KUNGL TEKNISKA HÖGSKOLAN

VETENSKAP OCH KONST

DYNAMICAL STUDIES IN SEVERAL COMPLEX VARIABLES

by

MATTIAS JONSSON

DOCTORAL THESIS, 1997

DEPARTMENT OF MATHEMATICS
ROYAL INSTITUTE OF TECHNOLOGY
STOCKHOLM, SWEDEN
mjo@math.kth.se

Akademisk avhandling som med tillstånd av Kungl Tekniska Högskolan framläg- ges till offentlig granskning för avläggande av filosofie doktorsexamen i matematik fredagen den 28 november 1997, kl 14.00 i sal E1, Kungl Tekniska Högskolan, Lindstedtsvägen 3, Stockholm.

TRITA 1997-MA-30
ISSN 1401-2278
Högskoletryckeriet, KTH

utterly impossible as are all these events they are probably as like those which may have taken place as any others which never took person at all are ever likely to be.

J. Joyce

PREFACE

By writing these words I complete a doctoral thesis, thereby passing a milestone in my life. It is a pleasure to express my gratitude to the people who helped me on the way:

- My advisor Michael Benedicks for professional assistance and candid enthusiasm all along.
- All the people at the Department of Mathematics for providing a pleasant atmosphere to work in, and the department itself for its generous support during my time as a graduate student.
- Nessim Sibony for introducing me to the field of dynamics in several complex variables.
- John Erik Fornæss and Berit Stensønes for hosting my visit to the University of Michigan.
- Eric Bedford for many fruitful discussions, for reading large parts of the manuscript, and for kindly consenting to having our joint work being included in this thesis.
- My family, especially my older sisters Monika and Elinor for, allegedly, first teaching me mathematics.
- My dear friend Johanna for a great many great days together.

Stockholm, September 1997

Mattias Jonsson

ABSTRACT

This paper deals with different aspects of dynamical systems in several complex variables. It contains the following six papers.

I. **Hyperbolic dynamics of endomorphisms.** We provide a written account of semilocal and global results for hyperbolic dynamics of endomorphisms.

II. **Holomorphic motions of hyperbolic sets** (submitted for publication). We study how hyperbolic sets of holomorphic automorphisms and endomorphisms vary under holomorphic perturbations of the map.

III. **Some properties of 2-critically finite holomorphic maps of $\mathbf{P}^2$** (to appear in *Ergodic Theory Dynam. Systems*). We sharpen previous results by Fornaess and Sibony and by Ueda, by showing that repelling periodic points, as well as the preimages of any given point, are dense in $\mathbf{P}^2$ for a 2-critically finite map.

IV. **Dynamics of polynomial skew products on $\mathbf{C}^2$: exponents, connectedness and expansion.** Polynomial skew products on $\mathbf{C}^2$ are holomorphic maps of $\mathbf{P}^2$ whose dynamics resemble that of a one-dimensional polynomial. We study the relation between the critical set, connectedness of Julia sets, Lyapunov exponents, and expansion.

V. **Sums of Lyapunov exponents for some polynomial maps of $\mathbf{C}^2$** (accepted by *Ergodic Theory Dynam. Systems*). Using a laminar structure for the invariant current, we prove a formula for the sum of the Lyapunov exponents of some polynomial maps of $\mathbf{C}^2$ with respect to an invariant measure of maximal entropy.

VI. **Regular polynomial endomorphisms of $\mathbf{C}^k$** (with E. Bedford). We study the dynamics of polynomial endomorphisms of $\mathbf{C}^k$ that extend holomorphically to $\mathbf{P}^k$; these are called regular. Using techniques from pluripotential theory and hyperbolic dynamics we prove results analogous to those for polynomial mappings of $\mathbf{C}$.

CONTENTS

Paper V: Sums of Lyapunov exponents for some polynomial maps of $\mathbf{C}^2$

Paper VI: Regular polynomial endomorphisms of $\mathbf{C}^k$

SUMMARY

0. Introduction

This thesis contains six papers, each of which deals with dynamical systems in several complex variables. I have chosen to group the papers into the following three categories.

1. General dynamics in several complex variables (Papers I-II).
2. Critically finite maps (Paper III).
3. Polynomial endomorphisms of $\mathbf{C}^k$ (Papers IV-VI).

In section 1 below I give some general background to each of these categories above, rather than describing my own results. This background is not intended to cover everything known in the field; I apologize for any omissions. Brief introductions to the six papers in the thesis are then given in section 2. The papers are not presented in chronological order. A certain overlap between sections 1 and 2 is unavoidable; I hope that the reader will benefit from it.

1. Background

1.1. **General complex dynamics.** In this thesis a dynamical system in several complex variables means a holomorphic mapping $f : M \to M$, where M is a complex manifold. Understanding the dynamics of f means understanding the asymptotic behavior of orbits in M under f.

1.1.1. *Complex dynamics in one variable.* Complex dynamics is often illustrated by Newton's method for solving polynomial equations. Let $p(z)$ be a polynomial in one variable. Newton's method is a numerical algorithm for finding the zeros of p; it goes as follows. Let w be a zero of p, and let z_0 be an approximation of w. By the definition of derivative, $p(z) \approx p(z_0) + p'(z_0)(z - z_0)$, the complex number $z_1 := z_0 - p(z_0)/p'(z_0)$ should be a better approximation of w. By repeating the procedure we obtain complex numbers $z_2, z_3, \dots$, which, hopefully, converge to the exact zero w.

Define the rational function

$$f(z) = z - p(z)/p'(z). \tag{1.1}$$

Then the successive approximations above are given by $z_n := f^n(z_0)$, where $f^n = f \circ \cdots \circ f$. Hence, understanding the behavior of Newton's method means understanding the dynamics of the rational function f.

More generally, one studies iterations of a general rational map f on the Riemann sphere $\hat{\mathbf{C}}$, i.e. not necessarily of the type $f(z) = z - p(z)/p'(z)$. A good reference for this is [CG]. It turns out that the sphere naturally divides into two parts: the *Fatou set* F, where the dynamics is "tame" and the *Julia set* J, where the dynamics is "chaotic". More precisely, F is the largest open set where the family $\{f^n\}$ of iterates is a normal family, and J is the complement of F. The Julia set can be characterized in many other ways, for instance as the closure of the repelling periodic points of f. A quite different characterization was found by Lyubich [L] and by Freire, Lopez, Mañé [FLM]. Namely, f has a unique invariant probability measure μ of maximal entropy and the support of μ is exactly J. Further, μ describes the distribution of periodic points, i.e.

$$\lim_{n\to\infty} \frac{1}{d^n+1} \sum_{f^n(a)=a} \delta_a = \mu. \tag{1.2}$$

In addition, μ describes the distribution of preimages of points. This means that if $a \in \hat{\mathbf{C}}$ is any point (with two possible exceptions), then

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{f^n(b)=a} \delta_b = \mu. \tag{1.3}$$

If f is a polynomial map of $\mathbf{C}$, then J can be described as the boundary of the set of points with bounded orbits. See section 1.3.1 for more details.

1.1.2. *Polynomial automorphisms of* $\mathbf{C}^2$. As mentioned above, the most general kind of complex dynamical system is for us a holomorphic mapping $f : M \to M$, where M is a complex manifold. We still define the Fatou set F of f to be the largest open set where the family $\{f^n\}$ is normal. As for the Julia set, the definition $J = M - F$ is only one out of many, inequivalent, possibilities.

In any case the above situation is too general for obtaining a lot of interesting results. So far, most results in dynamical systems in several complex variables deal with two classes of mappings: polynomial automorphisms of $\mathbf{C}^2$ and holomorphic endomorphisms of complex projective space $\mathbf{P}^k$.

The study of polynomial automorphisms of $\mathbf{C}^2$ has been pursued by many authors, including Friedland and Milnor [FM], Hubbard and Oberste-Vorth [HO1], [HO2], Fornæss and Sibony [FS1], and Bedford, Lyubich and Smillie [BS1], [BS2], [BS3], [BS4], [BS5], [BLS2], [BLS1]. It is not possible to give a survey on the theory in the limited space available here, so we will contend ourselves with indicating a few definitions and results.

An important property of a polynomial automorphism f of $\mathbf{C}^2$ is that it is invertible and that its inverse is a polynomial automorphism as well. This means that may consider both positive and negative iterates f^n, $n \in \mathbf{Z}$. Most concepts and results for f therefore come in pairs: one for the positive iterates of f and one for the negative iterates.

The polynomial automorphisms of $\mathbf{C}^2$ of degree $d \geq 2$ fall naturally into two groups. The first one is the set $\mathcal{E}_d$ of elementary maps. These are simple to understand dynamically and we will say nothing more about them. The second group is denoted $\mathcal{H}_d$; its elements are affinely conjugate to finite compositions of maps of the type

$$f(z,w) = (p(z) + aw, z),$$

where p is a monic polynomial of degree $d \geq 2$ and $a \neq 0$. For simplicity we will call an element of $\mathcal{H}_d$ a *Hénon map* (in [FM] the elements in $\mathcal{H}_d$ are called compositions of generalized Hénon transformations).

Many results on Hénon maps are inspired by dynamics of polynomial maps of $\mathbf{C}$ (see section 1.3.1. Two very important objects in the theory are the positive closed currents μ^+ and μ^- of bidegree (1,1), which are invariant for f and f^{-1}, respectively. Their supports J^+ and J^- are exactly the Julia sets of f and f^{-1}, i.e. the complements of the largest open sets where $\{f^n\}_{n\geq 0}$ and $\{f^n\}_{n\leq 0}$ form normal families. In fact, Fornæss and Sibony showed that μ^+ and μ^- are the unique positive closed currents of unit mass supported on J^+ and J^-, respectively. The currents μ^+ and μ^- have a property similar to (1.3): if X is any algebraic curve in $\mathbf{C}^2$ of degree r, then

$$\lim_{n\to+\infty} \frac{1}{d^n r}[f^{-n}(X)] = \mu^+, \tag{1.4}$$

where $[X]$ denotes the current of integration over X. A similar formula holds for μ^-.

The wedge product $\mu := \mu^+ \wedge \mu^-$ is well-defined and μ is an invariant probability measure on $\mathbf{C}^2$. In fact, μ is the pluricomplex equilibrium measure of the compact set K consisting of points in $\mathbf{C}^2$ with bounded forward and backward orbits. Further, μ describes the distribution of periodic points in the sense of (1.2).

1.1.3. *Complex dynamics on* $\mathbf{P}^k$. The second widely studied class of dynamical systems in higher dimensions are endomorphisms of complex projective space $\mathbf{P}^k$ for $k \geq 1$. Note that $\mathbf{P}^1$ can be identified with the Riemann sphere $\hat{\mathbf{C}}$, so endomorphisms of $\mathbf{P}^k$ may be viewed as generalizations of rational maps on $\hat{\mathbf{C}}$. The main difference between a polynomial automorphism of $\mathbf{C}^2$ and an endomorphism of $\mathbf{P}^k$ is that the latter is not invertible.

Just as for Hénon maps, pluripotential theory has been a key idea to the understanding of the dynamics of endomorphisms of $\mathbf{P}^k$. The study of dynamics on $\mathbf{P}^k$ using pluripotential theory was initiated by Hubbard and Papadopol [HP], and developed more generally and systematically by Fornæss and Sibony [FS4], [FS5], [FS3].

An important object in the theory is a positive invariant closed current T of bidegree $(1,1)$ on $\mathbf{P}^k$. This corresponds to the current μ^+ for Hénon maps and in the case $k=1$, T is the exactly the measure μ of maximal entropy described in section 1.1.1. One way of viewing T is as the asymptotic distribution of preimages of algebraic hypersurfaces. This means that if f is a holomorphic map of $\mathbf{P}^k$ of degree $d \geq 2$ (with some restrictions on the dynamics) and X is an algebraic hypersurface of $\mathbf{P}^k$ of degree r, then

$$\lim_{n\to+\infty} \frac{1}{d^n r}[f^{-n}(X)] = T. \tag{1.5}$$

The support of T, which we denote by J_1, is the Julia set of f in the sense that $\{f^n\}$ is a normal family exactly on $\mathbf{P}^k - J_1$. However, if $k > 1$, then J_1 does not carry all the properties of the Julia set in one dimension. For example, the periodic points of f will not be dense in J_1 in general. For this reason one defines the wedge products $T^l = T \wedge \cdots \wedge T$, $l = 1, \ldots, k$. These are well defined positive closed currents and a result by Russakovskii and Shiffman shows that T^l can be viewed as the asymptotic distribution of preimages of algebraic varieties of codimension l. For the exact statement we refer to [RSh]; see also [RSo]. We write $J_l := \mathrm{supp}(T^l)$. It is clear that $J_k \subset \cdots \subset J_1$. The question of characterizing J_l in terms of normal families is only partly solved.

Of special importance is the measure $\mu := T^k$. Fornæss and Sibony [FS3] proved that it is mixing and of maximal entropy and that it describes the distribution of preimages of points in the following sense: there is a pluripolar set $E \subset \mathbf{P}^k$ such that if $a \notin E$, then

$$\lim_{n\to\infty} \frac{1}{d^{kn}} \sum_{f^n(b)=a} \delta_b = \mu. \tag{1.6}$$

More recently, Briend [Bri2] showed that μ describes the distribution of periodic points, i.e. a formula similar to (1.2) holds.

1.1.4. *Hyperbolic rational maps on $\hat{\mathbf{C}}$.* Many results in dynamical systems require a priori assumptions on the dynamics; these often involve (uniform) *hyperbolicity.*

In the case of rational maps on $\hat{\mathbf{C}}$ this is fairly easy to describe. Namely, a rational map f is *hyperbolic* if there exists an $n \geq 1$ such that $|Df^n(z)| > 1$ for all $z \in J$. This condition has a characterization in terms of the critical points of f: f is hyperbolic if and only if all critical points of f are in basins of attracting periodic points. Let us give some examples to show how the hyperbolicity assumption affects the dynamics of f.

The first example concerns Fatou components. There is a classification of Fatou components for a general rational map, but if f is hyperbolic, then the description is simpler: the Fatou set consists of basins of attracting periodic points.

Second, the hyperbolicity of f affects the geometry of the Julia set: if f is hyperbolic, then the Hausdorff dimension of J satisfies $0 < HD(J) < 2$.

A third consequence is that a hyperbolic map f is *J-stable*, meaning that if g is sufficiently close to f, then g is hyperbolic, and there exists a homeomorphism $\phi : J_f \to J_g$ such that $g \circ \phi = \phi \circ f$. This result is due to Mañé, Sad and Sullivan [MSS]. We will have more to say about it in section 1.1.7.

1.1.5. *Axiom A maps.* An interesting class of dynamical systems are the *Axiom A maps.* These include, for instance, the hyperbolic rational maps on $\hat{\mathbf{C}}$ discussed in section 1.1.4. However, the definition of Axiom A has nothing to do with complex structure, so let us for a moment consider a smooth mapping $f : M \to M$, where M is a finite-dimensional, smooth, Riemannian manifold.

To define Axiom A we need the notion of the nonwandering set and of a hyperbolic set. A point $x \in M$ is nonwandering if it has no neighborhood U such that $f^n(U) \cap U = \emptyset$ for all $n \geq 1$. The *nonwandering set* is the set of nonwandering points; it is a closed set and we denote it by Ω.

The definition of a *hyperbolic set* is most easily formulated for diffeomorphisms, but since we will study maps which are not invertible we consider the general case. Suppose that Λ is a compact subset of M such that $f(\Lambda) = \Lambda$. If Λ is a single point, i.e. a fixed point, then Λ is hyperbolic if and only if the tangent space at Λ splits into two invariant subspaces on which Df is expanding and contracting, respectively. This definition can be generalized to any compact set Λ with $f(\Lambda) = \Lambda$. The details can be found in Paper I. Suffice it to say that the definition involves the set

$$\hat{\Lambda} = \{(x_i)_{i \leq 0}; x_i \in \Lambda, f(x_i) = x_{i+1}\}.$$

of histories in Λ. The set $\hat{\Lambda}$ is sometimes called the natural extension or the inverse limit space of Λ.

