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A classification of postcritically finite Newton maps

by

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Abstract

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One of the most important open problems in rational dynamics is understanding the structure of the space of rational functions. Newton maps of polynomials form an interesting subset of the space of rational maps that is more accessible for studying than the full space of rational maps.

For every postcritically finite Newton map N_p of a polynomial p we construct a finite connected graph that contains the postcritical set of N_p. We show that such graphs characterize Newton maps uniquely up to Möbius conjugation. Conversely, we show that every graph with an associated map that satisfies particular conditions is realized by a unique postcritically finite Newton map.

We show that there is a mapping from the set of postcritically finite Newton maps up to Möbius equivalence to the set of abstract extended Newton graphs with the corresponding equivalence relation on them. We show that this mapping is one to one, giving thereby a combinatorial classification of postcritically finite Newton maps in terms of finite connected graphs.

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Notations

$\mathbb{C}$ — the set of complex numbers

$\mathbb{C}_f$ — the set of critical points of a topological branched covering f

$\mathbb{D}$ — the complex unit disk, $\{z \in \mathbb{C} : |z| < 1\}$

$H(M)$ — the tree generated by the set M, i.e. the allowable hull $[M]_K$ (see page 20)

H^* — the extended Hubbard tree (see page 20)

Δ — the channel diagram (see page 26)

Δ_n — the Newton graph at level n (see page 26)

$K(f)$ — the filled Julia set of a polynomial f

$J(f)$ — the Julia set of a polynomial f, $J(f) = \partial K(f)$

N_p — the Newton map of a polynomial p, i.e. $N_p(z) = z - p(z)/p'(z)$

P_f — the postcritical set of a topological branched covering f

$P_f(D)$ — the set of postcritical points of f in D, i.e. $P_f(D) = P_f \cap D$

$\mathbb{S}^1$ — the unit circe, $\mathbb{S}^1 = \partial\mathbb{D}$

$[X]_K$ — the allowable hull of $X \subset K(f)$ (see page 20)

$\simeq_X$ — the isotopy relation relative to the set X (see page 15)

$[1, n]$ — the set of integers $1, 2, \ldots, n$

Chapter 1

Introduction

1.1 Overview

A rational function of one complex variable (i.e. a ratio of two polynomials) is among the simplest and most basic objects in algebra. However, an extremely rich and complicated structure is revealed when one starts to iterate a rational function, i.e. consider it from the point of view of dynamical systems. A general dynamical theory of rational functions is currently at a very early stage of development. However, some particular parameter families are rather well understood. A well-known example is the family of polynomials. The combinatorial structure of it is much simpler than for general rational functions, although there are still many open questions. Much less is known however about dynamics of rational functions.

At this point, a general combinatorial theory of all rational functions is beyond immediate reach. In this thesis we will confine our study of the parameter space of Newton maps. The study of Newton maps is very naturally motivated and has practical impact; Newton maps have thus become some of the most important classes of iterated rational maps. There are many long-standing open questions regarding this particular family, notably the question of Smale on a classification of periodic basins posed some 25 years ago [Sm85]. Newton maps form a large family of rational functions. They have some rather specific combinatorial and dynamical properties that are of substantial help in the study. The parameter space we consider has long served as motivations and driving force but continue to be mysteries.

In this thesis we construct a mapping from the set of postcritically finite Newton maps of a given degree up to Möbius equivalence to the set of abstract extended Newton graphs with a corresponding equivalence relation on them. We show that this map is one to one, giving thereby a classification of postcritically finite Newton maps in terms of finite graphs.

Newton maps of degree 1 and 2 are trivial, and we exclude these cases from our investigation. Let us make precise what we mean by a Newton map.

A rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 3$ is called a *Newton map* if ∞ is a repelling fixed point of f and for each fixed point $\xi \in \mathbb{C}$, there exists an integer $m \geq 1$ such that $f'(\xi) = (m-1)/m$.

This definition is motivated by the observation which goes back to [He, Proposition 2.1.2]), which is a special case of [RS, Proposition 2.8] (the case of superattracting fixed points, i.e. every $m = 1$): a rational map f of degree $d \geq 3$ is a Newton map if and only if there exists a polynomial $p : \mathbb{C} \to \mathbb{C}$ such that for $z \in \mathbb{C}$, $f(z) = z - p(z)/p'(z)$.

Observe that f and p have the same degree d if and only if p has d distinct roots. Note also that some rational maps arise as Newton maps of entire functions, for example for $h = pe^q$ where p, q are complex polynomials, the map N_h is a rational function. For an overview of the dynamics of Newton maps for entire transcendental functions see, for example, [BR, MS, RS].

In [MR] the classification of all postcritically fixed Newton maps was given. A Newton map is called postcritically fixed if all its critical points are mapped onto fixed points after finitely many iterations. In this thesis we extend the results in [MR] beyond the postcritically fixed case and allow critical points have just finite orbits, not necessarily landing at one of the fixed points eventually. Such Newton maps are called postcritically finite.

If f is a postcritically finite Newton map, then, similarly to [MR], we construct the channel diagram Δ of f, which is the union of the accesses from finite fixed points of f to ∞ (see Section 2.6) and Δ_n the Newton graph of f which is a connected preimage component of $f^{-n}(\Delta)$ containing ∞. For each free critical point of f which never lands on the Newton graph we construct extended Hubbard trees containing the forward orbit of the free critical point. The extended Hubbard trees are connected to the Newton graph via Newton rays (see Section 2.6) that are preimages of edges of the Newton graph landing at repelling periodic points on the Hubbard trees. The *extended Newton graph* (see Section 2.6) is a Newton graph together with the union of extended Hubbard trees constructed for each free critical point of f and Newton rays connecting the Newton graph with extended Hubbard trees.

In this way for every postcritically finite Newton map f we construct a finite graph containing the whole postcritical set, similarly to the Hubbard tree of a postcritically finite polynomial.

We introduce the notion of an *abstract extended Newton graph*, which is a pair (Σ, f) of a map f acting on a graph Σ that satisfies certain conditions (see Definition 4.0.8). In particular, the conditions on (Σ, f) allow f to be extended to the branched covering $\overline{f}$ of the whole sphere $\mathcal{S}^2$, such that $\overline{f}$ is injective when restricted to each component of $\mathcal{S}^2 \setminus \Sigma$.

We show that for every *abstract extended Newton graph* there exists a postcritically finite Newton map realizing it. Moreover, this Newton map is unique up to affine conjugacy.

The assignments of a Newton map to an abstract extended Newton graph and vice versa are injective and inverse to each other on equivalence classes of Newton maps and abstract extended Newton graphs, so we give a combinatorial classification of postcritically finite Newton maps by way of abstract extended Newton graphs. Our main results are the following (see Chapters 4 and 5 and for the precise definitions).

Theorem 1.1.1 (Newton Maps Generate Extended Newton Graphs)**.** *For every postcritically finite Newton map N_p there exists an extended Newton graph Δ_N^* so that (Δ_N^*, N_p) is an abstract extended Newton graph.*

Such graphs distinguish postcritically finite Newton maps, i.e. if (Δ_{1N}^*, N_{p_1}) and (Δ_{2N}^*, N_{p_2}) are Thurston equivalent abstract extended Newton graphs associated to Newton maps N_{p_1} and N_{p_2}, then the Newton maps N_{p_1} and N_{p_2} are affine conjugate.

Theorem 1.1.2 (Abstract Extended Newton Graphs Are Realised)**.** *Every abstract extended Newton graph is realized by a postcritically finite Newton map. This Newton map is unique up to affine conjugacy. More precisely, let (Σ, f) be an abstract extended Newton graph. Then, there exists a postcritically finite Newton map N_p with extended Newton graph Δ_N^* such that $(\overline{f}, \Sigma')$ and $(N_p, (\Delta_N^*)')$ are Thurston equivalent as marked branched coverings.*

Moreover, if N_p realizes two abstract extended Newton graphs (Σ_1, f_1) and (Σ_2, f_2), then the two abstract extended Newton graphs are Thurston equivalent.

Denote by $\mathcal{N}$ the set of postcritically finite Newton maps with the equivalence relation $\sim_{\mathcal{N}}$ defined by the affine conjugacy. In other words, $N_{p_1} \sim_{\mathcal{N}} N_{p_2}$ if N_{p_1} and N_{p_2} are affine conjugate. By $\mathcal{G}$ we denote the set of abstract extended Newton graphs with the equivalence relation $\sim_{\mathcal{G}}$ defined by Thurston equivalence (the precise

Definition 4.0.9 is given in Chapter 4). We say that $(\Sigma_1, f_1) \sim_{\mathcal{G}} (\Sigma_2, f_2)$ if (Σ_1, f_1) and (Σ_2, f_2) are Thurston equivalent. It follows from Theorem 1.1.1 and Theorem 1.1.2 that there exist well defined injective mappings $\mathcal{F} : \mathcal{N} \to \mathcal{G}$ and $\mathcal{F}' : \mathcal{G} \to \mathcal{N}$.

Theorem 1.1.3 (Bijective Correspondence)**.** *The mappings $\mathcal{F}$ and $\mathcal{F}'$ are bijective and inverse to each other, i.e. $\mathcal{F} \circ \mathcal{F}' = Id$ and $\mathcal{F}' \circ \mathcal{F} = Id$.*

This thesis is structured as follows. The introduction to the thesis is given in Chapter 1. In Chapter 2, we introduce the necessary terminology and notations. The dynamical properties of Newton maps are discussed in Section 2.1. In Sections 2.2 and 2.3 we review some aspects of Thurston's theory and give an introduction to the combinatorics of arc systems and possibilities of their intersections with Thurston obstructions. The notions of a Hubbard tree, extended Hubbard tree, their abstract analogs and other related results are given in Section 2.4. In Section 2.5 we review the results on extending maps on finite graphs to spheres that will be used later. We introduce the notions of a channel diagram, Newton graph and their abstract counterparts in Section 2.6. Chapter 2 ends with the review of the polynomial-like theory in Section 2.7.

Chapter 3 deals with the construction of an extended Newton graph for a given postcritically finite Newton map. First, in Section 3.1 the renormalization domains for Newton maps are constructed. Newton rays connecting Newton graph with fixed points of the polynomial-like mappings arising from renormalization domains are defined in Section 3.2. In Section 3.3 two examples with the detailed construction of periodic Newton rays are given. Chapter 3 ends with the construction of extended Newton graphs in Section 3.4.

In Chapter 4 the abstract analog of extended Newton graphs is given.

The main results of the thesis (Theorems 1.1.1, 1.1.2 and 1.1.3) are proven in Chapter 5.

Chapter 6 is devoted to the overview of possible extensions of results presented in the thesis.

1.2 Newton's method: root finding method and a dynamical system

Newton's method is perhaps the best known iterative method used to locate the roots of polynomials. If p is a polynomial and $N_p(z) = z - p(z)/p'(z)$ its Newton map, then starting with an initial guess x_1, one calculates the root $x_2 = N_p(x_1) = x_1 - \frac{p(x_1)}{p'(x_1)}$ of the linear map tangent to f at x_1. This tangent linear function approximates p well near x_1, and it is reasonable to assume that x_2 will be a better approximation of a root ξ than x_1. Indeed, it is a well know fact that if x_1 was sufficiently close to some root ξ of p, then the sequence obtained iteratively by $x_{n+1} = N_p(x_n) = x_n - \frac{p(x_n)}{p'(x_n)}$ converges to ξ. If it happens, we say that the starting value x_1 *finds* the root ξ. Newton's method is an extremely powerful technique — in general the convergence is quadratic: if ξ is a simple root, then $|x_{n+1} - \xi| = O\left(|x_n - \alpha|\right)^2$. Thus near a root, the Newton's method is a very efficient way to obtain approximations to the root.

While the local dynamical properties of Newton's method are well understood, the global dynamical properties is rather a difficult question. An example of possible difficulties is the following: in many cases there exist open sets of starting values for Newton's method that do not find any roots. Consider $p(x) = x^3 - 2x^2 + 2$. The Newton map $N_p(x) = x - (x^3 - 2x^2 + 2)/(3x^2 - 2)$ has a periodic cycle $0 \to 1 \to 0$ so that the sequence $\{(x_n)\}_{n\in\mathbb{N}}$ with $x_1 = 0$ and $x_{n+1} = N_p(x_n)$ will not converge. This example of a Newton's method has a periodic critical point that doesn't find a root of p (Figure 1.1).

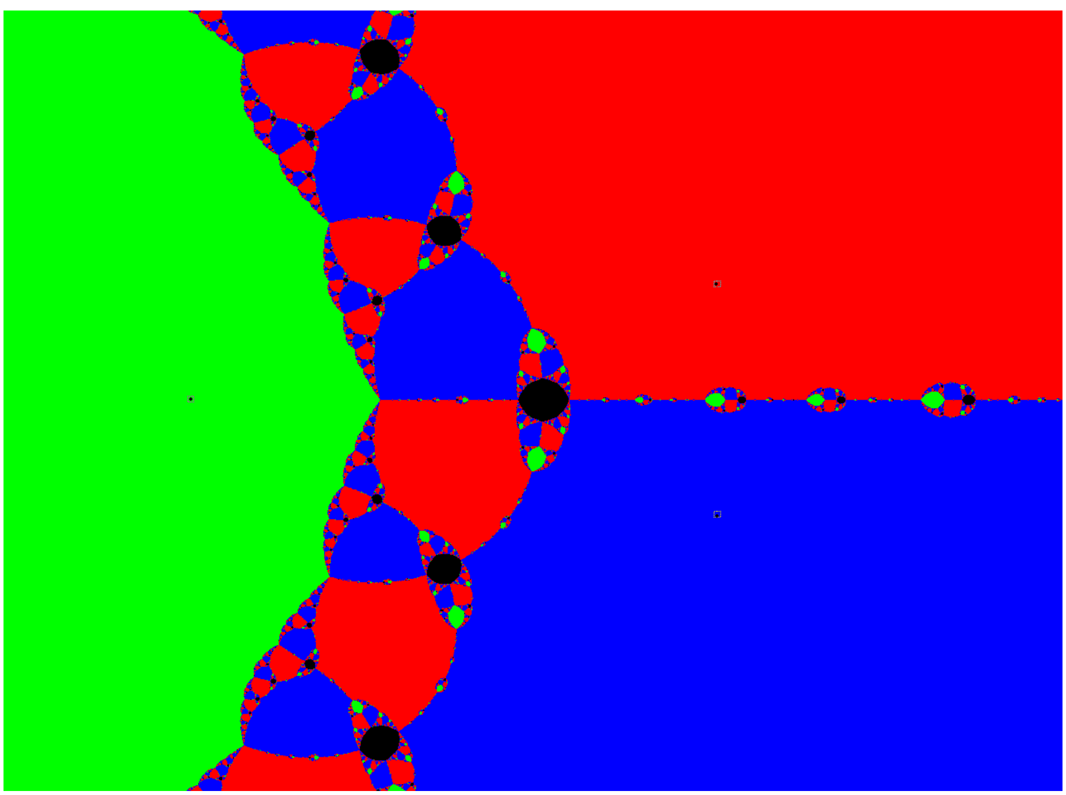

Figure 1.1: Newton map of the polynomial $p(z) = z^3 - 2z + 2$ has a superattracting cycle of period 2: $0 \to 1 \to 0$.

In the following we review known results on Newton's method in both directions: numerical aspects of the Newton's method as a root finding method and properties of Newton's method as a dynamical system.

The properties of a Newton's method as a root finding algorithm were investigated by Smale in [Sm85]. He studied local properties of Newton maps and asked for a classification of all polynomial Newton maps that have periodic attracting cycles.

In 1992 Manning has shown how to find at least one root of a polynomial using Newton's method: he constructed an explicit set S_d, $d \geq 10$, of starting points that would find at least one root of any properly normalized polynomial of degree d.

This result was extended in 2001 by Hubbard, Schleicher and Sutherland [HSS]. They showed that, given a degree d, there exists an explicit set S_d, $d \geq 2$, of starting points so that each complex polynomial p, normalized so that all its roots are contained in the unit disk, satisfies the property that for every root of p there exists at least one point in S_d that converges to this root under the Newton iteration. This set S_d consists of $1.1d(\log d)^2$ points.

In 2002, Schleicher [Sch02] gave an upper bound on the number of Newton iterations necessary for points in S_d to approach roots within a distance of $\varepsilon > 0$. The upper bound in terms of d and ε, given in [Sch02], grows exponentially in d. The imporvement for this upper bound from exponential in d to polynomial of low degree was announced by Schleicher [Sch08].

A number of people have studied Newton maps and used combinatorial models to structure the parameter spaces of Newton maps. Let d be the degree of a Newton map f. If $d = 2$, the dynamics of Newton maps is very easy: there is only one quadratic Newton map up to Möbius conjugation and the space of such maps reduces to a point (this was mentioned in perhaps one of the first papers on Newton's method as a dynamical system by Cayley in 1876, see [PSH]). For $d = 3$ the dynamics of the Newton maps is already very complicated. In 1987 Janet Head [He] introduced the *Newton tree* to characterize postcritically finite cubic Newton maps. Tan Lei [TL] built in 1997 upon the thesis of Janet Head's work and gave a classification of postcritically finite cubic Newton maps in terms of matings and captures. She constructed an isomorphism between the space of cubic Newton maps and $\mathbb{C}$. The parameter plane of cubic Newton maps is shown on Figure 1.2. Tan Lei gave also another combinatorial classification of the Newton cubic family by abstract graphs. More presicely, every postcritically finite cubic Newton map gives rise to a forward invariant finite connected graph, which contains the orbit of the critical points. And converesely, every abstract graph which satisfies certain properties is realized, i.e. there exists a unique postcritically finite cubic Newton map whose graph is homeomorphic to the given one. In [TL] Tan Lei describes exactly when two graphs are realized by the same Newton map.

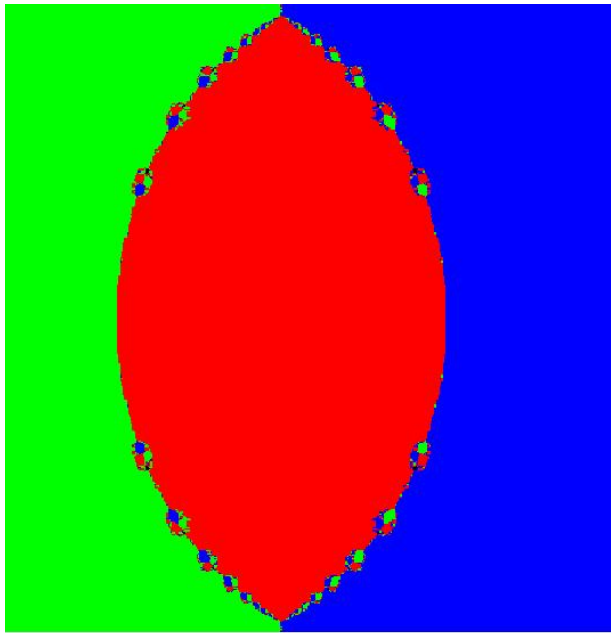

Figure 1.2: The parameter space of cubic Newton maps up to Möbius conjugation. The points are colored depending on the fixed point the free critical value converges to. Every component of the red, green, blue color is a hyperbolic component which contains the unique post-critically fixed cubic Newton map. The black regions are little Mandelbrot set where the free critical value converges to a non-trivial attracting cycle.

For the case $d > 3$ not much is currently known. Jiaqi Luo [Lu] in his thesis extended some of the results of in [TL] to "unicritical" Newton maps, i.e. Newton maps of arbitrary degree with only one *free* (non-fixed) critical value. For such maps, Luo constructs a forward-invariant, finite topological graph, the so-called *Newton graph* that contains the orbit of critical values of the Netwon map under consideration. Following the work of Tan Lei [TL], Luo studies the branched covers which are topological models of "unicritical" Newton maps and calles them *topological Newton maps*. Luo proves that if the unique free critical value is either periodic or eventually lands on a fixed critical point, then the topological Newton map is Thurston equivalent to a Newton map.

In 2006 Johannes Rückert [Rü] gave the classification of all *postcritically fixed* Newton maps for arbitrary values of degree d (see also the paper [MR] reworked by the author). A Newton map is called *postcritically fixed* if all its critical points are mapped onto fixed points after finitely many iterations. The work in [MR] can be seen as an extension of the results of Luo [Lu] beyond the setting of a single free critical value. The main differences to this setting are that the channel diagram is in general not a tree anymore and that in the presence of more than one non-fixed critical value, the iterated preimages of the channel diagram may be disconnected. In [MR] for every postcritically fixed Newton map a connected forward-invariant finite graph that contains the whole postcritical set of the Newton map is constructed. The notion of an abstract Newton graph is introduced afterwards. Finally it is shown that abstract Newton graphs materialize and can be realized uniquely by postcritically fixed Newton maps.

In this thesis we extend the results of [MR] beyond the postcritically fixed case and allow critical points have just finite orbits, not necessarily landing at one of the fixed points eventually. The purpose of this thesis is to construct a map from the set of postcritically finite Newton maps of a given degree up to Möbius equivalence to the set of abstract extended Newton graphs with a corresponding equivalence relation. We show that this map is one to one, giving thereby a classification of postcritically finite Newton maps in terms of finite connected graphs.

Chapter 2

Preliminaries

2.1 Dynamics of Newton maps

Besides their application to root-finding, Newton maps form a class of functions that is interesting to study in its own right. From one hand, the space of Newton maps of polynomials forms a large enough and interesting sub-class of rational functions. On the other hand, it seems to have enough structure to make a classification possible. A classification of all Newton maps will suggest general methods and conjectures covering larger classes of rational functions. Hence a classification of Newton maps might provide an important intermediate step towards the major goal of the whole complex dynamics: a classification of all rational functions.

Let us remind the definition of a Newton map from Section 1.1.

Definition 2.1.1 (Newton Map). *A rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 3$ is called a* Newton map *if ∞ is a repelling fixed point of f and for each fixed point $\xi \in \mathbb{C}$, there exists an integer $m \geq 1$ such that $f'(\xi) = (m-1)/m$.*

This definition is motivated by the following observation, which is a special case of [RS, Proposition 2.8] (the case of superattracting fixed points, i.e. every $m = 1$, goes back to [He, Proposition 2.1.2]).

Proposition 2.1.2 (Head's Theorem). *A rational map f of degree $d \geq 3$ is a Newton map if and only if there exists a polynomial $p : \mathbb{C} \to \mathbb{C}$ such that for every $z \in \mathbb{C}$, $f(z) = z - p(z)/p'(z)$.*

Let $p(z) = (z-a_1)^{m_1}(z-a_2)^{m_2} \dots (z-a_k)^{m_k}$ be a monic polynomial of degree

d with complex coefficients. It is easy to check that the map

$$N_p(z) = z - \frac{p(z)}{p'(z)} \tag{2.1}$$

satisfies the conditions of Definition 2.1.1 and N_p is a Newton map. Simple calculations show that a_i is an attracting fixed point of N_p with the multiplier $\frac{m_i-1}{m_i}$ and ∞ is the only non-attracting fixed point of N_p. Therefore, each fixed point $\xi \in \mathbb{C}$ of N_p has a neighborhood U_ξ such that for every $z \in U_\xi$, the sequence $N_p^{\circ k}(z)$ converges to ξ as $k \to \infty$. If ξ is a simple root of p, i.e. if $m_i = 1$, where $\xi = a_i$, then Böttcher's theorem [Mi2, Theorem 9.1] implies that this convergence is at least quadratic. If $\xi = a_i$ is a multiple root, i.e. if $m_i > 1$, then Koenig's theorem [Mi2, Theorem 8.2] implies the linear convergence. If N_p has an attracting cycle of period higher than one, the basin of attraction of this cycle is an open set of starting values for Newton's method that do not find any roots of p. The existence of such cycles explains the nature of black regions on Figure 2.1.

Shishikura [Sh] proved that the Julia set of a rational map is connected if there is only one repelling fixed point. Hence the result of Shishikura implies the following important dynamical property.

Proposition 2.1.3 (Julia sets of Newton maps are connected)**.** *The Julia set $J(N_p)$ of a Newton map N_p is connected.*

We summarize now a few basic facts about Newton maps:

- It follows from (2.1) that the roots of $p(z)$ correspond to the finite superattracting fixed points of $N_p(z)$.
- The point at infinity is a fixed point of N_p and $N_p'(\infty) = d/(d-1)$, hence ∞ is a repelling fixed point of N_p.
- The derivative of N_p is
$$N_p'(z) = \frac{p(z)p''(z)}{(p'(z))^2}.$$
Therefore the simple roots of p are superattracting fixed points of N_p. The rate of attraction of the Newton method in a neighborhood of simple roots of p is at least quadratic.
- Multiple roots of p are attracting fixed points of N_p. If a_i has the multiplicity m_i as a root of p, then $N_p'(a_i) = (m_i - 1)/m_i$. The rate of attraction in a neighborhood of multiple roots is linear.

- If p is a polynomial of degree d, then N_p is a rational function of degree at most d. When p has multiple roots, the degree of N_p is strictly less than d.

Let $f : \mathcal{S}^2 \to \mathcal{S}^2$ be an orientation-preserving branched covering. A critical point of f is a point z, where the local degree $\deg_z f$ is greater than 1. For a Newton map f it means that $z \in \mathbb{C}$ and $f'(z) = 0$, since ∞ is never a critical point for a Newton map.

Definition 2.1.4. *Set $C_f = \{$critical points of $f\} = \{x | \deg_x f > 1\}$ and*

$$P_f = \bigcup_{n \geq 1} f^n(C_f).$$

The map f is called a postcritically finite branched covering *if P_f is finite. We say that f is* postcritically fixed *if there exists $N \in \mathbb{N}$ such that for each $x \in C_f$, $f^{\circ N}(x)$ is a fixed point of f.*

For a domain $D \subset \mathbb{C}$ denote $P_f(D) = P_f \cap D$.

