different rotation disks are contained in different components of the filled Julia set K_f, each resulting Siegel map has only one rotation disk cycle, of period one and with bounded type rotation number (Thus the number of these Siegel maps is $n_{RD}(f)$). Then the conclusion follows from the Decomposition Theorem, Thurston's Theorem, Zhang's Theorem and Theorem 6.9.2. □

6.10.3 Characterization of rational maps with Herman rings

As another application, we can give a characterization of a class of rational maps with Herman rings, as follows:

Theorem 6.10.4. *Let (f, P) be a non-parabolic map, with only one rotation annulus cycle which is of period one and has rotation number of bounded type, and without rotation disks. Then (f, P) is c-equivalent to a rational map (R, Q) if and only if (f, P) has no Thurston obstructions. Moreover, the Lebesgue measure of the Julia set $J(R)$ is zero, and (R, Q) is unique up to Möbius conjugation.*

Proof. By the decomposition procedure, (f, P) admits finitely many decompositions (h_k, P_k), $k \in [1, n]$. Two are Siegel maps and the rest are Thurston maps. The theorem follows from the Decomposition Theorem, Thurston's Theorem, Zhang's Theorem and Theorem 6.9.4. □

We can further generalize Thurston's Theorem to rational maps with many Herman ring cycles but satisfying 'sperate configuration'.

Let (f, P) be a Herman map, $\mathcal{A}$ be the collection of all rotation annuli of f. For each $A \in \mathcal{A}$, we associate an analytic curve γ_A such that $\gamma_A \cap P_f = \emptyset$ and $f(\gamma_A) = \gamma_{f(A)}$. Let $\Gamma_f = \{\gamma_A; A \in \mathcal{A}\}$. We call Γ_f a (f, P)-invariant curve system.

Definition 6.10.1. (Sperate configuration) *We say that a Herman map (f, P) satisfies 'sperate configuration', if there is a (f, P)-invariant curve system Γ_f such that for any two different rotation annuli A_α and A_β, there is a curve $\gamma_{\alpha\beta} \in \cup_{k\geq 1} f^{-k}(\Gamma_f)$ that separates A_α and A_β.*

Theorem 6.10.5. *Let (f, P) be a Herman map without rotation disk cycle and the rotation numbers of all rotation annuli cycles are of bounded type. Suppose (f, P) satisfies sperate configuration. Then (f, P) is c-equivalent to a rational map (R, Q) if and only if (f, P) has no Thurston obstructions. Moreover, the Lebesgue measure of the Julia set $J(R)$ is zero, and (R, Q) is unique up to Möbius conjugation.*

Proof. By the decomposition procedure, (f, P) admits finitely many decompositions (h_k, P_k), $k \in [1, n]$. The sperate configuration implies that each Siegel map has only one rotation disk cycle, of period one and with bounded type rotation number. The theorem then follows from the Decomposition Theorem, Marked Thurston Theorem, Zhang's Theorem and Theorem 6.9.4. □

6.11 No Thurston obstructions vs rational realization

In this section, we will exhibit many examples of non-parabolic maps which have no Thurston obstructions but are not c-equivalent to rational maps. By Theorem 6.10.1, such non-parabolic map necessarily has at least one cycle of rotation domain. We first construct a Siegel map by mating two quadratic Siegel polynomials and show that it has no Thurston obstructions but is not c-equivalent to a rational map. Then we will use it to construct more such non-parabolic maps by surgery.