We now say that a map $f : M \to M$ is Axiom A if

(i) Ω is compact.
(ii) The periodic points for f are dense in Ω.
(iii) f is hyperbolic on Ω.

Every Axiom A map admits a *spectral decomposition.* This means that the nonwandering set Ω of f can be written in a unique way as a finite disjoint union of compact invariant sets Ω_i on which f is topologically transitive. The sets Ω_i are called the *basic sets* of f.

Define an ordering on the basic sets by saying that $\Omega_i > \Omega_j$ if there exists a complete orbit $(x_n)_{n \in \mathbf{Z}}$ such that $x_n \to \Omega_i$ as $n \to -\infty$ and $x_n \to \Omega_j$ as $n \to +\infty$. We say that f satisfies the *no-cycle condition* if there is no nontrivial cycle for the ordering $>$.

The no-cycle condition has ramifications for the global stability of f. Let us say that f is *$\hat{\Omega}$-stable* if $\hat{g}|_{\hat{\Omega}_g}$ is conjugate to $\hat{f}|_{\hat{\Omega}_f}$ for g close to f. Here $\hat{f}$ is the shift map $\hat{f}((x_i)) = (f(x_i))$. The *$\hat{\Omega}$-stability theorem* asserts that if f is an open Axiom A mapping of a compact manifold and f has no cycles, then f is $\hat{\Omega}$-stable. See Paper I for more details.

1.1.6. *Axiom A in complex dynamics.* We will now discuss Axiom A in the setting of holomorphic endomorphisms of $\mathbf{P}^k$, $k \geq 1$ and Hénon maps on $\mathbf{C}^2$.

To begin with, a rational map on $\hat{\mathbf{C}} \simeq \mathbf{P}^1$ is Axiom A if and only if it is hyperbolic. In this case, the basic sets are J and a finite number of attracting periodic points.

In [BS1], Bedford and Smillie did a similar analysis for Hénon maps. They showed that $f \in \mathcal{H}_d$ is Axiom A if and only if f is hyperbolic on J. In this case, the basic sets of f are J and a finite number of attracting (or repelling) periodic points. Moreover, f is topologically mixing on J and satisfies the no-cycles condition. Thus f is Ω-stable.

Hyperbolicity for a Hénon map also has consequences for the Julia sets J^+ and J^- and for the currents μ^+ and μ^- defined above. For example, J^+ is foliated by global stable manifolds of the points of J, and each leaf (i.e. each stable manifold) is dense in J^+. This laminar structure passes over to the current μ^+, which is a *uniformly laminar current*. This means, loosely speaking, that μ^+ is locally of the form

$$\mu^+ = \int [M_a]\, \nu(a),$$

where $[M_a]$ is the current of integration over a complex disk M_a and ν is a positive finite measure, which is called the *transversal measure* for the stable foliation. The laminar structure of μ^+ and μ^- also give a geometric interpretation of how these two currents intersect to form the measure $\mu = \mu^+ \wedge \mu^-$. In fact Bedford, Lyubich and Smillie [BLS2] were able to carry many of the above ideas through even without the Axiom A assumption.

The theory of Axiom A holomorphic endomorphisms of $\mathbf{P}^k$ is still not fully developed. Most results are known only for $k = 2$. Even so, the theory is not as complete as for rational maps on $\hat{\mathbf{C}}$ or for Hénon maps, and many questions remain open. What is mentioned here is mainly due to Fornaess and Sibony [FS6], who were the first to study hyperbolicity for endomorphisms of $\mathbf{P}^2$.

Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be an Axiom A endomorphism. By spectral decomposition we may write its nonwandering set as $\Omega = S_0 \cup S_1 \cup S_2$, where S_i is the union of basic sets of unstable index i (S_0 is attracting, S_1 is of saddle type and S_2 is repelling). These three sets are all nonempty.

We have that J_2 is a basic set of f and $J_2 \subset S_2$. It would be interesting to know whether $J_2 = S_2$. This is equivalent to that all repelling periodic points belong to J_2. In fact Hubbard and Papadopol [HP] have given an example of a holomorphic map on $\mathbf{P}^2$ with a repelling periodic point outside J_2, but is not clear whether their example can be made Axiom A.

It would also be interesting to know whether an Axiom A map on $\mathbf{P}^2$ can have any cycles. In fact, as far as the author knows, there is no known example of an Axiom A map with two different basic sets S_1^1, S_1^2 in S_1, such that $S_1^1 > S_1^2$ for the ordering defined above.

In [FS6], Fornæss and Sibony were particularly interested in the dynamics of so called *s-hyperbolic maps*. These are Axiom A maps satisfying additional assumptions (in particular that $S_2 = J_2$). They obtained results similar to those for hyperbolic Hénon maps, of which we mention a few. First, they showed that the Julia set of f is the union of J_2 and the stable set of S_1, i.e. $J_1 = J_2 \cup W^s(S_1)$. Then they considered a basic set S_1^1 in S_1 which is *minimal* for the ordering $>$ above, and proved that T is a laminar current near S_1^1. Further, they constructed an unstable current σ near S_1^1 similar to μ^- for Hénon maps. The support of this current is exactly the unstable set of S_1^1 and σ has a laminar structure. Finally, they showed how T and σ intersect at S_1^1 to form an ergodic invariant measure $\nu := T \wedge \sigma$ whose support is exactly S_1^1.

1.1.7. *Holomorphic motions.* In section 1.1.4 we mentioned that a hyperbolic rational map f on $\hat{\mathbf{C}}$ is J-stable. This means, in particular, that if f_a, $a \in \mathbf{D}$ is a holomorphic family of rational maps with $f_0 = f$ hyperbolic, then there is an $r > 0$ and for all $a \in \mathbf{D}_r$ a homeomorphism $h_a : J(f) \to J(f_a)$ such that $f_a \circ h_a = h_a \circ f$. Moreover, $h_0 = \mathrm{id}$ and f_a is a hyperbolic rational map. Now consider a repelling periodic point $z \in J$ of f. The conjugacy h_a must map z to a repelling periodic point z_a of f_a and the implicit function theorem implies that z_a depends holomorphically on a

In fact, the whole set J_a depends holomorphically on a. The proper way to express this is via holomorphic motions. A general definition goes as follows. Let $\mathbf{D}$ be the unit disk, M a complex manifold and X a subset of M. Then a *holomorphic motion* of X parameterized by $\mathbf{D}$ is a continuous map $\phi : \mathbf{D} \times X \to M$ such that:

(i) $\phi(0, \cdot) = \mathrm{id}$.
(ii) $\phi(\cdot, x) : \mathbf{D} \to M$ is holomorphic for every $x \in X$.
(iii) $\phi(a, \cdot) : X \to M$ is injective for every $a \in \mathbf{D}$.

Holomorphic motions were introduced by Mañé, Sad and Sullivan [MSS], who proved that if $\{f_a\}_{a \in \mathbf{D}} : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is a holomorphic family of rational functions, such that $f = f_0$ is hyperbolic, then there is an $r > 0$ and a holomorphic motion $h : \mathbf{D}_r \times J \to M$ such that for each $a \in \mathbf{D}_r$

(i) $J_a := h(a, J)$ is the Julia set for f_a and f_a is hyperbolic.
(ii) The map $h_a := h(a, \cdot) : J \to J_a$ is a homeomorphism and $f_a \circ h_a = h_a \circ f$.

In fact, the assumption on f in [MSS] is (potentially) weaker than hyperbolicity. Holomorphic motions in $\hat{\mathbf{C}}$, such as those in the result by Mañé, Sad and Sullivan, exhibit strong geometric properties. For instance, each map $\phi(a, \cdot)$ above is quasiconformal, and the continuity assumption of ϕ is redundant. These properties do not hold in higher dimension.

However, the result that the Julia set of a hyperbolic rational map is a holomorphic motion is true in a more general context. This is the content of Paper II of this thesis, where we prove that a hyperbolic set of a diffeomorphism, or an expanding set of an endomorphism moves holomorphically with the parameter. The precise statement can be found in the summary of Paper II.

As for hyperbolic sets of endomorphisms, the situation is slightly more complicated. It can be formulated in terms of *strongly analytic multifunctions*, but again we refer to the summary of Paper II for details.

1.2. **Critically finite maps.**

1.2.1. *Construction of chaotic maps.* One approach to critically finite maps is the following: given a complex manifold M, how do we find an everywhere chaotic, holomorphic dynamical system $f : M \to M$? The question is vague, because it is not clear what is meant by chaotic. However, let us agree that a mapping f is chaotic if either

(i) The Fatou set of f is empty.
(ii) The repelling periodic points of f are dense in M.

We will be concerned with the case $M = \mathbf{P}^k$, $k = 1, 2$, but let us start with the much easier situation when M is a torus, $M = \mathbf{C}/\Gamma$, where Γ is a lattice in $\mathbf{C}$. Then the map $g(u) = 2u$ clearly satisfies (i) and (ii). From g we may construct a map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ as follows. Let $\mathfrak{p} : \mathbf{T} \to \hat{\mathbf{C}}$ be the Weierstrass function. This is a branched cover of the sphere of degree 2 such that $\mathfrak{p}(-u) = \mathfrak{p}(u)$ for all $u \in \mathbf{T}$. Since g preserves the fibers of $\mathfrak{p}$, it follows that there is a holomorphic map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ of degree 4, such that $f \circ \mathfrak{p} = \mathfrak{p} \circ g$. Hence f satisfies (i) and (ii) as well. Such rational maps were first considered by Lattès, and are therefore called Lattès examples.

1.2.2. *1-critically finite maps on* $\hat{\mathbf{C}}$. Let us now try to nail down the dynamical characteristics of a Lattés example f which makes it a chaotic map. It follows from the formula $f \circ \mathfrak{p} = \mathfrak{p} \circ g$ that the critical values of f are also critical values of $\mathfrak{p}$. Let C be the critical set of f and define the sets $D := \bigcup_{n\geq 1} f^n(C)$ and $E := \bigcap_{j\geq 0} f^j(D)$. Then D and E are *finite* sets and the points in E are repelling periodic points for f. Let us call a map f satisfying the conditions in the last sentence a *1-critically finite map* (or a *Thurston map*).

It is known that any 1-critically finite map satisfies (i) and (ii). There are many proofs of this result. In fact, Thurston [Th] classified 1-critically finite maps and showed that they have even stronger properties than (i) and (ii). For instance, the measure μ of maximal entropy is equivalent to Lebesgue measure on $\hat{\mathbf{C}}$.

1.2.3. *2-critically finite maps on* $\mathbf{P}^2$. The natural generalizations to dimension 2 of 1-critically finite maps of $\hat{\mathbf{C}}$ are the *2-critically finite maps of* $\mathbf{P}^2$. For the definition see the summary of Paper III. Examples of 2-critically finite maps of $\mathbf{P}^2$ include those constructed from 1-critically finite maps of $\hat{\mathbf{C}}$; see [U2] for details. In fact, it is not easy to find examples which are not of this type, but they do exist [FS2].

Fornæss and Sibony [FS2],[FS4] initiated the study of critically finite maps on $\mathbf{P}^2$ and showed, among other things, that the Fatou set of a 2-critically finite map is empty. To be exact, they proved this result under an additional technical assumption. Ueda [U2] later gave a proof in the general case.

The above result shows that a 2-critically finite map of $\mathbf{P}^2$ is chaotic in the sense of (i), but it does not imply (ii). In Paper III it is shown that in fact (ii) does hold. More precisely, if $f : \mathbf{P}^2 \to \mathbf{P}^2$ is 2-critically finite, then

(i) If $E \subset \mathbf{P}^2$ is a nonempty closed subset with $f^{-1}(E) \subset E$, then $E = \mathbf{P}^2$.
(ii) Repelling periodic points of f are dense in $\mathbf{P}^2$.

Later on Briend [Bri1] sharpened (ii) by proving that the repelling periodic points are distributed according to the measure $\mu := T^2$. By (i) the support of μ is all of $\mathbf{P}^2$. It is unknown to the author whether μ is always equivalent to Lebesgue measure.

Ueda [U1] has given another result in the same direction. Namely, he proved that if Z is a connected complex space and $\psi : Z \to \mathbf{P}^2$ is a holomorphic map such that the family $\{f^n \circ \psi\}$ is normal on Z, then ψ is constant.

1.3. **Polynomial endomorphisms of $\mathbf{C}^k$.**

1.3.1. *Polynomial maps in one variable.* What distinguishes a polynomial mapping $p(z) = z^d + \dots$ of $\mathbf{C}$ of degree $d \geq 2$ from a general rational mapping of $\hat{\mathbf{C}}$ is the presence of a completely invariant point, namely ∞. This point affects the dynamics of p in several ways.

First, the dynamics near infinity is easily described in terms of the *Böttcher coordinate*. This is the unique holomorphic function φ defined near ∞ such that $\varphi(z) = z + O(1)$ as $z \to \infty$ and $\varphi \circ p = \varphi^d$. Hence φ conjugates p to the homogeneous polynomial $\zeta \to \zeta^d$.

Second, the Julia set J of p is given as $J = \partial K$, where K is the set of points with bounded orbits. The maximum principle shows that $\mathbf{C} - K$ is connected, so J is connected if and only if K is connected.

Third, the dynamics of p may be studied by using potential theory on $\mathbf{C}$. This approach was pioneered by Brolin [Bro], and developed further by Sibony (see [CG]) and Tortrat [To]. The connection between dynamics and potential theory is given by the function $G(z) := \lim_{n\to\infty} d^{-n} \log^+ |p^n(z)|$. Indeed, G is the *Green function* of the compact set K and the measure μ of maximal entropy is exactly harmonic measure on J, i.e. $\mu = \frac{1}{2\pi} dd^c G$. In addition, $G = \log|\varphi|$ whenever φ is defined.

The Böttcher coordinate is useful for studying the connectivity of J. Using the equation $p \circ \varphi = \varphi^d$ we may try to extend φ to all of $\hat{\mathbf{C}} - K$. This will work as long as we do not encounter any critical values of p. In fact, a careful analysis shows that J is connected if and only if no critical point of p is in the basin of attraction of ∞.

In the important case when J is connected, φ extends to a conformal equivalence of $\hat{\mathbf{C}} - K$ onto $\hat{\mathbf{C}} - \bar{\mathbf{D}}$. This leads naturally to the study of J in terms of *external rays*, a powerful method introduced by Douady and Hubbard [DH]. The external rays are images under $\psi := \varphi^{-1}$ of a ray in $\hat{\mathbf{C}} - \bar{\mathbf{D}}$. Thus the set of external rays can be identified with the circle S^1. Further, the radial limit $e(\theta) := \lim_{r\to 1+} \varphi(re^{i\theta})$ exists for almost every $\theta \in S^1$ and $e_*(\frac{d\theta}{2\pi}) = \mu$, where $\frac{d\theta}{2\pi}$ denotes normalized Lebesgue measure on the circle.

In fact, it is possible to define external rays even if J is not connected. The endpoint map e is still well-defined and has the same property as above.

Suppose J is connected. It is natural to ask whether the external rays land continuously on J, i.e. if e maps S^1 continuously onto J. This is equivalent to J being locally connected, something which is not always true. However, it does hold in some situations, e.g. if p is hyperbolic.

We close this section by considering Lyapunov exponents. The general definition of Lyapunov exponents can be found in the summary of Paper V. Suffice it to say that there is a number $\lambda(p)$, called the *Lyapunov exponent* of p with respect to μ, such that $\lim_{n\to\infty} \frac{1}{n} \log |Dp^n(z)| = \lambda(p)$ for μ-a.e. $z \in J$. There is an interesting formula for $\lambda(p)$, formulated by Przytycki [P].

$$\lambda(p) = \log d + \sum_{p'(c)=0} G(c),$$

where d is the degree of p. Papers IV-VI contain various generalizations of this equation. An interesting consequence of the above formula is that the following statements are equivalent for a polynomial mapping p of $\mathbf{C}$ of degree $d \geq 2$:

(i) J is connected.
(ii) All critical points of p have bounded orbits.
(iii) $\lambda = \log d$.