Definition 2.1.5 (Immediate Basin)**.** *Let N_p be a Newton map and $\xi \in \mathbb{C}$ a fixed point of N_p. Let $B_\xi = \{z \in \mathbb{C} : \lim_{n\to\infty} N_p^n(z) = \xi\}$ be the basin (of attraction) of ξ. The connected component of B_ξ containing ξ is called the* immediate basin *of ξ and denoted U_ξ.*

In 1989 Przytycki [Pr] showed that U_ξ is simply connected and unbounded. This result was strengthened by Shishikura [Sh]: he showed that every component of the Fatou set is simply connected, not just immediate basins.

Definition 2.1.6 (Invariant access to ∞)**.** *Let ξ be an attracting fixed point of N_p and U_ξ its immediate basin. An* access *of ξ to ∞ is a homotopy class of curves within U_ξ that begin at ξ, land at ∞ and are homotopic with fixed endpoints.*

Another important dynamical property of Newton maps is that the mutual location of immediate basins is determined by the number of critical points in immediate basins. This will provide us with the first-level combinatorial data for Newton maps.

Proposition 2.1.7 (Accesses to infinity in immediate basins)**.** *[HSS]. Let m_ξ be the number of critical points of a Newton map N_p in the immediate basin U_ξ, counted with multiplicity. Then $N_p|_{U_\xi}$ is a covering map of degree $m_\xi + 1$, and U_ξ has exactly m_ξ accesses to ∞.*

Let us consider a disk $\mathbb{D}$ which contains all the fixed points of a Newton map N_p. A *channel of a fixed point* ξ is an unbounded component of $U_\xi \setminus \mathbb{D}$ [HSS]. If for a Newton map N_p there exists an immediate basin U_ξ which has at least two accesses to ∞ we say that N_p has *multiple channels* and call such a basin U_ξ *multiple*. Otherwise we say that N_p has no multiple channels.

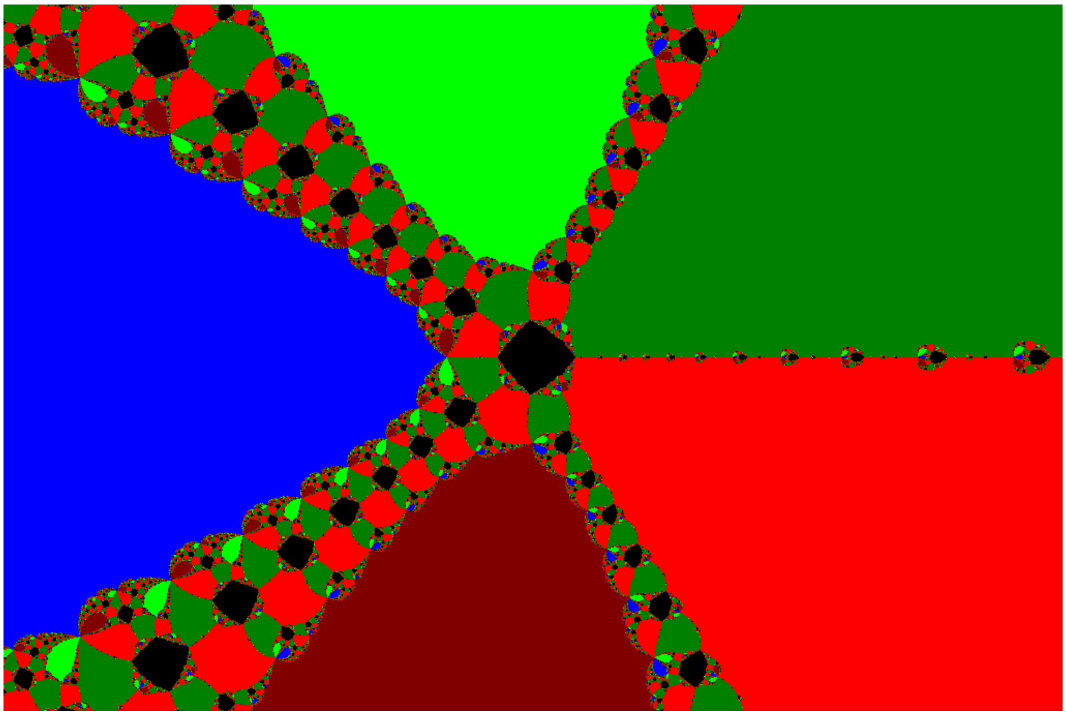

Figure 2.1: The Newton map of degree 5 for the polynomial $p(z) = z^5 - 4z + 4$ with the superattracting 2-cycle $0 \mapsto 1 \mapsto 0$. Colors indicate to which of the five roots of $p(z)$ a given starting point converges; black indicate starting points converging to no root but to the superattracting cycle $0 \mapsto 1 \mapsto 0$ instead.

Lemma 2.1.8. *Let N_p be a postcritically finite Newton map, $\xi \in \mathbb{C}$ a fixed point of N_p and U_ξ the immediate basin of ξ. Then ξ is a superattracting fixed point of N_p and there is no critical point in U_ξ except ξ.*

Proof. Since ξ is a fixed point of N_p, ξ is a critical point of N_p. Let k be the multiplicity of ξ as a critical point of N_p and suppose there exists a critical point in U_ξ. It follows from [Mi2, Theorem 9.3] that there exists a maximal number $0 < r < 1$ so that the local Böttcher map ψ_ε near ξ extends to a conformal isomorphism ψ from the open disk $\mathbb{D}_r$ centered at zero of radius r onto an open subset $W = \psi(\mathbb{D}_r) \subset U_\xi$ such that $\overline{W}$ is compactly contained in U_ξ and the boundary $\partial W \subset U_\xi$ contains at least one critical point c. Since the sequence $\{N_p^i(c)\}_{i=0}^{\infty}$ converges to ξ and this sequence is by assumption finite, there exists a positive integer n such that

$N_p^{\circ n}(c) = \xi$. On the other hand

$$\psi(N_p(z)) = \psi(z)^k \quad \text{for every} \quad z \in W \tag{2.2}$$

and if we take a sequence $z_m \in W$ with $\lim_{m\to\infty} z_m = c$ we get a contradiction taking limit $m \to \infty$ in (2.2). $\square$

2.2 Thurston's theory on branched coverings

Definition 2.2.1. *A* marked branched covering *is a pair (f, X), where $f : \mathcal{S}^2 \to \mathcal{S}^2$ is a branched covering and X is a finite set containing P_f such that $f(X) \subset X$.*

Definition 2.2.2 (Thurston Equivalence)**.** *Let (f, X) and (g, Y) be two marked branched coverings. We say that they are* Thurston equivalent *if there are two homeomorphisms ϕ_0, $\phi_1 : \mathcal{S}^2 \to \mathcal{S}^2$ such that*

$$\phi_0 \circ f = g \circ \phi_1$$

and there exists an isotopy $\Phi : [0,1] \times \mathcal{S}^2 \to \mathcal{S}^2$ with $\Phi(0,\cdot) = \phi_0$ and $\Phi(1,\cdot) = \phi_1$ such that $\Phi(t,\cdot)|_X$ is constant in $t \in [0,1]$ with $\Phi(t, X) = Y$.

Let γ be a simple closed curve. We will call γ *essential* if both components of the complement $\mathcal{S}^2 \setminus \gamma$ contain at least two points of X. We say that γ is a simple closed curve in $(\mathcal{S}^2, X)$ if $\gamma \subset \mathcal{S}^2 \setminus X$.

Definition 2.2.3 (Isotopic curves)**.** *Let γ_0, γ_1 be two simple closed curves in $(\mathcal{S}^2, X)$. We say that γ_0 and γ_1 are* isotopic relative to X*, written $\gamma_0 \simeq_X \gamma_1$, if there exists a continuous, one-parameter family γ_t, $t \in [0,1]$, of simple closed curves joining γ_1 and γ_2. We denote the isotopy class of a simple closed curve γ by $[\gamma]$.*

Definition 2.2.4 (Multicurve)**.** *By a multicurve $\Pi = \{\gamma_1, \ldots, \gamma_n\}$ we denote a collection of mutually disjoint and pairwise non-isotopic essential simple closed curves in $(\mathcal{S}^2, X)$. A multicurve Π is f-stable if for every $\gamma \in \Pi$ every essential connected component of $f^{-1}(\gamma)$ is isotopic relative to X to some element of* Π.

We regard elements of Π as the basis vectors of $\mathbb{R}^\Pi$. For every f-stable multicurve Π we can consider the corresponding Thurston linear transform (matrix) $f_\Pi : \mathbb{R}^\Pi \to \mathbb{R}^\Pi$ as follows: define

$$f_\Pi(\gamma_j) = \sum_{\gamma' \subset f^{-1}(\gamma_j)} \frac{1}{\deg(f|_{\gamma'} : \gamma' \to \gamma_j)} [\gamma']_\Pi,$$

where $[\gamma']_\Pi$ denotes the element of Π isotopic to γ', if it exists. If there are no such elements, the sum is taken to be zero. It is easy to see that the transformation f_Π depends only on the isotopy classes of curves relative to X and the transformation f_Π can be iterated.

Since any Thurston matrix has real non-negative elements, its largest eigenvalue is real non-negative, and there exists an eigenvector with real non-negative entries corresponding to this eigenvalue by the Perron-Frobenius theorem [Gan, Chapter XIII]. Denote the largest eigenvalue of Π by λ_Π.

From now on we assume that Π is a stable multicurve. A multicurve Π is called a *(Thurston) obstruction* if $\lambda_\Pi \geq 1$. A square matrix $A \in M_n(\mathbb{R})$ is called *irreducible* if for every pair (i, j), $i, j \in [1, n]$ there exists an integer $k > 0$ such that $(A^k)_{i,j} > 0$.

A multicurve Π is said to be *irreducible* if the matrix representing the linear transform f_Π is irreducible. In other words for any (i, j) there is an integer k and a component γ' in $f^{-k}(\gamma_j)$ isotopic to γ_i relative to X.

An irreducible multicurve Π is called an *irreducible (Thurston) obstruction* if $\lambda_\Pi \geq 1$.

Definition 2.2.5 (Hyperbolic Orbifold)**.** *Let $f : \mathcal{S}^2 \to \mathcal{S}^2$ be a marked branched covering of degree $d > 1$ with the postcritical set P_f. The orbifold O_f is a pair (f, v_f), where $v_f : P_f \to \mathbb{N} \cup \{\infty\}$ is the smallest function such that $v_f(x)$ is a multiple of $v_f(y) \deg_y f$ for each $y \in f^{-1}(\{x\})$.*

The orbifold O_f is called hyperbolic *if the Euler charactersitic*

$$\chi(O_f) = 2 - \sum_{x \in P_f} \left(1 - \frac{1}{v_f(x)}\right)$$

is negative.

Remark 2.2.6. In most cases O_f is hyperbolic. There are few exceptions which can be easily studied [DH]. If f has at least three fixed branched points, then it will have hyperbolic orbifold. For Newton maps we consider in this thesis it is always the case. In general, $\#P_f \geq 5$ suffices to make the orbifold of f hyperbolic.

Theorem 2.2.7 (Thurston's theorem)**.** *A marked branched covering (f, X) with hyperbolic orbifold is Thurston equivalent to a marked rational map (R, Y) if and only if (f, X) has no Thurston obstruction, i.e. if $\lambda_\Pi < 1$ for every f-stable multicurve Π. In this case the rational map R is unique up to a conjugation by a Möbius transformation.*

A typical application of Thurston's theorem works as follows: one gives a combinatorial classification of all maps under consideration (for example, all postcritically finite polynomials of a given degree), uses the combinatorial data to build a topological branched cover which is specified by the combinatorics uniquely up to Thurston equivalence, then proves that there is no Thurston obstruction so there is a unique rational map (up to conformal conjugation) realizing this combinatorics. Conversely, every such rational map is associated to some set of combinatorial data.

Since we are interested in a combinatorial classification of a particular set of rational maps, namely Newton maps, Thurston's theorem is a very useful and powerful tool for us. There is a particular type of Thurston obstructions called Levy cycles that is usually much easier to detect combinatorially.

Definition 2.2.8 (Levy cycle)**.** *A multicurve* $\Pi = \{\gamma_1, \ldots, \gamma_n\}$ *is a* Levy cycle *if for* $i = 1, 2, \ldots, n$ *the curve* $\gamma_{i-1} \simeq_X \gamma_i'$ $(\gamma_0 = \gamma_n)$*, where* γ_i' *is a component of* $f^{-1}(\gamma_i)$*, and* $f : \gamma_i' \to \gamma_i$ *is a homeomorphism.*

One important method which often helps to analyse a Thurston obstruction or a Levy cycle is to look at dynamics of connected components of $\mathcal{S}^2 \setminus \Pi$ under f^{-1}. An important specific case is given in the following.

Definition 2.2.9 (Degenerate Levy cycles)**.** *Let* $\Pi = \{\gamma_1, \ldots, \gamma_n\}$ *be a Levy cycle of a marked branched covering* (f, X)*. We say that* Π *is a* degenerate Levy cycle *if for every* $i = 1, 2 \ldots, n$ *there exists one of the disk components* B_i *of* $S^2 \setminus \gamma_i$ *such that each* $f^{-1}(B_{i+1})$ *has a component* B_i' *isotopic to* B_i *relative to* X *and* $f : B_i' \to B_{i+1}$ *is of degree one for* $i = 1, 2, \ldots, n$ *with* $B_{n+1} = B_1$*.*

2.3 Arcs intersecting obstructions

We present a theorem of Kevin Pilgrim and Tan Lei [PT] that is useful to show that certain marked branched coverings are equivalent to rational maps. Again, we first need to introduce some notation.

Let (f, X) be a marked branched covering of degree $d \geq 3$.

Definition 2.3.1 (Arc System)**.** *An* arc *in* $(\mathcal{S}^2, X)$ *is a map* $\alpha : [0,1] \to \mathcal{S}^2$ *such that* $\alpha(\{0,1\}) \subset X$, $\alpha((0,1)) \cap X = \emptyset$, *$\alpha$ is a continuous mapping, injective on* $(0,1)$*. The notion of* isotopy relative to X *extends to arcs and is also denoted by* $\simeq$*.*

A set of pairwise non-isotopic arcs in $(\mathcal{S}^2, X)$ *is called an* arc system*. Two arc systems* Λ, Λ' *are* isotopic *if each curve in* Λ *is isotopic relative to* X *to a unique element of* Λ' *and vice versa.*

Unless otherwise stated, by $\simeq$ we always assume the isotopy relative to X and omit the sub-index X in $\simeq_X$.

Note that arcs connect marked points (the endpoints of an arc need not be distinct) while simple closed curves run around them. We will see that this leads to intersection properties that will give us some control over the location of possible Thurston obstructions. Since arcs and curves are only defined up to isotopy, we make precise what we mean by arcs and curves intersecting.

Definition 2.3.2 (Intersection Number)**.** *Let α and β each be an arc or a simple closed curve in $(\mathcal{S}^2, X)$. Their* intersection number *is*

$$\alpha \cdot \beta := \min_{\alpha' \simeq \alpha,\, \beta' \simeq \beta} \#\{(\alpha' \cap \beta') \setminus X\} \ .$$

The intersection number extends bilinearly to arc systems and multicurves.

If λ is an arc in $(\mathcal{S}^2, X)$, then the closure of a component of $f^{-1}(\lambda \setminus X)$ is called a *lift* of λ. Each arc clearly has d distinct lifts. If Λ is an arc system, an arc system $\tilde{\Lambda}$ is called a *lift* of Λ if each $\tilde{\lambda} \in \tilde{\Lambda}$ is a lift of some $\lambda \in \Lambda$.

If Λ is an arc system, we introduce a linear map f_Λ on the real vector space $\mathbb{R}^\Lambda$ similar as for multicurves: for $\lambda \in \Lambda$, set

$$f_\Lambda(\lambda) := \sum_{\lambda' \subset f^{-1}(\lambda)} [\lambda'] \ ,$$

where $[\lambda']$ denotes the isotopy class of λ' relative X. Again, the sum is taken to be zero if λ has no preimages in the isotopy class of λ'. We say that Λ is *irreducible* if the matrix representing f_Λ is.

Denote by $\tilde{\Lambda}(f^{\circ n})$ the union of those components of $f^{-n}(\Lambda)$ that are isotopic to elements of Λ relative X, and define $\tilde{\Pi}(f^{\circ n})$ in an analogous way. Note that if Λ is irreducible, each element of Λ is isotopic to an element of $\tilde{\Lambda}(f^{\circ n})$.

The following theorem is Theorem 3.2 of [PT]. It shows that up to isotopy, irreducible Thurston obstructions cannot intersect the preimages of irreducible arc systems (except possibly the arc systems themselves). We will use this theorem to show that the extended map of an abstract extended Newton graph is Thurston equivalent to a rational map.

Theorem 2.3.3 (Arcs Intersecting Obstructions)**.** *[PT] Let (f, X) be a marked branched covering, Π an irreducible Thurston obstruction and Λ an irreducible arc system. Suppose furthermore that $\#(\Pi \cap \Lambda) = \Pi \cdot \Lambda$. Then, exactly one of the* *following is true:*

1. $\Pi \cdot \Lambda = 0$ *and* $\Pi \cdot f^{-n}(\Lambda) = 0$ *for all* $n \geq 1$.
2. $\Pi \cdot \Lambda \neq 0$ *and for* $n \geq 1$, *each component of* Π *is isotopic to a unique component of* $\tilde{\Pi}(f^{\circ n})$. *The mapping* $f^{\circ n} : \tilde{\Pi}(f^{\circ n}) \to \Pi$ *is a homeomorphism and* $\tilde{\Pi}(f^{\circ n}) \cap (f^{-n}(\Lambda) \setminus \tilde{\Lambda}(f^{\circ n})) = \emptyset$. *The same is true when interchanging the roles of* Π *and* Λ. □

2.4 Hubbard trees

It is a frequent observation in complex dynamics that many dynamical properties can be encoded in symbolic terms. Douady and Hubbard [DH84/85] introduced a combinatorial description of the dynamics of postcritically finite polynomials using the association to each filled Julia set a tree, called Hubbard tree. In particular they showed that for polynomials whose critical points are all (pre-)periodic the dynamical behavior is completely encoded in the so called *Hubbard tree*.

Let f be a complex polynomial. The point at infinity is a superattracting fixed point for f. This allows one to define the *filled Julia set* $K(f)$ as the complement of the basin of attraction of infinity. In other words, $K(f)$ consists of those $z \in \mathbb{C}$ such that the forward orbit $f^n(z)$ is bounded for all positive integers n. The Julia set $J(f)$ is equal to the boundary of $K(f)$.

Recall that a tree is a topological space which is uniquely arcwise connected and homeomorphic to a union of finitely many copies of the closed unit interval. We assume here that all trees are embedded into $\mathcal{S}^2$.

The interest of Hubbard trees is that they contain all the combinatorial information about the dynamics of the polynomials. Indeed, Douady and Hubbard showed that if we retain the dynamics of a polynomial and the local degree of f on the set of vertices, the embedding of the tree in the complex plane and a little bit of extra information, then different postcritically finite polynomials give rise to different Hubbard trees. A variation of the converse is also true and was proved in a general version by A. Poirier [Po].

We recall some facts about the dynamics of postcritically finite polynomials from [Mi2]. If f is a postcritically finite polynomial, then the filled Julia set $K(f)$ is a connected and locally connected compact set [DH84/85]. For each Fatou component U_i, there is exactly one point $x \in U_i$ such that $f^n(x) \in P_f$ for some non-negative integer n. Denote by U_j the Fatou component containing $f(x)$.

A classical theorem of Böttcher implies that there are holomorphic isomorphisms $\phi_i : (\mathbb{D}, 0) \to (U_i, x)$, $\phi_j : (\mathbb{D}, 0) \to (U_j, f(x))$ such that for all $z \in \mathbb{D}$:

$$\phi_j(z^{k_i}) = f(\phi_i(z)),$$

where k_i is the local degree degree of f near x. Since $J(f)$ is locally connected, each Fatou component has locally connected boundary, and by Caratheodory's theorem the map ϕ_i extends continuously to $\mathbb{S}^1$. Let $R(t) = \{r \exp(2\pi i t) | 0 \leq r \leq 1\}$. The image $R_i(t) = \phi_i(R(t))$ is called the *ray of angle t in U_i*. If $x = \infty$, the ray $R_i(t)$ is called an *external ray*, otherwise it is called *internal ray*. Each periodic point $x \in J(f)$ is the landing point of at least one and at most finitely many external rays (see [Mi2]).

Definition 2.4.1. *An arc $\gamma \subset K(f)$ is called* allowable *if for every Fatou component U_i, $\phi_i(\gamma \cap \overline{U_i})$ is contained in the union of two rays of $\overline{\mathbb{D}}$.*

For every z, z' in $K(f)$ there is a unique allowable arc joining them. We denote this arc by $[z, z']_{K(f)}$. We say that a subset $X \subset K(f)$ is *allowably connected* if for every $z_1, z_2 \in X$ we have $[z_1, z_2]_{K(f)} \subset X$. Note that a union of a family of allowably connected subsets having a common point is allowably connected. The intersection of a family of allowably connected subsets is allowably connected. We define the *allowable hull* $[X]_K$ of $X \subset K(f)$ as the intersection of all the allowably connected subsets of $K(f)$ containing X.

Definition 2.4.2. *For a finite invariant set M, containing the set C_f of critical points of f, we denote by $H(M)$ the tree generated by M, i.e. the allowable hull $[M]_K$. The minimal tree $H(M_0)$, is the tree generated by $M_0 = P_f$, the postcritical set. This last tree $H = H(M_0)$ is usually called in the literature* the Hubbard Tree *of f.*

Let $H^* = H(M)$, where $M = P_f \cup \{z \in \mathbb{C} : f(z) = z\}$, the set containing the postcritical set of f. This tree H^* we call an *extended Hubbard tree* which will be used in our construction of an extended Newton graph.

Remark 2.4.3. If f is a polynomial of degree 1, then the tree $H^* = H(M)$ consists of only one point, we call it degenerate.

Definition 2.4.4 (Valency). *Given a point $z \in H(M)$, the* valency $\nu_{H(M)}(z)$ *of $H(M)$ at the point z is the number of connected components of $H(M) - \{z\}$. In other words $\nu_{H(M)}(z)$ is equal to the number of branches of $H(M)$ that are incident at z.*

A point $z \in H(M)$ for which $\nu_{H(M)}(z) > 2$ is called a *branched point* of $H(M)$, a point $z \in H(M)$ for which $\nu_{H(M)}(z) = 1$ is called an *endpoint*.

Similar definitions are valid for points in the Julia sets of polynomials. Namely, for a postcritically finite polynomial p a point $z \in J(p)$ is called *biaccessible* if there are at least two external rays landing at z. The relation between the number of external rays landing at points from the Julia set and valency at points from Hubbard trees is the following.

Proposition 2.4.5. *[Po] Let p be a postcritically finite polynomial and $z \in J(p)$ is a periodic branched point of $J(p)$. Then $z \in H(M)$ for any finite invariant set M that contains the postcritical set of p. Furthemore, $v_{H(M)}(z)$ is independent of M and equals the number of components of $J(p) - \{z\}$. In particular there are exactly $v_{H(M)}(z)$ external rays landing at z.*

For the proof of Proposition 2.4.5 see [Po, Proposition 3.3].

In the sequel we will need to axiomatize the notions of a Hubbard tree and an extended Hubbard tree. Here we present the necessary background from [Po].

Definition 2.4.6 (Angled Tree)**.** Angled tree H *is a finite connected tree together with an angle function $l, l' \mapsto \angle(l, l') = \angle_v(l, l') \in \mathbb{Q}/\mathbb{Z}$ which for each pair of edges l, l' meeting at the vertex v assignes a rational modulo 1. The angle function is skew-symmetric with $\angle(l, l') = 0$ if and only if $l = l'$ and $\angle_v(l, l'') = \angle_v(l, l') + \angle_v(l, l'')$ for every triple of edges l, l', l'' meeting at the vertex v. Such an angle function determines an isotopy class of embeddings of the tree H into $\mathbb{C}$.*

Now we introduce the dynamics on H: let $\tau : H \to H$ be a continuous map sending V to V and injective on the closures of edges, where V is the vertex set of H. We also specify a *local degree function* $\delta : V \to \mathbb{Z}$ which assigns a positive integer $\delta(v)$ to each vertex $v \in V$. The number

$$\deg(\delta) = 1 + \sum_{v \in V} (\delta(v) - 1)$$

is called the degree of H. We require that $\deg(\delta) > 1$. A vertex $v \in V$ is called *critical* if $\delta(v) > 1$ and *non-critical* otherwise. By $C_\delta = \{v \in V : \ v \text{ is critical}\}$ we denote the *critical set* which by our assumptions is always non empty.

Remark 2.4.7. In fact it is enough to know the dynamics of τ only on the vertex set V. One can extend the vertex map $\tau|_V$ from the set of vertices V to the whole tree in the following way: each edge of H maps homeomorphically onto the shortest path joining the images under τ of the endpoints of this edge. This way one obtains the extended map $\tau' : H \to H$. The map τ' turns out to be equivalent to τ with the equivalence relation defined later in Definition 2.4.12.

For a vertex v denote by E_v the set of edges of H incident at v. The maps τ and δ must be related in the following way: for every pair of edges $l, l' \in E_v$:

$$\angle_{\tau(v)}(\tau(l), \tau(l')) = \delta(v)\angle_v(l, l').$$

A vertex $v \in V$ is *periodic* if for some positive integer n we have $\tau^{\circ n}(v) = v$. The orbit of a periodic critical point is called *a critical cycle*. We say that a vertex $v \in V$ is a *Fatou vertex* if it eventually maps into a critical cycle. Otherwise, if it eventually maps to a non-critical cycle it is called a *Julia vertex*.

One can also define a distance function d_H on the tree H. For a pair of vertices v, v' denote by $d_H(v, v')$ the number of the edges in the shortest path in H between v and v'.