We begin with the definition of the mating of two quadratic polynomials. The notations here follow from [YaZ]. Let ⓒ denote the complex plane $\mathbb{C}$ compactified by adjoining a circle of directions at infinity $\{\infty \cdot e^{2\pi i t}; t \in \mathbb{R}/\mathbb{Z}\}$ with the natural topology. Each $f_i = z^2 + c_i$ extends continuously to a copy of ⓒ$_i$, acting as the squaring map $z \mapsto z^2$ on the circle at infinity. Gluing the disks ⓒ$_i$ together via the equivalence relation $\sim_\infty$ identifying the point $\infty \cdot e^{2\pi i t} \in$ ⓒ$_1$ with $\infty \cdot e^{-2\pi i t} \in$ ⓒ$_2$, we obtain a 2-sphere (ⓒ$_1 \sqcup$ ⓒ$_2$)$/ \sim_\infty$. The well-defined map $f_1 \sqcup f_2$ on this sphere given by f_i on ⓒ$_i$ is a degree 2 branched covering of the sphere with an invariant equator. We shall refer to this map as the formal mating of f_1 and f_2.

For any quadratic polynomial $z \mapsto e^{2\pi i\theta} z + z^2$, we can conjugate it to the normal form

$$f_\theta(z) = z^2 + c_\theta,\ c_\theta = \frac{e^{2\pi i\theta}}{2}\Big(1 - \frac{e^{2\pi i\theta}}{2}\Big).$$

We may assume that θ is an irrational number of bounded type. It's known that f_θ has a Siegel disk whose boundary is a quasicircle containing the critical point 0. The mating $f = f_\theta \sqcup f_{-\theta}$ is a Siegel map. Let D_θ and $D_{-\theta}$ be two rotation disks of f, where D_θ (resp. $D_{-\theta}$) inherits the complex structure of the Siegel disk of f_θ (resp. $f_{-\theta}$).

Lemma 6.11.1. *The Siegel map $(f, P) := (f_\theta \sqcup f_{-\theta}, \overline{D_\theta} \cup \overline{D_{-\theta}})$ has no Thurston obstructions but is not c-equivalent to a rational map.*

Remark 6.11.1. *One the other hand, given two irrational numbers θ_1 and θ_2 of bounded type, with $\theta_1 + \theta_2 \neq 0 \mod \mathbb{Z}$, Yampolsky and Zakeri [YaZ] proved that the mating $f_{\theta_1} \sqcup f_{\theta_2}$ is c-equivalent to a unique quadratic rational map up to Möbius conjugation.*

Proof. Let Γ be a (f, P)-stable multicurve. If Γ is non-empty, then Γ necessarily consists of one curve $\gamma \subset \overline{\mathbb{C}} - \overline{D_\theta} \cup \overline{D_{-\theta}}$. Consider the annulus A bounded by γ and the boundary of the rotation disk D_θ. The preimage $f^{-1}(A)$ is an annulus B, and $\deg(f : B \to A) = 2$. One boundary curve of B is a figure eight curve while the other boundary curve δ of B is the preimage of γ, and $\deg(f : \delta \to \gamma) = 2$. Since δ is homotopic to γ in $\overline{\mathbb{C}} - P$, we have that $\lambda(\Gamma, f) = \frac{1}{2}$. Thus (f, P) has no Thurston obstructions.

If (f, P) is c-equivalent to a rational map, say (R, Q). Then (R, Q) has three fixed points. Two fixed points are the centers of Siegel disks, with multipliers $e^{2\pi i\theta}$ and $e^{-2\pi i\theta}$. The third fixed point is necessarily a repelling fixed point, since a quadratic rational map has at most two non-repelling cycles. We denote the multiplier of the repelling fixed point by λ. Then by holomorphic index formula ([M1]),

$$\frac{1}{1 - e^{2\pi i\theta}} + \frac{1}{1 - e^{-2\pi i\theta}} + \frac{1}{1 - \lambda} = 1.$$

It follows that $\lambda = \infty$, which is a contradiction. □

In the following, we will use the Siegel map $(f_\theta \sqcup f_{-\theta}, \overline{D_\theta} \cup \overline{D_{-\theta}})$ to produce more non-parabolic maps without Thurston obstructions but not c-equivalent to rational maps.