1.3.2. *Polynomial endomorphisms of* $\mathbf{C}^k$. All polynomial maps of $\mathbf{C}$ extend to holomorphic mappings of $\hat{\mathbf{C}} \simeq \mathbf{P}^1$. The corresponding statement is false in higher dimension. For instance, no Hénon map in $\mathbf{C}^2$ of degree $d \geq 2$ extends continuously to $\mathbf{P}^2$.

In this thesis we focus on polynomial endomorphisms of $\mathbf{C}^k$ that do extend continuously (and thus holomorphically) to $\mathbf{P}^k$. We call them *regular*. They can also be characterized by the growth condition $\liminf_{|x|\to\infty} |f^n(x)|/|x|^d > 0$, where d is the (algebraic) degree of f.

Some results from one dimension continue to hold for a regular polynomial endomorphism f of $\mathbf{C}^k$. For instance, let K be the set of bounded orbits of f, and let F be the Fatou set, i.e. the largest open set where $\{f^n\}$ forms a normal family. Then K is compact, $\partial K \cap F = \emptyset$ and $\mathrm{int}(K) \subset F$.

Other results are true with slight modifications. For instance, define $G := \lim_{n\to\infty} d^{-n} \log^+ |f^n|$. Then G is the pluricomplex *Green function* of K. The measure μ of maximal entropy satisfies $\mu = (\frac{1}{2\pi} dd^c G)^k$ and is therefore the pluricomplex equilibrium measure of K. Its support J_k is exactly the Shilov boundary of K.

It is also possible to generalize the concept of Böttcher coordinates and external rays to polynomial endomorphisms of $\mathbf{C}^k$. This is done in Paper VI; will only describe a few of the ideas here. The hyperplane $\Pi := \mathbf{P}^k - \mathbf{C}^k \simeq \mathbf{P}^{k-1}$

is completely invariant and the restriction of f to Π is a holomorphic mapping f_Π. For the purposes of Papers V and VI, the roles played in one dimension of ∞ and $\hat{\mathbf{C}} - K$ are played by J_Π and $W^s(J_\Pi)$ in $\mathbf{P}^k$. Here J_Π is the $(k-1)$th Julia set of f_Π in the sense of section 1.1.3 and $W^s(J_\Pi)$ is the stable set of J_Π, i.e. the set of points in $\mathbf{P}^k$ attracted to J_Π. Note that if $k = 2$, then $\Pi \simeq \hat{\mathbf{C}}$, so f_Π can be viewed as a rational map and J_Π is the (one and only) Julia set of f_Π.

If f is a homogeneous regular polynomial endomorphism of $\mathbf{C}^k$, then $W^s(J_\Pi)$ is contained in the complex homogeneous cone over J_Π in $\mathbf{C}^k - \{0\}$. Further, $W^s(J_\Pi)$ has the structure of a Riemann surface lamination, whose leaves are disks contained in complex lines through the origin. The Green function G is harmonic on each leaf and we may define the external rays in $W^s(J_\Pi)$ as gradient lines of G on the leaves of the lamination.

In Paper VI we do a similar construction for a nonhomogeneous polynomial endomorphism f under the assumption that f_Π is (uniformly) expanding on J_Π. Then $W^s(J_\Pi)$ may not be a Riemann surface lamination, but it is a lamination outside a small closed set. On the remaining part of $W^s(J_\Pi)$ we may define external rays. These rays land, in a measure theoretic sense, on J_k, the Shilov boundary of K. For details on this, see paper VI.

The main tool for studying the properties of external rays is the current T^{k-1}. Let A be the basin of attraction of Π. Then the support of $T^{k-1} \llcorner A$, i.e. the restriction of T^{k-1} to A, is exactly $W^s(J_\Pi)$ and the laminar structure of the set $W^s(J_\Pi)$ parallels that of the current $T^{k-1} \llcorner A$.

The condition in one dimension that no critical point is attracted to ∞ has an equivalent in $\mathbf{C}^k$, namely that the critical set of f does not intersect $W^s(J_\Pi)$. Under this condition, the stable set $W^s(J_\Pi)$ is a Riemann surface lamination, where each leaf is a complex disk W_a properly embedded in A. It is an interesting question whether the disks W_a land continuously on J, i.e. if the embeddings of $\hat{\mathbf{C}} - \bar{\mathbf{D}}$ defining W_a extend continuously to S^1. In Paper VI we prove that if the dimension k is two and f satisifies suitable hyperbolicity assumptionsm then the disks do land continuously.

2. Summary of results

Paper I. The purpose of Paper I is to provide a written account of some results in hyperbolic dynamics for endomorphisms. We consider both semilocal and global dynamics. All the material is very well known for diffeomorphisms, but harder to find in the noninvertible case. Paper I will hopefully improve the situation, although some or all of the results (and proofs) are previously known. One out of many good references for dynamics of diffeomorphisms is [S]. For the setting of this paper, the main references are [R] and [PS]. Many proofs are taken from, or inspired by, [R].

The building blocks in hyperbolic dynamics are *hyperbolic sets*. These are generalizations of hyperbolic fixed points. We consider a C^∞ mapping f of a finite-dimensional C^∞ manifold M. Let Λ be a compact subset of M with

$f(\Lambda) = \Lambda$ and denote by $\hat{\Lambda}$ the set of *histories* in Λ, i.e.

$$\hat{\Lambda} = \{(x_i)_{i\leq 0}; x_i \in \Lambda, f(x_i) = x_{i+1}\}.$$

Then $\hat{\Lambda}$ is compact and the shift $\hat{f}((x_i)) := (f(x_i))$ defines a homeomorphism $\hat{f}$ of $\hat{\Lambda}$. The set $\hat{\Lambda}$ or the pair $(\hat{\Lambda}, \hat{f})$ is sometimes called the natural extension or the inverse limit of Λ. We will use the notation $\hat{x}$ for a history $(x_i)_{i\leq 0}$ in Λ. Define the tangent bundle $T_{\hat{\Lambda}}$ as the set of pairs $(\hat{x}, v)$, where $\hat{x} \in \hat{\Lambda}$ and $v \in T_{x_0}M$. The derivative of f lifts to a map $D\hat{f}$ of $T_{\hat{\Lambda}}$.

We say that f is *hyperbolic on* Λ or that Λ is a *hyperbolic set* if $T_{\hat{\Lambda}}$ splits into a direct sum $E^u \oplus E^s$ of continuous invariant subbundles such that

$$|D\hat{f}^n(v)| \geq c\lambda^n|v| \quad v \in E^u$$
$$|D\hat{f}^n(v)| \leq c^{-1}\lambda^{-n}|v| \quad v \in E^s,$$

for some constants $c > 0$ and $\lambda > 1$ and for all $n \geq 1$.

Many properties of hyperbolic fixed points generalize to hyperbolic sets. Let us mention two of these. First, hyperbolic sets are *persistent*. This means that if g is C^1-close to f, then g has a hyperbolic set Λ_g close to $\Lambda = \Lambda_f$ and there is a homeomorphism conjugating $\hat{f}|_{\hat{\Lambda}_f}$ to $\hat{g}|_{\hat{\Lambda}g}$. Second, a hyperbolic set has local stable and unstable manifolds. Define

$$W^s_\delta(p) = \{x \in M; d(f^i(x), f^i(p)) < \delta \quad \forall i \geq 0\}$$
$$W^u_\delta(\hat{q}) = \{x \in M; \exists \hat{x}, x_0 = x, d(x_i, q_i) < \delta \quad \forall i \leq 0\},$$

for $p \in \Lambda$, $\hat{q} \in \hat{\Lambda}$ and $\delta > 0$. The *stable manifold theorem* asserts that if δ is small enough, then $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ are smooth embedded disks, varying continuously with p and $\hat{q}$, respectively. For more details see Theorem 1.2 in the paper.

The local (un)stable manifolds allow us to analyze the dynamics of f near a hyperbolic set Λ, especially if an additional condition is satisfied. Suppose we have a point $p \in \Lambda$, a history $\hat{q} \in \hat{\Lambda}$ and an orbit $(x_i)_{i\in\mathbf{Z}}$ in M such that $d(x_i, f^i(p)) < \delta$ for all $i \geq 0$ and $d(x_i, q_i) < \delta$ for all $i \leq 0$. If, under these conditions, the orbit (x_i) is contained in Λ, then we say that $\hat{\Lambda}$ has *local product structure*. An important consequence of local product structure is *shadowing*.

Corollary 2.5. *Assume that $\hat{\Lambda}$ has local product structure. For every $\epsilon > 0$ there exists an $\eta > 0$ such that if $(x_i)_{i\in\mathbf{Z}}$ is a sequence of points in Λ with $d(f(x_i), x_{i+1}) < \eta$ for all i, then there is an f-orbit $(y_i)_{i\in\mathbf{Z}}$ in Λ such that $d(y_i, x_i) < \epsilon$ for all i.*

We also have a shadowing result for $\hat{\Lambda}$. In fact, we first prove that $\hat{\Lambda}$ admits shadowing and then deduce the above result as a corollary.

We now focus on global dynamics. A point $x \in M$ is *nonwandering* if it has no neighborhood U with $f^n(U) \cap U = \emptyset$ for all $n \geq 1$. An endomorphism f is *Axiom A* if (1) Ω is compact, (2) periodic points for f are dense in Ω, and (3) f

is hyperbolic on Ω. We will mainly consider mappings that are *open*. The first result on Axiom A maps is

Proposition 3.3. *If f is an open Axiom A endomorphism, then $\hat{\Omega}$ has local product structure.*

Thus we may do shadowing in Ω or $\hat{\Omega}$. Further, we have the following *spectral decomposition theorem.*

Corollary 3.5. *The nonwandering set Ω of an open Axiom A endomorphism f can be uniquely decomposed into a finite union of compact invariant sets Ω_i on which f is topologically transitive.*

The sets Ω_i are called the *basic sets* of f. We prove Corollary 3.5 by lifting the situation to $\hat{\Omega}$ and applying standard arguments for diffeomorphisms.

Finally we consider Axiom A endomorphisms with no cycles. To explain this, let f be an open Axiom A endomorphism of a compact manifold M and define an ordering on the basic sets of f by declaring that $\Omega_j > \Omega_k$ if there is an orbit $(x_i)_{i \in \mathbf{Z}}$ such that $x_i \to \Omega_j$ as $i \to -\infty$ and $x_i \to \Omega_k$ as $i \to \infty$. The mapping f is said to have *no cycles* if there are no nontrivial cycles for the ordering $>$. The reason for introducing the no-cycle condition is the following *$\hat{\Omega}$-stability theorem.*

Theorem 4.3. *If f is an open Axiom A endomorphism of a compact manifold M with no cycles, then f is $\hat{\Omega}$-stable, i.e. if g sufficiently close to f, then Ω_g is close to Ω_f, and $\hat{f}|_{\hat{\Omega}_f}$ and $\hat{g}|_{\hat{\Omega}_g}$ are conjugate.*

Paper II. We consider a holomorphic family $f_a : M \to M$, $a \in \mathbf{D}$, of holomorphic endomorphisms of a complex Hermitian manifold M parameterized by the unit disk $\mathbf{D}$.

For terminology and results on hyperbolic dynamics we refer to Paper I. Assume that $f = f_0$ has a hyperbolic subset $\Lambda = \Lambda_0$ (in the paper we use K instead of Λ). We ask ourselves in what way Λ is persistent when we vary the parameter a. From general hyperbolic dynamics we know that for a small enough, there will be a hyperbolic set Λ_a close to Λ such that $\hat{f}_a|_{\hat{\Lambda}_a}$ is conjugate to $\hat{f}|_{\hat{\Lambda}}$.

If Λ is a hyperbolic fixed point, then the implicit function theorem implies that $a \to \Lambda_a$ defines a holomorphic curve in M. For general Λ, the assignment $a \to \Lambda_a$ defines a *strongly analytic multifunction* in the following sense.

Theorem B. *Let f_a be as above. Then there is an $r > 0$ and a continuous map $h : \mathbf{D}_r \times \hat{\Lambda} \to M$ such that*

(1) *For each $a \in \mathbf{D}_r$, $\Lambda_a := h_a(\hat{\Lambda})$ is a hyperbolic set for f_a, where $h_a = h(a, \cdot)$.*
(2) *For each $a \in \mathbf{D}_r$, the map h_a satisfies the relation $f_a \circ h_a = h_a \circ \hat{f}$ and lifts to a homeomorphism $\hat{h}_a : \hat{\Lambda} \to \hat{\Lambda}_a$, which is just the identity for $a = 0$.*
(3) *The map $h(\cdot, \hat{x}) : \mathbf{D}_r \to M$ is holomorphic for each $\hat{x} \in \hat{\Lambda}$.*

(4) *The set* $\bigcup_{a\in\mathbf{D}_r}\{a\}\times\Lambda_a$ *in* $\mathbf{D}_r\times M$ *is the union of graphs of holomorphic maps from* $\mathbf{D}_r$ *to* M.

There are several, inequivalent, definitions of analytic multifunctions in higher dimension (see [A] for a discussion) but the notion of strongly analytic multifunctions defined in Paper II is stronger than them all.

Theorem B simplifies in two cases. First, if the mappings f_a are invertible, then $\hat{\Lambda}_a$ may be identified with Λ_a and the assignment $a\to\Lambda_a$ is a *holomorphic motion* of Λ.

Theorem A. *Let* $\{f_a\}$ *be a holomorphic family of holomorphic automorphisms of a Hermitian manifold* M *parameterized by* $\mathbf{D}$. *Suppose that* $f=f_0$ *has a hyperbolic subset* Λ. *Then* Λ *moves holomorphically with the parameter* a *at* $a=0$. *More precisely, there exists an* $r>0$ *and a continuous map* $h:\mathbf{D}_r\times\Lambda\to M$ *such that*

(1) $\Lambda_a:=h(a,\Lambda)$ *is a hyperbolic subset for* f_a *for all* $a\in\mathbf{D}_r$.
(2) *The map* $h_a:=h(a,\cdot):\Lambda\to\Lambda_a$ *is a homeomorphism and* $f_a\circ h_a=h_a\circ f$ *for all* $a\in\mathbf{D}_r$.
(3) *The map* $h(\cdot,x):\mathbf{D}_r\to M$ *is holomorphic for all* $x\in\Lambda$.

The second case is when Λ is an *expanding* set. This means that there exist constants $c>0$ and $\lambda>1$ such that

$$|Df^n(x)v|\geq c\lambda^n|v|$$

for all $x\in\Lambda$, $v\in T_xM$ and all $n\geq1$. The assignment $a\to\Lambda_a$ is a holomorphic motion in this situation too.

Theorem C. *If* $\{f_a\}$ *is a holomorphic family of endomorphisms and* Λ *is an expanding set for* $f=f_0$, *then* Λ *moves holomorphically with* a *at* $a=0$ *in the sense of Theorem A.*

Paper III. In the third paper in this thesis we consider *2-critically finite maps* of the complex projective plane $\mathbf{P}^2$. The definition of these involve three conditions, which we now describe. If f is a holomorphic map of $\mathbf{P}^2$, then we let $C=C_1$ denote its critical set and define

$$D_1:=\bigcup_{n\geq1}f^n(C_1)$$
$$E_1:=\bigcap_{j\geq0}f^j(D_1).$$

The first condition is that the set D_1 is algebraic or, equivalently, that the union defining D_1 is finite. In this case E_1 is algebraic too. The second condition is that E_1 and C_1 has no common irreducible component. This means that there is no component of the critical set which is mapped into itself by some iterate of f. A map f satisfying these two conditions is called *1-critically finite*. If f is 1-critically finite, then we define

$$C_2 := C_1 \cap E_1$$
$$D_2 := \bigcup_{n\geq 1} f^n(C_2)$$
$$E_2 := \bigcap_{j\geq 0} f^j(D_2).$$

It is a result by Ueda [U2], that these are finite sets. Now the third condition on a 2-critically finite map is that $C_2 \cap E_2 = \emptyset$. This means that f has no periodic critical point.