We say that (H, V, τ, δ) is *expanding* if for any edge l with endpoints v, v' which are Julia vertices there is an integer $n \geq 1$ such that $d_H(\tau^{\circ n}(v), \tau^{\circ n}(v')) > 1$.

Definition 2.4.8 (Abstract Hubbard Tree)**.** *An* abstract Hubbard tree *is an expanding angled tree $H = ((H, V, \tau, \delta), \angle)$. An* abstract extended Hubbard tree *is an expanding angled tree $H^* = ((H^*, V, \tau, \delta), \angle)$ such that among vertices of H^* there are* $\deg(\delta)$ *points that are fixed under the map $\tau : H^* \to H^*$.*

Remark 2.4.9. A postcritically finite polynomial f and a finite invariant set $M \supset C_f$ naturally define an abstract Hubbard tree $(H(M), \angle)$. The angle function is defined as follows. For a periodic Fatou vertex v the edges of $H(M)$ having v as their common endpoint are segments of constant arguments in Böttcher coordinates in a neighborhood of v and the angle between two such edges is defined as the difference of their coordinates. For other Fatou points the angle function can be defined by appropriate pullbacks of the coordinates at $\tau(v)$. If v is a Julia vertex, then $J(f)\setminus\{v\}$ consists of finitely many components, denote the number of them by m. The "angle" between these components is defined to be a multiple of $1/m$. Since the edges of the tree locally correspond to the components of $J(f) \setminus \{v\}$, the angle between the components defines the angle between the edges at v.

Similarly, if we set $M = P_f \cup \{z \in \mathbb{C} : f(z) = z\}$, then the tree $(H^*, \angle) = (H(M), \angle)$ generated by M is an abstract extended Hubbard tree.

Definition 2.4.10. *Let f be a postcritically finite polynomial and H^* its extended Hubbard tree. A fixed point $\omega \in \mathbb{C}$ of f is said to be* a β-fixed point *of f if $H^* \setminus \{\omega\}$*

is connected.

Definition 2.4.11 (Hubbard Tree Extension)**.** *Given two abstract (extended) Hubbard trees H, H' of the same degree $n = \deg(H) = \deg(H') > 1$, we say that H' is an* extension *of H (in symbols $H \preceq H'$), if there is a dynamically compatible orientation-preserving embedding $\phi : H \to H'$ such that the following conditions are satisfied:*

1. $\phi(V) \subset V'$;
2. $\tau'(\phi(v)) = \phi(\tau(v))$ *for all* $v \in V$;
3. $\delta(v) = \delta'(\phi(v))$ *for all* $v \in V$;
4. $\angle_v(l, l') = \angle'_{\phi(v)}(\phi(l), \phi(l'))$ *for all* $l, l' \in E_v$.

Definition 2.4.12 (Equivalence Relation)**.** *Two abstract (extended) Hubbard trees H and H' are* equivalent *if $H \preceq H'$ and $H' \preceq H$. Or, two abstract (extended) Hubbard trees are equivalent if there is an orientation-preserving homeomorphism of the plane to itself carrying (H, V, τ, δ) to (H', V', τ', δ'), conjugating the dynamics of vertices, and preserving the local degree functions.*

This determines an equivalence relation between abstract (extended) Hubbard trees. The set of abstract (extended) Hubbard trees equivalent to H is denoted by $[H]$. In [Po, Proposition 2.9] the following statement if proven.

Proposition 2.4.13. *[Po, Proposition 2.9] Every abstract Hubbard tree H contains a unique minimal tree $\min([H])$. This unique minimal tree is the tree generated by the orbit of the critical set. Every abstract extended Hubbard tree H^* contains a unique minimal abstract extended Hubbard tree $\min([H^*])$. This unique minimal tree is the tree generated by the orbit of the critical set and the fixed points of $\tau : H \to H$.*

We present the basic existence and uniqueness theorem for abstract Hubbard trees.

Theorem 2.4.14 (Realization of Abstract Hubbard Trees)**.** *[Po, Theorem 4.7] Any abstract Hubbard tree H^* can be realized as a tree associated with a postcritically finite polynomial f. Equivalently, there exists a unique postcritically finite polynomial f, and an invariant set $M \supset C_f$ such that $H(M) \in [H]$. The polynomial f is unique up to affine conjugacy.*

Now follows the analog of Theorem 2.4.14 for abstract *extended* Hubbard trees.

Theorem 2.4.15 (Realization of Abstract Extended Hubbard Trees)**.** *Any abstract extended Hubbard tree H^* can be realized as a tree associated with a postcritically finite polynomial f. Equivalently, there exists a unique postcritically finite polynomial f, and an invariant set $M \supset C_f$ such that $H(M) \in [H^*]$. The polynomial f is unique up to affine conjugacy.*

In Remark 2.4.9 the Hubbard trees H and H^* were constructed for a postcritically finite polynomial f such that H satisfies the properties of an abstract Hubbard tree and H^* — the properties of an abstract extended Hubbard tree. In fact the following theorem holds.

Theorem 2.4.16 (Bijective Correspondence)**.** *[Po, Theorem 4.8] The set of affine conjugacy classes of postcritically finite polynomials of degree at least two is in bijective correspondence with the set of equivalence classes of minimal abstract Hubbard trees of degree at least two.*

Similarly, the result on the bijective correspondence between postcritically finite polynomials and abstract extended Hubbard trees applies.

Theorem 2.4.17. *The set of affine conjugacy classes of postcritically finite polynomials of degree at least two is in bijective correspondence with the set of equivalence classes of minimal abstract extended Hubbard trees of degree at least two.*

2.5 Extending maps on finite graphs

Here we discuss a way to extend maps on finite graphs to maps on the whole sphere, giving a criterion of extendibility. For this, we first need to introduce some notation regarding maps on embedded graphs and their extensions to $\mathcal{S}^2$, compare [BFH, Chapter 6]. We assume in the following without explicit mention that all graphs are embedded into $\mathcal{S}^2$.

The main idea for such an extension lies in the following lemma [BFH, Chapter 6] (the so-called "Alexander trick").

Lemma 2.5.1. *Let $h : \mathcal{S}^1 \to \mathcal{S}^1$ be an orientation-preserving homeomorphism. Then there exists an orientation preserving homeomorphism $\overline{h} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $\overline{h}|_{\mathcal{S}} = h$. The map $\overline{h}$ is unique up to isotopy relative $\mathcal{S}$.*

Definition 2.5.2 (Finite Graph)**.** *An edge of a graph is an arc homeomorphic to a closed interval. A* finite graph Γ *is the quotient of a finite disjoint union of edges of the graph, by an equivalence relation on the set of endpoints of these arcs. The* *images of the endpoints under the equivalence relation are called* vertices *of the graph. A* finite embedded graph *is a homeomorphic image of a finite graph into* $\mathcal{S}^2$*.*

Definition 2.5.3 (Graph Map)**.** *Let* Γ_1, Γ_2 *be connected finite embedded graphs. A map* $f : \Gamma_1 \to \Gamma_2$ *is called a* graph map *if it is continuous and injective on each edge of the graph* Γ_1 *so that forward images of vertices are vertices .*

Definition 2.5.4 (Regular Extension)**.** *Let* $f : \Gamma_1 \to \Gamma_2$ *be a graph map. An orientation-preserving branched covering map* $\overline{f} : \mathcal{S}^2 \to \mathcal{S}^2$ *is called a* regular extension *of* f *if* $\overline{f}|_{\Gamma_1} = f$ *and* $\overline{f}$ *is injective on each component of* $\mathcal{S}^2 \setminus \Gamma_1$.

It follows from Definition 2.5.4 that every regular extension $\overline{f}$ may have critical points only at the vertices of Γ_1.

Lemma 2.5.5 (Isotopic Graph Maps)**.** [BFH, Corollary 6.3] *Let* $f, g : \Gamma_1 \to \Gamma_2$ *be two graph maps that coincide on the vertices of* Γ_1 *such that for each edge* $e \subset \Gamma_1$ *we have* $f(e) = g(e)$ *as a set. Suppose that* f *and* g *have regular extensions* $\overline{f}, \overline{g} : \mathcal{S}^2 \to \mathcal{S}^2$*. Then there exists a homeomorphism* $\psi : \mathcal{S}^2 \to \mathcal{S}^2$*, isotopic to the identity relative the vertices of* Γ_1*, such that* $\overline{f} = \overline{g} \circ \psi$*.*

Let $f : \Gamma_1 \to \Gamma_2$ be a graph map. For the next proposition, for each vertex v of Γ_1 choose a neighborhood $U_v \subset \mathcal{S}^2$ such that all edges of Γ_1 that enter U_v terminate at v; we may assume without loss of generality that in local cordinates, U_v is a round disk of radius 1 that is centered at v, that all edges entering U_v are radial lines and that $f|_{U_v}$ is length-preserving. We make analogous assumptions for Γ_2. Then, we can extend f to each U_v as in [BFH]: for a vertex $v \in \Gamma_1$, let γ_1 and γ_2 be two adjacent edges ending there. In local coordinates, these are radial lines at angles, say, ϑ_1, ϑ_2 such that $0 < \vartheta_2 - \vartheta_1 \le 2\pi$ (if v is an endpoint of Γ_1, then set $\vartheta_1 = 0$, $\vartheta_2 = 2\pi$). In the same way, choose arguments ϑ_1', ϑ_2' for the image edges in $U_{f(v)}$ and extend f to a map $\tilde{f}$ on $\Gamma_1 \cup \bigcup_v U_v$ by setting

$$(\rho, \vartheta) \mapsto \left(\rho, \frac{\vartheta_2' - \vartheta_1'}{\vartheta_2 - \vartheta_1} \cdot \vartheta\right),$$

where (ρ, ϑ) are polar coordinates in the sector bounded by the rays at angles ϑ_1 and ϑ_2. In particular, sectors are mapped onto sectors in an orientation-preserving way. Then, the following holds.

Proposition 2.5.6 (Regular Extension)**.** [BFH, Proposition 6.4] *A graph map* $f : \Gamma_1 \to \Gamma_2$ *has a regular extension if and only if for every vertex* $y \in \Gamma_2$ *and every* *component U of $\mathcal{S}^2 \setminus \Gamma_1$, the extension $\tilde{f}$ is injective on*

$$\bigcup_{v \in f^{-1}(\{y\})} U_v \cap U \, .$$

2.6 The Newton graph

The combinatorial classification of postcritically fixed Newton maps was given in [MR] in terms of Newton graphs. Newton graphs are defined as follows. Consider a postcritically finite Newton map N_p with attracting fixed points $a_1, a_2, \ldots, a_d$. For such N_p the points $a_1, a_2, \ldots, a_d$ must in fact be superattracting (see Lemma 2.1.8). Let U_i denote the immediate basin of a_i. Each U_i has a global Böttcher coordinate $\phi_i : (\mathbb{D}, 0) \to (U_i, a_i)$ with the property that $N_p(\phi_i(z)) = \phi_i(z^{k_i})$ for each $z \in \mathbb{D}$ (the complex unit disk), where $k_i - 1 \geq 1$ is the multiplicity of a_i as a critical point of N_p. The map $z \to z^{k_i}$ fixes $k_i - 1$ internal rays in $\mathbb{D}$. Under ϕ_i, these map to $k_i - 1$ pairwise disjoint injective curves $\Gamma_i^1, \Gamma_i^2, \ldots, \Gamma_i^{k_i - 1} \subset U_i$ that connect a_i to ∞, are pairwise non-homotopic (with homotopies fixing the endpoints) and are invariant under N_p. They represent all accesses to ∞ of U_i (see Proposition 2.1.7). The union

$$\Delta = \bigcup_i \bigcup_{j=1}^{k_i - 1} \overline{\Gamma_i^j}$$

forms a connected graph in $\widehat{\mathbb{C}}$ that is called the *channel diagram*. It follows from the definition that $N_p(\Delta) = \Delta$. The channel diagram records the mutual locations of the immediate basins of N_p and provides a first-level combinatorial information about the dynamics of the Newton map. For any $n \geq 0$, denote by Δ_n the connected component of $N_p^{-n}(\Delta)$ that contains Δ. The pair (Δ_n, N_p) of a graph Δ_n and a Newton map N_p acting on it is called the *Newton graph* of N_p at level n. Newton graphs give more precise combinatorial data than channel diagrams, and by results from [MR] they are very useful as combinatorial models for Newton maps. For postcritically fixed Newton maps all critical points are eventually fixed and contained in the Newton graph at sufficiently high level. Moreover, the following result is proven in [MR].

Theorem 2.6.1. [MR, Theorem 3.4] *There exists a positive integer N so that Δ_N contains all poles of N_p.*

It follows from Theorem 2.6.1 that for any prepole there exists sufficiently large m such that this prepole is in Δ_m. The following theorem proven in [MR] allows to structure the basins of attraction of finite fixed points of N_p.

Theorem 2.6.2. [MR, Theorem 1.4] *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a Newton map with attracting fixed points $a_1, \ldots, a_d \in \mathbb{C}$, and let U_0' be a component of some B_i, the basin of attraction of a_i. Then, U_0' can be connected to ∞ by the closures of finitely many components $U_1', \ldots, U_k'$ of $\bigcup_{i=1}^d B_i$.*

More precisely, there exists a curve $\gamma : [0,1] \to \widehat{\mathbb{C}}$ such that $\gamma(0) = \infty$, $\gamma(1) \in U_0'$ and for every $t \in [0,1]$, there exists $m \in \{0, 1, \ldots, k\}$ such that $\gamma(t) \in \overline{U}_m'$.

One of the main results in [MR] states that to a *postcritically fixed* Newton map N_p (see Definition 2.1.4) one can associate a Newton graph that is unique up to specific equivalence relation and different Newton graphs (with respect to equivalence relation) distinguish different Newton maps (up to affine conjugacy) they are associated to.

Since we are talking here about postcritically fixed and postcritically finite Newton maps we distinguish between the two different abstract combinatorial objects modelling the Newton graphs constructed for postcritically fixed and postcritically finite Newton maps. In the terminology of [MR] the abstract Newton graphs associated to postcritically fixed Newton maps are called here *abstract postcritically fixed Newton graphs*. The following theorem is proved in [MR].

Theorem 2.6.3. *[MR, Theorem 1.5]. Every postcritically fixed Newton map f gives rise to an abstract postcritically fixed Newton graph. There exists a unique $N \in \mathbb{N}$ such that (Δ_N, f) is an abstract postcritically fixed Newton graph.*

If f_1 and f_2 are Newton maps with channel diagrams Δ_1 and Δ_2 such that $(\Delta_{1,N}, f_1)$ and $(\Delta_{2,N}, f_2)$ are equivalent as abstract postcritically fixed Newton graphs, then f_1 and f_2 are affinely conjugate. Hence there exists a well defined injective mapping $\mathcal{F}$ from the set of postcritically fixed Newton maps up to affine conjugacy to the set of abstract postcritically fixed Newton graphs up to Thurston equivalence.

In fact, the results in [MR], in particular Theorem 2.6.2, imply that the Newton graph can be constructed for *any* Newton map, not necessarily postcritically fixed. The abstract counterparts of such Newton graphs we call abstract Newton graphs. We will need the following theorem for the future purpose.

Theorem 2.6.4. *Every postcritically finite Newton map f gives rise to an abstract Newton graph. This graph is unique up to Thurston equivalence. More precisely, there exists a unique $N \in \mathbb{N}$ such that (Δ_N, f) is an abstract Newton graph.*

In order to axiomatize the graphs that are extracted for Newton maps, in [MR] there were developped the notions of an *abstract channel diagram* and an *abstract postcritically fixed Newton graph* which classify postcritically fixed Newton maps (note that the latter is called abstract Newton graph in [MR], since the Newton graph construction was done in [MR] only for postcritically fixed Newton maps).

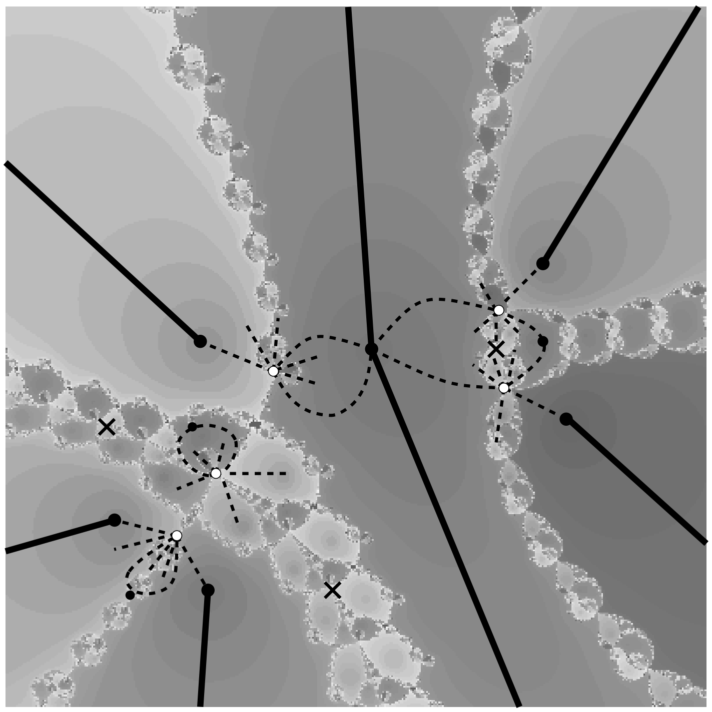

Figure 2.2: A Newton map of degree 6 with its channel diagram: the solid lines represent accesses to ∞ of the immediate basins, the black dots correspond to the fixed points (the vertex at ∞ is not visible). The dashed lines show the first preimage of the channel diagram: white circles represent poles, a cross is a free critical point.

Definition 2.6.5 (Abstract Channel Diagram)**.** *An* abstract channel diagram *of degree $d \geq 3$ is a graph $\Delta \subset \mathcal{S}^2$ with vertices $v_0, \ldots, v_d$ and edges $e_1, \ldots, e_l$ that satisfies the following properties:*

(1) $l \leq 2d - 2$;

(2) each edge joins v_0 to a v_i, $i > 0$;

(3) each v_i is connected to v_0 by at least one edge;

(4) if e_i and e_j both join v_0 to v_k, then each connected component of $\mathcal{S}^2 \setminus \overline{e_i \cup e_j}$ contains at least one vertex of Δ.

It is not difficult to check that the channel diagram Δ constructed for a Newton map N_p above satisfies conditions of Definition 2.6.5. Indeed by construction, Δ has at most $2d-2$ edges and it satisfies (2) and (3). Finally, Δ satisfies (4), because for any immediate basin U_ξ of N_p, every component of $\mathbb{C} \setminus U_\xi$ contains at least one fixed point of N_p [RS, Corollary 5.2].

Definition 2.6.6 (Abstract Postcritically Fixed Newton Graph)**.** *Let $\Gamma \subset \mathcal{S}^2$ be a connected finite graph, Γ' the set of its vertices and $f : \Gamma \to \Gamma$ a graph map. The pair (Γ, f) is called an* abstract postcritically fixed Newton graph *if it satisfies the following conditions:*

(1) There exists $d_\Gamma \geq 3$ and an abstract channel diagram $\Delta \subsetneq \Gamma$ of degree d_Γ such that f fixes each vertex and each edge of Δ.

(2) For every vertex $y \in \Gamma'$ and every component U of $\mathcal{S}^2 \setminus \Gamma$, the extension $\tilde{f}$ is injective on

$$\bigcup_{v \in f^{-1}(\{y\})} U_v \cap U \,.$$

Hence by Proposition 2.5.6, f can be extended to a branched covering $\overline{f} : \mathcal{S}^2 \to \mathcal{S}^2$.

(3) If $v_0, \ldots, v_{d_\Gamma}$ are the vertices of Δ, then $v_i \in \overline{\Gamma \setminus \Delta}$ if and only if $i \neq 0$. Moreover, there are exactly $\deg_{v_i}(\overline{f}) - 1 \geq 1$ edges in Δ that connect v_i to v_0 for $i \neq 0$, where $\deg_x(\overline{f})$ denotes the local degree of $\overline{f}$ at $x \in \Gamma'$.

(4) $\sum_{x \in \Gamma'} \big(\deg_x(\overline{f}) - 1\big) = 2d_\Gamma - 2$.

(5) There exists N_Γ such that $f^{\circ(N_\Gamma - 1)}(x) \in \Delta$ for all $x \in \Gamma'$ with $\deg_x(\overline{f}) > 1$. And if N_Γ is minimal with this property, then $f^{\circ N_\Gamma}(\Gamma) \subset \Delta$.

(6) The graph $\overline{\Gamma \setminus \Delta}$ is connected.

(7) Γ equals the component of $\overline{f}^{-N_\Gamma}(\Delta)$ that contains Δ.

If (Γ, f) is an abstract Newton graph, f can be extended to a branched covering map $\overline{f} : \mathcal{S}^2 \to \mathcal{S}^2$ by Condition (2) and Proposition 2.5.6. It is used implicitly in (7). Condition (4) and the Riemann-Hurwitz formula ensure that $\overline{f}$ has degree d_Γ. An immediate consequence of Lemma 2.5.5 is that $\overline{f}$ is unique up to Thurston equivalence.

Definition 2.6.7 (Abstract Newton Graph)**.** *Let $\Gamma \subset \mathcal{S}^2$ be a connected finite graph, Γ' the set of its vertices and $f : \Gamma \to \Gamma$ a graph map. The pair (Γ, f) is called an* abstract Newton graph *if it satisfies the following conditions:*

(1) There exists $d_\Gamma \geq 3$ *and an abstract channel diagram* $\Delta \subsetneq \Gamma$ *of degree* d_Γ *such that* f *fixes each vertex and each edge of* Δ*.*

(2) The graph map f *can be extended to a branched covering* $f : \mathcal{S}^2 \to \mathcal{S}^2$ *such that the following conditions (3) – (6) are satisfied.*

(3) If $v_0, \ldots, v_{d_\Gamma}$ *are the vertices of* Δ*, then* $v_i \in \overline{\Gamma \setminus \Delta}$ *if and only if* $i \neq 0$*. Moreover, there are exactly* $\deg_{v_i}(\overline{f}) - 1 \geq 1$ *edges in* Δ *that connect* v_i *to* v_0 *for* $i \neq 0$*, where* $\deg_x(\overline{f})$ *denotes the local degree of* $\overline{f}$ *at* $x \in \Gamma'$*.*

(4) $\sum_{x \in \Gamma'} \big(\deg_x(\overline{f}) - 1\big) \leq 2d_\Gamma - 2$*.*

(5) The graph $\overline{\Gamma \setminus \Delta}$ *is connected.*

(6) Γ *equals the component of* $\overline{f}^{-N_\Gamma}(\Delta)$ *that contains* Δ*.*

It follows from [MR] that, if N_p is a postcritically finite Newton map, then the pair (Δ_{N_Γ}, N_p) satisfies all conditions of Definition 2.6.7, where N_Γ is the minimal positive integer n such that all critical points of N_p that are eventually fixed (under some iterate of N_p) are mapped onto its channel diagram Δ under n'th iterate of N_p. Therefore each postcritially finite Newton map N_p gives rise to an abstract Newton graph.

Two abstract (postcritically fixed) Newton graphs (Γ_1, f_1) and (Γ_2, f_2) are said to be *equivalent* if there exists a graph homeomorphism $g : \Gamma_1 \to \Gamma_2$ that preserves the cyclic order of edges at each vertex of Γ_1 and conjugates f_1 to f_2.

It is proved in [MR] that every abstract postcritically fixed Newton graph is realized by a postcritically fixed Newton map.

Theorem 2.6.8. *[MR, Theorem 1.6] Every abstract postcritically fixed Newton graph is realized by a postcritically fixed Newton map. This Newton map is unique up to affine conjugacy. More precisely, let* (Γ, g) *be an abstract postcritically fixed Newton graph. Then, there exists a postcritically fixed Newton map* f *with channel diagram* $\hat{\Delta}$ *such that* $(\overline{g}, \Gamma')$ *and* $(f, \hat{\Delta}'_{N_\Gamma})$ *are Thurston equivalent as marked branched coverings (here* Γ' *denotes the set of vertices of the graph* Γ*).*

If f *realizes two abstract postcritically fixed Newton graphs* (Γ_1, g_1) *and* (Γ_2, g_2)*, then the two abstract postcritically fixed Newton graphs are equivalent. Hence there exists a well defined injective mapping* $\mathcal{F}'$ *from the set of abstract Newton graphs up to Thurston equivalence to the set of postcritically fixed Newton maps up to affine conjugacy.*

Moreover, it follows from [MR] that the maps $\mathcal{F}$ and $\mathcal{F}'$ from Theorem 2.6.3 and Theorem 2.6.8 are bijective and inverse to each other, i.e. $\mathcal{F} \circ \mathcal{F}' = Id$ and $\mathcal{F}' \circ \mathcal{F} = Id$.

The main result of this thesis is the construction of analogous mappings $\mathcal{F}$ and $\mathcal{F}'$ between the postcritically finite Newton maps and abstract extended Newton graphs, and proving that they are inverse to each other.

2.7 Polynomial-like mappings and renormalization

Polynomial-like mappings play an important role in complex dynamics. They were introduced by Douady and Hubbard [DH3] among other things to explain the partial self-similarity of the Mandelbrot set. Later polynomial-like mappings were used to study the local connectivity of Julia sets and the Mandelbrot set, rigidity of polynomials, etc.

Definition 2.7.1. *A* polynomial-like *map of degree d is a triple (f, U, V) where U, V are topological disks in $\mathbb{C}$ such that $\overline{U}$ is a compact subset of V and $f : U \to V$ is a proper holomorphic map such that every point in V has d preimages in U when counted with multiplicities.*

Definition 2.7.2. *Let $f : U \to V$ be a polynomial-like map. The* filled Julia set *of f is defined as the set of points in U that never leave V under iteration by f, i.e.*

$$K(f) = \bigcap_{n=1}^{\infty} f^{-n}(V).$$

As for polynomials we define the Julia set *as $J(f) = \partial K(f)$.*

Remark 2.7.3. If $d = 1$ in Definition 2.7.1, then we call the map f a *degenerate polynomial-like map.*

The simplest example of polynomial-like mappings is a restriction of any polynomial: for a polynomial p of degree $d \geq 2$ let $V = \{z \in \mathbb{C} : |z| < R\}$ for sufficiently large R and $U = f^{-1}(V)$. Then $p : U \to V$ is a polynomial-like mapping of degree d.