Theorem 6.11.1. *Given nonnegative integers n_A, n_{RD}, n_{RA}, d satisfying*

$$n_A + n_{RD} + 2n_{RA} \leq 2d - 2, \ 1 \leq n_{RA} \leq d - 2, \ n_{RD} + n_{RA} \geq 2.$$

There exists a non-parabolic map (f, P) of degree d, such that

1. $n_A(f) = n_A, n_{RD}(f) = n_{RD}, n_{RA}(f) = n_{RA}$, and the rotation number of each rotation cycle is of bounded type.

2. (f, P) has no Thurston obstructions.

3. (f, P) is not c-equivalent to a rational map.

The idea of the proof of Theorem 6.11.1 is to glue some well-chosen rational maps with the Siegel map (f, P) defined as in Lemma 6.11.1. To obtain these rational maps as candidates, we need a result of Shishikura ([Sh1]):

Theorem 6.11.2. (Shishikura) *Given nonnegative integers* n_{AB}, n_{PB}, n_{SD}, n_{HR}, n_{cremer} *and* d *satisfying*

$$n_{AB} + n_{PB} + n_{SD} + 2n_{HR} + n_{cremer} \leq 2d - 2,\ n_{HR} \leq d - 2,$$

there exists a rational function f *of degree* d*, such that the numbers of attracting cycles, parabolic cycles, Siegel disk cycles, Herman ring cycles and Cremer cycles are* n_{AB}, n_{PB}, n_{SD}, n_{HR} *and* n_{cremer}*, respectively.*

Proof of Theorem 6.11.1

We first consider the case $n_{RD} \geq 1$. Then Shishikura's Theorem and quasiconformal surgery guarantee the existence of a non-parabolic rational map g with $(n_A(g), n_{RD}(g), n_{RA}(g), \deg(g)) = (n_A, n_{RD}, n_{RA} - 1, d - 1)$. Choose a Siegel disk cycle $D_0 \mapsto D_1 \mapsto \cdots \mapsto D_p = D_0$ of g. For each D_i we choose an analytic curve $\gamma_i \subset D_i$ such that $g(\gamma_i) = \gamma_{i+1}$ and the disk $\Delta_i \subset D_i$ bounded by γ_i contains at most one point in the postcritical set P_g (Notice that the only possible point in $\Delta_i \cap P_g$ is necessarily the center of the Siegel disk D_i). Let θ be the rotation number of the Siegel disks cycle.

We consider the Siegel map $(S_\theta, P_\theta) := (f_\theta \sqcup f_{-\theta}, \overline{D_\theta} \cup \overline{D_{-\theta}})$. We can view it as the composition of p maps $S_\theta = f_{p-1} \circ \cdots \circ f_0$ with $f_i : \overline{\mathbb{C}}_i \to \overline{\mathbb{C}}_{i+1}, 0 \leq 1 \leq p$ and $f_0 = S_\theta, f_1 = \cdots = f_{p-1} = id, \overline{\mathbb{C}}_p = \overline{\mathbb{C}}_0$. Let δ_0 be a S_θ-invariant curve in $D_{-\theta}$ and $\delta_i = f_i \circ \cdots \circ f_0(\delta_0)$ for $1 \leq i \leq p - 1$. We cut $\Delta_0 \cup \cdots \cup \Delta_{p-1}$ off for g and cut the disk $U_i \subset f_i \circ \cdots \circ f_0(D_{-\theta})$ bounded by δ_i for $f_i, 0 \leq i \leq p - 1$. Then we glue each $f_i|_{\overline{\mathbb{C}}_i - U_i}$ with $g|_{\overline{\mathbb{C}} - \Delta_0 \cup \cdots \cup \Delta_{p-1}}$ along the boundary curves δ_i and γ_i. We can assume that

a). The gluing procedure preserves the complex structure of the rotation domains.

b). The center of the rotation disk $f_i \circ \cdots \circ f_0(D_{-\theta})$ replace the center of the Siegel disk D_i.