In a similar way one defines k-critically finite maps of $\mathbf{P}^k$, but we will stick to the case $k = 2$.

The purpose of Paper III is to show that a 2-critically finite map f is "everywhere chaotic". To motivate this, we recall the result by Fornæss and Sibony [FS4], and Ueda [U2], saying that a 2-critically finite map has empty Fatou set. In Paper III we prove more. The first main result is the following.

Theorem 2.2. *If f is 2-critically finite and $E \subset \mathbf{P}^2$ is a closed, nonempty set with $f^{-1}(E) \subset E$, then $E = \mathbf{P}^2$.*

Another way of saying this is that *the preimages of any point are dense in* $\mathbf{P}^2$. If we apply this result with $E = J_1$, the Julia set of f, then we see that $J_1 = \mathbf{P}^2$, i.e. the result by Fornæss and Sibony, and Ueda. However, that result is used in the proof of Theorem 2.2. On the other hand, we may apply Theorem 2.2 to $E = J_2$, where $J_2 = \mathbf{P}^2$ is the support of the measure μ of maximal entropy. Hence we get that $J_2 = \mathbf{P}^2$ for any 2-critically finite map.

Note that the notation in Paper III deviates slightly from the one in this summary. In particular, J_1 and J_2 are called J_0 and J_1, respectively.

The second main result in Paper III is that *repelling periodic points are dense in* $\mathbf{P}^2$ for any 2-critically finite map of $\mathbf{P}^2$. This has later been strengthened by Briend [Bri1], [Bri2], who showed that the repelling periodic points of f are distributed according to the measure μ, which, by the above remark, is supported on all of $\mathbf{P}^2$.

Paper IV. A (polynomial) skew product on $\mathbf{C}^2$ is a map of the form

$$f(z,w) = (p(z), q(z,w)),$$

where p and q are polynomials of the same degree $d \geq 2$ and $p(z) = z^d + O(z^{d-1}))$, $q(z,w) = w^d + O(w^{d-1})$. Such a map f extends to a holomorphic map of $\mathbf{P}^2$ and maps any vertical line $\{z\} \times \mathbf{C} \simeq \mathbf{C}$ to another vertical line $\{p(z)\} \times \mathbf{C} \simeq \mathbf{C}$ by a polynomial map q_z. There are two reasons for studying the dynamics of skew products on $\mathbf{C}^2$. First, they provide a good source of examples of holomorphic maps on $\mathbf{P}^2$. Second, they can be viewed as compositions of different polynomial maps of $\mathbf{C}$. The importance of considering polynomial skew products was also stressed by Heinemann [He1], [He2].

To the map f of $\mathbf{C}^2$ (or $\mathbf{P}^2$) we can associate a Green function G, measuring the rate of escape to infinity, a positive closed current $T = dd^cG$, and an invariant probability measure $\mu = T \wedge T$ (see section 1.3.2). The component p of f also has a Green function G_p and an invariant measure $\mu_p = dd^cG_p$. Its Julia set is $J_p = \mathrm{supp}(\mu_p)$. Finally, for each vertical line $\{z\} \times \mathbf{C}$ we can define a Green function R_z, a probability measure μ_z and a Julia set J_z.

The first result concerns the relation between μ, μ_p and μ_z.

Theorem 2.2. *For any skew product f we have*

$$\mu = \int \mu_z \, \mu_p(z).$$

This gives a partial dynamical characterization of the set $J_2 = \mathrm{supp}(\mu)$.

Proposition 3.2.

$$J_2 = \overline{\bigcup_{z \in J_p} \{z\} \times J_z}.$$

To the measure μ we can associate two Lyapunov exponents λ_1 and λ_2, measuring the average growth of expansion of f^n. See the summary of Paper V for more details.

Theorem 2.6. *For any skew product f on $\mathbf{C}^2$ of degree $d \geq 2$ we have*

$$\lambda_1 = \log d + \sum_{p'(c)=0} G_p(c)$$

$$\lambda_2 = \log d + \int \left(\sum_{\frac{\partial q}{\partial w}(z,c)=0} G(z,c) \right) \mu_p(z).$$

From Theorem 2.6 we see that $\lambda_1, \lambda_2 \geq \log d$, something which is not generally true for polynomial maps of $\mathbf{C}^2$.

For a polynomial map p of $\mathbf{C}$ there is an interesting relationship between the Lyapunov exponent $\lambda(p)$, the value of G_p at the critical points of p and the connectedness of J_p, see section 1.3.1. The following result generalizes this to a skew product.

Theorem 4.10. *If f is a skew product on $\mathbf{C}^2$ of degree $d \geq 2$, then the following are equivalent*

(i) *J_p is connected and J_z is connected for all $z \in J_p$.*
(ii) *$G_p(c) = 0$ for all critical points c of p and $G(z,c) = 0$ for all $(z,c) \in J_p \times \mathbf{C}$ with $\frac{\partial q}{\partial w}(z,c) = 0$.*
(iii) *$\lambda_1 = \lambda_2 = \log d$.*

The set of all skew products on $\mathbf{C}^2$ of a given degree $d \geq 2$ can be identified with $\mathbf{C}^N$, $N = N(d)$. Let M_d be the subset of $\mathbf{C}^N$ corresponding to skew products satisfying (i)–(iii) above. This set should be thought of as a *connectedness locus*.

Theorem 5.2. *M_d is compact in $\mathbf{C}^N$.*

Finally we give a criterion for expansion of f on J_2. Let PC be the closure of the postcritical set of f, i.e. $PC = \overline{\bigcup_{n\geq 1} f^n(C)}$.

Theorem 6.3. *Let f be a skew product on $\mathbf{C}^2$. Then f is expanding on J_2 if and only if $J_2 \cap PC = \emptyset$.*

It is unknown whether Theorem 6.3 holds for general holomorphic endomorphisms of $\mathbf{P}^k$, $k \geq 2$.

As a consequence of the proof of Theorem 6.3 we obtain.

Corollary 6.5. *If f is a skew product on $\mathbf{C}^2$ and f is expanding on J_2, then*

$$J_2 = \bigcup_{z\in J_p} \{z\} \times J_z.$$

This result was previously known in special cases [He1].

Paper V. An important result in smooth ergodic theory is Oseledec's theorem, concerning the existence of *Lyapunov exponents*. For simplicity we formulate in only for an ergodic measure. See e.g. [Y] for a more general treatment.

Theorem (Oseledec)**.** *Let f be a smooth mapping of a Riemannian manifold M of dimension k and let μ be an ergodic invariant probability measure. Then there exists a set $E \subset M$ with $\mu(E) = 1$, positive integers $m_1, \dots m_l$ with $\sum m_i = k$, and real numbers $\lambda_1 < \cdots < \lambda_l$ with the following properties. For each $x \in E$ there is a sequence of subspaces*

$$\emptyset = M_0(x) \subsetneq M_1(x) \subsetneq \cdots \subsetneq M_l(x) = T_xM,$$

such that $\dim(M_i) = m_i$ *and*

$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)v| = \lambda_i$$

for $v \in M_i - M_{i-1}$.

The numbers λ_i are called the Lyapunov exponents of f with respect to the measure μ.

In Paper V we study the situation when f is a regular polynomial endomorphism of $\mathbf{C}^2$ of degree $d \geq 2$. As invariant measure we use $\mu = T \wedge T$, which Fornæss and Sibony proved to be ergodic. See section 1.1.3 for more details. Let Λ be the sum of the Lyapunov exponents of f with respect to μ. Thus

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \log |\det Df^n(x)|$$

for μ-a.e. $x \in \mathbf{C}^2$.

The main result in Paper V is a formula for Λ. Recall that the restriction f_Π of f to Π is a rational map. Let μ_Π denote its measure of maximal entropy (see section 1.1.1) and $\lambda(f_\Pi)$ the Lyapunov exponent of f_Π with respect to μ_Π.

Theorem 3.2. *If f is a regular polynomial endomorphism of $\mathbf{C}^2$ of degree $d \geq 2$, such that f is sufficiently close to the mapping $(z, w) \to (z^d, w^d)$, then*

$$\Lambda = \log d + \lambda(f_\Pi),$$

where $\lambda(f_\Pi)$ is the Lyapunov exponent of f_Π with respect to μ_Π.

The main tool in the proof is a laminar description of the current T. More precisely we prove that $W^s(J_\Pi)$ is the disjoint union of complex disks W_a, each of which is properly embedded in A. Here $A = \mathbf{P}^2 - K$ is the basin of attraction of Π. The disks W_a have uniformly bounded area and $T \llcorner A$ is a laminar current of the form $T \llcorner A = \int [W_a]\, \mu_\Pi(a)$.

Paper V was written before Paper VI. Theorem 3.2 above is proved in a more general version in Paper VI and the notation used in this summary follows the latter paper.

Paper VI. In the sixth and last paper of this thesis we undertake an investigation of the dynamics of a regular polynomial endomorphism f of $\mathbf{C}^k$. The general idea is to study the dynamics of f in a similar way as the dynamics of a polynomial mapping of $\mathbf{C}$, i.e. through the Green function, the Böttcher coordinate and external rays. We use the notation introduced in section 1.3.2 in the summary.

The restriction $f_\Pi = f|_\Pi$ is a holomorphic endomorphism of the hyperplane $\Pi \simeq \mathbf{P}^{k-1}$ at infinity. Let A denote the basin of attraction of Π. We are concerned with the invariant measure μ_Π of maximal entropy for f_Π and its support J_Π. An assumption that we impose is that f_Π is (uniformly) expanding on J_Π; see the summary of Paper II for the definition. The expansion implies that f has local stable manifolds near J_Π.

We are able to carry out the construction of Böttcher coordinates and external rays in $\mathbf{C}^k$. The set J_Π naturally takes the place of ∞ and $W^s(J_\Pi)$ the place of $\mathbf{C} - K$. A key to the understanding of external rays is the current $T^{k-1} \llcorner A$, which denotes the restriction of T^{k-1} to A, the basin of attraction of Π. We show that the support of this current is exactly $W^s(J_\Pi)$. The latter set has the structure of a Riemann surface lamination $\mathcal{W}^s(J_\Pi)$ outside a small closed set, and this lamination goes hand in hand with a laminar structure of $T^{k-1} \llcorner A$.

Using the laminar structure we prove a formula for the sum of the Lyapunov exponents of f, generalizing the corresponding result in Paper V. To state this, we need a critical measure, which we define as $\mu_c := T^{k-1} \wedge [\mathcal{C}]$, where $\mathcal{C}$ is the critical set of f. Let $\Lambda(f)$ and $\Lambda(f_\Pi)$ be the sums of the Lyaounov exponents of

f and f_Π with respect to the measures μ and μ_Π, respectively. Then we have.

$$\Lambda(f) = \log d + \Lambda(f_\Pi) + \int G\,\mu_c.$$

We define external rays as gradient lines of the Green function G restricted to the leaves of $\mathcal{W}^s(J_\Pi)$. The set $\mathcal{E}$ of external rays can be identified with $J_\Pi \times S^1$ and carries the natural measure $\mu_\Pi \otimes \frac{d\theta}{2\pi}$. Let $e : \mathcal{E} \to \partial K$ be the endpoint map, defined by following the rays along decreasing values of G. We show that e is a.e. well defined, measurable, and pushes forward the measure $\mu_\Pi \otimes \frac{d\theta}{2\pi}$ to μ.

The one-dimensional Böttcher coordinate conjugates a polynomial to its homogeneous part of highest degree. We show that if f is a regular polynomial endomorphism of $\mathbf{C}^k$, then the restriction of f to a neighborhood of Π in $W^s(J_\Pi)$ is conjugate to the homogeneous part f_0 of highest degree restricted to a neighborhood of Π inside the complex homogeneous cone $C(J_\Pi)$.

In the special case when $k = 2$ and no critical point of f is in $W^s(J_\Pi)$, then the latter set is a disjoint union of complex disks, each of which is properly embedded in A. Under suitable hyperbolicity assumptions we show that the endpoint map e maps $J_\Pi \times S^1$ Hölder continuously onto J_2. The main crux in showing that the endpoint map e is continuous is to show that the boundaries of the disks W_a accumulate only on J_2. In particular, we must show that no disk W_a intersects an unstable manifold $W^u(\hat{q})$ of a history $\hat{q}$ belonging to a hyperbolic set outside J_2. Compare with the discussion in Section 1.1.6.

That e maps $J_\Pi \times S^1$ continuously onto J_2 implies that J_2 is a topological quotient of $J_\Pi \times S^1$. It would be interesting to know what identifications e can introduce.

References

[A] Aupetit, B. Analytic multifunctions and their applications. In Gauthier, P. M., Sabidussi, G., editor, *Complex Potential Theory*, pages 1–74. Kluwer Academic Publishers, 1994.

[Bri1] Briend, J. Exposants de Liapounoff et points périodiques d'endomorphismes holomorphes de $\mathbf{CP}^k$. *C. R. Acad. Sci. Paris Sér. I Math.* **323**(1996), 805–808.

[Bri2] Briend, J. Exposants de Liapounoff des endomorphismes holomorphes de $\mathbf{CP}^k$. Preprint.

[Bro] Brolin, H. Invariant sets under iteration of rational functions. *Ark. Mat.* **6**(1965), 103–144.

[BLS1] Bedford, E., Lyubich, M., Smillie, J. Distribution of periodic points of polynomial diffeomorphisms of $\mathbf{C}^2$. *Invent. Math.* **114**(1993), 277–288.

[BLS2] Bedford, E., Lyubich, M., Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$ **IV**: The measure of maximal entropy and laminar currents. *Invent. Math.* **112**(1993), 77–125.

[BS1] Bedford, E., Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$: Currents, equilibrium measure and hyperbolicity. *Invent. Math.* **103**(1991), 69–99.

[BS2] Bedford, E., Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$ **II**: Stable manifolds and recurrence. *J. Am. Math. Soc.* **4**(1991), 657–679.

[BS3] Bedford, E., Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$ **III**: Ergodicity, exponents and entropy of the equilibrium measure. *Math. Ann.* **294**(1992), 395–420.

[BS4] Bedford, E., Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$ **V**: Critical points and Lyapunov Exponents. To appear in J. Geom. Anal.

[BS5] Bedford, E., Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$ **VI**: Connectivity of J. Preprint.

[CG] Carleson, L., Gamelin, T. W. *Complex Dynamics.* Springer-Verlag, 1993.

[DH] Douady, A., Hubbard, J. H. Étude dynamique des polynômes complexes I–II. *Publications Mathmatiques d'Orsay* 1984 and 1985.

[FLM] Freire, A.,Lopez, A., Mañé, R. An invariant measure for rational maps. *Bol. Soc. Bras. Mat.* **14**(1983), 45–62.

[FM] Friedland, S.,Milnor, J. Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dynam. Systems* **9**(1989), 67–99.

[FS1] Fornæss, J.E., Sibony, N. Complex Hénon mappings in $\mathbf{C}^2$ and Fatou-Bieberbach domains. *Duke. Math. J.* **65**(1992), 345–380.

[FS2] Fornæss, J.E., Sibony, N. Critically finite rational maps on $\mathbf{P}^2$. In *The Madison Symposium on Complex Analysis*, volume 137 of *Contemp. Math.*, pages 245–260. Amer. Math. Soc., Providence, RI, 1992.

[FS3] Fornæss, J.E., Sibony, N. Complex dynamics in higher dimension. In Gauthier, P. M., Sabidussi, G., editor, *Complex Potential Theory*, pages 131–186. Kluwer Academic Publishers, 1994.