Remark 2.7.4. In general, for a triple (f, U, V) with $U \subset V$ and $f : U \to V$ a proper holomorphic map we denote by

$$K(f, U, V) = \bigcap_{n=1}^{\infty} f^{-n}(V)$$

the set of points in U that never leave V under iteration by f.

Two polynomial-like maps f and g are *hybrid equivalent* if there is a quasiconformal conjugacy ψ between f and g, defined on a neighborhood of their respective filled Julia sets, such that $\bar{\partial}\psi = 0$ on $K(f)$.

The crucial relation between polynomial-like maps and polynomials is explained in the following theorem, due to Douady and Hubbard [DH3]. They showed that a polynomial-like map behaves dynamically like a polynomial.

Theorem 2.7.5 (The Straightening Theorem)**.** *Let $f : U' \to U$ be a polynomial-like map of degree d. Then f is hybrid equivalent to a polynomial P of degree d. Moreover, if $K(f)$ is connected, then P is unique up to affine conjugation.*

Now we define the notion of renormalization of rational functions. Let R be a rational function of degree d and z_0 a critical point of R, i.e. $z_0 \in C_R$.

Definition 2.7.6. *R^n is called* renormalizable about z_0 *if there exist open disks $U, V \subset \mathbb{C}$ satisfying the following conditions:*

1. *z_0 lies in U.*
2. *(R^n, U, V) is a polynomial-like map of degree at least two with connected filled Julia set.*
3. *$n \geq 2$ or C_R is not a subset of U.*

A *renormalization* is a polynomial-like restriction $\rho = (R^n, U, V)$ as above. We call n the *period* of the renormalization ρ.

For a renormalization ρ, its period n and $i = 1, 2, \ldots, n$ denote $n(\rho) = n$, $U(\rho) = U$, $V(\rho) = V$. The filled Julia set of ρ is denoted by $K(\rho)$, the Julia set $J(\rho)$, and the critical and postcritical sets by $C(\rho)$ and $P(\rho)$ respectively. The *i'th small filled Julia set* $K(\rho, i) = R^i(K(\rho))$ and the *i'th Julia set* $J(\rho, i) = R^i(J(\rho))$. The *$i$'th small critical set* $C(\rho, i) = K(\rho, i) \cap C(R)$. Clearly $C(\rho, i)$ may be empty for $0 < i < n$. However, by definition $C(\rho, n)$ is nonempty.

The filled Julia sets of the renormalization depend only on its period and small critical sets. Namely the following is true:

Theorem 2.7.7 (Uniqueness of renormalization)**.** *Let $\rho = (R^n, U, V)$ and $\rho' = (R^n, U', V')$ be two renormalizations of the same period. If $C(\rho, i) = C(\rho', i)$ for all $1 \leq i \leq n$, then the filled Julia sets $K(\rho) = K(\rho')$.*

Proof. It follows from [MC, Theorem 6.13] that $K = K(\rho) \cap K(\rho')$ is connected. Let U'' be the component of $U \cap U'$ containing K and $V'' = R^n(U'')$. Since by definition V'' contains $f(K) = K$ we have $U'' \subset V''$. Using [MC, Theorem 5.11] we obtain that (R^n, U'', V'') is a polynomial-like map with a filled Julia set K. Since the sets of critical points of the three maps $R^n : U \to V, R^n : U' \to V'$ and $R^n : U'' \to V''$ are equal we obtain the desired $K(\rho) = K(\rho') = K$. □

In Section 2.4 the notion of (extended) Hubbard trees for a given postcritically finite polynomials was introduced. Note that the same construction applies to polynomial-like mappings $f : U' \to U$ with connected filled Julia set. In the following we implicitly use (extended) Hubbard trees associated to postcritically finite polynomial-like mappings.

Chapter 3

Extended Newton graph

3.1 Renormalization of Newton maps

The Newton graph Δ_N of N_p defined in Section 2.6 divides the complex plane into finitely many pieces, and each critical point has an itinerary with respect to this partition. In this Section for every periodic critical point of N_p we construct the renormalization domain containing the chosen critical point. Then we obtain a polynomial-like mapping that describes the dynamics of periodic critical points that never fall onto Δ_N in terms of a polynomial. Having polynomial-like map associated to a free periodic critical point we obtain an extended Hubbard tree associated to each of them, which will serve for us as a combinatorial model for the dynamical behavior of free periodic critical points.

Lemma 3.1.1. *For a postcritically finite Newton map N_p there exists a positive integer N and finitely many pairs of domains (U_k, V_k), $U_k \subset V_k$, $k \in [1, M]$, for some non-negative integer M, which satisfy the following properties:*

1. *$\partial U_k, \partial V_k \subset \Delta_N$ for every $k \in [1, M]$, where Δ_N is the Newton graph at level N.*

2. *For each pair of domains (U_k, V_k) there exists a positive integer $m(k)$ such that $N_p^{m(k)} : U_k \to V_k$ is a proper map of degree $d_k \geq 1$.*

3. *Different components U_l, U_k, $l \neq k$, are disjoint.*

4. *For every periodic point $z_1 \in P_{N_p}$ of period at least two there exists k such that $z_1 \in U_k$. Moreover, $P_{N_p} \cap V_k \subset K(N_p^{m(k)}, U_k, V_k)$ ($K(N_p^{m(k)}, U_k, V_k)$ being defined as in Remark 2.7.4).*

Proof. Let N be the level of the Newton graph from Theorem 2.6.4. For any periodic point $z_1 \in P_{N_p}$ of period at least two note that z_1 cannot lie on the graph Δ_N, since the only periodic points on Δ_N are the fixed points of N_p. Hence z_1 falls into one of the complementary components of Δ_N, denote it by V_1'. Let m be the period of z_1 that is greater or equal than N (an integer multiple of the minimal period of z_1) and let $m(1) = m$. We construct a neighborhood V_1, whose boundary consists of edges in the Newton graph, such that V_1 contains z_1 and every point from the postcritical set of N_p in V_1 has a finite orbit under N_p^m which always stays in V_1.

Let V_1'' be a preimage of V_1' under N_p^m such that $z_1 \in V_1''$. Then $V_1'' \subset V_1'$, since $\Delta_N \subset N_p^{-m}(\Delta_N)$ and therefore every preimage of V_1' under N_p^m is either disjoint from V_1' or its subset.

Denote $F = N_p^m : V_1'' \to V_1'$. The map N_p is postcritically finite, hence there exists a positive integer n such that $P_F(F^{-n}(V_1')) \subset K(F, V_1'', V_1')$. Among such integers n choose the minimal one and denote it by $n(1)$. Let V_1 be the component of $F^{-n(1)}(V_1')$ that contains z_1. It follows from the construction that V_1 satisfies the property that every point from $P_{N_p}(V_1)$ has a finite orbit lying in V_1 under the map F. Let $U_1 = F^{-1}(V_1) \cap V_1$. The same argument as for $V_1'' \subset V_1'$ mentioned above implies that $U_1 \subset V_1$. Note that $\partial V_1 \subset \Delta_{N_1}$ with $N_1 = N + n(1)m(1)$.

Now considering the graph Δ_{N_1} and any periodic postcritical point z_2 of period at least two which doesn't lie in V_1 we construct the domains (U_2, V_2) in a similar way. Again, denote by N_2 the minimal level of the Newton graph that contains the boundary of V_2 and so on... We analogously construct the required set of domains $(U_k, V_k)_{k \in [1,M]}$, where M is the total number of constructed components numerated by the corresponding periodic points from P_{N_p}. □

Let $N' = N + \sum_{k=1}^{M} n(k)m(k)$ be the level of the Newton graph that contains the boundaries of all components V_k for $k \in [1, M]$. Denote $F_k = N_p^{m(k)} : U_k \to V_k$. In the following we prove that slightly shrinking and enlarging the boundaries of U_k and V_k the mapping F_k between the modified components U_k and V_k can be made to be polynomial-like. Before proving the presice statement let us introduce several notions.

Definition 3.1.2 (The ε-neighborhood). *Let K be a compact subset of $\widehat{\mathbb{C}}$. The* ε-neighborhood *of K is the set of points $x \in \widehat{\mathbb{C}}$ such that $d(x, K) < \varepsilon$, where d is the spherical metric in $\widehat{\mathbb{C}}$.*

Definition 3.1.3 (Fatou and Julia vertices). *Let N_p be a Newton map and $\Delta_{N'}$ its*

Newton graph at level N'. A vertex $v \in \Delta_{N'}$ is called a Fatou-vertex *if it belongs to the Fatou set of N_p. Otherwise, it is called* a Julia-vertex.

It is easy to see that a vertex $v \in \Delta_{N'}$ is a Fatou-vertex if and only if it is eventually mapped by N_p onto one of the superattracting fixed points of N_p.

Before modifying the boundaries of U_k and V_k we first prove that in case $\partial U_k \cap \partial V_k \neq \emptyset$ there are no branched vertices on ∂U_k and ∂V_k, i.e. every vertex on ∂U_k, ∂V_k has exactly two edges starting at it.

Proposition 3.1.4. *Let $e = [v, v_1]$ be an edge of $\Delta_{N'+m(k)}$ such that $e \subset \partial U_k \cap \partial V_k$ and v is a Julia-vertex, v_1 is a Fatou-vertex. Then v has exactly two edges in ∂U_k having v as an endpoint.*

Proof. Suppose that there exist at least three edges in ∂U_k with v being their common endpoint, denote them by

$$[v, v_1],\ [v, v_2],\ [v, v_3].$$

Since $e = [v, v_1] \subset \partial U_k \cap \partial V_k$, we have that

$$[v, v_1] \subset \Delta_{N'} \cap \Delta_{N'+m(k)}.$$

Let N be the level of the Newton graph Δ_N of N_p from Theorem 2.6.4. Since $(\Delta_{N'}, N_p)$ satisfies the conditions of Definition 2.6.7, in particular its Condition (6), and $m(k) \geq N$ by the construction, it follows $\Delta_{N'+m(k)} \setminus \Delta_{N'}$ is connected and there exists a path consisting of edges in $\Delta_{N'+m(k)} \setminus \{v\}$ that connects v_1 and v_2. This contradicts to the fact that U_k is a complementary component of the Newton graph $\Delta_{N'+m(k)}$ and has edges $[v, v_1]$, $[v, v_2]$ on its boundary. $\square$

In our construction we will constantly be using the following fact from elementary topology.

Proposition 3.1.5. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a continuous map and U, V subsets of $\widehat{\mathbb{C}}$ such that U is compactly contained in V. Then $f(U)$ is compactly contained in $f(V)$, i.e. $\overline{f(U)} \subset f(V)$.*

Proof. Since $\widehat{\mathbb{C}}$ is compact, the map f is closed, i.e. it maps closed subsets of $\widehat{\mathbb{C}}$ to closed subsets of $\widehat{\mathbb{C}}$. Hence

$$\overline{f\left(\overline{U}\right)} = f\left(\overline{U}\right)$$

Therefore, since $U \subset \overline{U} \subset V$ and $f\left(\overline{U}\right) \subset f(V)$, we have

$$\overline{f(U)} \subset \overline{f(\overline{U})} \quad \text{and} \quad \overline{f(U)} \subset f(V).$$

□

Lemma 3.1.6. *For every proper map $F_k : U_k \to V_k$ constructed in Lemma 3.1.1 there exists a polynomial-like map $\widehat{F_k} : \widehat{U_k} \to \widehat{V_k}$, where $\widehat{F_k} = N_p^{m(k)}|_{\widehat{U_k}}$ so that $\deg \widehat{F_k} = \deg F_k$ and $K(\widehat{F_k}, \widehat{U_k}, \widehat{V_k}) = K(F_k, U_k, V_k)$.*

Proof. Fix some $k \in [1, M]$. It follows from Lemma 3.1.1 that $U_k \subset V_k$. However it might happen that $\partial U_k \cap \partial V_k \neq \emptyset$. If $\partial U_k \cap \partial V_k = \emptyset$, then let $\widehat{V_k} = V_k$ and $\widehat{U_k} = U_k$. Otherwise we need to slightly modify V_k to $\widehat{V_k}$ near the boundary (shrink or enlarge) so that $\widehat{U_k}$ is compactly contained in $\widehat{V_k}$, where $\widehat{U_k} = N_p^{-m(k)}(\widehat{V_k}) \cap \widehat{V_k}$. Note that the boundary ∂U_k consists of edges in the Newton graph $\Delta_{N'+m(k)}$ joining Fatou and Julia-vertices. In the following we consider the case $\partial U_k \cap \partial V_k \neq \emptyset$.

Case 1. The boundaries of U_k and V_k intersect over at least one edge. Denote by $e = [v, v_1]$ an edge in the intersection $\partial U_k \cap \partial V_k$, where v is a Julia-vertex and v_1 is a Fatou-vertex. It follows from Proposition 3.1.4 that v has exactly two edges in ∂U_k having v as an endpoint.

The construction of $\widehat{U_k}$ and $\widehat{V_k}$ is done in several steps, where we deal with vertices and edges separately. In order to keep the notation short in the following we write m instead of $m(k)$.

Step 1. *Julia vertices:* fix $\varepsilon > 0$ sufficiently small. Since ∞ is a repelling fixed point of N_p^m, for small enough $\varepsilon > 0$ there exists a disk neighborhood $\Omega(\infty)$ of radius ε with the center at ∞ such that $\overline{\Omega(\infty)} \subset N_p^m(\Omega(\infty))$. Let $\Omega'(\infty) = \Omega(\infty)$.

For every prepole $J^1 \in N_p^{-m}(\infty)$ let $\Omega_1(J^1)$ be the connected component of $N_p^{-m}(\Omega(\infty))$ that contains J^1. There exists a domain $\Omega_1'(J^1)$ in the ε-neighborhood (see Definition 3.1.2) of $\Omega_1(J^1)$ that compactly contains $\Omega_1(J^1)$, i.e. $\overline{\Omega_1(J^1)} \subset \Omega_1'(J^1)$. Hence by Proposition 3.1.5

$$\overline{\Omega'(\infty)} \subset N_p^m(\Omega_1'(J^1)). \tag{3.1}$$

For every prepole $J^k \in N_p^{-km}(\infty)$ for an integer $k > 1$ one can similarly construct domains $\Omega_k'(J^k)$ so that

$$\overline{\Omega_{k-1}'(N_p^m(J^k))} \subset N_p^m(\Omega_k'(J^k)), \quad k > 1. \tag{3.2}$$

In this way we construct domains $\Omega'(\infty)$ and $\Omega'_k(J^k)$, $k \geq 1$, around infinity and the prepoles respectively.

Step 2. *Edges*: first we construct a required neighborhood around a fixed edge e_i of the Newton graph $\Delta_{N'}$ that connects a_i and ∞. As follows from Section 2.6 the dynamics of N_p^m on U_i is conjugate to the dynamics of the map $z \mapsto z^{d_i^m}$ on the unit disk $\mathbb{D}$, where $d_i - 1 \geq 1$ is the multiplicity of a_i as a critical point of N_p (recall that by U_i we denote the immediate basin of a_i). Since we have already constructed a neighborhood $\Omega'(\infty)$ of ∞, in the following we construct a neighborhood of e_i not in the whole immediate basin U_i, but in a sufficiently small subset B_i of U_i, which is properly contained in U_i and contains $\widetilde{e_i} = e_i \setminus \Omega'(\infty)$. The dynamics of N_p^m on B_i is conjugate to the dynamics of $z \mapsto z^{d_i^m}$ on some proper subset of the unit disk. Without loss of generality we can assume that this proper subset is itself a disk of radius $1 - \varepsilon$. Within such a disk of radius $r = 1 - \varepsilon$ the mapping $\tau : z \to z^{d_i^m}$ satisfies the property that the image of any disk of radius $r' < r$ under τ is properly contained in itself. Hence there exists an ε-neighborhood $\Omega(\widetilde{e_i})$ of $\widetilde{e_i}$ in B_i such that

$$\overline{N_p^m(\Omega(\widetilde{e_i}))} \subset \Omega(\widetilde{e_i}).$$

Let $\Omega'(\widetilde{e_i}) = \Omega(\widetilde{e_i})$.

Now we construct such neighborhoods for the preimages of fixed edges under the map N_p^m. For every preimage F_i^1 of $\widetilde{e_i}$ under N_p^m let $\Omega_1(F_i^1)$ be the connected component of $N_p^{-m}(\Omega'(\widetilde{e_i}))$ that contains F_i^1. There exists a domain $\Omega'_1(F_i^1)$ in the ε-neighborhood of F_i^1 such that that $\Omega'_1(F_i^1)$ is compactly contained in $\Omega_1(F_i^1)$, i.e. $\overline{\Omega'_1(F_i^1)} \subset \Omega_1(F_i^1)$. Hence

$$\overline{N_p^m(\Omega'_1(F_i^1))} \subset \Omega'(\widetilde{e_i}). \tag{3.3}$$

For every preimage F_i^k of $\widetilde{e_i}$ under N_p^{km} for an integer $k > 1$ in the same way taking pullbacks of already constructed neighborhoods and shrinking them within the ε-neighborhood of F_i^k one can construct domains $\Omega'_k(F_i^k)$ such that

$$\overline{N_p^m(\Omega'_k(F_i^k))} \subset \Omega'_{k-1}\left(N_p^m\left(F_i^k\right)\right), \quad k > 1. \tag{3.4}$$

Note that in the construction described above we choose ε sufficiently small so that the following property holds: for every Julia-vertex J^k and F_i^k which is a part of a Fatou-edge that has J^k as one of its endpoint we have that $\partial\Omega'_k\left(F_i^k\right)$ intersects $\partial\Omega'(J^k)$ at precisely two different points.

It can happen that

$$\Omega'_k(F_i^k) \cap \Omega'_p(F_i^p) \neq \emptyset \quad \text{for some} \quad k \neq p$$

in some Fatou component U_i^k (here by U_i^k we denote a preimage component of the immediate basin U_i under $N_p^{-km}(U_i)$). If this case we let

$$\Omega'(F_i^k \cup F_i^p) = \Omega'_k(F_i^k) \cup \Omega'_p(F_i^p).$$

This gives us a set of neighborhoods around the edges (see Figure 3.1).

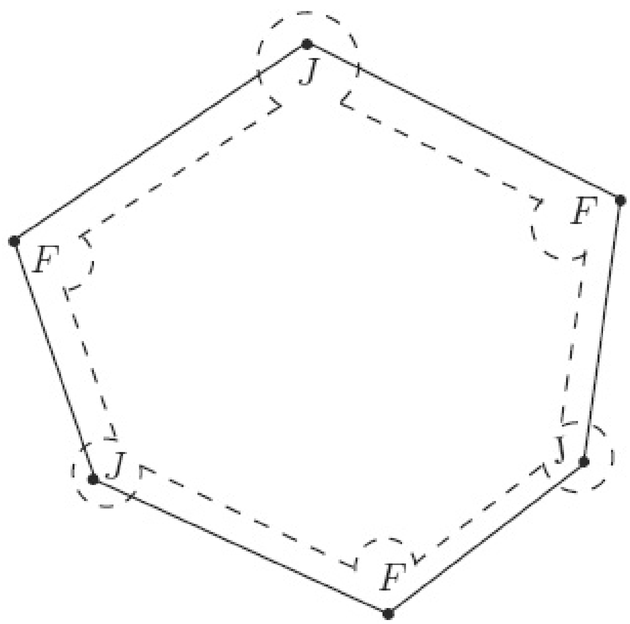

Figure 3.1: Thickening the domain U_k around edges and Julia vertices. The vertices of the Newton graph from the Fatou set are denoted by letter F, the ones from the Julia set — by J. The boundary of the modified domain is indicated by a dashed line.

Step 3. Now we construct the required neighborhood $\widehat{U_k}$ of U_k. Let $\Omega'(F)$ be the union of all neighborhoods around Fatou-edges in ∂U_k constructed in Step 2 and denote $l(F) = \partial\Omega'(F) \cap U_k$. For every neighborhood $\Omega'_k(J^k)$ of Julia-vertex $J^k \in \partial U_k$ there exist two connected components of $\partial\Omega'_k(J^k) \setminus l(F)$. Denote by $l(J^k)$ the connected component such that

$$l(J^k) \cap \widehat{\mathbb{C}} \setminus U_k \neq \emptyset$$

and by $l(J)$ denote the union of $l(J^k)$ for all Julia-vertices $J^k \in \partial U_k$. Let

$$l(U_k) = l(F) \cup l(J).$$

It follows that $l(U_k)$ is a closed curve surrounding Julia-vertices in ∂U_k. Finally, let $\widehat{U_k}$ be the component of $\widehat{\mathbb{C}} \setminus l(U_k)$ that contains the Julia-vertices of ∂U_k. It follows from (3.1) – (3.4) that $\widehat{U_k}$ is compactly contained in $\widehat{V_k}$, where $\widehat{V_k} = \widehat{F_k}\left(\widehat{U_k}\right)$ and $\widehat{F_k} = N_p^m|_{\widehat{U_k}}$. Choosing sufficiently small $\varepsilon > 0$ such that

$$P_{N_p}\left(\widehat{U_k}\right) = P_{N_p}\left(U_k\right)$$

we make sure that

$$K\left(\widehat{F_k}, \widehat{U_k}, \widehat{V_k}\right) = K\left(F_k, U_k, V_k\right) \quad \text{and} \quad \deg(\widehat{F_k}) = \deg(F_k).$$

(Since N_p is postcritically finite it is always possible to choose sufficiently small $\varepsilon > 0$ so that these properties are satisfied).

Case 2. The boundaries of U_k and V_k intersect only over vertices of $\Delta_{N'}$.

Step 4. In this case we only need to modify the boundaries of U_k and V_k at vertices in ∂U_k and ∂V_k. Similarly as in Case 1, if ∂U_k and ∂V_k intersect at a Julia-vertex v we thicken U_k slightly in the ε-neighborhood of v and leave the edges the same as in ∂U_k outside of such neighborhood. If ∂U_k and ∂V_k intersect at a Fatou-vertex v we replace ∂U_k by an arc in U_k of a sufficiently small disk within the ε-neighborhood of v. All the edges in $\partial\widehat{U_k}$ outside such an arc we leave the same. Similarly to the previous case $\widehat{U_k}$ is compactly contained in $\widehat{V_k}$, where $\widehat{V_k} = \widehat{F_k}\left(\widehat{U_k}\right)$ and $\widehat{F_k} = N_p^m|_{\widehat{U_k}}$. Choosing sufficiently small $\varepsilon > 0$ such that

$$P_{N_p}\left(\widehat{U_k}\right) = P_{N_p}\left(U_k\right)$$

we make sure that

$$K\left(\widehat{F_k}, \widehat{U_k}, \widehat{V_k}\right) = K\left(F_k, U_k, V_k\right) \quad \text{and} \quad \deg(\widehat{F_k}) = \deg(F_k).$$

□

Remark 3.1.7. Note that for each polynomial-like map $\widehat{F_k} : \widehat{U_k} \to \widehat{V_k}$ constructed in Lemma 3.1.6 and associated with a periodic postcritical point z_k we can construct an extended Hubbard tree $H^*(U_k, z_k)$ which contains the postcritical set of $\widehat{F_k}$ in $\widehat{U_k}$ and all the fixed points of $\widehat{F_k}$ in $\widehat{U_k}$. It follows from Lemma 3.1.1 and Lemma

3.1.6 that the trees $H^*(U_k, z_k)$ don't depend on the choice of the renormalization domains $\widehat{U_k}$, $\widehat{V_k}$. Trees $H^*(U_k, z_k)$ and $H^*(U_l, z_l)$ for $k \neq l$ are disjoint except possible intersection at ∞.

The trees $H^*(U_k, z_k)$ and $H^*(U_l, z_l)$ lie in domains $\widehat{U_k}$ and $\widehat{U_l}$ respectively. The only possible intersection of $\widehat{U_k}$ and $\widehat{U_l}$ could be at vertices of $\partial U_k \cup \partial V_k$ within $\left(\widehat{U_k} \setminus U_k\right) \cup \left(\widehat{V_k} \setminus V_k\right)$, since $U_k \cap V_k = \emptyset$. The only such vertex could be ∞. Hence any two trees $H^*(U_k, z_k)$ and $H^*(U_l, z_l)$ are disjoint except possibly at ∞.

Definition 3.1.8. *For a given postcritically finite Newton map N_p, let N be the minimal integer n such that no two different extended Hubbard trees $H^*(U_k, z_k)$, $k \in [1, M]$ fall into the same complementary component of Δ_n. The graph Δ_N is called* the Newton graph *of N_p. Note that such N exists due to Lemma 3.1.1, Lemma 3.1.6 and Remark 3.1.7.*

3.2 Newton rays

In order to construct a graph containing the whole postcritical set of N_p we connect extended Hubbard trees, described in the previous chapter, to the Newton graph through chains of Fatou components of N_p, from which we obtain so-called *Newton rays* starting at infinity and landing at repelling periodic points on Hubbard trees. Such chains of Fatou components are known as bubble rays, which have been used in the literature in several situations [YZ, Ro, Lu].