In this way, we get a non-parabolic map f with $(n_A(f), n_{RD}(f), n_{RA}(f), \deg(f)) = (n_A(g), n_{RD}(g), n_{RA}(g) + 1, \deg(g) + 1) = (n_A, n_{RD}, n_{RA}, d)$, as required.

Now we consider the case $n_{RD} = 0$. In this case, $n_{RA} \geq 2$. First, it follows from Shishikura's Theorem that there is a non-parabolic rational map g with

$$(n_A(g), n_{RD}(g), n_{RA}(g), \deg(g)) = \begin{cases} (n_A, 1, n_{RA} - 2, d - 2), & \text{if } n_A = 0, 1, \\ (n_A - 1, 1, n_{RA} - 2, d - 2), & \text{if } n_A \geq 2. \end{cases}$$

Let θ be the rotation number of the Siegel disk cycle of g. We mate the Siegel map $(S_\theta, P_\theta) = (f_\theta \sqcup f_{-\theta}, \overline{D_\theta} \cup \overline{D_{-\theta}})$ with the quadratic Siegel polynomial $f_{-\theta}$

in the following way: Cut the S_θ-fixed disk $\Delta_\theta \Subset D_\theta$ off for S_θ and cut the $f_{-\theta}$-fixed disk $U_\theta \Subset D_{-\theta}$ off for $f_{-\theta}$, then we glue $S_\theta|_{\overline{\mathbb{C}}-\Delta_\theta}$ and $f_{-\theta}|_{\overline{\mathbb{C}}-U_\theta}$ along the boundary curves $\partial\Delta_\theta$ and ∂U_θ. We get a Herman map (H_θ, Q_θ). This Herman map has one fixed rotation disk with rotation number $-\theta$ and one rotation annulus with rotation number θ. Moreover, $\deg(H_\theta) = \deg(S_\theta) + \deg(f_{-\theta}) - 1 = 3$. By performing the same mating procedure as above, we can mate (g, P_g) with (H_θ, Q_θ) and obtain a non-parabolic map (f, P) with $(n_A(f), n_{RD}(f), n_{RA}(f), \deg(f)) = (n_A(g), n_{RD}(g) + n_{RD}(H_\theta) - 2, n_{RA}(g) + n_{RA}(H_\theta) + 1, \deg(g) + \deg(H_\theta) - 1) = (n_A(g), 0, n_{RA}, d)$. If $n_A = 0, 1$, then $n_A(g) = n_A$, then map (f, P) is as required; if $n_A \geq 2$, then $n_A(f) = n_A - 1$. In this case, notice that the superattracting fixed point ∞ of $f_{-\theta}$ descends to a 'superattracting' cycle of (f, P), we can change this superattracting cycle of (f, P) to be locally holomorphic and attracting whose multiplier satisfies $0 < |\lambda| < 1$ by quasiconformal surgery and get the required map.

To finish, we will show that in either case, (f, P) has no Thurston obstructions but is not c-equivalent to a rational map. By Decomposition Theorem, there is a (f, P)-stable multicurve Γ and finitely many Siegel maps or Thurston maps $(h_k, P_k), k \in [1, n]$ whose combinatorics and rational realizations dominate the original one. The construction of (f, P) guarantees that

- The (f, P)-stable multicurve Γ is in fact a stable multicurve of $g|_{\overline{\mathbb{C}}-\Delta_0\cup\cdots\cup\Delta_{p-1}}$. So it follows from Marked McMullen Theorem that $\lambda(\Gamma, f) = \lambda(\Gamma, g) < 1$.
- One of the resulting maps of $(h_k, P_k), k \in [1, n]$ is the Siegel map $(S_\theta, P_\theta) = (f_\theta \sqcup f_{-\theta}, \overline{D_\theta} \cup \overline{D_{-\theta}})$ while the rest resulting maps all have rational realizations.

Then by the Decomposition Theorem, (f, P) has no Thurston obstructions but is not c-equivalent to a rational map.

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