[FS4] Fornæss, J.E., Sibony, N. Complex dynamics in higher dimension. I. *Astérisque* **222**(1994), 201–231.

[FS5] Fornæss, J.E., Sibony, N. Complex dynamics in higher dimension. II. In Bloom, T. et al, editor, *Modern Methods in Complex Analysis*, number 137 in Annals of Mathematics Studies, pages 135–182. Princeton University Press, 1995.

[FS6] Fornæss, J.E., Sibony, N. Hyperbolic maps on $\mathbf{P}^2$. Preprint.

[He1] Heinemann, S.-M. Julia sets for holomorphic endomorphisms of $\mathbf{C}^n$. *Ergodic Theory Dynam. Systems* **16**(1996), 1275–1296.

[He2] Heinemann, S.-M. Dynamische Aspekte holomorpher Abbildungen in $\mathbf{C}^n$. PhD thesis, Universität Göttingen, 1994.

[HO1] Hubbard, J.H, Oberste-Vorth, R.W. Henon mappings in the complex domain. I. The global topology of dynamical space. *Inst. Hautes tudes Sci. Publ. Math* **79**(1994), 5–46.

[HO2] Hubbard, J.H., Oberste-Vorth, R.W. Hénon mappings in the complex domain II. In Branner, B., Hjorth, P., editor, *Real and complex dynamical systems*, pages 89–132. Kluwer Academic Publishers, 1995.

[HP] Hubbard, J.H, Papadopol, P. Superattractive fixed points in $\mathbf{C}^n$. *Indiana Univ. Math. J.* **43**(1994), 321–365.

[L] Lyubich, M. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dynam. Systems* **3**(1983), 351–385.

[MSS] Mañé, R., Sad, P., Sullivan, D. On the dynamics of rational maps. *Ann. Sci. Éc. Norm. Sup.* **16**(1983), 193–217.

[P] Przytycki, F. Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map. *Invent. Math.* **80**(1985), 161–179.

[PS] Pugh, C, Shub, M. Ergodic attractors. *Trans. Amer. Math. Soc.* **312**(1989), 1–54.

[R] Ruelle, D. *Elements of differentiable dynamics and bifurcation theory.* Academic Press, 1989.

[RSh] Russakovskii, A., Shiffman, B. Value distribution for sequences of rational mappings and complex dynamics. Preprint.

[RSo] Russakovskii, A., Sodin, M. Equidistribution for sequences of polynomial mappings *Indiana Univ. Math. J.***44**(1995), 851–852.

[S] Shub, M. *Global stability of dynamical systems.* Springer-Verlag, 1987.

[Th] Thurston, W. On the combinatorics and dynamics of rational maps. Preprint.

[To] Tortrat, P. Aspects potentialistes de l'itération des polynômes. In *Séminaire de Théorie du Potentiel, Paris, No. 8*, number 1235 in Lecture Notes in Math., pages 195–209. Springer, 1987.

[U1] Ueda, T. Personal Communication.

[U2] Ueda, T. Critical orbits of holomorphic maps on projective spaces. *To appear in J. Geom. Anal.*

[Y] Young, L.S. Ergodic theory of differentiable dynamical systems. In Branner, B., Hjorth, P., editor, *Real and complex dynamical systems*, pages 293–336. Kluwer Academic Publishers, 1995.

Dept. of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden
E-mail address: `mjo@math.kth.se`

Paper I

HYPERBOLIC DYNAMICS OF ENDOMORPHISMS

MATTIAS JONSSON

Abstract. We present the theory of hyperbolic dynamics of endomorphisms in. Topics covered are hyperbolic sets, stable manifolds, local product structure, shadowing, spectral decomposition and $\hat{\Omega}$-stability.

0. Introduction

In this paper we study a smooth mapping f of a manifold M as a dynamical system. We will discuss both semilocal and global dynamical properties of f, but always under some hyperbolicity assumption. The main examples we have in mind are holomorphic endomorphisms of complex projective space $\mathbf{P}^k$, $k \geq 1$ but we will state the results in greater generality.

There are many excellent and detailed expositions on differentiable dynamics, e.g. [S], but they usually consider only invertible systems, such as diffeomorphisms of a compact manifold. As for noninvertible maps, the attitude seems to be that most results for diffeomorphisms continue to hold when interpreted correctly, but it is difficult to find a detailed written account; the purpose of this paper is to improve upon that. We do not claim that our results are new. Our main references are [R] and [PS].

The building blocks in hyperbolic dynamics are *hyperbolic sets*. These are generalizations of hyperbolic fixed points, i.e. fixed points where the derivative has no eigenvalue of modulus one. For the precise definition of what it means for a compact, invariant set Λ to be hyperbolic, we refer to section 1, but the definition involves the set

$$\hat{\Lambda} = \{(x_i)_{i\leq 0}; x_i \in \Lambda, f(x_i) = x_{i+1}\}.$$

of *histories* in Λ.

A hyperbolic set Λ has *local stable and unstable manifolds* at each point; see Theorem 1.2 for details. Another basic feature of hyperbolic sets is *persistence* under perturbations. This means that if f is hyperbolic on $\Lambda = \Lambda_f$ and g is close to f, then g has a hyperbolic set Λ_g close to Λ_f such that $\hat{f}|_{\hat{\Lambda}_f}$ and $\hat{g}|_{\hat{\Lambda}_g}$ are conjugate. Here $\hat{f}$ is the shift $f((x_i)) = (f(x_i))$. For more details

1991 *Mathematics Subject Classification.* 58F15.

Key words and phrases. Hyperbolic dynamics, shadowing, local product structure, spectral decomposition,stability.

see Proposition 1.4. Note that the sets Λ_f and Λ_g themselves need not be homeomorphic.

Many results on the dynamics near a hyperbolic set Λ are best formulated in terms of $\hat{\Lambda}$. With this in mind we introduce the concept of *local product structure* for $\hat{\Lambda}$. The definition says that if $(\hat{p}^{(i)})_{i\in\mathbf{Z}}$ and $(\hat{q}^{(i)})_{i\in\mathbf{Z}}$ are orbits in $\hat{\Lambda}$ and $(\hat{x}^{(i)})_{i\in\mathbf{Z}}$ is an orbit which follows $(\hat{p}^{(i)})$ in positive time and follows $(\hat{q}^{(i)})$ in negative time, then $\hat{x}^{(i)}$ is in fact an orbit in $\hat{\Lambda}$.

Under the assumptions of local product structure for $\hat{\Lambda}$ we prove *shadowing* results for $\hat{\Lambda}$ and Λ, saying that an approximate orbit in $\hat{\Lambda}$ (Λ) is always close to an honest orbit in $\hat{\Lambda}$ (Λ). It seems difficult to prove this result for Λ without first proving it for $\hat{\Lambda}$.

Hyperbolicity of a compact set Λ is a semilocal condition, only involving the dynamics in a neighborhood of Λ. Axiom A, however, is a global condition, i.e. a condition on the dynamics of f on all of M. For most results on Axiom A maps we will make two assumptions, namely that M is compact, and that f is an open mapping. These assumptions are needed in some of the proofs; they are always satisfied for nonconstant holomorphic endomorphisms of $\mathbf{P}^k$.

The *nonwandering set* Ω of f is, by definition, the set of points $x \in M$ having no neighborhood U such that $f^n(U) \cap U = \emptyset$ for all $n \geq 1$. If M is compact, then all orbits of f converge to Ω in forward and backward time. We say that f is Axiom A if periodic points are dense in Ω and f is hyperbolic on Ω.

The first consequence of Axiom A is that $\hat{\Omega}$ has local product structure; thus the shadowing results mentioned above apply. We use this to prove versions of Smale's *spectral decomposition theorem* for $\hat{\Omega}$ and Ω, saying that $\hat{\Omega}$ (Ω) is the finite disjoint union of compact invariant sets, called basic sets, on which $\hat{f}$ (f) is topologically transitive. Again it seems difficult to prove this for Ω without going via $\hat{\Omega}$.

Finally we address stability. An endomorphism f is called *$\hat{\Omega}$-stable* if $\hat{f}|_{\hat{\Omega}_f}$ is conjugate to $\hat{g}|_{\hat{\Omega}_g}$ for all g sufficiently close to f. Define a relation on the basic sets of an Axiom A endomorphism f by saying that $\Omega_j > \Omega_k$ if there is an orbit $(x_i)_{i\in\mathbf{Z}}$ such that $x_i \to \Omega_j$ as $i \to -\infty$ and $x_i \to \Omega_k$ as $i \to \infty$. Then f is said to have *no cycles* if there is no nontrivial sequence of basic sets $\Omega_{i_0} < \Omega_{i_1} < \cdots < \Omega_{i_k} = \Omega_{i_0}$ We prove that if f is Axiom A and has no cycles, then f is $\hat{\Omega}$-stable. Axiom A in itself does not imply $\hat{\Omega}$-stability.

The paper starts by recalling the definition of a hyperbolic set for an endomorphism and stating some basic properties, including the stable manifold theorem and persistence. This is done in section 1. The proofs here are only sketched, as the (long) details can be found elsewhere. In section 2 we consider local product structure for a hyperbolic set and prove shadowing results. Then, in section 3, we define Axiom A endomorphisms, show that their nonwandering sets have the suitable local product structure and prove the spectral decomposition theorem.

Finally, in the last section we study $\hat{\Omega}$-stability and prove that an open Axiom A endomorphism f of a compact manifold M with no cycles is $\hat{\Omega}$-stable.

1. Hyperbolic sets and the stable manifold theorem

In this section we will give the definition of a hyperbolic set and state some basic facts about them. In particular we will be concerned with persistence under perturbations and existence of local stable and unstable manifolds.

Suppose f is a C^∞ endomorphism of a C^∞ finite-dimensional Riemannian manifold M. Let Λ be a compact subset of M with $f(\Lambda) = \Lambda$ and define $\hat{\Lambda}$ to be the set of histories in Λ, i.e.

$$\hat{\Lambda} = \{(x_i)_{i \le 0}; x_i \in \Lambda, f(x_i) = x_{i+1}\}.$$

Then $\hat{\Lambda}$ is a closed subset of $\Lambda^{\mathbf{N}}$, hence compact. We will often use the notation $\hat{x}$ for a point $(x_i)_{i \le 0}$ in $\hat{\Lambda}$. Every distance d on Λ defines a distance on $\hat{\Lambda}$, also denoted by d, by

$$d(\hat{x}, \hat{y}) = \sum_{i \le 0} 2^i d(x_i, y_i).$$

The restriction $f|_\Lambda$ lifts to a homeomorphism $\hat{f}$ of $\hat{\Lambda}$ given by $\hat{f}((x_i)) = (x_{i+1})$. There is a natural projection π from $\hat{\Lambda}$ to Λ sending $\hat{x}$ to x_0 and the pullback under π of the restriction to Λ of the tangent bundle of M is a bundle on $\hat{\Lambda}$ which we call the tangent bundle $T_{\hat{\Lambda}}$. Explicitly, a point in $T_{\hat{\Lambda}}$ is of the form $(\hat{x}, v)$ where $\hat{x} \in \hat{\Lambda}$ and v is a tangent vector in $T_{x_0}M$. The derivative Df lifts to a map $D\hat{f}$ of $T_{\hat{\Lambda}}$ in a natural way.

Now f is said to be *hyperbolic* on Λ, or that Λ is a *hyperbolic set*, if there exists a continuous splitting $T_{\hat{\Lambda}} = E^u \oplus E^s$ which is invariant under $D\hat{f}$ and such that $D\hat{f}$ is expanding on E^u and contracting on E^s. More precisely, $D\hat{f}(E^{u/s}) \subset E^{u/s}$ and there exist constants $c > 0$ and $\lambda > 1$ such that for all $n \ge 1$

$$\begin{aligned} |D\hat{f}^n(v)| &\ge c\lambda^n |v| \quad v \in E^u \\ |D\hat{f}^n(v)| &\le c^{-1}\lambda^{-n} |v| \quad v \in E^s. \end{aligned}$$

Remark 1.1. It is possible to make a smooth change of metric in a neighborhood of Λ and obtain $c = 1$ in the equation above.

Note that whereas the fiber of the unstable bundle E^u at a point $\hat{x} \in \hat{\Lambda}$ depends on the whole history $\hat{x}$ of x_0, the fiber of E^s at $\hat{x}$ depends only on the point x_0. Hence the dimension of the fiber of E^u at a point $\hat{x}$ depends only on x_0, so the dimensions of the fibers of the bundles E^u and E^s are locally constant.

As a special case of the above we say that f is *expanding* on Λ if the bundle E^s is trivial. This means that there exist constants $c > 0$ and $\lambda > 1$ such that $|D\hat{f}^n(x)v| \ge c\lambda^n |v|$ for all $x \in \Lambda$, $v \in T_xM$ and all $n \ge 1$.

Perhaps the most fundamental result in hyperbolic dynamics is the stable manifold theorem. For each point p in Λ and each history $\hat{q}$ in $\hat{\Lambda}$, we define *local stable and unstable manifolds* by

$$W^s_\delta(p) = \{y \in M; d(f^i(y), f^i(p)) < \delta \ \forall i \geq 0\}$$
$$W^u_\delta(\hat{q}) = \{y \in M; \exists \hat{y}, \pi(\hat{y}) = y, d(y_i, q_i) < \delta \ \forall i \leq 0\}$$

for small $\delta > 0$. The following theorem asserts that the (un)stable manifolds are indeed nice objects.

Theorem 1.2. ***(Stable Manifold Theorem)*** *If δ is small enough, then*

(i) *For all $p \in \Lambda$ and all $\hat{q} \in \hat{\Lambda}$, $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ are embedded C^∞ disks in M tangent to $E^s(p)$ and $E^u(\hat{q})$ at p and q_0, respectively.*
(ii) *$W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ depend continuously on p and $\hat{q}$, respectively.*
(iii) *If $x \in W^s_\delta(p)$, then $d(f^n(x), f^n(p)) \to 0$ exponentially fast as $n \to \infty$. Similarly, every point x in $W^u_\delta(\hat{q})$ has a unique history $\hat{x}$ such that $x_j \in W^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \leq 0$ and $d(x_j, q_j) \to 0$ exponentially fast as $j \to -\infty$.*

Let us sketch a proof of Theorem 1.2. The idea is to consider the set $B(\hat{\Lambda}, M)$ of bounded maps of $\hat{\Lambda}$ into M. This is a Banach manifold modeled on the Banach space of bounded sections of $T_{\hat{\Lambda}}$. Define a map $\mathcal{F}$ of $B(\hat{\Lambda}, M)$ by $\mathcal{F}(h) = f \circ h \circ \hat{f}^{-1}$. Then the projection π is a fixed point of $\mathcal{F}$ and the assumption that f was hyperbolic on Λ means exactly that π is a hyperbolic fixed point. By a general stable manifold theorem for hyperbolic fixed points in Banach spaces it follows that $\mathcal{F}$ has a local (un)stable manifold. The (un)stable manifolds of f are then obtained as $\{h(x)\}$, where h runs over the (un)stable manifold of $\mathcal{F}$. To do all of this precisely, and to verify that (i)–(iii) holds, requires a nontrivial amount of work, which we will not go into here. A proof of a more general theorem can be found in [PS].

A special case of a hyperbolic set Λ is a *hyperbolic fixed point p*. This means that $f(p) = p$ and Df_p has no eigenvalue of modulus one. Theorem 1.2 is then easier to prove and the method of proof yields the following "Lambda Lemma" or "Inclination Lemma". For an outline of the proof see [R].