Definition 3.2.1. *A* bubble *of $K(N_p)$ is a Fatou component $A \subset \mathring{K}(N_p)$ in the basin of attraction of one of the fixed critical points of N_p. The* generation *of a bubble A is the smallest non-negative integer $n = Gen(A)$ such that $N_p^n(A) = A_i$, where A_i is the immediate basin of a_i for some $i \in [1, d]$. The* center *of a bubble A is the preimage $N_p^{-Gen(A)}(a_i)$ in A.*

Let us now construct a sequence of bubbles growing from infinity. Denote by

$$\mathcal{A}^0 = \bigcup_{1 \leq i \leq d} A_i.$$

Let $\mathcal{A}^1 \supset \mathcal{A}^0$ be the union of bubbles in the preimage $N_p^{-1}(\mathcal{A}^0)$ that are attached to $\mathcal{A}^0$, in other words a bubble $B \in \mathcal{A}^1$ if $\overline{B} \cap \overline{\mathcal{A}^0} \neq \emptyset$ (note that there might be bubbles $B \in N_p^{-1}(\mathcal{A}^0)$ having (pre-)poles on the boundary that are not on the boundaries of immediate basins A_i, $i \in [1, d]$).

Similarly, for an integer $j > 1$ denote by $\mathcal{A}^j$ the union of bubbles in the preimage

$N_p^{-1}(\mathcal{A}^{j-1})$ that are attached to $\mathcal{A}^{j-1}$.

Definition 3.2.2. *Consider a sequence of bubbles $B_0 \in \mathcal{A}^0$, $B_1 \in \mathcal{A}^1, \ldots, B_j \in \mathcal{A}^j, \ldots,$ such that $Gen(B_j) > Gen(B_{j-1})$ and $\overline{B_j} \cap \overline{B_{j-1}} \neq \emptyset$ for all $j \geq 1$. The closure of the union*

$$\mathcal{B} = \overline{\bigcup_{j\geq 0} B_j}$$

is called a bubble ray*.*

A bubble ray consisting of finitely many bubbles B_0, B_1, ..., B_m is called a *finite bubble ray.* In such case the bubble B_m is called the *end* of the bubble ray $\overline{\bigcup_{0\leq j\leq m} B_j}$. The generation $\mathrm{Gen}(B_m)$ is called the *generation of the bubble ray* $\overline{\bigcup_{0\leq j\leq m} B_j}$.

Subhyperbolicity of N_p implies that the diameters of B_j decay exponentially as j increases and the tail of $\mathcal{B}$ converges to a unique point which we denote by $t(\mathcal{B})$. Hence

$$\mathcal{B} = \bigcup_{j\geq 0} \overline{B_j} \cup \{t(\mathcal{B})\}.$$

For each bubble B_j it follows from the construction that $N_p^{\mathrm{Gen}(B_j)}(B_j) = A_i$ for some $i \in [1, d]$. As was mentioned in Section 2.6 each immediate basin A_i has a global Böttcher coordinate $\phi_i : (\mathbb{D}, 0) \to (A_i, a_i)$ such that $N_p(\phi_i(z)) = \phi_i(z^{k_i})$, $z \in \mathbb{D}$, where $k_i - 1$ is the multiplicity of a_i as a critical point of N_p. The map ϕ_i allows us to define the notion of internal rays at angles ϑ in each A_i as images $\phi_i(re^{i\vartheta})$, $0 < r < 1$. Lifting the map ϕ_i to B_j we can define internal rays in B_j.

Let Δ_N be the Newton graph of N_p from Definition 3.1.8.

For a bubble ray

$$\mathcal{B} = \overline{\bigcup_{j\geq 0} B_j}$$

defined above one can construct a path connecting ∞ to $t(\mathcal{B})$ consisting of closures of internal rays in closures of bubbles of $\mathcal{B}$ along the edges of Δ_N and their preimages under N_p. Denote by $\mathcal{R}^*(\mathcal{B})$ the union of the closures of internal rays in $\bigcup_{j\geq 0} \overline{B_j}$ joining the centers of bubbles B_j (see Figure 3.2).

Definition 3.2.3. *We call $\mathcal{R}^*(\mathcal{B})$* the extended Newton ray *associated with the bubble ray $\mathcal{B}$. Then we say that $\mathcal{R}^*(\mathcal{B})$* lands *at $t(\mathcal{B})$. The part $\mathcal{R}(\mathcal{B}) \subset \mathcal{R}^*(\mathcal{B})$ such that the intersection $\mathcal{R}(\mathcal{B}) \cap \Delta_N$ is precisely one vertex of Δ_N is called* the Newton ray *associated with $\mathcal{B}$.*

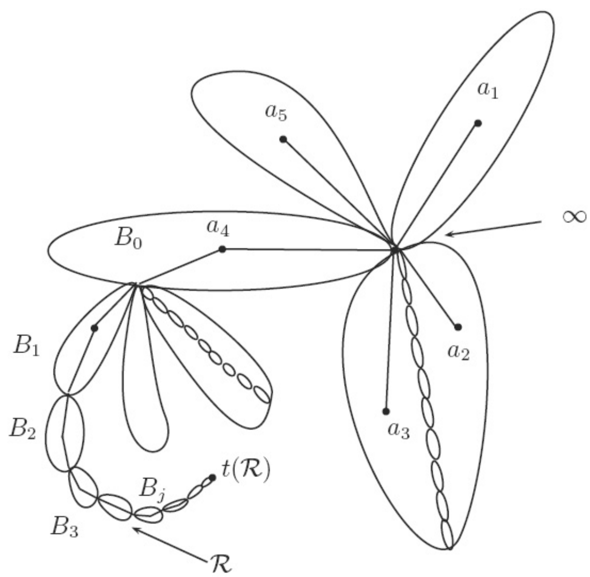

Figure 3.2: Schematic construction of a bubble ray and the Newton ray associated to it for the Newton map of degree 5.

Similarly, one can define an extended Newton ray and a Newton ray for a finite bubble ray as a finite union of closures of internal rays in closures of bubbles.

Definition 3.2.4. *An extended Newton ray $\mathcal{R}^*(\mathcal{B})$ is said to be* periodic *if there exists an integer $m \geq 1$ such that $N_p^m(\mathcal{R}^*(\mathcal{B})) = \mathcal{R}^*(\mathcal{B}) \cup \mathcal{E}$, where $\mathcal{E} \subset \Delta_N$ is a union of edges of Δ_N (note here that since Δ_N is a finite graph, the union $\mathcal{E} \subset \Delta_N$ is finite). The smallest such m is* the period *of $\mathcal{R}^*(\mathcal{B})$. A Newton ray $\mathcal{R}(\mathcal{B})$ is said to be* periodic *if there exists an integer $m \geq 1$ such that $N_p^m(\mathcal{R}(\mathcal{B})) = \mathcal{R}(\mathcal{B}) \cup \mathcal{E}$, where $\mathcal{E} \subset \Delta_N$ is a union of edges of Δ_N. The smallest such m is* the period *of $\mathcal{R}(\mathcal{B})$.*

Now we show that for each of the repelling fixed points of the map on each extended Hubbard tree constructed in the previous section, there exists a periodic (extended) Newton ray that lands at this point. More generally, the following lemma holds.

Lemma 3.2.5. *Let ω be a repelling periodic point of period $m > 1$ of N_p. Then there exists a periodic (extended) Newton ray $\mathcal{R}$ which lands at ω. The period of $\mathcal{R}$ is an integer multiple of m.*

Proof. Denote $f = N_p^m$. Since ω is repelling, there exists a neighborhood Y of ω such that Y contains a bubble $B \subset \bigcup_{i\geq 0} N_p^{-i}(\mathcal{A}^0)$ and ω is an attracting fixed point for some branch $h = f^{-1}$ with $h(Y) \subset Y$. Theorem 2.6.2 implies that the bubble B is the end of some finite union of bubble rays and their preimages, in other words there exist an integer n, a sequence of integers $k_i \geq 0$ and bubbles $B_i \in \mathcal{A}^i$ for $0 \leq i \leq n$, such that the sequence of bubbles $\mathcal{B} = \bigcup_{0 \leq i \leq n} N_p^{-k_i}(B_i)$ ends at B.

Taking the preimages of $\mathcal{B}$ under f we obtain a sequence of bubble rays preimages $\mathcal{B}_k = f^{-k}(\mathcal{B})$, $k \geq 0$. Since ω is an attracting fixed point of h, the ends of finite bubble rays $\mathcal{B}_k$ converge to ω. From any finite bubble ray of generation $\leq G$ in $\mathcal{B}$ or one of its preimages we obtain a finite bubble ray of generation $\leq G$ in $\mathcal{B}_k$ for every integer $k \geq 1$. Since there are only finitely many of finite bubble rays of generation $\leq G$ for every positive integer G, there must be a bubble ray $\mathcal{B}'_0$ contained in infinitely many of $\mathcal{B}_k$. Let $\mathcal{B}_{k_0}$ be the $\mathcal{B}_k$ containing $\mathcal{B}'_0$ that has the lowest generation and $\mathcal{B}_{k_1}$ — the second lowest. Then $f^r(\mathcal{B}_{k_1}) = \mathcal{B}_{k_0}$ for some $r \geq 1$ and the preimage of $\mathcal{B}'_0$ under f^r is a longer bubble ray $\mathcal{B}'_1 \supset \mathcal{B}'_0$ with $\mathcal{B}'_1 \subset \mathcal{B}_{k_1}$. Taking further preimages under the same branch of $g = f^{-r}$ we get a sequence of nested finite bubble rays $\mathcal{B}'_n$ such that $\mathcal{B}'_0 \subset \mathcal{B}'_1 \subset \ldots \subset \mathcal{B}'_n \subset \ldots$ with $\mathcal{B}'_n \subset \mathcal{B}_{k_n}$. For sufficiently big generation G the postcritial set of f is disjoint from the bubbles in $\mathcal{B}_{k_n} \setminus \mathcal{B}'_n$ and there exists a neighborhood of all bubbles in $\mathcal{B}_{k_0} \setminus \mathcal{B}'_0$ where g^n is defined for all $n \geq 0$.

Hence subhyperbolicity of f implies that the bubbles in $\mathcal{B}_{k_n} \setminus \mathcal{B}'_n$ shrink to points as n increases. On the other hand, the ends of $\mathcal{B}_{k_n}$ converge to ω as n increases. Therefore the bubbles in $\mathcal{B}_{k_n} \setminus \mathcal{B}'_n$ shrink to ω as n goes to infinity and we conclude that the union

$$\mathcal{B}' = \bigcup_{n \geq 0} \mathcal{B}'_n$$

converges to ω. However it might happen that $\mathcal{B}'_0 \subset \mathcal{B}'$ is not a bubble ray according to Definition 3.2.3, since it might contain preimages of bubble rays which don't belong to $\mathcal{A}^i$ for all $i \geq 0$. Recall that $\mathcal{B}$ is a finite union of bubbles $N_p^{-k_i}(B_i)$, where $B_i \in \mathcal{A}^i$, $0 \leq i \leq n$. Let $K = \max\{k_0, k_1, \ldots, k_n\}$. Then, the image $\mathcal{B}'' = N_p^{Km}(\mathcal{B}')$ lands at ω and $\mathcal{B}''$ is a bubble ray. Moreover, $\mathcal{B}''$ is periodic of period an integer multiple of m.

According to the discussion above, the extended Newton ray $\mathcal{R}^* = \mathcal{R}^*(\mathcal{B}'')$ associated with the bubble ray $\mathcal{B}''$ gives a required periodic extended Newton ray that lands at ω and has the period that is an integer multiple of m. The part $\mathcal{R}(\mathcal{B}) \subset \mathcal{R}^*(\mathcal{B})$ such that $\mathcal{R}(\mathcal{B}) \cap \Delta_N$ is a vertex of Δ_N is the corresponding Newton ray that lands at ω and has the period that is an integer multiple of m. □

In Section 3.1 the polynomial-like mappings $\widehat{F}_k : \widehat{U}_k \to \widehat{V}_k$, $k \in [1, M]$, of periods

$m(k)$ were constructed. Let us fix k and from now on let ω be a β-fixed point of $\widehat{F}_k$. Since ω is a β-fixed point of $\widehat{F}_k$, the extended Hubbard tree $H^*(U_k, z_k)$ associated to $\widehat{F}_k$ has exactly one edge that has ω as one of its endpoints (otherwise $H^*(U_k, z_k) \setminus \{\omega\}$ would be disconnected), denote this edge by E_ω.

Remark 3.2.6. One of the properties of (extended) Newton rays that plays an important role in the following discussion is that for any (extended) Newton ray $\mathcal{R}$ that lands at $t(\mathcal{R})$ and for any extended Hubbard tree $H^*(U_k, z_k)$, $k \in [1, M]$,

$$\mathcal{R} \setminus \{t(\mathcal{R})\} \cap H^*(U_k, z_k) = \emptyset.$$

Indeed, any point $x \in \mathcal{R} \setminus \{t(\mathcal{R})\}$ is eventually mapped onto Δ_N after some iterate of N_p. On the other hand, every tree $H^*(U_k, z_k)$, $k \in [1, M]$, is invariant under appropriate iterate of N_p and all forward images of $H^*(U_k, z_k)$ under N_p are disjoint from Δ_N.

In the following we prove that there exists a periodic Newton ray that lands at ω and has period $m = m(k)$. It follows from Lemma 3.2.5 that there exists an integer r and a periodic extended Newton ray $\mathcal{R}_{1\omega}$ of period mr that lands at ω. Since $N_p^m(\omega) = \omega$, the images $\mathcal{R}_{i\omega} = N_p^{im}(\mathcal{R}_{1\omega})$, $1 \le i \le r$, form a cycle of periodic extended Newton rays that all land at ω.

Proposition 3.2.7. *Let $C = \{\mathcal{R}_{1\omega}, \mathcal{R}_{2\omega}, \dots, \mathcal{R}_{r\omega}\}$ be a cycle of periodic extended Newton rays of period mr that land at the β-fixed point ω of $\widehat{F}_k$. If $r > 1$, then any two rays $\mathcal{R}_{i\omega}$, $\mathcal{R}_{j\omega}$, $i \neq j$, $i, j \in [1, r]$, intersect at infinitely many points. The intersections are locally finite, except at ω, i.e. for every intersection point v of $\mathcal{R}_{i\omega}$ and $\mathcal{R}_{j\omega}$ different from ω, there exists a neighborhood of v such that the only intersection point of $\mathcal{R}_{i\omega}$ and $\mathcal{R}_{j\omega}$ in this neighborhood is v.*

Proof. Let Y be a small enough neighborhood of ω such ω is an attracting fixed point for some branch $h = N_p^{-m}$, $h(Y) \subset Y$ and such that there are no postcritical points of N_p in Y. Assume that, on the contrary, there exists a pair of extended Newton rays $\mathcal{R}_{i\omega}$, $\mathcal{R}_{j\omega}$ that have at most finitely many intersection points. Taking possibly further restriction of Y to its subset we can assume that $\mathcal{R}_{i\omega}$, $\mathcal{R}_{j\omega}$ don't intersect inside Y. Then any two rays in the cycle C will also be disjoint in Y. Without loss of generality assume that the cyclic order around ω in Y is $\mathcal{R}_{1\omega}, \mathcal{R}_{2\omega}, \dots, \mathcal{R}_{r\omega}$. Let S_r be the "sector" bounded by $\mathcal{R}_{r\omega}$, $\mathcal{R}_{1\omega}$ and contained in Y. The image $S_1 = N_p^m(S_r)$ is the "sector" bounded by $\mathcal{R}_{1\omega}$, $\mathcal{R}_{2\omega}$ and contained in $N_p^m(Y) \supset Y$. Since

$$H^*(U_k, z_k) \cap S_r \neq \emptyset$$

this would imply that

$$H^*(U_k, z_k) \cap S_1 \neq \emptyset,$$

because

$$N_p^m\left(H^*(U_k, z_k)\right) = H^*(U_k, z_k).$$

Note that $H^*(U_k, z_k)$ is disjoint from $\mathcal{R}_{i\omega} \setminus \{\omega\}$, $i \in [1, r]$, and therefore this would imply that $H^*(U_k, z_k) \setminus \{\omega\}$ is disconnected which is impossible, since ω is a β-fixed point of $\widehat{F}_k$.

Local finiteness of intersections of $\mathcal{R}_{i\omega}$ and $\mathcal{R}_{j\omega}$ follows from the fact that every intersection point v of $\mathcal{R}_{i\omega}$ and $\mathcal{R}_{j\omega}$ is a preimage under N_p of a vertex from Δ_N and there exists a sufficiently small neigborhood of v such that the only intersection point of $\mathcal{R}_{i\omega}$ and $\mathcal{R}_{j\omega}$ in this neighborhood is v. □

Note that any two (extended) Newton rays can intersect only at preimages of vertices of Δ_N, over preimages of edges of Δ_N under N_p or at their landing point (in case it is the same for both rays).

Let us fix now for the rest of the thesis the counterclockwise orientation in $\mathcal{S}^2$.

Definition 3.2.8 (Newton Rays Order)**.** *Let $\mathcal{R}'$, $\mathcal{R}''$ be (extended) Newton rays landing at ω and $E_w \subset H^*(U_k, z_k)$ be the edge of $H^*(U_k, z_k)$ with endpoint ω. Assume that $\mathcal{R}'$ and $\mathcal{R}''$ don't cross-intersect, i.e. they satisfy the following property: if l is a curve disjoint from $\mathcal{R}'$, $\mathcal{R}''$ and connecting the endpoints of $\mathcal{R}''$ and E_w different from ω, then $\mathcal{R}'$ lies in one complementary component of $\widehat{\mathbb{C}} \setminus \left(E_\omega \cup l \cup \mathcal{R}''\right)$. We say that $\mathcal{R}' \succeq \mathcal{R}''$ if there exists a neighborhood Y of ω such that for some branch $h = N_p^{-m}$, $h(Y) \subset Y$ and the cyclic order around ω is*

$$\mathcal{R}'(Y),\ \mathcal{R}''(Y),\ E_w(Y),$$

where

$$\mathcal{R}'(Y) = \mathcal{R}' \cap Y, \quad \mathcal{R}''(Y) = \mathcal{R}'' \cap Y, \quad E_w(Y) = E_w \cap Y.$$

Remark 3.2.9. Note that for any neighborhood $Y' \subset Y$ of ω the cyclic order of $\mathcal{R}''(Y')$, $\mathcal{R}'(Y')$, $E_w(Y')$ around ω is the same as the cyclic order of $\mathcal{R}''(Y)$, $\mathcal{R}'(Y)$, $E_w(Y)$ around ω. Hence the relation $\mathcal{R}' \succeq \mathcal{R}''$ is well defined and doesn't depend on the choice of the neighborhood Y in Definition 3.2.8.

Now for given two periodic Newton rays $\mathcal{R}_1$, $\mathcal{R}_2$ landing at a β-fixed point ω of $\widehat{F}_k$ we construct a Newton ray $\mathcal{R} = RE(\mathcal{R}_1, \mathcal{R}_2)$ such that

$$\mathcal{R} \succeq \mathcal{R}_1,\ \mathcal{R} \succeq \mathcal{R}_2 \quad \text{and} \quad \mathcal{R} \quad \text{lands at} \quad \omega.$$

Let Y be a neighborhood of ω such that for some branch $h = N_p^{-m}$, $h(Y) \subset Y$, $E_w(Y) \subsetneqq E_w$ and

$$\partial Y \cap \mathcal{R}_1 = \{v_1\}, \quad \partial Y \cap \mathcal{R}_2 = \{v_2\},$$

where v_1, v_2 are iterated preimages of vertices of Δ_N under N_p. Denote by $Y_1, Y_2, \ldots$ the connected components of

$$Y \setminus (\mathcal{R}_1 \cup \mathcal{R}_2 \cup E_w)$$

(there might be infinitely many of such domains according to Proposition 3.2.7). Without loss of generality assume that

$$E_w(Y) \subset \partial Y_1 \cap \partial Y_2$$

and the cyclic order around ω is Y_2, E_w, Y_1. Denote

$$RE_Y(\mathcal{R}_1, \mathcal{R}_2) = \partial Y_1 \setminus (E_w \cup \partial Y)\,.$$

It follows from the construction that

$$RE_Y(\mathcal{R}_1, \mathcal{R}_2) \subset \mathcal{R}_1(Y) \cup \mathcal{R}_2(Y).$$

Let i be the smallest positive integer i such that

$$N_p^{im}(RE_Y(\mathcal{R}_1, \mathcal{R}_2)) \cap \Delta_N \neq \emptyset.$$

Denote

$$\mathcal{R} = RE(\mathcal{R}_1, \mathcal{R}_2) = N_p^{im}(RE_Y(\mathcal{R}_1, \mathcal{R}_2)).$$

Note that if two Newton rays intersect, then the bubble rays associated to them must have common bubbles containing the intersection points or edges over which the Newton rays intersect. Vice versa, every intersection point or a common edge of two Newton rays is contained in a bubble that is a common part of bubble rays associated to both of the Newton rays. Therefore, if the condition of decreasing generations of bubbles containing the edges of Newton rays $\mathcal{R}_1$ and $\mathcal{R}_2$ is satisfied, then this condition is also satisfied for the ray $\mathcal{R} = RE(\mathcal{R}_1, \mathcal{R}_2)$. Hence $\mathcal{R}$ is a periodic Newton ray that lands at ω such that

$$\mathcal{R} \succeq \mathcal{R}_1,\, \mathcal{R} \succeq \mathcal{R}_2 \quad \text{and} \quad \mathcal{R} \subset \mathcal{R}_1 \cup \mathcal{R}_2.$$

The ray $\mathcal{R} = RE(\mathcal{R}_1, \mathcal{R}_2)$ is said to be *the right envelope* of Newton rays $\mathcal{R}_1, \mathcal{R}_2$ (see

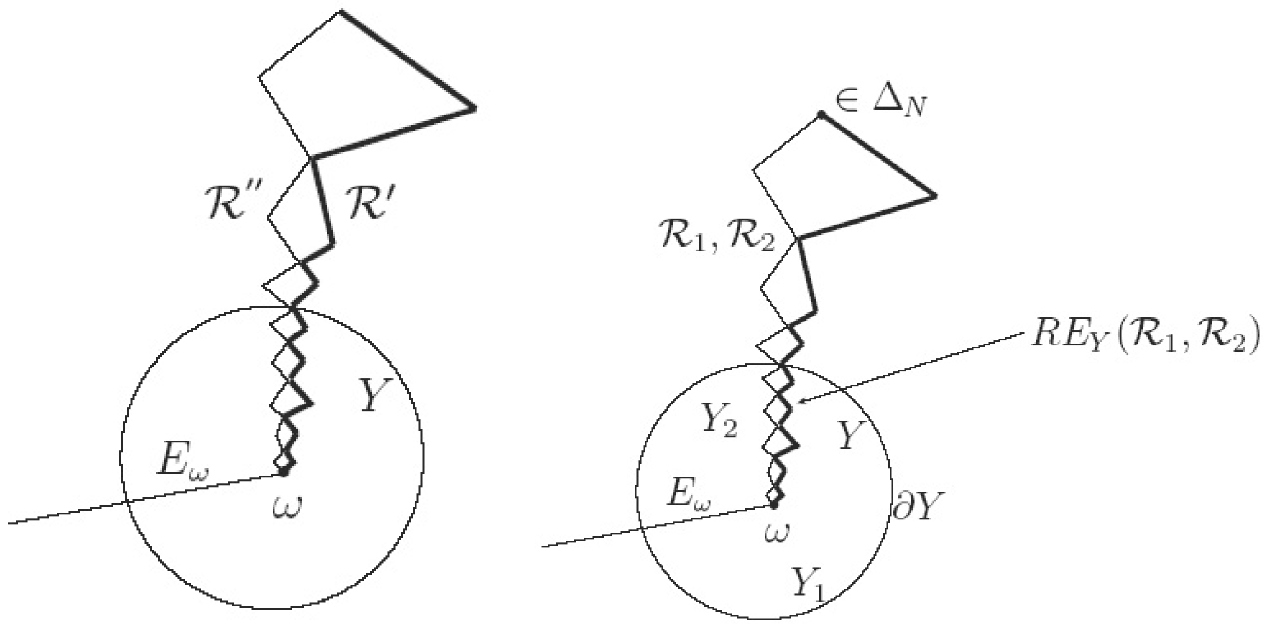

Figure 3.3: Construction of the right envelope of two Newton rays $\mathcal{R}_1$, $\mathcal{R}_2$. On the left-hand side two Newton rays $\mathcal{R}'$ and $\mathcal{R}''$ such that $\mathcal{R}' \succeq \mathcal{R}''$ are shown. On the right-hand side for $\mathcal{R}_1$ and $\mathcal{R}_2$ their right envelope $RE_Y(\mathcal{R}_1, \mathcal{R}_2)$ in a neighborhood Y of ω is shown.

Figure 3.3). Analogously the right envelope $RE(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n)$ of finitely many Newton rays $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$ is constructed.

Remark 3.2.10. Note that for any neighborhood $Y' \subset Y$ of ω

$$RE_{Y'}(\mathcal{R}_1, \mathcal{R}_2) \subset RE_Y(\mathcal{R}_1, \mathcal{R}_2)$$

and

$$\bigcup_{i \geq 0} N_p^{im}\left(RE_{Y'}(\mathcal{R}_1, \mathcal{R}_2)\right) = \bigcup_{i \geq 0} N_p^{im}\left(RE_Y(\mathcal{R}_1, \mathcal{R}_2)\right).$$

Hence the construction of the Newton ray $RE(\mathcal{R}_1, \mathcal{R}_2)$ doesn't depend on the choice of Y.

Lemma 3.2.11. *For any β-fixed point ω of the polynomial-like mapping $\widehat{F}_k : \widehat{U}_k \to \widehat{V}_k$ of period m there exists a Newton ray of period m that lands ω. There exist only finitely many Newton rays of period m landing at ω.*

Proof. It follows from Lemma 3.2.5 that there exists a positive integer r and a Newton ray $\mathcal{R}_{1\omega}$ of period mr that lands at ω. Let $\mathcal{R}_{i\omega} = N_p^{im}(\mathcal{R}_{1\omega})$ for $i \in [1, r]$ and

$$\mathcal{R} = RE(\mathcal{R}_{1\omega}, \mathcal{R}_{2\omega}, \ldots, \mathcal{R}_{r\omega})$$

be the right envelope of $\mathcal{R}_{1\omega}, \mathcal{R}_{2\omega}, \ldots, \mathcal{R}_{r\omega}$. Then $\mathcal{R}$ is a Newton ray that lands at ω.