Proposition 1.3. *If p is a* hyperbolic fixed point *of f and Σ is an embedded C^1 submanifold of M intersecting $W^s_\delta(p)$ transversely near p, then for n large enough $f^n(\Sigma)$ contains an embedded manifold Σ_n, which is C^1-close to $W^u_\delta(\hat{p})$, where $\hat{p} = (\dots, p, p)$. Similarly, if Σ' is an embedded C^1 submanifold of M intersecting $W^u_\delta(\hat{p})$ transversely near p, then $f^{-n}(\Sigma')$ contains a submanifold Σ'_n, which is C^1-close to $W^s_\delta(p)$ for large n.*

We close this section by stating a persistence property for hyperbolic sets.

Proposition 1.4. *If f is hyperbolic on $\Lambda = \Lambda_f$ and g is C^1-close to f, then there exists a continuous map $h : \hat{\Lambda} \to M$ close to the projection $\pi(\hat{x}) = x_0$ such that $g \circ h = h \circ \hat{f}$ and that g is hyperbolic on $\Lambda_g := h(\Lambda_f)$. The map h lifts to a* *homeomorphism* $\hat{h} : \widehat{\Lambda_f} \to \widehat{\Lambda_g}$ *with* $\hat{g} \circ \hat{h} = \hat{h} \circ \hat{f}$*, and* h *depends continuously on* g *in the* C^r *topology,* $1 \leq r \leq \infty$.

Let us sketch a proof of this. Consider the Banach manifold $C(\hat{\Lambda}, M)$ of continuous maps of $\hat{\Lambda}$ into M and define a selfmap $\mathcal{F}_g$ of $C(\hat{\Lambda}, M)$ for each g by $\mathcal{F}_g(h) = g \circ h \circ \hat{f}^{-1}$. Again π is a hyperbolic fixed point of $\mathcal{F}_f$, so for g sufficiently close to f, $\mathcal{F}_g$ has a hyperbolic fixed point h_g, depending continuously on g. This is the map h above.

2. Local product structure and shadowing

We now use the local stable and unstable manifolds to analyze the dynamics near a hyperbolic set Λ. In particular we will define the notion of local product structure on $\hat{\Lambda}$ and show how this implies that pseudoorbits in $\hat{\Lambda}$ (Λ) can be shadowed by real orbits in $\hat{\Lambda}$ (Λ).

Let Λ be a hyperbolic set for an endomorphism f. If δ is small enough, then by continuity $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ are almost flat, i.e. C^1-close to the tangent at p and q_0, respectively for all $p \in \Lambda$ and all $\hat{q} \in \hat{\Lambda}$. Therefore, by the continuity of E^u and E^s, $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ intersect in at most one point. In particular, if $p = q_0$, then $W^s_\delta(p) \cap W^u_\delta(\hat{q}) = \{q_0\}$, which implies

Proposition 2.1. *If* f *is hyperbolic on* Λ*, then* $f|_\Lambda$ *is expansive, i.e. there is a* $\delta > 0$ *such that if* $(x_i)_{i \in \mathbf{Z}}$ *and* $(y_i)_{i \in \mathbf{Z}}$ *are two orbits in* Λ *with* $d(x_i, y_i) < \delta$ *for all* i*, then* $x_i = y_i$ *for all* i*. The same result holds if only* (x_i) *is assumed to be in* Λ.

More generally we say that Λ has *local product structure* if δ can be chosen so that $W^s_\delta(p) \cap W^u_\delta(\hat{q}) \subset \Lambda$.

If Λ has local product structure, if $p \in \Lambda$, $\hat{q} \in \hat{\Lambda}$ and if p,q_0 are sufficiently close, then $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ intersect in exactly one point $x \in \Lambda$ and x has a history $\hat{x}$ such that $x_j \in W^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \leq 0$. It is not a priori clear that $\hat{x} \in \hat{\Lambda}$, i.e. that $x_j \in \Lambda$ for all $j \leq 0$. It will be useful in the sequel to assume this, so we state the following definition.

Definition 2.2. We say that $\hat{\Lambda}$ has local product structure if δ can be chosen so that if the intersection $W^s_\delta(p) \cap W^s_\delta(\hat{q})$ is nonempty, then it consists of a unique point $x \in \Lambda$ and the unique history $\hat{x}$ of x with $x_j \in W^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \leq 0$ is completely contained in $\hat{\Lambda}$. See Figure 1.

If $\hat{\Lambda}$ has local product structure, then there exist $\delta' > 0$ and $\kappa > 0$ such that if $p \in \Lambda$, $\hat{q} \in \hat{\Lambda}$ and $d(p, q_0) < \delta'$, then there is a unique history $\hat{x} \in \hat{\Lambda}$ such that $x_0 \in W^s_\delta(p) \cap W^u_\delta(\hat{q})$ and $x_j \in W^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \leq 0$. Furthermore,

$$d(x_0, p) \leq \kappa d(p, q_0), \tag{2.1}$$

$$d(\hat{x}, \hat{q}) \leq \kappa d(p, q_0) \tag{2.2}$$

We define $[p, \hat{q}]$ to be this history $\hat{x}$.

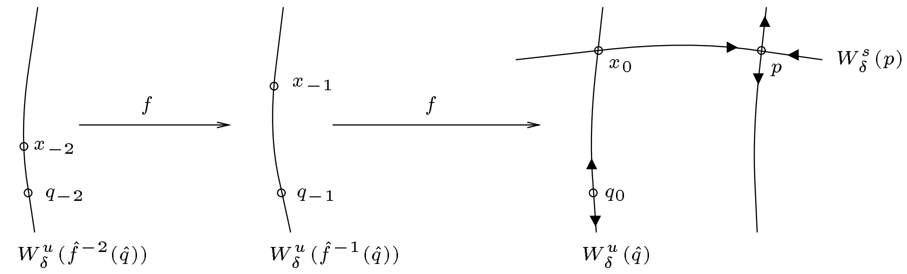


FIGURE 1. Local product structure for $\hat{\Lambda}$.

Definition 2.3. Let $\eta > 0$. An η-pseudoorbit in M is a sequence $(x_i)_{[t_1,t_2]}$, where $-\infty \le t_1 < t_2 \le \infty$, such that $d(f(x_i), x_{i+1}) < \delta$ for $t_1 \le i < t_2$. An η-pseudoorbit $(x_i)_{[t_1,t_2]}$ is ϵ-shadowed by an orbit $(y_i)_{[t_1,t_2]}$ if $d(y_i, x_i) < \epsilon$ for all $i \in [t_1, t_2]$. In a similar way we define (shadowing of) pseudoorbits in $\hat{M}$ or $\hat{\Lambda}$.

Theorem 2.4. ***(Shadowing Lemma for*** $\hat{\Lambda}$***).*** *If Λ is a hyperbolic set for f and $\hat{\Lambda}$ has local product structure, then for each $\epsilon > 0$ there exists an $\eta > 0$ such that every η-pseudoorbit in $\hat{\Lambda}$ can be ϵ-shadowed by an orbit in $\hat{\Lambda}$.*

Proof. Since $\hat{f}$ is uniformly continuous on $\hat{\Lambda}$ it suffices to prove the result for an iterate of f (we may have to shrink η). Let $(\hat{x}^{(i)})_{[t_1,t_2]}$ be a η-pseudoorbit in $\hat{\Lambda}$, where $\hat{x}^{(i)} = (x_j^{(i)})_{j\le 0}$. Using the compactness of $\hat{\Lambda}$ and a diagonal process we may assume that $-\infty < t_1 < t_2 < \infty$. After relabeling, then, we may assume that $t_2 = 0$ and $-\infty < t_1 < 0$.

We will construct points $\hat{y}^{(i)} \in \hat{\Lambda}$ for $t_1 \le i \le 0$ such that

$$(\hat{y}^{(i)}, \hat{f}(\hat{y}^{(i)}), \dots, \hat{f}^i(\hat{y}^{(i)}))$$

ϵ-shadows the η-pseudoorbit

$$(\hat{x}^{(i)}, \hat{x}^{(i+1)}, \dots, \hat{x}^{(0)}).$$

We define $\hat{y}^{(i)} = (y_j^{(i)})_{j\le 0}$ inductively by

$$\begin{aligned} \hat{y}^{(0)} &= \hat{x}^{(0)}, \\ \hat{y}^{(i-1)} &= \hat{f}^{-1}([y_0^{(i)}, \hat{f}(\hat{x}^{(i-1)})]), \end{aligned}$$

see Figure 2. The idea behind this is that $\hat{f}^k(\hat{y}^{(i-1)})$ is close to $\hat{f}^{k-1}(\hat{y}^{(i)})$ for $k \ge 1$ and close to $\hat{f}^k(\hat{x}^{(i-1)})$ for $k \le 0$.

We have to check that the definition above makes sense. Let δ', δ and κ be the constants in (2.1) and (2.2). After replacing f by an iterate we may assume that there exists an $\alpha < 1/2$ such that f contracts stable directions by a factor α and expands unstable directions by a factor $\max(\kappa, 1)/\alpha$. Choose η,$\epsilon_0 > 0$ so

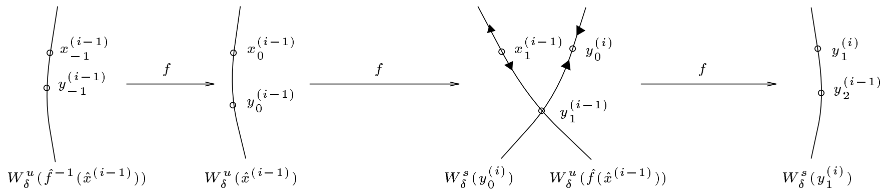


FIGURE 2. Definition of the shadowing orbit.

small that

$$\begin{aligned}\eta + \epsilon_0 &< \delta' \\ \alpha(\eta + \epsilon_0) &< \epsilon_0.\end{aligned}$$

Assume inductively that $t_1 < i \leq 0$, that $\hat{y}^{(i)}$ is well-defined, and that

$$d(\hat{x}^{(i)}, \hat{y}^{(i)}) = \sum_{j \leq 0} 2^j d(x_j^{(i)}, y_j^{(i)}) < \epsilon_0.$$

Then

$$\begin{aligned}d(y_0^{(i)}, x_1^{(i-1)}) &< \eta + d(x_0^{(i)}, y_0^{(i)}) \\ &< \eta + \epsilon_0 \\ &< \delta',\end{aligned}$$

so $[y_0^{(i)}, \hat{f}(\hat{x}^{(i-1)})]$, and hence $\hat{y}^{(i-1)}$, is well-defined. Since $x_j^{(i-1)}$ and $y_j^{(i-1)}$ belong to the same local unstable manifold for all $j \leq 0$ it follows that

$$\begin{aligned}d(\hat{x}^{(i-1)}, \hat{y}^{(i-1)}) &\leq \frac{\alpha}{\kappa} d(\hat{f}(\hat{x}^{(i-1)}), \hat{f}(\hat{y}^{(i-1)})) \\ &\leq \alpha d(x_1^{(i-1)}, \hat{y}_0^{(i)}) \\ &\leq \alpha(d(x_1^{(i-1)}, x_0^{(i)}) + d(x_0^{(i)}, y_0^{(i)})) \\ &\leq \alpha(\eta + \epsilon_0),\end{aligned}$$

which by assumption is less than ϵ_0. Hence it follows inductively that $\hat{y}^{(i)}$ is well-defined for $t_1 \leq i \leq 0$.

We now complete the proof by showing that $(\hat{y}^{(i)}, \ldots, \hat{f}^i(\hat{y}^{(i)}))$ ϵ-shadows $(\hat{x}^{(i)}, \ldots, \hat{x}^{(0)})$. First, if $t_1 < i \leq 0$, then it follows from (2.1) that

$$\begin{aligned}d(y_0^{(i)}, y_1^{(i-1)}) &\leq \kappa d(y_0^{(i)}, x_1^{(i-1)}) \\ &\leq \kappa(d(y_0^{(i)}, x_0^{(i)}) + d(x_0^{(i)}, x_1^{(i-1)})) \\ &\leq \kappa(\epsilon_0 + \eta).\end{aligned}$$

Now we let $i \le t \le 0$ and estimate

$$d(\hat{x}^{(t)}, \hat{f}^{t-i}(\hat{y}^{(i)})) \le d(\hat{x}^{(t)}, \hat{y}^{(t)}) + \sum_{j=0}^{t-i-1} d(\hat{f}^j(\hat{y}^{(t-j)}), \hat{f}^{j+1}(\hat{y}^{(t-j-1)})).$$

The first term is bounded by ϵ_0. The terms in the last sum can be written as

$$\begin{aligned} d(\hat{f}^j(\hat{y}^{(t-j)}), \hat{f}^{j+1}(\hat{y}^{(t-j-1)})) &= \sum_{s \le 0} 2^s d(y_{s+j}^{(t-j)}, y_{s+j+1}^{(t-j-1)}) \\ &= \sum_{-j \le s \le 0} + \sum_{s<-j}. \end{aligned}$$

Note that $y_{s+j}^{(t-j)}$ and $y_{s+j+1}^{(t-j-1)}$ are on the same local stable manifold if $s+j \ge 0$, so the first sum is bounded by

$$\sum_{-j \le s \le 0} 2^s \alpha^{s+j} d(y_0^{(t-j)}, y_1^{(t-j-1)}) \le \frac{2^{-j}}{1-2\alpha} \kappa(\epsilon_0 + \eta).$$

The second sum is bounded by

$$\begin{aligned} &\sum_{s<-j} 2^s (d(y_{s+j}^{(t-j)}, x_{s+j}^{(t-j)}) + d(x_{s+j}^{(t-j)}, x_{s+j+1}^{(t-j-1)}) + d(x_{s+j+1}^{(t-j-1)}, y_{s+j+1}^{(t-j-1)})) \\ &\le 2^{-j}(d(\hat{y}^{(t-j)}, \hat{x}^{(t-j)}) + d(\hat{x}^{(t-j)}, \hat{f}(\hat{x}^{(t-j-1)})) + d(\hat{x}^{(t-j-1)}, \hat{y}^{(t-j-1)})) \\ &\le 2^{-j}(\epsilon_0 + \eta + \epsilon_0). \end{aligned}$$

Thus $d(\hat{x}^{(t)}, \hat{f}^{t-i}(\hat{y}^{(i)})) < \epsilon$, where $\epsilon = 5\epsilon_0 + 2\eta + 2\kappa(\epsilon_0+\eta)/(1-2\alpha)$ can be made arbitrarily small by choosing η and ϵ_0 appropriately. $\square$

Once we can shadow orbits in $\hat{\Lambda}$ it is fairly easy to do shadowing in Λ.

Corollary 2.5. ***(Shadowing Lemma for*** Λ***).*** *Suppose that $\hat{\Lambda}$ has local product structure. Then for each $\epsilon > 0$ there exists an $\eta > 0$ such that every η-pseudoorbit in Λ can be ϵ-shadowed by an orbit in Λ.*

Proof. By Theorem 2.4 there exists an $\eta' > 0$ such that every η'-pseudoorbit in $\hat{\Lambda}$ can be $(\epsilon/2)$-shadowed by an orbit in $\hat{\Lambda}$. Fix $m > 0$ so that $2^{1-m}\mathrm{diam}(\hat{\Lambda}) < \eta'/2$. Let $A \ge 2$ be larger than the Lipschitz constant for f on Λ and let $\eta < A^{-m-1}\min(\eta', \epsilon)/2$.

Now suppose $(x_i)_{[t_1,t_2]}$ is an η-pseudoorbit in Λ. If $t_2 < \infty$, then we define $x_i = f^{i-t_2}(x_{t_2})$ for $i \ge t_2$ and if $t_1 > -\infty$, then we pick any history $\hat{q}$ of x_{t_1} in $\hat{\Lambda}$ and declare $x_i = q_{i-t_1}$ for $i \le t_1$. In this way we obtain an η-pseudoorbit $(x_i)_{i \in \mathbf{Z}}$ in Λ.

Define a sequence $(\hat{x}^{(i)})_{i \in \mathbf{Z}}$ of points in $\hat{\Lambda}$ by

$$\hat{x}^{(i)} = (\hat{z}^{(i)}, f(x_{i-m}), \dots, f^{m-1}(x_{i-m}), f^m(x_{i-m})),$$

where $\hat{z}^{(i)}$ is any history of x_{i-m} in $\hat{\Lambda}$. We claim that $(\hat{x}^{(i)})$ is an η'-pseudoorbit in $\hat{\Lambda}$. Indeed, for any $i \in \mathbf{Z}$ we have

$$\begin{aligned}
d(\hat{f}(\hat{x}^{(i-1)}), \hat{x}^{(i)}) &\leq 2^{1-m} d(\hat{f}(\hat{z}^{(i-1)}), \hat{z}^{(i)}) + \sum_{1-m \leq j \leq 0} 2^j d(x_{j+1}^{(i-1)}, x_j^{(i)}) \\
&\leq 2^{1-m} \operatorname{diam}(\hat{\Lambda}) + \sum_{1 \leq k \leq m} 2^{k-m} d(f^{k+1}(x_{i-m-1}), f^k(x_{i-m})) \\
&< \eta'/2 + \sum_{1-m \leq j \leq 0} 2^j A^{m+j} d(f(x_{i-m-1}), x_{i-m}) \\
&\leq \eta'/2 + A^{m+1}\eta \\
&\leq \eta'.
\end{aligned}$$

By Theorem 2.4 we can find an orbit $(\hat{y}^{(i)})_{i \in \mathbf{Z}}$ in $\hat{\Lambda}$ which $\epsilon/2$-shadows $(\hat{x}^{(i)})$. If we let $y_i = y_0^{(i)}$, then y_i is an orbit in Λ and we have

$$\begin{aligned}
d(y_i, x_i) &\leq d(y_0^{(i)}, x_0^{(i)}) + d(x_0^{(i)}, x_i) \\
&\leq d(\hat{y}^{(i)}, \hat{x}^{(i)}) + d(f^m(x_{i-m}), x_i) \\
&\leq \epsilon/2 + \sum_{j=1}^{m-1} d(f^j(x_{i-j}), f^{j-1}(x_{i-j+1})) \\
&\leq \epsilon/2 + \sum_{j=1}^{m-1} A^{j-1}\eta \\
&< \epsilon.
\end{aligned}$$

Hence (y_i) ϵ-shadows (x_i) and we are done. □

Using shadowing we can control the orbits of f staying near Λ in positive or negative time. A neighborhood U of Λ with the properties in the following corollary will be called a *fundamental neighborhood*.

Corollary 2.6. ***(Fundamental neighborhood)***. *Let Λ be a hyperbolic set for a map f such that $\hat{\Lambda}$ has local product structure. Then, for any sufficiently small neighborhood U of Λ in M we have*

(i) *If $x \in U$ and $f^j(x) \in U$ for all $j \geq 0$, then $x \in W^s_\delta(p)$ for some $p \in \Lambda$.*
(ii) *If $x \in U$ and x has a history $\hat{x}$ with $x_i \in U$ for all $i \leq 0$, then $x \in W^u_\delta(\hat{q})$ for some $\hat{q} \in \hat{\Lambda}$.*
(iii) *If $(x_i)_{i \in \mathbf{Z}}$ is a complete orbit in U then $x_i \in \Lambda$ for all i.*
(iv) *If g is C^1-close to f, then the set Λ_g in Proposition 1.4 is given by*

$$\Lambda_g = \{x_0; (x_i)_{i \in \mathbf{Z}} \text{ is a } g\text{-orbit completely contained in } U\}.$$

In particular, $\hat{\Lambda}_g$ has local product structure.

Proof. We will apply Corollary 2.5 with $\epsilon = \delta/2$. Assume that $\eta \le \delta$ and define $U := \{x \in M; d(x, \Lambda) < \eta/2\}$, with η from Corollary 2.5.

(i) Pick points z_i in Λ for $i \ge 0$ with $d(x_i, z_i) < \eta/2$. Then $(z_i)_{i\ge 0}$ is an η-pseudoorbit in Λ so by Corollary 2.5 there is an orbit $(p_i)_{i\ge 0}$ in Λ which $\delta/2$-shadows (z_i). It follows that $d(p_i, x_i) < \eta/2 + \delta/2 \le \delta$ for all $i \ge 0$ so $x \in W^s_\delta(p)$, where $p = p_0$.

(ii) As in (i) we construct an η-pseudoorbit $(z_i)_{i\le 0}$ in Λ such that $d(x_i, z_i) < \eta/2$ for all $i \le 0$. Corollary 2.5 provides us with a point $\hat{q} \in \hat{\Lambda}$ such that $d(q_i, x_i) < \eta/2 + \delta/2 \le \delta$ for all i, so $x \in W^u_\delta(\hat{q})$.

(iii) From (i) and (ii) we find $p \in \Lambda$ and $\hat{q} \in \hat{\Lambda}$ such that $x \in W^s_\delta(p) \cap W^u_\delta(\hat{q})$. Since $\hat{\Lambda}$ has local product structure this implies that $x \in \Lambda$.

(iv) We have $\Lambda_g \subset U$ by Proposition 1.4 so we only have to prove the reverse inclusion. Let $(x_i)_{i\in\mathbf{Z}}$ be a g-orbit completely contained in U. By shrinking U we may assume that if g is close to f, then (x_i) may be $\delta/2$-shadowed by an f-orbit (y_i) in Λ and hence δ-shadowed by the g-orbit (z_i) in Λ_g coming from the conjugacy in Proposition 1.4. Thus

$$x_0 \in W^s_\delta(z_0) \cap W^u_\delta((z_i)_{i\le 0}) = \{z_0\} \in \Lambda_g.$$

□

3. Axiom A endomorphisms

The results up to now have been of a semilocal nature, i.e. they concern the dynamics near a compact set. In order to study global dynamical properties we now restrict our attention to Axiom A endomorphisms. Our goal here is to prove the spectral decomposition theorem, which allows us to understand the dynamics of f near its nonwandering set. For the proof we will assume that f is an open mapping.

Let f be an C^∞ endomorphism of a C^∞ manifold M. A point $x \in M$ is *nonwandering* if it has no neighborhood V such that $f^n(V) \cap V = \emptyset$ for all $n \ge 1$. The *nonwandering set* Ω of f is the set of all nonwandering points; it is a closed set.

Definition 3.1. f is said to satisfy Axiom A if its nonwandering set satisfies

(i) Ω is compact.

(ii) Periodic points are dense in Ω.

(iii) f is hyperbolic on Ω.

Remark 3.2. If Ω satisfies (i) and (ii), then $f(\Omega) = \Omega$, so (iii) makes sense. Also, if f is Axiom A, then periodic points (under $\hat{f}$) are dense in $\hat{\Omega}$.

The following proposition shows that the results in section 2 apply to open Axiom A endomorphisms.

Proposition 3.3. *If f is an open Axiom A map, then $\hat{\Omega}$ has local product structure.*

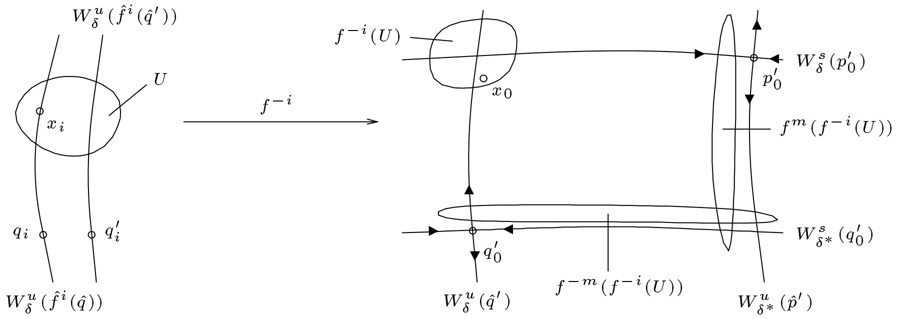


FIGURE 3. Local product structure for $\hat\Omega$.

Proof. Choose $\delta^* > \delta > 0$ so small that if $\hat p, \hat q \in \hat\Omega$ and $W^s_\delta(p_0)$ and $W^u_\delta(\hat q)$ intersect in a unique point, then $W^s_{\delta^*}(q_0)$ and $W^u_{\delta^*}(\hat p)$ intersect in a unique point. Now let $\hat p$ and $\hat q$ be any two points in $\hat\Omega$ such that $W^s_\delta(p_0)$ and $W^u_\delta(\hat q)$ intersect in a unique point x. Then x has a history $\hat x$ such that $x_j \in W^u_\delta(\hat f^j(\hat q))$ for all $j \le 0$. We have to prove that $\hat x \in \hat\Omega$.

We first consider the case when $\hat p$ and $\hat q$ are periodic, say of periods l and m, respectively. Let $g = f^{lm}$ and let U be any neighborhood of x. By Proposition 1.3, $g^j(U)$ contains a manifold C^1-close to $W^u_\delta(\hat q)$ and $g^{-j}(U)$ contains a manifold C^1-close to $W^s_\delta(p_0)$ for all large j. Therefore $g^j(U)$ and $g^{-j}(U)$ intersect in a point near $x^* := W^s_{\delta^*}(q_0) \cap W^u_{\delta^*}(\hat p)$ for all large j, so x is nonwandering, i.e. $x \in \Omega$.

For general $\hat p$, $\hat q$ let $\hat x$ be the history defined above, let $i \le 0$ and let U be any neighborhood of x_i. Then $f^{-i}(U)$ is a neighborhood of x, because f is open. Since periodic points are dense in $\hat\Omega$ we may find periodic points $\hat p'$, $\hat q'$ in $\hat\Omega$ close to $\hat p$, $\hat q$ such that $W^s_\delta(p_0')$ intersects $W^u_\delta(\hat q')$ in $f^{-i}(U)$ and $W^u_\delta(\hat f^i(\hat q'))$ intersects U. Then the above argument shows that $f^k(U)$ intersects $f^{-k}(U)$ for infinitely many $k \ge 0$. Hence x_i is nonwandering for all $i \le 0$, so $\hat x \in \hat\Omega$. See Figure 3 for an illustration of the proof. $\square$

Theorem 3.4. ***(Spectral decomposition of*** $\hat\Omega$***).*** *If f is an open Axiom A endomorphism, then $\hat\Omega$ can be written in a unique way as a disjoint union $\hat\Omega = \cup_{i=1}^l \hat\Omega_i$, where each $\hat\Omega_i$ is compact, satisfies $\hat f(\hat\Omega_i) = \hat\Omega_i$ and $\hat f$ is transitive on $\hat\Omega_i$. The sets $\hat\Omega_i$ are called the basic sets of $\hat f$. Morover, each $\hat\Omega_i$ can be further decomposed into a finite disjoint union $\hat\Omega_i = \cup_{1\le j\le n_i} \hat\Omega_{i,j}$, where $\Omega_{i,j}$ is compact, $\hat f(\hat\Omega_{i,j}) = \hat\Omega_{i,j+1}$ ($\hat\Omega_{i,n_i+1} = \hat\Omega_{i,1}$) and $\hat f^{n_i}$ is mixing on each $\hat\Omega_{i,j}$.*

Proof. From Proposition 3.3 we know that $\hat\Omega$ has local product structure. Choose $\delta, \delta' > 0$ as in the discussion preceding (2.1) and (2.2). If $\hat p \in \hat\Omega$ is a periodic history, say of period l, then we let $\hat W^u_\delta(\hat p)$ be the set of histories $\hat x \in \hat\Omega$ such that $d(x_i, p_i) < \delta$ for all $i \le 0$. Similarly, we let $\hat W^u(\hat p)$ be the set of histories

$\hat{x} \in \hat{\Omega}$ such that $d(x_i, p_i) \to 0$ as $i \to -\infty$. Then $\hat{W}^u(\hat{p}) = \cup_{j\geq 0} \hat{f}^{jl}(\hat{W}^u_\delta(\hat{p}))$. Let $X_{\hat{p}}$ be the closure of $\hat{W}^u(\hat{p})$ in $\hat{\Omega}$.

Suppose that $\hat{p} \in \hat{\Omega}$ is periodic of period l. We first prove that if $\hat{y} \in \hat{\Omega}$ and $d(\hat{y}, X_{\hat{p}}) < \delta'$, then $\hat{y} \in X_{\hat{p}}$. We may assume that $\hat{y}$ is periodic, say of period m. Take any point $\hat{x} \in \hat{W}^u(\hat{p})$ with $d(\hat{y}, \hat{x}) < \delta'$ and let $\hat{z} = [y_0, \hat{x}]$. Then $\hat{z} \in \hat{W}^u(\hat{p})$, which implies that $\hat{f}^j(\hat{z}) \in \hat{W}^u(\hat{p})$ if $j \geq 0$ and l divides j. But $\hat{f}^j(\hat{z})$ is close to $\hat{y}$ if j is large and m divides j, so $\hat{y} \in X_{\hat{p}}$.

The next step is to prove that if $\hat{p}$ and $\hat{q}$ are two periodic points in $\hat{\Omega}$ of periods l and m, respectively, then either $X_{\hat{p}} = X_{\hat{q}}$ or $X_{\hat{p}} \cap X_{\hat{q}} = \emptyset$. First suppose $\hat{q} \in X_{\hat{p}}$. By the preceding paragraph $X_{\hat{p}}$ is open, so we may find $\gamma \in (0, \delta)$ such that $\hat{W}^u_\gamma(\hat{q}) \subset X_{\hat{p}}$. Then $f^{jlm} \hat{W}^u_\gamma(\hat{q}) \subset X_{\hat{p}}$ for all $j \geq 0$, so $X_{\hat{q}} \subset X_{\hat{p}}$. On the other hand, $X_{\hat{q}}$ is open and intersects $X_{\hat{p}}$, so we may find $\hat{x} \in X_{\hat{q}} \cap \hat{W}^u(\hat{p})$. But it is easy to see that $\hat{f}^m(X_{\hat{q}}) = X_{\hat{q}}$ so $\hat{f}^{-jlm}(\hat{x}) \in X_{\hat{q}}$ for all $j \geq 0$, which implies that $\hat{p} \in X_{\hat{q}}$. Therefore $\hat{q} \in X_{\hat{p}}$ implies $X_{\hat{p}} = X_{\hat{q}}$. Now suppose $\hat{p}$ and $\hat{q}$ are periodic and that $X_{\hat{p}}$ and $X_{\hat{q}}$ are not disjoint. Then they intersect in an open set, which contains a periodic history $\hat{r}$, so the previous argument shows that $X_{\hat{p}} = X_{\hat{r}} = X_{\hat{q}}$.

The different sets $X_{\hat{p}}$ form a disjoint open covering of the compact set $\hat{\Omega}$ so they are finite in number. It is clear that $\hat{f}(X_{\hat{p}}) = X_{\hat{f}(\hat{p})}$ so $\hat{f}$ induces a permutation of the different sets $X_{\hat{p}}$. Let $\hat{\Omega}_{i,j}$, $i = 1, \dots, l$, $j = 1, \dots, n_i$ be the distinct sets $X_{\hat{p}}$, labeled so that $\hat{f}(\hat{\Omega}_{i,j}) = \hat{\Omega}_{i,j+1}$, for $j = 1, \dots, n_i$ where $\hat{\Omega}_{i,n_i+1} = \hat{\Omega}_{i,1}$. Let $\hat{\Omega}_i = \cup_{i=1}^{n_i} \hat{\Omega}_{i,j}$ for $i = 1, \dots, l$. Then $\hat{f}(\hat{\Omega}_i) = \hat{\Omega}_i$ and $\hat{f}^{n_i}(\hat{\Omega}_{i,j}) = \hat{\Omega}_{i,j}$ for all i, j.