Denote by Y the neighborhood of ω such that for some branch $h = N_p^{-m}$, $h(Y) \subset Y$ and let Y_1 be the connected component of

$$Y \setminus \bigcup_{i=1}^{r} \mathcal{R}_{i\omega}$$

such that

$$\mathcal{R} \cap \partial Y_1 \neq \emptyset \quad \text{and} \quad E_\omega \cap \partial Y_1 \neq \emptyset.$$

Since the map N_p is orientation preserving,

$$N_p^m(Y_1) \cap Y_1 = Y_1.$$

Thus $N_p^m(\mathcal{R}) = \mathcal{R} \cup \mathcal{E}$, where $\mathcal{E}$ is a union of edges of Δ_N. Therefore the right envelope $\mathcal{R}$ of the rays from the cycle $C = \{\mathcal{R}_{1\omega}, \mathcal{R}_{2\omega}, \ldots, \mathcal{R}_{r\omega}\}$ is a periodic Newton ray of period m.

It is left to prove that there exist only finitely many Newton rays of period m that land at ω. Indeed, for any periodic Newton ray $\mathcal{R}$ of period m

$$N_p^m(\mathcal{R}) = \mathcal{R} \cup \mathcal{E}$$

and $\mathcal{R}$ consists of iterated preimages of $\mathcal{E}$ under N_p^m. Hence the union $\mathcal{E}$ of edges in Δ_N determines the ray $\mathcal{R}$ uniquely and the number of such rays $\mathcal{R}$ is bounded above by $(\#\Delta_N)^m$, where $\#\Delta_N$ is the number of edges of the Newton graph Δ_N. □

It follows from Lemma 3.2.11 that for every cycle C_i of periodic Newton rays landing at ω there exists the right envelope $\mathcal{R}_i$ of rays in C_i such that $\mathcal{R}_i$ is a Newton ray of period m landing at ω. Moreover, there exist only finitely many of such rays $\mathcal{R}_i$. Let q be the number of them.

Definition 3.2.12. *Denote by* $\mathcal{R}_\omega = RE\,(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_q)$ *the right envelope of the rays* $\mathcal{R}_i$, $i \in [1, q]$. *By* $\mathcal{R}_\omega^* \supset \mathcal{R}_\omega$ *we denote the corresponding extended Newton ray.*

3.3 Two examples

Example 1. On Figure 3.4 the dynamical plane of the Newton map N_p of degree 5 with $p(z) = z^5 - 4z + 4$ is shown (see also Figure 2.1). It easy to compute that

$$N_p(z) = z - \frac{p(z)}{p'(z)} = \frac{4(z^5 - 1)}{5z^4 - 4}.$$

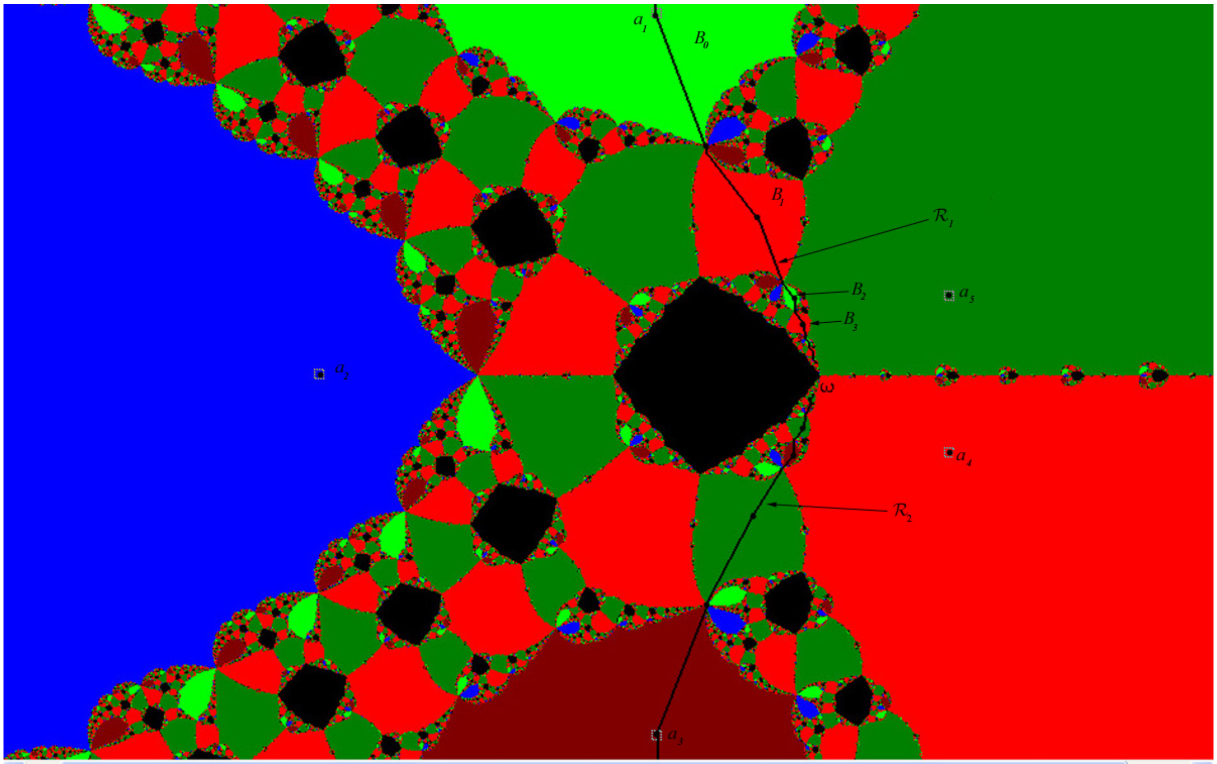

Figure 3.4: The Newton map N_p of degree 5 for the polynomial $p(z) = z^5 - 4z + 4$ with the superattracting 2-cycle $0 \mapsto 1 \mapsto 0$ and point ω such that $N_p^2(\omega) = \omega$ with two periodic extended Newton rays $\mathcal{R}_1$, $\mathcal{R}_2$ of period 2 landing at ω.

The origin 0 is a superattracting periodic point of period 2 for N_p with the superattracting 2-cycle

$$0 \mapsto 1 \mapsto 0.$$

The roots of the polynomial p are denoted by $a_1, a_2, \ldots, a_5$. Each point is colored according to the root to which iterations of N_p converge for this starting value. The open black regions indicate starting values that do no converge to any root. The big black open region in the center of Figure 3.4 indicates the immediate basin of the superattracting periodic point 0. Every point in this basin converges to the cycle $0 \mapsto 1 \mapsto 0$ under the iterates of N_p.

The boundary of the immediate basing of 0 contains a repelling periodic point $\omega \in \mathbb{R}$, $\omega \approx 0.468$ such that

$$N_p^2(\omega) = \omega.$$

In the following we explain the construction of two periodic Newton rays $\mathcal{R}_1$ and $\mathcal{R}_2$ landing at ω. Let B_0 be the immediate basin of a_1 and B_1 be the first preimage of the immediate basin of a_4 such that B_0 and B_1 touch at the point $i\sqrt[4]{4/5}$, one of the poles of N_p. Denote by B_2 the second preimage of B_0 under N_p (i.e. $N_p^2(B_2) = B_0$)

such that

$$N_p^2\left(\overline{B_1} \cap \overline{B_2}\right) = \infty.$$

Similarly, B_3 is the bubble touching B_2 such that $N_p^2(B_3) = B_1$ and so on ....

As was mentioned in Section 2.6, each immediate basin A_i of a_i for $i \in [1,5]$ has a global Böttcher coordinate $\phi_i : (\mathbb{D}, 0) \to (A_i, a_i)$ such that

$$N_p(\phi_i(z)) = \phi_i(z^2), \quad z \in \mathbb{D}.$$

The map ϕ_i allows us to define the notion of internal rays at angles ϑ in each A_i as images $\phi_i(re^{i\vartheta})$, $0 < r < 1$. Denote such internal ray by $R_i(\vartheta)$ and its landing point on ∂A_i by $l_i(\vartheta)$. Let us fix the Böttcher coordinate maps ϕ_i such that $l_i(0) = \infty$ for $i \in [1,5]$. Since $N_p(B_1) = A_4$, the map ϕ_4 can be lifted to B_1, because

$$N_p : B_1 \to A_4 \quad \text{is conformal.}$$

This allows us to define internal rays and their landing points in B_1. An internal ray in B_1 at angle ϑ defined in this way we denote by $R_4^1(\vartheta)$ and its landing point on ∂B_1 by $l_4^1(\vartheta)$. Similarly we define internal rays in B_2, $B_3, \ldots$. Let

$$\mathcal{R}_1 = \overline{R_1(0)} \cup \overline{R_1\left(1/2\right)} \cup \overline{R_4^1(0)} \cup \overline{R_4^1\left(1/2\right)} \cup \overline{R_1^2(0)} \cup \overline{R_1^2\left(1/2\right)} \cup \ldots.$$

It follows from the construction that

$$N_p^2(\mathcal{R}_1) = \mathcal{R}_1 \cup \overline{R_4(0)},$$

and $\mathcal{R}_1$ is a periodic extended Newton ray of period 2 that lands at ω. Similarly

$$\mathcal{R}_2 = \overline{R_3(0)} \cup \overline{R_3\left(1/2\right)} \cup \overline{R_5^1(0)} \cup \overline{R_5^1\left(1/2\right)} \cup \overline{R_3^2(0)} \cup \overline{R_3^2\left(1/2\right)} \cup \ldots$$

is a periodic extended Newton ray of period 2 landing at ω such that

$$N_p^2(\mathcal{R}_2) = \mathcal{R}_2 \cup \overline{R_5(0)}.$$

Example 2. The dynamical plane of the Newton map N_p of degree 6 with

$$p(z) = 17z^6 - z^5 - 89z + 89$$

is shown on Figure 3.5. The map N_p has two free critical points that are not fixed under N_p: the origin 0 and $\frac{2}{51}$. The origin 0 is a superattracting periodic point of

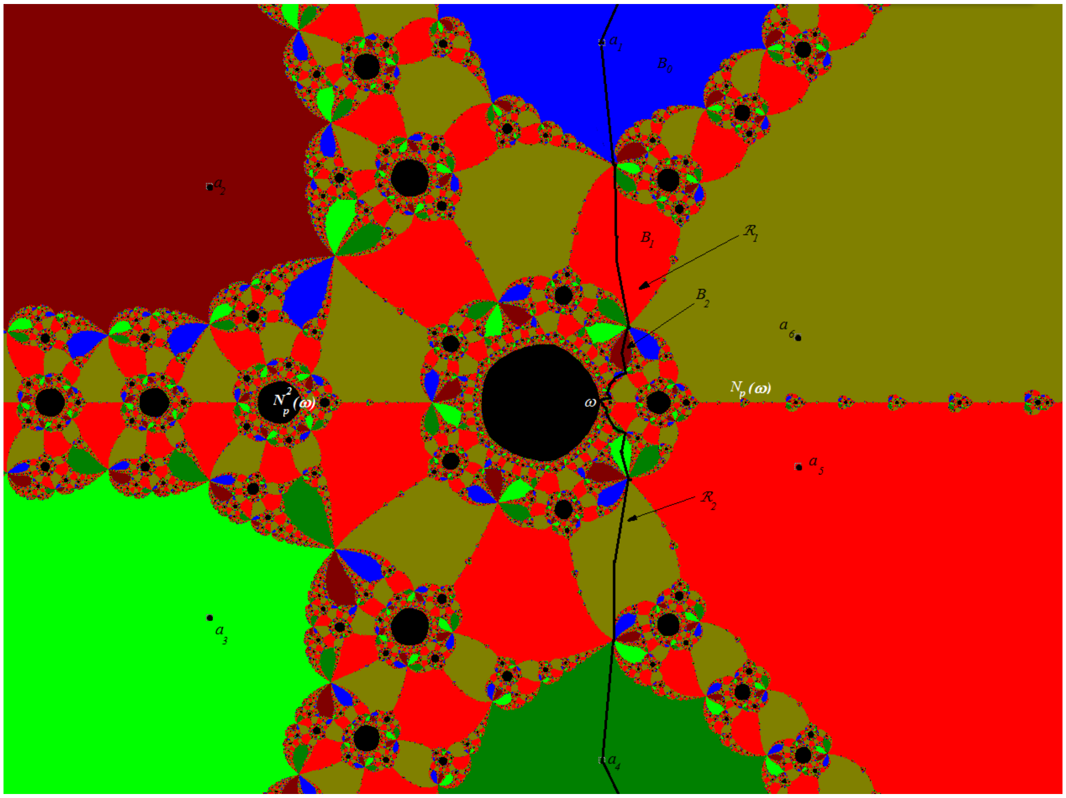

Figure 3.5: The Newton map N_p of degree 6 for the polynomial $p(z) = 17z^6 - z^5 - 89z + 89$ with the superattracting 3-cycle $0 \mapsto 1 \mapsto -1 \mapsto 0$ and point ω such that $N_p^3(\omega) = \omega$ with two periodic extended Newton rays $\mathcal{R}_1$, $\mathcal{R}_2$ of period 3 landing at ω.

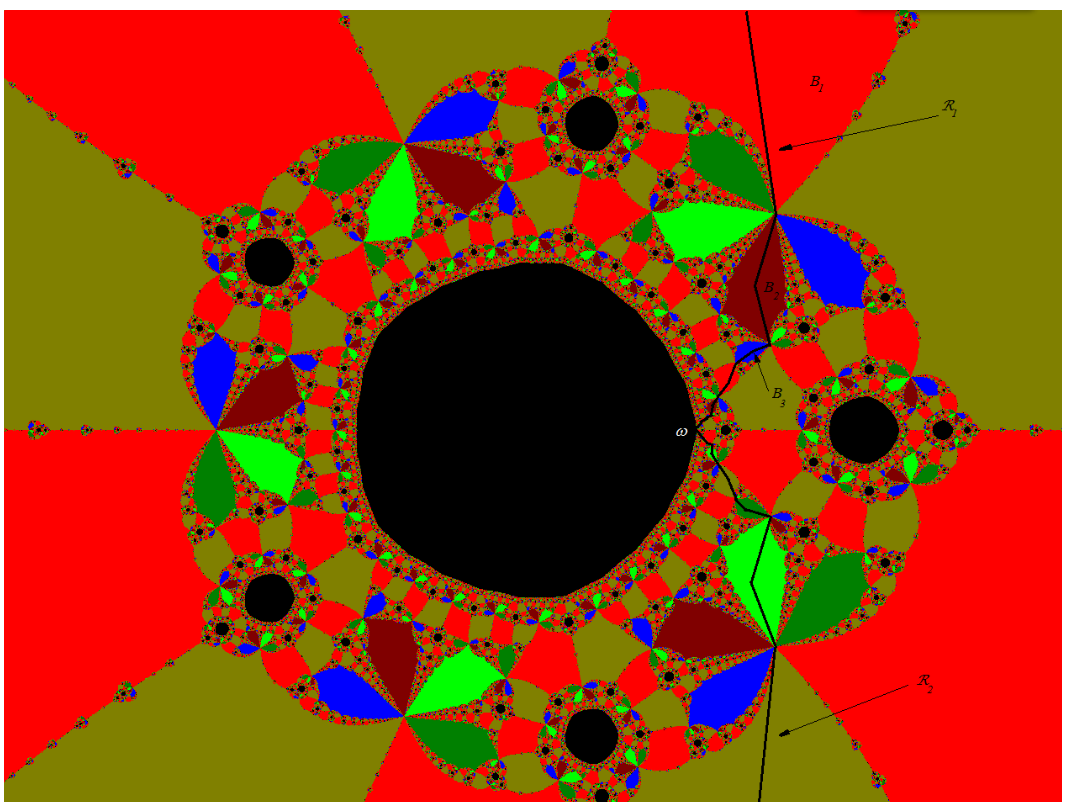

Figure 3.6: Magnification of Figure 3.5 around ω.

period 3 for N_p with the superattracting 3-cycle

$$0 \mapsto 1 \mapsto -1 \mapsto 0.$$

The other free critical point $\frac{2}{51}$ is contained in the immediate basin of 0 and its orbit converges to the cycle $0 \mapsto 1 \mapsto -1 \mapsto 0$ under the iterates of N_p. The map N_p is no longer postcritically finite as in the previous example, but it is hyperbolic. The roots of the polynomial p are denoted by $a_1, a_2, \ldots, a_6$ and the corresponding immediate basins by $A_1, A_2, \ldots, A_6$. Each point is colored according to the root to which iterations of N_p converge for this starting value. The open black regions indicate starting values that do not converge to any root but converge to the superattracting 3-cycle $0 \mapsto 1 \mapsto -1 \mapsto 0$.

The boundary of the immediate basin of 0 contains a repelling periodic point $\omega \approx 0.255$ such that

$$N_p^3(\omega) = \omega.$$

The images $N_p(\omega)$ and $N_p^2(\omega)$ lie on the boundaries of the immediate basins of points 1 and -1 respectively, they are indicated on Figure 3.5.

Now we explain the construction of two periodic extended Newton rays $\mathcal{R}_1$ and $\mathcal{R}_2$ landing at ω. Let $B_0 = A_2$ be the immediate basin of a_2 and B_1 be the first preimage of A_5 under N_p such that $\overline{B_0}$ and $\overline{B_1}$ touch at the pole of N_p that belongs to ∂B_0. Denote by B_2 the second preimage of A_2 under N_p (i.e. $N_p^2(B_2) = A_2$) such that

$$N_p^2\left(\overline{B_1} \cap \overline{B_2}\right) = \infty.$$

By B_3 denote the bubble such that

$$N_p^3\left(B_3 \cap B_2\right) = \infty \quad \text{and} \quad N_p^3(B_3) = B_0$$

and so on ....

Keeping the same notation for internal rays in bubbles as in the previous example, let

$$\mathcal{R}_1 = \bigcup_{n \geq 0} \overline{R_1^{3n}(0) \cup R_1^{3n}(1/2)} \cup \overline{R_5^{3n+1}(0) \cup R_5^{3n+1}(1/2)} \cup \overline{R_2^{3n+2}(0) \cup R_2^{3n+2}(1/2)}$$

with the convention that $R_1^0(0) = R_1(0)$ and $R_1^0(1/2) = R_1(1/2)$. It follows from the construction that

$$N_p^3(\mathcal{R}_1) = \mathcal{R}_1 \cup \overline{R_5(0)} \cup \overline{R_2(0)}$$

and $\mathcal{R}_1$ is a periodic extended Newton ray of period 3 that lands at ω. Similarly, the extended Newton ray $\mathcal{R}_2$ is constructed

$$\mathcal{R}_2 = \bigcup_{n\geq 0} \overline{R_4^{3n}(0) \cup R_4^{3n}(1/2)} \cup \overline{R_6^{3n+1}(0) \cup R_6^{3n+1}(1/2)} \cup \overline{R_3^{3n+2}(0) \cup R_3^{3n+2}(1/2)}$$

and

$$N_p^3(\mathcal{R}_2) = \mathcal{R}_2 \cup \overline{R_6(0)} \cup \overline{R_3(0)}.$$

Hence $\mathcal{R}_2$ is a periodic extended Newton ray of period 3 landing at ω.

3.4 Construction of extended Newton graphs

In Sections 3.1 and 3.2 for a Newton map N_p the Newton graph Δ_N at level N, extended Hubbard trees $H^*(U_k, z_k)$, $k \in [1, M]$, and periodic Newton rays were constructed. Here we prove that this combinatorial information is enough in order to capture the behavior of the postcritical set of N_p.

Theorem 3.4.1. *For a given postcritically finite Newton map N_p, let Δ_N be the Newton of N_p (see Definition 3.1.8). There exists a finite connected graph Δ_N^* that contains Δ_N, is invariant under N_p and contains the whole postcritical set of N_p such that every edge of Δ_N^* is eventually mapped by N_p either onto Δ_N, onto an extended Hubbard tree or onto a periodic Newton ray.*

Proof. Let Δ be the channel diagram of Δ_N. The graph Δ_N captures the behavior of postcritical points of N_p which eventually fall onto Δ. In the following construction we take care of postcritical points of N_p which never fall onto Δ.

Let z_k be a periodic point of period $m(k) \geq 2$ from the postcritical set of N_p, $k \in [1, M]$, such that z_k belongs to a critical cycle of N_p. It follows from Lemma 3.1.1 and Lemma 3.1.6 that there exist domains $(\widehat{U_k}, \widehat{V_k})$, $\widehat{U_k} \subset \widehat{V_k}$, such that $z_k \in \widehat{U_k}$ and

$$N_p^{m(k)} : \widehat{U_k} \to \widehat{V_k}$$

is a polynomial-like mapping. Suppose that the mapping $N_p^{m(k)}$ has degree at least two. As was mentioned in Remark 3.1.7 one can associate an extended Hubbard tree $H^*(U_k, z_k)$ to such a pair $(\widehat{U_k}, \widehat{V_k})$. In order to obtain a connected graph we connect the Newton graph Δ_N to each of the extended Hubbard trees $H^*(U_k, z_k)$ for $k \in [1, M]$. The role of this connection will be played by Newton rays constructed in the Section 3.2.

Consider a polynomial-like map $\widehat{F_k} : \widehat{U_k} \to \widehat{V_k}$. It follows from Lemma 3.2.11 that for each β-fixed point ω of $\widehat{F_k}$ in $H^*(U_k, z_k)$ there exists a periodic Newton ray of period $m(k)$ landing at ω. Let $\mathcal{R}_\omega$ be the right envelope of such Newton rays landing at ω and $\mathcal{R}^*_\omega \supset \mathcal{R}_\omega$ be the corresponding extended Newton ray (see Definition 3.2.12). Denote by $\gamma(U_k, z_k)$ the union of all $\mathcal{R}^*_\omega$ for all β-fixed points ω of the polynomial-like mapping $\widehat{F_k} : \widehat{U_k} \to \widehat{V_k}$. It follows from Lemma 3.2.11 that there are finitely many Newton rays of period $m(k)$ landing at ω, and since there are finitely many β-fixed points ω of $\widehat{F_k}$, $\gamma(U_k, z_k)$ is a finite union of extended Newton rays. Let

$$\Upsilon(U_k, z_k) = H^*(U_k, z_k) \cup \gamma(U_k, z_k)$$

and

$$\Upsilon(z_k) = \bigcup_{i \geq 1} N_p^i(\Upsilon(U_k, z_k)).$$

Since

$$\Upsilon(U_k, z_k) \subseteq N_p^{m(k)}(\Upsilon(U_k, z_k)),$$

by the construction, we have that

$$\Upsilon(z_k) = \bigcup_{i=1}^{m(k)} N_p^i(\Upsilon(U_k, z_k))$$

and $\Upsilon(z_k)$ is a finite connected forward invariant graph.

Assume now that the polynomial-like mapping $N_p^{m(k)} : \widehat{U_k} \to \widehat{V_k}$ is degenerate and has degree one. In this case $H^*(U_k, z_k)$ is a degenerate extended Hubbard tree that consists of only one point z_k. It follows from Lemma 3.2.5 that there exists a Newton ray landing at z_k, denote it by $\gamma(U_k, z_k)$. Similarly to the case considered above, let

$$\Upsilon(U_k, z_k) = H^*(U_k, z_k) \cup \gamma(U_k, z_k)$$

and

$$\Upsilon(z_k) = \bigcup_{i \geq 1} N_p^i(\Upsilon(U_k, z_k)).$$

Then $\Upsilon(z_k)$ is a finite connected forward invariant graph.

If a postcritical point z_l is strictly preperiodic, let k, m be the minimal positive integers such that

$$N_p^{k+m}(z_l) = N_p^k(z_l).$$

Consider the postcritical point $N_p^k(z_l)$, it is periodic of period m and it follows from Lemma 3.1.1 and Lemma 3.1.6 that there exist domains $\widehat{U_k}$, $\widehat{V_k}$ such that $z_k \in \widehat{U_k}$

and

$$N_p^m : \widehat{U_k} \to \widehat{V_k}$$

is a polynomial-like map (here $z_k = N_p^k(z_l)$). Denote by $H^*(U_l, z_l)$ the connected component of $N_p^{-k}(H^*(U_k, z_k))$ that contains z_l. We construct by induction on k the curves $\gamma(U_l, z_l)$ that connect $H^*(U_l, z_l)$ to ∞ such that

$$N_p^k(\gamma(U_l, z_l)) \subset \gamma(U_k, z_k) \cup \Delta_N. \quad (3.4.1)$$

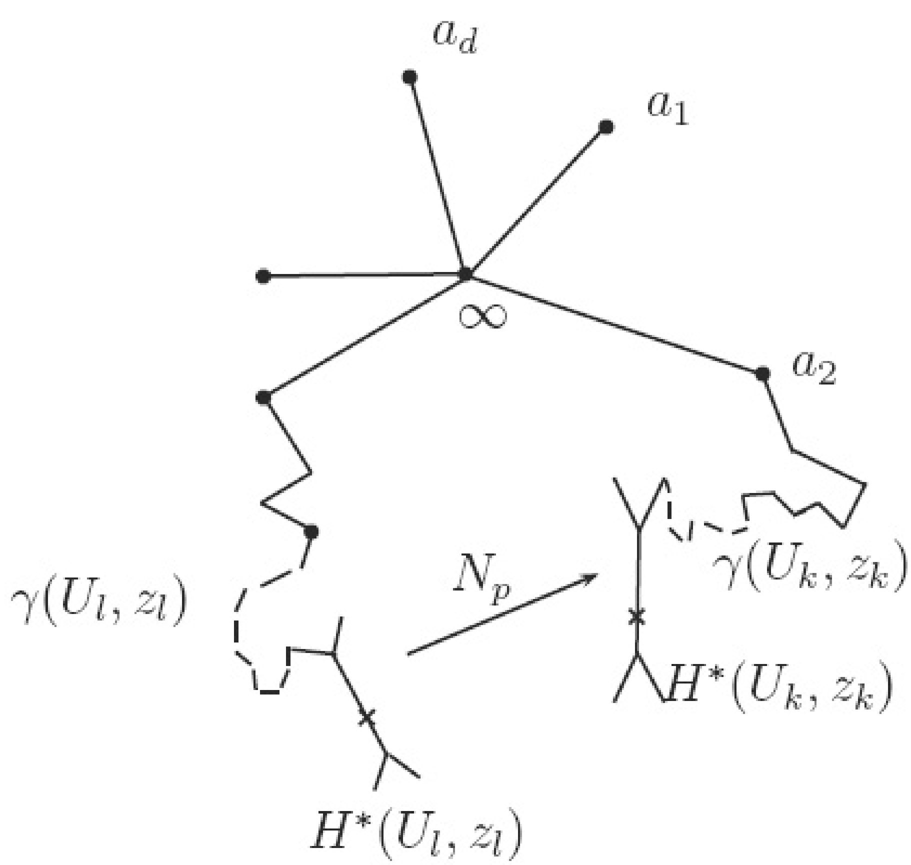

Figure 3.7: Extended Hubbard trees $H^*(U_k, z_k)$, $H^*(U_l, z_l)$ constructed for a preperiodic point z_l such that $N_p^{k+m}(z_l) = N_p^k(z_l)$ and connected to Δ_N via Newton rays $\gamma(U_k, z_k)$ and $\gamma(U_l, z_l)$: the case with $k = 1$. The edges from Δ_N are indicated by the thick lines and Newton rays outside Δ_N are indicated by dotted lines.

Recall that $\gamma(U_k, z_k)$ is the union of extended Newton rays connecting ∞ with points from $H^*(U_k, z_k)$. For $k = 1$ let $\gamma'(U_l, z_l)$ be the preimage component of $N_p^{-1}(\gamma(U_k, z_k))$ that connects $H^*(U_l, z_l)$ to points from the set $N_p^{-1}(\infty)$. Note that $N_p^{-1}(\infty) \in \Delta_N$ by construction. The union of $\gamma'(U_l, z_l)$ and all extended Newton rays that land at points in $N_p^{-1}(\infty)$ (denote the set of such extended Newton rays by Δ^1) consists of extended Newton rays that land at points in $H^*(U_l, z_l)$ (see Figure

3.7). Let $\gamma(U_l, z_l)$ be the union of the right envelopes of these extended Newton rays.

For $k > 1$ denote by $\gamma'(U_l, z_l)$ the preimage component of $N_p^{-1}(\gamma(U_1, z_1))$ that connects $H^*(U_l, z_l)$ to points from the set $N_p^{-1}(\infty)$. (Here $z_1 = N_p(z_l)$ and U_1 is the complementary component of Δ_N that contains z_1. By the induction hypothesis the curves $\gamma(U_1, z_1)$ are constructed). If, similarly to the case $k = 1$, we let $\gamma(U_l, z_l)$ be the union of the right envelopes of extended Newton rays in $\gamma'(U_l, z_l) \cup \Delta^1$, then the property (3.4.1) holds and the required $\gamma(U_l, z_l)$ is constructed.

Let

$$\Upsilon(U_l, z_l) = H^*(U_l, z_l) \cup \gamma(U_l, z_l).$$

Finally, the graph

$$\Delta_N^* = \Delta_N \bigcup_{z_k \in P_{N_p}} \Upsilon(U_k, z_k)$$

is finite, connected, forward invariant under N_p and contains the whole postcritical set of N_p. Moreover, every edge of Δ_N^* is eventually mapped by N_p either onto Δ_N, onto one of the extended Hubbard trees or onto a periodic Newton ray. $\square$

Definition 3.4.2. *The graph Δ_N^* constructed in Theorem 3.4.1 is said to be the* extended Newton graph *associated to N_p.*

Chapter 4

Abstract extended Newton graph

In order to axiomatize the graphs extracted from Newton maps and defined in Section 3.4 we introduce the notion of an *abstract extended Newton graph* which will be shown to be the combinatorial data which distinguishes postcritically finite Newton maps.

Abstract extended Newton graphs consists of abstract Newton graphs (see Definition 2.6.7), abstract extended Hubbard trees (see Definition 2.4.8) and abstract Newton rays that play the role of connections between the previous two. First we define the notion of (pre-)periodic abstract Newton rays.

Let Γ be a finite connected graph embedded in $\mathcal{S}^2$ and $f : \Gamma \to \Gamma$ a graph map (see Definition 2.5.4) such that f can be extended to a branched covering $\overline{f} : \mathcal{S}^2 \to \mathcal{S}^2$.

Definition 4.0.3. A periodic abstract Newton ray (with respect to (Γ, f)) $\mathcal{R}$ *is an arc in* $\mathcal{S}^2$ *that has one of its endpoints in* Γ *and the other one outside* Γ *such that there exists a positive integer* m *satisfying the following properties:*

- $\overline{f}^m(\mathcal{R}) = \mathcal{R} \cup \mathcal{E}$, *where* $\mathcal{E} \subset \Gamma$ *is a union of edges from* Γ*;*
- $\mathcal{R} \cap \mathcal{E} = \mathcal{R} \cap \Gamma$ *is a vertex of* Γ*;*
- *If* $\mathcal{R}^1$ *is an arc in* $\mathcal{S}^2$ *such that* $\mathcal{R}^1 \subset \mathcal{R}$, $\mathcal{R}^1 \neq \emptyset$, *and* $\overline{f}^m(\mathcal{R}^1) = \mathcal{R}^1 \cup \mathcal{E}^1$, *where* $\mathcal{E}^1 \subset \Gamma$ *is a union of edges from* Γ, *then* $\mathcal{R}^1 = \mathcal{R}$.

The smallest such integer m *is said to be* the period *of* $\mathcal{R}$. *The endpoint of* $\mathcal{R}$ *outside* Γ *we denote by* $t(\mathcal{R})$ *and say that* $\mathcal{R}$ lands *at* $t(\mathcal{R})$.

Note, that since Γ is a finite graph, $\mathcal{E}$ is a finite union of edges from Γ.

Definition 4.0.4. A strictly preperiodic abstract Newton ray (with respect to (Γ, f)) $\mathcal{R}'$ *is an arc in* $\mathcal{S}^2$ *that has one of its endpoints that eventually lands in* Γ *under iterates of* $\overline{f}$ *and the other one outside* Γ *such that there exists a positive integer* l *such that* $\overline{f}^l(\mathcal{R}') = \mathcal{R} \cup \mathcal{E}$, *where* $\mathcal{E} \subset \Gamma$ *and* $\mathcal{R}$ *is a periodic abstract Newton ray (with respect to* (Γ, f)*).*

The smallest such positive integer l *is* the preperiod *of* $\mathcal{R}'$. *The endpoint of* $\mathcal{R}'$ *outside* Γ *and never lands in* Γ *under iterates of* $\overline{f}$ *we denote by* $t(\mathcal{R}')$ *and say that* $\mathcal{R}'$ lands *at* $t(\mathcal{R}')$.

The following proposition implies that periodic abstract Newton rays consist of preimages of edges of Γ under $\overline{f}$. As a consequence it follows that the same applies to strictly preperiodic abstract Newton rays.

Proposition 4.0.5. *Let* $\mathcal{R}$ *be a periodic abstract Newton ray (with respect to* (Γ, f)*). Then for every* $x \in \mathcal{R} \setminus t(\mathcal{R})$, *there exists a positive integer* k *such that* $\overline{f}^k(x) \in \Gamma$.

Proof. It follows from Definition 4.0.3 that there exists a positive integer m such that

$$\overline{f}^m(\mathcal{R}) = \mathcal{R} \cup \mathcal{E}, \quad \text{where} \quad \mathcal{E} \subset \Gamma.$$

Let

$$\mathcal{P}_1 = \overline{f}^{(-m)}(\mathcal{E}) \cap \mathcal{R}, \quad \mathcal{P}_i = \overline{f}^{(-m)}(\mathcal{R}_{i-1}) \cap \mathcal{R}, \quad i > 1$$

and $\mathcal{R}^1 = \bigcup_{i=1}^{\infty} \mathcal{P}_i$. If $\mathcal{R}^1 \neq \mathcal{R}$ then readily $\mathcal{R}^1 \subset \mathcal{R}$ and $\overline{f}^m(\mathcal{R}^1) = \mathcal{R}^1 \cup \mathcal{E}$.

We get a contradiction with Definition 4.0.3. Hence $\mathcal{R}^1 = \mathcal{R}$ and for every $x \in \mathcal{R} \setminus t(\mathcal{R})$ there exists a positive integer i such that $x \in \mathcal{P}_i$ and $\overline{f}^{mi}(x) \in \mathcal{E} \subset \Gamma$. □

Recall that we always assume that the orientation in $\mathcal{S}^2$ is counterclockwise as it was fixed in Section 3.2.

Definition 4.0.6 (Abstract Newton Rays Order)**.** *Let* $\mathcal{R}_1$, $\mathcal{R}_2$ *be abstract Newton rays landing at a* β*-fixed point* ω *of an abstract extended Hubbard tree* $H^* \subset \mathcal{S}^2$. *Let* $E_w \subset H^*$ *be the edge of* H^* *with endpoint* ω. *Assume that* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *don't cross-intersect. We say that* $\mathcal{R}_1 \succeq \mathcal{R}_2$ *if there exists a neighborhood* Y *of* ω *such that the cyclic order around* ω *is* $\mathcal{R}_1(Y)$, $\mathcal{R}_2(Y)$, $E_w(Y)$, *where*

$$\mathcal{R}_1(Y) = \mathcal{R}_1 \cap Y, \quad \mathcal{R}_2(Y) = \mathcal{R}_2 \cap Y, \quad E_w(Y) = E_w \cap Y.$$

Remark 4.0.7. Note that for any neighborhood $Y' \subset Y$ of ω the cyclic order of $\mathcal{R}_1(Y')$, $\mathcal{R}_2(Y')$, $E_w(Y')$ around ω is the same as the cyclic order of $\mathcal{R}_1(Y)$, $\mathcal{R}_2(Y)$, $E_w(Y)$

around ω. Hence the relation $\mathcal{R}_1 \succeq \mathcal{R}_2$" is well defined and doesn't depend on the choice of the neighborhood Y in Definition 4.0.6.

Now we are ready to introduce the concept of an abstract extended Newton graph. Later we prove that this graph carries enough information to characterize postcritically finite Newton maps.

Definition 4.0.8 (Abstract Extended Newton Graph)**.** *Let $\Sigma \subset \mathcal{S}^2$ be a finite connected graph, $f : \Sigma \to \Sigma$ is a graph map and Σ' the set of vertices of Σ. A pair (Σ, f) is called an* abstract extended Newton graph *if it satisfies the following conditions:*

- ***Abstract Newton Graph.***

  (1) There exists a positive integer N and an abstract Newton graph Γ at level N such that $\Gamma \subseteq \Sigma$.

- ***Uniquely extendable up to Thurston equivalence. Degree of the extension.***

  (2) For every vertex $y \in \Sigma'$, every component U of $\mathcal{S}^2 \setminus \Sigma$ and every $v \in f^{-1}(\{y\})$ there exists a neighborhood U_v of v such that all edges of Σ that enter U_v terminate at v and the extension $\tilde{f}$ (as defined in Section 2.5) is injective on

  $$\bigcup_{v \in f^{-1}(\{y\})} U_v \cap U.$$

  It follows from Proposition 2.5.6 that f can be extended to a branched covering $\overline{f} : \mathcal{S}^2 \to \mathcal{S}^2$. An immediate consequence of Lemma 2.5.5 is that $\overline{f}$ is unique up to Thurston equivalence.

  (3) $\sum_{x \in \Sigma'} \big(\deg_x(\overline{f}) - 1\big) = 2d_\Gamma - 2$, where d_Γ is the degree of the abstract channel diagram $\Delta \subset \Gamma$.

- ***Abstract Extended Hubbard Trees.***

  (4) There exist a non-negative integer M, abstract extended Hubbard trees $H_i^* \subset \Sigma$, $i \in [1, M]$, and positive integers m_i such that for each $i \in [1, M]$,

  $$H_i^* \cap \Gamma = \emptyset, \quad \overline{f}^{m_i}(H_i^*) = H_i^* \quad \text{and} \quad \overline{f}^{k}(H_i^*) = H_j^*$$

  for $1 \leq k < m_i$ and some $j \in [1, M]$. The smallest such m_i is the period of H_i^* and the trees H_i^* are said to be periodic *of period m_i.*

(5) *There exist a non-negative integer K and trees $H_i^{*'} \subset \Sigma$, $i \in [1, K]$, such that for each $i \in [1, K]$ there exists $j(i) \in [1, M]$ and $\overline{f}^{l(i)}\left(H_i^{*'}\right) = H_{j(i)}^*$ for some positive integer $l(i)$. The smallest such $l(i)$ is the preperiod of $H_i^{*'}$ and the trees $H_i^{*'}$ are said to be* preperiodic *of preperiod $l(i)$.*

(6) *Any two different abstract extended Hubbard trees H_i^*, H_j^*, $i \neq j$, $i, j \in [1, M]$, lie in different complementary components of Γ.* ***Minimality:*** *The level N of the abstract Newton graph Γ is minimal such that this property holds.*

- ***Connections between the abstract Newton graph and abstract extended Hubbard Trees via abstract Newton rays.***

  We assume in the following that whenever we say (pre-)periodic abstract Newton ray we mean (pre-)periodic abstract Newton ray with respect to (Γ, f).

(7) *For every periodic abstract extended Hubbard tree H_i^* of period m_i and every fixed point ω_i of H_i^* there exists a periodic abstract Newton ray $\mathcal{R}_i$ such that $t(\mathcal{R}_i) = \omega_i$ (note that it is not required that $\mathcal{R}_i \subset \Sigma$). For every β-fixed point ω_i of H_i^*, the graph Σ contains one and only one periodic abstract Newton ray $\mathcal{R}_i$ of period m_i such that $t(\mathcal{R}_i) = \omega_i$. The ray $\mathcal{R}_i$ is the rightmost periodic abstract Newton ray of period m_i landing at ω_i, i.e. for any other periodic abstract Newton ray $\mathcal{R}$ of period m_i landing at ω_i, $\mathcal{R}_i \succeq \mathcal{R}$ (see Definition 4.0.6). If H_i^* is degenerate and consists of only one point, then any periodic abstract Newton ray landing at ω_i is by default the rightmost.*

The set of all periodic abstract Newton rays of period m_i landing at different β-fixed points $\omega_i \in H_i^*$ and contained in Σ is denoted by $\mathcal{R}(H_i^*, m_i)$.

(8) *For every preperiodic tree $H_i^{*'}$ of preperiod l_i such that $\overline{f}^{l_i}\left(H_i^{*'}\right) = H_j^*$, where H_j^*, $j \in [1, M]$, is a periodic abstract extended Hubbard tree of period m_j, the following condition holds: for every $\omega_i' \in H_i^{*'}$ such that $\omega_j = \overline{f}^{l_i}(\omega_i')$ is a β-fixed point of H_j^*, there exists one and only one preperiodic abstract Newton ray $\mathcal{R}_i'$ contained in Σ with $t(\mathcal{R}_i') = \omega_i'$ and $\overline{f}^{l_i}(\mathcal{R}_i') \in \mathcal{R}(H_j^*, m_j)$. The set of all such abstract Newton rays $\mathcal{R}_i'$ landing at $H_i^{*'}$ is denoted by $\mathcal{R}'\left(H_i^{*'}, l_i\right)$.*

- ***Types of edges in Σ.***

(9) *Every edge of Σ is of one of the following types:*

  - *Type N: Newton edges — edges in the abstract Newton graph Γ;*

– *Type H: Hubbard edges — edges in trees* H_i^*, $i \in [1, M]$, *or* $H_j^{*'}$, $j \in [1, K]$;

– *Type R: periodic and preperiodic abstract Newton rays (as defined in Definitions 4.0.3, 4.0.4).*

Any two different edges are either disjoint or they intersect at a common endpoint in Σ'.

If (Σ, f) is an abstract extended Newton graph, f can be extended to a branched covering map $\overline{f} : \mathcal{S}^2 \to \mathcal{S}^2$ by Condition (2) and Proposition 2.5.6. Condition (3) and the Riemann-Hurwitz formula ensure that $\overline{f}$ has degree d_Γ.

The abstract extended Hubbard trees H_i^*, $H_i^{*'}$ from Conditions (4) and (5) of Definition 4.0.8 could in fact be degenerate and consist of only one point. In such case this point would correspond either to a non-critical periodic cycle of $\overline{f}$ or would eventually be mapped onto such cycle by an iterate of $\overline{f}$. Let $M' \leq M$ and $K' \leq K$ be non-negative integers such that the trees H_i^*, $i \in [1, M']$, and $H_j^{*'}$, $j \in [1, K']$, are degenerate and consist of only one point.

Denote by

$$\Sigma^d = \bigcup_{i=1}^{M'} \mathcal{R}\left(H_i^*, m_i\right) \bigcup_{j=1}^{K'} \mathcal{R}'\left(H_j^{*'}, l_j\right)$$

the union of degenerate abstract extended Hubbard trees and abstract Newton rays connecting them to Γ (recall that every abstract Newton ray is an arc by definition and contains its landing point).

Before giving the following definition we recall that all graphs we consider in this thesis are embedded into $\mathcal{S}^2$.

Definition 4.0.9 (Thurston Equivalence)**.** *Let* (Σ_1, f_1) *and* (Σ_2, f_2) *be two abstract extended Newton graphs with self-maps* $f_i \colon \Sigma_i \to \Sigma_i$, $i = 1, 2$. *We say that* (Σ_1, f_1) *and* (Σ_2, f_2) *are* Thurston equivalent *if there exist two homeomorphisms* $\phi_1, \phi_2 \colon \Sigma_1 \setminus \Sigma_1^d \to \Sigma_2 \setminus \Sigma_2^d$ *such that they preserve the cyclic order of edges at all the vertices of* $\Sigma_1 \setminus \Sigma_1^d$, $\Sigma_2 \setminus \Sigma_2^d$ *and so that* $\phi_1 \circ f_1 = f_2 \circ \phi_2$ *on* $\Sigma_1 \setminus \Sigma_1^d$. *Moreover we require that* ϕ_1 *is isotopic to* ϕ_2 *relative to the vertices of* $\Sigma_1 \setminus \Sigma_1^d$.

Chapter 5

Proof of the main results

5.1 Proof of Theorem 1.1.1

Theorem 1.1.1 (Newton Maps Generate Extended Newton Graphs). *For every postcritically finite Newton map N_p there exists an extended Newton graph Δ_N^* that satisfies properties of Definition 4.0.8 so that (Δ_N^*, N_p) is an abstract extended Newton graph.*

Such graphs distinguish postcritically finite Newton maps, i.e. if (Δ_{1N}^*, N_{p_1}) and (Δ_{2N}^*, N_{p_2}) are Thurston equivalent abstract extended Newton graphs associated to Newton maps N_{p_1} and N_{p_2}, then the Newton maps N_{p_1} and N_{p_2} are affine conjugate.

Proof. For a given Newton map N_p consider the extended Newton graph Δ_N^* as constructed in Theorem 3.4.1. We show that (Δ_N^*, N_p) is an abstract extended Newton graph.

Let us verify the conditions (1) – (9) of Definition 4.0.8.

(1) Let Δ_N be the Newton graph of N_p as in Definition 3.1.8. Then (Δ_N, N_p) satisfies the properties of an abstract Newton graph (see also [MR]).

(2) It follows from Theorem 3.4.1 that the whole postcritical set of N_p is contained in the extended Newton graph Δ_N^*. Hence N_p is injective on every complementary component of Δ_N^*. Condition (2) follows.

(3) Since all critical points of N_p are among the vertices of Δ_N^* and any vertex v of Δ_N^* with $\deg_v(N_p) > 1$ is a critical point of N_p, Condition (3) follows from Riemann–Hurwitz formula.

(4) The extended Hubbard trees $H_k^* = H^*(U_k, z_k)$ constructed in Theorem 3.4.1

for periodic postcritical points z_k of N_p are periodic, disjoint from Δ_N^* and satisfy the properties of abstract extended Hubbard trees as follows from Theorem 2.4.16 (see also [Po, Theorem A]).

(5) According to the construction of Δ_N^*, the trees H_k^* associated to preperiodic postcritical points z_k of N_p are preimages of periodic Hubbard trees under iterates of N_p, hence Condition (5) is satisfied.

(6) This condition follows from the fact that different trees H_i^*, H_j^* lie in domains U_i, U_j which are separated by an appropriate preimage of Δ_N under N_p as implied by Lemma 3.1.6.

(7), (8) Every Newton ray (see Definition 2.6.7) is an abstract Newton ray. The properties follow now from the construction in Theorem 3.4.1, Lemma 3.2.5 and Lemma 3.2.11.

(9) It follows from the construction of Δ_N^* in Theorem 3.4.1 that every edge of Δ_N^* is either

— of type N, if it belongs to the Newton graph $\Delta_N \subset \Delta_N^*$;

— of type H, if it belongs to one of the extended Hubbard trees H_i^*, $i \in [1, M]$, or is eventually mapped onto it by N_p;

— of type R, if it belongs to one of the periodic Newton rays $\mathcal{R}$ or is eventually mapped by an iterate of N_p onto a periodic Newton ray union finitely many edges of Δ_N.

Every extended Hubbard tree H_i^* is invariant under appropriate iterate of N_p and every edge of Δ_N is eventually mapped onto Δ by N_p, where Δ is the channel diagram of Δ_N. Since the interior of any edge from H_i^* is disjoint from Δ as follows from Lemma 3.1.1, edges of type H can intersect with edges of type N only at their common endpoints.

It follows from Remark 3.2.6 that the interiors of edges of type H are also disjoint from edges of type R.

Finally, by Definition 3.2.3, the edges of type N and edges of type R can only intersect at vertices of Δ_N. Such vertices belong to $(\Delta_N^*)'$.

Let's prove that equivalent abstract extended Newton graphs can only be produced by affine conjugate Newton maps. Denote by P_{1N}^d and P_{2N}^d the union of degenerate Hubbard trees consisting of only one point for N_{p_1} and N_{p_2}. By R_{1N}^d and R_{2N}^d denote the union of Newton rays landing at points from P_{1N}^d and P_{2N}^d

respectively. Let

$$\Delta^d_{1N} = P^d_{1N} \cup R^d_{1N} \quad \text{and} \quad \Delta^d_{1N} = P^d_{1N} \cup R^d_{1N}.$$

Suppose now that there exist graph homeomorphisms $\phi_1, \phi_2 : \Delta^*_{1N} \setminus \Delta^d_{1N} \to \Delta^*_{2N} \setminus \Delta^d_{2N}$ such that

$$\phi_1(N_{p_1}(z)) = N_{p_2}(\phi_2(z))$$

for all $z \in \Delta^*_{1N} \setminus \Delta^d_{1N}$ and ϕ_1, ϕ_2 preserve the cyclic order of edges at all the vertices of $\Delta^*_{1N} \setminus \Delta^d_{1N}$. The graph $\Delta^*_{1N} \setminus \Delta^d_{1N}$ is connected, all its complementary components are disks and using Lemma 2.5.1 and Lemma 2.5.5 one can extend ϕ_1 and ϕ_2 to homeomorphisms $\overline{\phi_1}, \overline{\phi_2} : \mathcal{S}^2 \to \mathcal{S}^2$ up to isotopy relative to $\Delta^*_{1N} \setminus \Delta^d_{1N}$ such that

$$\overline{\phi_1}(N_{p_1}(z)) = N_{p_2}(\overline{\phi_2}(z)) \tag{5.1.1}$$

for all $z \in \mathcal{S}^2$ and $\overline{\phi_1}$ is isotopic to $\overline{\phi_2}$ relative to vertices of $\Delta^*_{1N} \setminus \Delta^d_{1N}$. Any point from P^d_{1N} falls into one of the complementary components of the graph $\Delta^*_{1N} \setminus \Delta^d_{1N}$. Moreover, it is the only point of Δ^*_{1N} in such complementary component since the Condition (6) of Definition 4.0.8 is satisfied as was proven above. We can choose representatives in isotopy classes of $\overline{\phi_1}$ and $\overline{\phi_2}$ relative to $\Delta^*_{1N} \setminus \Delta^d_{1N}$ such that (5.1.1) holds and

$$\overline{\phi_1}|_{P^d_{1N}} = \overline{\phi_2}|_{P^d_{1N}}.$$

Since the mapping class group of a punctured disk is trivial (see, for example, [FM11]), $\overline{\phi_1}$ and $\overline{\phi_2}$ are isotopic relative to the vertices of Δ^*_{1N}.

By Theorem 2.2.7, N_{p_1} and N_{p_2} are conjugate by a Möbius transformation that fixes ∞, i.e. they are affine conjugate and we are done. □

5.2 Proof of Theorem 1.1.2

Theorem 1.1.2 (Abstract Extended Newton Graphs Are Realised) *Every abstract extended Newton graph is realized by a postcritically finite Newton map which is unique up to affine conjugacy. More precisely, let (Σ, f) be an abstract extended Newton graph. Then, there exists a postcritically finite Newton map N_p with an extended Newton graph Δ^*_N such that $\left(\overline{f}, \Sigma'\right)$ and $\left(N_p, (\Delta^*_N)'\right)$ are Thurston equivalent as marked branched coverings, where $(\Delta^*_N)'$ is the set of vertices of Δ^*_N.*

Moreover, if N_p realizes two abstract extended Newton graphs (Σ_1, f_1) and (Σ_2, f_2), then the two abstract extended Newton graphs are equivalent.

Proof. Here we will be using Theorem 2.3.3 due to [PT], which was discussed in Chapter 2.3.

Let (Σ, f) be an abstract extended Newton graph. It follows from Condition (1) of Definition 4.0.8 that there exists an abstract Newton graph $\Gamma \subseteq \Sigma$. Denote by Δ the abstract channel diagram of Γ (see Definition 2.6.7).

In order to prove Theorem 1.1.2 it suffices to show that the marked branched covering $(\overline{f}, \Sigma')$ doesn't have Thurston obstructions. Let Π be its Thurston obstruction on the contrary and $\gamma \in \Pi$. It is easy to see that any edge $\lambda \in \Delta$ forms an irreducible arc system (see Definition 2.3.1). Therefore, using Theorem 2.3.3 we have that

$$\gamma \cdot (\overline{f}^{-n}(\lambda) \setminus \lambda) = 0.$$

This is true for any edge $\lambda \in \Delta$, hence

$$\gamma \cdot (\Gamma \setminus \Delta) = 0 \quad \text{for any} \quad \gamma \in \Pi.$$

There are two cases: either $\Pi \cdot \Delta \neq 0$ or $\Pi \cdot \Delta = 0$.

Case 1. Suppose that $\Pi \cdot \Delta \neq 0$. Note that any edge $e \subset \Sigma$ of type R is either a periodic abstract Newton ray or a preperiodic abstract Newton ray that consists of preimages of Δ under $\overline{f}$ as follows from Proposition 4.0.5 and Condition (7) of Definition 2.6.7. Moreover, by Definiton 4.0.3 and Definition 4.0.4, any (pre-)periodic abstract Newton ray is disjoint from Δ. Hence $\gamma \cdot e = 0$ for any edge $e \subset \Sigma$ of type R and $\gamma \in \Pi$. It follows from Theorem 2.3.3 that Π is a Levy cycle,

$$\Pi \cdot (\Gamma \setminus \Delta) = 0$$

and since $\Pi \cdot \Delta = 0$, there exist $\gamma \in \Pi$ and $\lambda \in \Delta$ such that $\gamma \cdot \lambda \geq 1$, hence $\gamma \cap \lambda \neq \emptyset$. Let $\Pi = \{\gamma_1, \ldots, \gamma_k\}$ and γ'_j be the component of $\overline{f}^{-1}(\gamma_j)$ that is isotopic to γ_{j-1} relative to $P_{\overline{f}}$ (with the convention that $\gamma_0 = \gamma_k$ and $\gamma_{k+1} = \gamma_1$).

Note that, according to Definition 2.6.7 and Definition 2.6.5, the graph Σ contains the distinguished vertex v_0 fixed under $\overline{f}$ and the only edges of Σ starting from v_0 are the fixed edges $[v_0, v_i]$, connecting v_0 through arcs in Δ to fixed branched points v_i, $i \in [1, d]$, of $\overline{f}$.

For a simple closed curve ν that doesn't pass through v_0 denote by $D(\nu)$ the complementary component of ν that doesn't contain v_0 and by $D_{v_0}(\nu)$ the one that contains v_0.

Proposition 5.2.1. *Every $\gamma \in \Pi$ separates the endpoints v_0 and v_i of the edge*

$[v_0, v_i]$ *for each* $i \in [1, d]$.

Proof. It follows from Definition 2.6.7 that the graph $\Gamma \setminus \Delta$ is connected. Since Π doesn't intersect $\Gamma \setminus \Delta$, the graph $\Gamma \setminus \Delta$ is contained in one of the complementary components of γ in $\mathcal{S}^2$ for every $\gamma \in \Pi$. On the other hand, in the case under consideration, $\Pi \cdot \Delta \neq 0$. Suppose, on the contrary, that there exist $j \in [1, k]$ and $i \in [1, d]$ such that $v_i \in D_{v_0}(\gamma_j)$. Then the whole abstract Newton graph $\Gamma \in D_{v_0}(\gamma_j)$, since otherwise γ_j would intersect $\Gamma \setminus \Delta$, because $\Gamma \setminus \Delta$ is connected and all the vertices of Γ except v_0 are contained in $\Gamma \setminus \Delta$. All d preimages of v_0 under $\overline{f}$ are thus contained in $D_{v_0}(\gamma'_{j+1})$, therefore

$$\Gamma \setminus \Delta \subset D_{v_0}(\gamma'_j) \quad \text{for every} \quad j \in [1, k]$$

and

$$\overline{f} : D(\gamma'_{j+1}) \to D(\gamma_{j+1}), \quad j \in [1, k],$$

is a homeomorphism. In particular $D(\gamma'_{j+1})$, $j \in [1, k]$, cannot contain any critical points of $\overline{f}$. Note that since γ_j is essential, there must be $x \in P_{\overline{f}}$ such that $x \in D(\gamma_j)$ (see Figure 5.1). Moreover, $x \in D(\gamma'_{j+1})$ because γ'_{j+1} is isotopic to γ_j relative to $P_{\overline{f}}$. On the other hand, if any $x \in P_{\overline{f}}$ is contained in $D(\gamma_j)$ then $\overline{f}(x)$ is contained in $D(\gamma_{j+1})$. Therefore $\bigcup_{j=1}^{k} D(\gamma_j)$ will contain all the iterates of x under $\overline{f}$. Since the orbit of x is finite, it must eventually iterate onto a periodic cycle which cannot contain critical points because

$$\overline{f} : D(\gamma'_{j+1}) \to D(\gamma_{j+1})$$

is a homeomorphism for every $j \in [1, k]$. It follows now from Condition (7) of Definition 4.0.8 that there must be an edge $\mathcal{R}$ of type R that connects x to Γ. This edge $\mathcal{R}$ intersects γ_j. This is impossible since $\gamma_j \cdot (\Delta_N \setminus \Delta) = 0$. □

It follows from Proposition 5.2.1 that every $\gamma \in \Pi$ separates v_0 and fixed branched points v_i, $i \in [1, d]$, and intersects all the fixed edges $[v_0, v_i]$, $i \in [1, d]$.

Proposition 5.2.2. Π *is a degenerate period one Levy cycle (consists of only one curve).*

Proof. As was shown in Proposition 5.2.1 the map

$$\overline{f} : D_{v_0}(\gamma'_j) \to D_{v_0}(\gamma_{j+1}), \quad j \in [1, k],$$

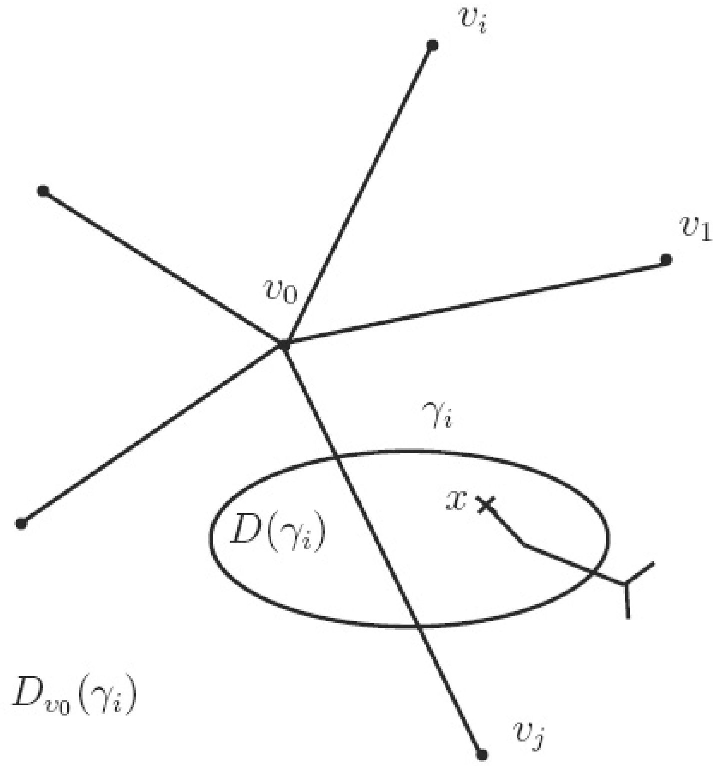

Figure 5.1: The abstract channel diagram of Σ, a curve $\gamma_j \in \Pi$ with $v_i \in D_{v_0}(\gamma_j)$ and a postcritical point $x \in D(\gamma_j)$.

is a homeomorphism and for every curve $\gamma_j \in \Pi$ there exists a component $D_{v_0}(\gamma_j)$ of $\mathcal{S}^2 \setminus \gamma_j$ that has a disk preimage under $\overline{f}$ that is mapped one-to-one to $D_{v_0}(\gamma_{j+1})$. Therefore Π is a degenerate Levy cycle.

Let's prove now that Π consists in fact of only one curve. Suppose, on the contrary, $k > 1$. Remember that for every j

$$\Sigma \setminus \Delta \subset D(\gamma_j)$$

and the curves γ_j in the Thurston obstruction Π don't intersect each other by the very definition. Hence the disks $D(\gamma_j)$, $j \in [1, k]$, and therefore $D_{v_0}(\gamma_j)$, $j \in [1, k]$ can be ordered by inclusion.

Without loss of generality assume that $D_{v_0}(\gamma_1)$ is the largest among $D_{v_0}(\gamma_j)$, $j \in [1, k]$ with respect to this order. Note that

$$\overline{f}(D_{v_0}(\gamma_1')) = D_{v_0}(\gamma_2)$$

and

$$\overline{f}(D_{v_0}(\gamma_k')) = D_{v_0}(\gamma_1).$$

On the other hand,

$$D_{v_0}(\gamma_2) \subset D_{v_0}(\gamma_1),$$

therefore there exists a curve $\gamma_1'' \subset D_{v_0}(\gamma_k')$ such that

$$\overline{f}(D_{v_0}(\gamma_1'')) = D_{v_0}(\gamma_2).$$

Note also that the curves γ_1' and γ_k' are disjoint, since the images γ_2 and γ_1 are disjoint and thus

$$D_{v_0}(\gamma_k') \subset D_{v_0}(\gamma_1').$$

Hence we obtain two nested disks $D_{v_0}(\gamma_1'') \subset D_{v_0}(\gamma_1')$ which are both mapped one-to-one by the map $\overline{f}$ onto the disk $D_{v_0}(\gamma_2)$. Contradiction with the fact that Π is degenerate. □

Now, we can assume that $\Pi = \{\gamma\}$ and γ has a preimage component γ' isotopic to itself relative to $P_{\overline{f}}$.

Each abstract extended Hubbard tree either belongs to $D(\gamma)$, $D_{v_0}(\gamma)$ or intersects γ. No abstract extended Hubbard tree H is in $D_{v_0}(\gamma)$, because otherwise there would be an edge of type R connecting H to $\Gamma \setminus \Delta$ which intersects γ. On the other hand, not all abstract extended Hubbard trees lie in $D(\gamma)$, because otherwise the curve γ would be non-essential (one of its complementary components $D_{v_0}(\gamma)$ would contain the only marked point v_0). Hence there exists at least one abstract extended Hubbard tree H which intersects γ (see Figure 5.2). Suppose H is periodic of period $m > 1$ and $\overline{f}^m(H) = H$. The edges $[v_0, v_i]$, $i \in [1, d]$, divide $D_{v_0}(\gamma')$ into d "sectors". Denote by S_i' the "sector" bounded by

$$[v_0, v_i],\ [v_0, v_{i+1}],\ i \in [1, d] \quad \text{and} \quad \gamma'$$

(with the convention $v_{d+1} = v_0$). Similarly, denote by S_i the sector bounded by

$$[v_0, v_i],\ [v_0, v_{i+1}],\ i \in [1, d], \quad \text{and} \quad \gamma.$$

It follows that $\overline{f}(S_i') = S_i$. Since H intersects γ and γ', it has common points with one of the sectors S_i' and

$$\overline{f}(H \cap S_i') \subset S_i.$$

The image $\overline{f}(H)$ is one of the abstract extended Hubbard trees in Σ (Condition (4) of Definition 4.0.8) distinct from H (because $m > 1$). Therefore $\overline{f}(H)$ can be separated from H via edges of type R according to Condition (6) of Definition 4.0.8.

The edges of type R that separate H and $f(H)$ must then intersect γ. Contradiction with the fact that

$$\gamma \cdot (\Gamma \setminus \Delta) = 0.$$

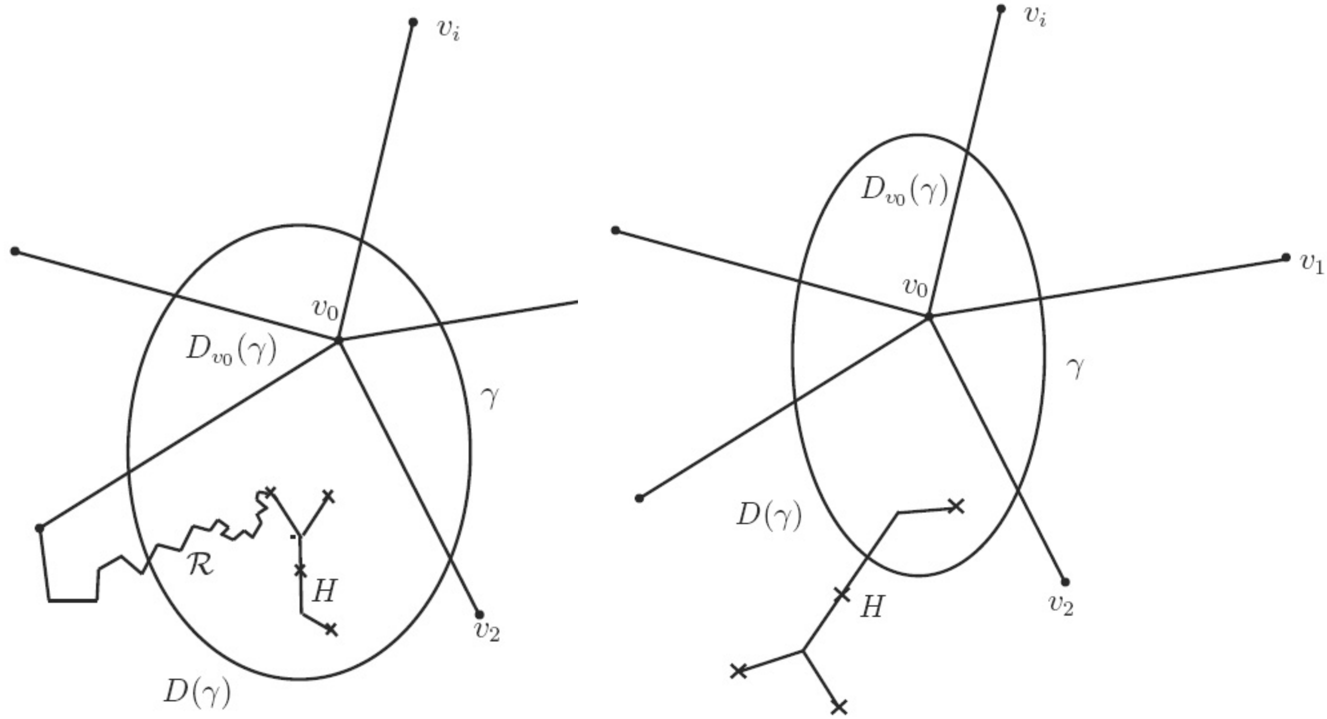

Figure 5.2: The edges $[v_0, v_i]$ of Σ and an abstract extended Hubbard tree $H \subset \Sigma$. On the left-hand side the case when H is completely contained in $D_{v_0}(\gamma)$ is shown. In this case there exists an edge $\mathcal{R}$ of type R connecting H to Γ. On the right-hand side the case when H intersects γ is shown.

Case 2. Suppose $\Pi \cdot \Delta = 0$.

In this case due to Theorem 2.3.3 we obtain that $\Pi \cap \Gamma = \emptyset$. It follows from Condition (6) of Definition 4.0.8 that every complementary component of Γ contains at most one abstract extended Hubbard tree.

This implies that every curve $\gamma \in \Pi$ lies in one of the complementary components of Γ. For every such component U denote

$$\Pi_U = \{\gamma \in \Pi : \gamma \in U\}.$$

Choose U such that $\Pi_U \neq \emptyset$. Without loss of generality we may assume that U contains exactly one periodic abstract Hubbard tree H_i of period m_i.

Consider the maps $F_U = (\overline{f}_U)^{m_i}$ and $F = \overline{f}^{m_i}$ (by $\overline{f}_U$ we denote the restriction of $\overline{f}$ to U). Since Π is a Thurston obstruction for $\overline{f}$ the multicurve Π is also a

Thurston obstruction for F and $\lambda(F_\Pi) \geq 1$. We want to show that $F_\Pi = (F_U)_\Pi$ which would contradict to the fact that the abstract Hubbard tree H_i associated to U can be realized by a polynomial [Po].

Indeed, one can extract an irreducible Thurston obstruction for F from Π. Let us still denote it by Π and assume that U still has the property that $\Pi_U \neq \emptyset$. Now we show that $\Pi \subset U$. Suppose there exists a complementary component $W \neq U$ of Γ with some $\gamma' \in W \cap \Pi$. Then due to the irreducibility of Π there exists a non-negative integer n, a component γ'' of $F^{-n}(\gamma')$ and γ'' is homotopic to γ. Since $\gamma'' \cap \Gamma = \emptyset$ we conclude that $\gamma'' \in U$ is homotopic to γ and $F^n(\gamma'') = \gamma' \in W$. The curve γ'' surrounds points from Σ' contained in U. By the construction those are contained in the abstract extended Hubbard tree $H_i \subset U$ and moreover $F(H_i) = H_i$ by Condition (4) of Definiton 4.0.8. If γ'' doesn't intersect the Hubbard tree H_i, then H_i must be contained in the complementary component of γ'' which is contained in U. On the other hand, the tree H_i is connected to the abstract Newton graph Γ via at least one edge $\gamma_i \in \Sigma$ of type R due to Condition (7) of Definition 4.0.8, the edge γ_i must intersect γ''. This is impossible by Theorem 2.3.3.

Therefore we conclude that $\gamma'' \cap H_i \neq \emptyset$. Hence

$$F^n(\gamma'') \cap F^n(H_i) \neq \emptyset.$$

Since $F(H_i) = H_i$ we obtain a contradiction with $\gamma' \cap H_i \neq \emptyset$ because $\gamma' \in W$ which is disjoint from U and γ' cannot intersect H_i. Contradiction with $W \neq U$.

The obtained contradictions in both cases prove the first claim of Theorem 1.1.2.

Let us now prove the last claim of Theorem 1.1.2. Suppose that the post-critically finite Newton map N_p realizes two abstract Newton graphs (Σ_1, f_1) and (Σ_2, f_2). Then $(\overline{f}_1, \Sigma_1')$, $(\overline{f}_2, \Sigma_2')$ and $(N_p, (\Delta_N^*)')$ are all Thurston equivalent as marked branched coverings. In particular, $(\overline{f}_1, \Sigma_1')$ and $(\overline{f}_2, \Sigma_2')$ are Thurston equivalent as marked branched coverings. Let $g : (\mathcal{S}^2, \Sigma_1') \to (\mathcal{S}^2, \Sigma_2')$ be a homeomorphism that conjugates $\overline{f}_1$ to $\overline{f}_2$ on Σ_1'. If e is an edge of Σ_1 with endpoints $x_1, x_2 \in \Sigma_1'$, then $g(e)$ connects $g(x_1)$ with $g(x_2)$. Moreover, g preserves the cyclic order at each vertex of Σ_1, because it is a homeomorphism of $\mathcal{S}^2$. So if $g' : g(\Sigma_1) \to \Sigma_2$ is a homeomorphism that maps each $g(e)$ to the edge of Σ_2 that connectes $g(x_1)$ and $g(x_2)$, then $g' \circ g$ realizes an equivalence between the two abstract extended Newton graphs (we put $\phi_1 = \phi_2 = g' \circ g$ in Definition 4.0.9). $\square$

5.3 Proof of Theorem 1.1.3

First let us denote by $\mathcal{N}$ the set of postcritically finite Newton maps with the equivalence relation $\sim_{\mathcal{N}}$ defined by the affine conjugacy. In other words, $N_{p_1} \sim_{\mathcal{N}} N_{p_2}$ if N_{p_1} and N_{p_2} are affine conjugate. The equivalence class of N_p we denote by $[N_p]$.

Let $\mathcal{G}$ be the set of abstract extended Newton graphs with the equivalence relation $\sim_{\mathcal{G}}$ defined by Thurston equivalence (see Definition 4.0.9). We say that $(\Sigma_1, f_1) \sim_{\mathcal{G}} (\Sigma_2, f_2)$ if (Σ_1, f_1) and (Σ_2, f_2) are Thurston equivalent. The equivalence class of an abstract extended Newton graph (Σ, f) we denote by $[(\Sigma, f)]$.

It follows from Theorem 1.1.1 and Theorem 1.1.2 that there exist well defined injective mappings $\mathcal{F} : \mathcal{N} \to \mathcal{G}$ and $\mathcal{F}' : \mathcal{G} \to \mathcal{N}$.

In this section we prove the following theorem.

Theorem 1.1.3 (Bijective Correspondence) *The mappings $\mathcal{F}$ and $\mathcal{F}'$ are bijective and inverse to each other, i.e. $\mathcal{F} \circ \mathcal{F}' = Id$ and $\mathcal{F}' \circ \mathcal{F} = Id$.*

Proof. Let $(\Sigma, f) \in \mathcal{G}$ be an abstract extended Newton graph. It follows from Theorem 1.1.2 that (Σ, f) is realized by a postcritically finite Newton map from $\mathcal{N}$, denote it by N_p. Thus

$$\mathcal{F}'([\Sigma, f]) = [N_p].$$

Denote by Δ_N^* the extended Newton graph from Theorem 1.1.1 so that

$$\mathcal{F}([N_p]) = [(\Delta_N^*, N_p)]\,.$$

Theorem 1.1.3 implies that $(\overline{f}, \Sigma')$ and $\left(N_p, (\Delta_N^*)'\right)$ are Thurston equivalent as marked branched coverings. Let

$$g : (\mathcal{S}^2, \Sigma') \to \left(\mathcal{S}^2, (\Delta_N^*)'\right)$$

be a homeomorphism that conjugates $\overline{f}$ to N_p on Σ'. If e is an edge of Σ with endpoints x_1, $x_2 \in \Sigma'$, then $g(e)$ connects $g(x_1)$ with $g(x_2)$. Moreover, g preserves the cyclic order at each vertex of Σ, because it is a homeomorphism of $\mathcal{S}^2$. So if $g' : g(\Sigma) \to \Delta_N^*$ is a homeomorphism that maps each $g(e)$ to the edge of Δ_N^* that connectes $g(x_1)$ and $g(x_2)$, then $g' \circ g$ realizes a Thurston equivalence between the two abstract extended Newton graphs (Σ, f) and (Δ_N^*, N_p) (we can put $\phi_1 = \phi_2 = g' \circ g$ in Definition 4.0.9). Hence, by Definition 4.0.9

$$[(\Sigma, f)] = [(\Delta_N^*, N_p)]$$

and

$$\mathcal{F} \circ \mathcal{F}' = Id.$$

Therefore the mapping $\mathcal{F} : \mathcal{N} \to \mathcal{G}$ is bijective and $\mathcal{F}' \circ \mathcal{F} = Id$. □

Chapter 6

Possible extensions of results

The extended Newton graph constructed in Subsection 3.4 partitions the Riemann sphere in a dynamically meaningful way, so that we can associate to each critical point its *itinerary* with respect to this partition. If all free critical points of N_p have periodic or preperiodic itineraries, they would eventually be caught in the domains of the polynomial-like maps, so the corresponding dynamical possibilities are still described in terms of products of polynomial parameter spaces (no longer necessarily postcritically finite). The principal difference to the postcritically finite case is that the polynomials themselves are no longer completely described by their Hubbard trees, while the Newton maps can presumably still be classified in terms of these polynomials.

Question 1. *Is it possible to extend the classification results in terms of abstract extended Newton graphs to the case of Newton maps for which all the free critical points have either periodic or preperiodic itineraries?*

Another direction of extensions of the results in this thesis could concern the possibility of classification of Newton maps as matings of two polynomials. Matings are a method to describe the (usually complicated) dynamics of a rational map in terms of the dynamics of two polynomials of equal degree (which is often viewed as something simpler; moreover, for polynomial maps there usually is good combinatorics available). Matings were first introduced by Douady and Hubbard [DH84/85]. Informally, a mating is a way to glue two polynomial Julia sets together. The result is a topological space (in many cases, homeomorphic to the 2-sphere) and a continuous self-map of this space, which may or may not be topologically conjugate to a rational map. The question is when this topological conjugacy exists. This question is usually decided using Thurston's Theorem (see Theorem 2.2.7).

The only classification results on Newton maps before introducing Newton graph first in [MR, Rü] were on cubic Newton maps [TL], and they were in terms of matings. It is thus natural to ask to which extent this previous work goes through for higher degree maps. John Hubbard raised the question whether Newton maps in all degrees are matings of particular polynomials.

Question 2. *Given two polynomials, is it possible to decide in combinatorial terms whether their formal mating is equivalent to a Newton map? Conversely, is it possible to develop a criterion on extended Newton graphs to decide whether the corresponding Newton maps are matings?*

Suppose a Newton map is a mating of two polynomials, specified in terms of their Hubbard trees. Both Hubbard trees, glued appropriately, can be embedded into the dynamical plane of the Newton map, so that there exists a finite graph in the dynamical plane of a Newton map containing the whole postcritical set (note that both polynomials must be hyperbolic, so we cannot have the notorious situation that two dendrite Julia sets glued together yield the sphere with its complex structure). Conversely, one could expect that if an extended Newton graph Δ_N^* of a Newton map N_p contains two trees that are invariant under N_p and satisfy natural properties of Hubbard trees, then N_p is the mating of two polynomials.

We expect that the set of Newton maps that are matings forms a rather small portion of all postcritically finite Newton maps.

Question 3. *Is it true, that if the mating of two polynomials f and g is a postcritically finite Newton map of degree d, then one of the two polynomials has $d-1$ superattracting fixed points and the other polynomial has one?*

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