We prove that $\hat{f}^{n_i}$ is mixing on $\hat{\Omega}_{i,j}$ for all i, j. Let U and V be two open sets in $\hat{\Omega}_{i,j}$. We have to show that $\hat{f}^{tn_i}(U) \cap V \neq \emptyset$ for all sufficiently large t. Let $\hat{p}$ be a periodic point in U, say of period l. Then $X_{\hat{f}^{sn_i}(\hat{p})} = \hat{\Omega}_{i,j}$ so we may find points $\hat{x}^{(s)}$ in $\hat{W}^u(\hat{f}^{sn_i}(\hat{p})) \cap V$ for $s = 0, \dots, l-1$. For every sufficiently large t we may then find $0 \leq s \leq l-1$ such that $\hat{f}^{-tn_i}(\hat{x}^{(s)}) \in U$ so $\hat{f}^{tn_i}(U) \cap V \neq \emptyset$. Hence $\hat{f}^{n_i}$ is mixing on $\hat{\Omega}_{i,j}$ for all i, j and this implies that $\hat{f}$ is transitive on Ω_i for all i. ☐

As we see next, the spectral decomposition of $\hat{\Omega}$ induces one of Ω.

Corollary 3.5. ***(Spectral decomposition of*** Ω***).*** *If f is an open Axiom A endomorphism, then Ω can be written in a unique way as a disjoint union $\Omega = \cup_{i=1}^{l} \Omega_i$, where each Ω_i is compact, satisfies $f(\Omega_i) = \Omega_i$ and f is transitive on Ω_i. The sets Ω_i are called the basic sets of f. Morover, each Ω_i can be further decomposed into a finite disjoint union $\Omega_i = \cup_{1\leq j\leq n_i} \Omega_{i,j}$, where $\Omega_{i,j}$ is compact, $f(\Omega_{i,j}) = \Omega_{i,j+1}$ ($\Omega_{i,n_i+1} = \Omega_{i,1}$) and f^{n_i} is mixing on each $\Omega_{i,j}$.*

Proof. We define $\Omega_{i,j} = \pi(\hat{\Omega}_{i,j})$, where $\pi : \hat{\Omega} \to \Omega$ is the projection. We claim that the $\Omega_{i,j}$'s are pairwise disjoint. If not, then there exist periodic points p and

q of periods l and m, respectively, such that $X_{\hat{p}} \cap X_{\hat{q}} = \emptyset$ but $\pi(X_{\hat{p}}) \cap \pi(X_{\hat{q}}) \neq \emptyset$. Let x be a point in Ω with two histories $\hat{x}^{(1)} \in X_{\hat{p}}$, $\hat{x}^{(2)} \in X_{\hat{q}}$. If $j \geq 0$, then $\hat{f}^{jlm}(\hat{x}^{(1)}) \in X_{\hat{p}}$, $\hat{f}^{jlm}(\hat{x}^{(2)}) \in X_{\hat{q}}$ and $d(\hat{f}^{jlm}(\hat{x}^{(1)}), \hat{f}^{jlm}(\hat{x}^{(1)})) \to 0$ as $j \to \infty$. This is a contradiction, because $d(\hat{X}_{\hat{p}}, \hat{X}_{\hat{q}}) \geq \delta'$.

Thus the sets $\Omega_{i,j}$ are pairwise disjoint. They are compact because $\Omega_{i,j}$ is compact for all i, j and π is continuous. It remains to be seen that f^{n_i} is mixing on $\Omega_{i,j}$. This is easy, because if U and V are two open subsets of $\Omega_{i,j}$, then $\hat{U} := \pi^{-1}(U)$ and $\hat{V} := \pi^{-1}(V)$ are open subsets of $\hat{\Omega}_{i,j}$ and $\hat{f}^{tn_i}(\hat{U}) \cap \hat{V} \neq \emptyset$ for sufficiently large t. It follows that $f^{tn_i}(U) \cap V \neq \emptyset$ for sufficiently large t, which completes the proof. □

It follows easily from the definition of the nonwandering set that if M is compact and $(x_i)_{i \in \mathbf{Z}}$ is a complete orbit in M, then $x_i \to \Omega$ as $i \to \pm\infty$. In the Axiom A case we can say more. Using the fact that the basic sets are compact, disjoint and f-invariant, we easily prove the following result.

Lemma 3.6. *Assume that M is compact and that f is an open Axiom A endomorphism. If $x \in M$, then there exists a basic set Ω_j such that $f^i(x) \to \Omega_j$ as $i \to \infty$. Similarly, if $\hat{x}$ is a history in $\hat{M}$, then there exists a (possibly different) basic set Ω_j such that $x_i \to \Omega_j$ as $i \to -\infty$.*

Combining Lemma 3.6 and Corollary 2.6 we obtain.

Proposition 3.7. *Assume that f is an open Axiom A endomorphism and that M is compact.*

1. *(i) If $x \in M$, then there exists a unique basic set Ω_j such that $f^j(x) \to \Omega_j$ as $j \to \infty$. Moreover, there exists a (not necessarily unique) $p \in \Omega_i$ such that $d(f^j(x), f^j(p)) \to 0$ as $j \to \infty$.*
2. *(ii) If $\hat{x} \in \hat{M}$, then there exists a unique basic set Ω_i such that $x_j \to \Omega_i$ as $j \to -\infty$. Moreover, there exists a (not necessarily unique) $\hat{q} \in \widehat{\Omega_i}$ such that $d(x_j, q_j) \to 0$ as $j \to -\infty$.*

4. $\hat{\Omega}$-STABILITY AND THE NO-CYCLE CONDITION

Given a dynamical system we may ask whether it is stable under perturbations. The answer to this fairly vague question depends on what we mean by stability. In this section we define the notion of $\hat{\Omega}$-stability and give sufficient conditions for it in terms of hyperbolicity.

Let $f : M \to M$ be an Axiom A endomorphism. For this section we will assume that f is open and M is compact. Let $\Omega = \bigcup_{1 \leq i \leq l} \Omega_i$ be the spectral decomposition for f. Define a relation $<$ among the basic sets Ω_i by declaring that $\Omega_i < \Omega_j$ if $W^s(\Omega_i) \cap W^u(\Omega_j) \neq \emptyset$. Here

$$W^s(\Omega_j) = \{x \in M; f^i(x) \to \Omega_j \text{ as } i \to \infty\}$$
$$W^u(\Omega_j) = \{x \in M; \exists \hat{x}, \pi(\hat{x}) = x, x_i \to \Omega_j \text{ as } i \to -\infty\}$$

Let us first show that there are no trivial cycles for the relation $<$.

Lemma 4.1. *For any i we have $W^s(\Omega_i) \cap W^u(\Omega_i) = \Omega_i$.*

Proof. The proof is similar to that of Proposition 3.3. Let $(x_k)_{k\in\mathbf{Z}}$ be a complete orbit with $x_k \to \Omega_i$ as $|k| \to \infty$. We have to show that $x_0 \in \Omega_i$ and it suffices to show that x_0 is nonwandering. Choose δ' as in the discussion preceding (2.1). By Proposition 3.7 there exist $k > 0$, $y \in \Omega_i$ and $\hat{z} \in \widehat{\Omega_i}$ such that $x_k \in W^s_\delta(y)$ and $x_{-k} \in W^u_\delta(\hat{z})$. Let U be an open neighborhood of x_0. Then $f^k(U)$ is open and intersects $W^s_\delta(y)$. Now f is transitive on Ω_i so we may find $j \geq 0$ and $y' \in \Omega_1$ such that $W^s_\delta(y') \cap f^k(U) \neq \emptyset$ and $d(f^j(y'), z_0) < \delta'$. We may replace $f^j(y')$ by a periodic point u of period m. Hence $W^s_\delta(u) \cap f^{k+j}(U) \neq \emptyset$. Similarly, we may find a periodic history $\hat{v} \in \widehat{\Omega_i}$ of period n such that $W^u_\delta(\hat{v}) \cap f^{-k}(U) \neq \emptyset$ and $d(v_0, u) < \delta'$. By Proposition 1.3 $f^{k+j+ml}(U)$ contains a manifold C^1-close to $W^u_\delta(\hat{v})$ and $f^{-k-nl}(U)$ contains a manifold C^1-close to $W^s_\delta(u)$ for large l. Hence $f^{2k+j+(m+n)l}(U) \cap U \neq \emptyset$ for large l, so x_0 is nonwandering. $\square$

We say that f satisfies the *no-cycle condition* or, simply, that f has no cycles if there is no nontrivial chain

$$\Omega_{i_1} < \Omega_{i_2} < \cdots < \Omega_{i_n} = \Omega_{i_1}.$$

Definition 4.2. An endomorphism $f : M \to M$ is $\hat{\Omega}$-stable if there exists a neighborhood U of f and for every $g \in U$ a homeomorphism $\phi : \hat{\Omega}_f \to \hat{\Omega}_g$ with $\hat{g} \circ \phi = \phi \circ \hat{f}$. Here Ω_f and Ω_g are the nonwandering sets of f and g respectively.

We now come to the main result in this section. For simplicity we restrict our attention to compact manifolds M.

Theorem 4.3. *If M is compact and $f : M \to M$ is an open Axiom A endomorphism with no cycles, then f is $\hat{\Omega}$-stable.*

Remark 4.4. The proof will show that the conjugacy ϕ can be chosen close to the identity. Note that the conjugacy takes place on the level of histories — the sets Ω_f and Ω_g need not be homeomorphic.

Let us make some observations before starting with the proof. By spectral decomposition, Ω is the disjoint union of the basic sets Ω_i, $1 \leq i \leq l$ and there are fundamental neighborhoods U_i of Ω_i in the sense of Corollary 2.6.

In particular, if g is C^1-close to f, then g has hyperbolic sets $\Omega_{i,g}$, $1 \leq i \leq l$ contained in U_i and there are homeomorphisms $\phi_i : \hat{\Omega}_{i,f} \to \hat{\Omega}_{i,g}$ conjugating $\hat{f}$ to $\hat{g}$. Thus $\Omega_{g,i}$ has local product structure, periodic points for g are dense in $\Omega_{i,g}$ and the restriction of g to $\Omega_{i,g}$ is transitive. In particular $\Omega_{i,g}$ is contained in the nonwandering set Ω_g of g. To prove that f is $\hat{\Omega}$-stable, it therefore suffices to prove that Ω_g is exactly the union of the sets $\Omega_{i,g}$. In general, there is no reason for this to be true. Picture 4 illustrates an Axiom A diffeomorphism f of, say, the two-dimensional sphere admitting an Ω-*explosion*, meaning that the

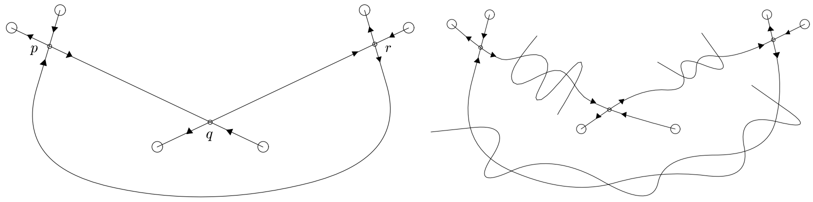


FIGURE 4. An Ω-explosion.

nonwandering set for the original map f (a finite set) is much smaller than the nonwandering set for the perturbed map g (an infinite set). The nonwandering set of f consists of six sources and sinks, marked with big circles, and three saddle points p, q and r. These are the basic sets of f. The nonwandering set of g contain perturbations of these nine points, but also all the transverse intersection between unstable and stable manifolds in the second picture.

The main tool in proving Theorem 4.3 is the existence of a *filtration*, which we now describe. If f is Axiom A and has no cycles, then we may label the basic sets of f in such a way that $\Omega_i > \Omega_j$ implies $i > j$.

Proposition 4.5. *Let $f : M \to M$ be an open Axiom A map with no cycles, where M is compact. Then there is an integer $m \geq 1$, fundamental neighborhoods U_j of Ω_j and compact sets $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_l = M$, such that $U_1 = int(M_1)$, $f^m(M_j) \subset int(M_j)$ for $1 \leq j \leq l$, and $f^m(M_j - U_j) \subset int(M_{j-1})$ for $2 \leq j \leq l$.*

We postpone the proof of Proposition 4.5 and show instead how to deduce $\hat{\Omega}$-stability.

Proof of Theorem 4.3. Let g be C^1-close to f. As mentioned above it suffices to show that the nonwandering set Ω_g of g is the union of the sets $\Omega_{j,g}$, $1 \leq j \leq l$, so let $(x_i)_{i \in \mathbf{Z}}$ be a g-orbit completely contained in Ω_g. If g is close enough to f, then Proposition 4.5 holds with f replaced by g. Hence there is a j, $1 \leq j \leq l$, such that $x_i \in U_j$ for all i. But then $x_i \in \Omega_{j,g}$ for all i by Corollary 2.6. $\square$

Thus it remains to construct the filtration in Proposition 4.5. Figure 5 illustrates the first two steps in the construction of the filtration. Here Ω_1 is an attracting set, by the labeling of Ω_i, and $M_1 = \overline{U}_1$ is a neighborhood of Ω_1. Next, $W^u(\Omega_2)$ is in the stable set of Ω_1 and M_2 is the union of M_1 and a neighborhood of $W^u(\Omega_2) - M_1$. It will take some care to define this neighborhood so that the properties in Proposition 4.5 hold.

We start the proof of Proposition 4.5 with a preliminary result.

Lemma 4.6. *The set $\Lambda_k := \bigcup_{i \leq k} W^u(\Omega_i)$ is compact and $\bigcup_{i \leq k} W^s(\Omega_i)$ is an open neighborhood of Λ_k for $1 \leq k \leq l$.*

Proof. We first show that Λ_k is closed, hence compact. Let $x \in \overline{W^u(\Omega_{i_0})}$ for some $i_0 \leq k$. We must show that $x \in W^u(\Omega_i)$ for some $i \leq k$. Pick histories

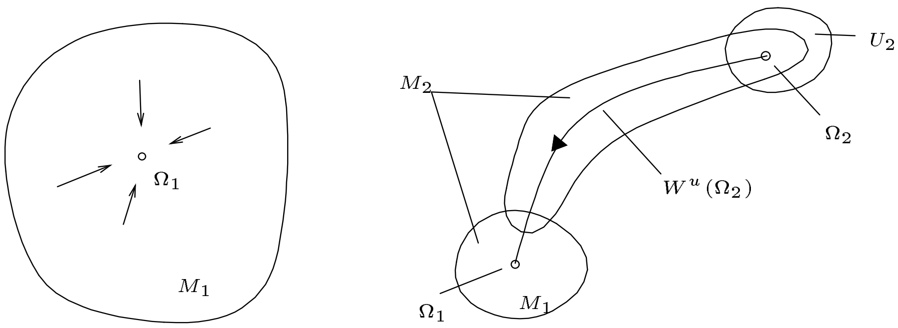


FIGURE 5. Construction of the filtration.

$\hat{y}^{(\mu)}$, $\mu \geq 1$, such that $y_0^{(\mu)} \to x$ as $\mu \to \infty$ and $y_s^{(\mu)} \to \Omega_{i_0}$ as $s \to -\infty$ for all μ. By passing to a subsequence we may assume that $\hat{y}^{(\mu)}$ converges to a history $\hat{z}$.

Let I be the set of i such that $\hat{y}^{(\mu)}$ accumulates on Ω_i as $\mu \to \infty$. More precisely, $i \in I$ if there exist $\mu_k \to \infty$ and $s_k \leq 0$ such that $y_{s_k}^{(\mu_k)} \to \Omega_i$ as $k \to \infty$. The proof now goes through a number of steps.

Lemma 4.7. *There is an $i \in I$ such that $x \in W^u(\Omega_i)$.*

Proof of Lemma 4.7. Recall that $\hat{y}^{(\mu)} \to \hat{z}$ as $\mu \to \infty$. We have $z_0 = x$ and there is an i such that $z_s \to \Omega_i$ as $s \to -\infty$. We claim that $i \in I$. To see this, pick s_k with $d(z_{s_k}, \Omega_i) < \frac{1}{k}$ for $k > 0$. If μ_k is large enough, then $d(y_{s_k}^{(\mu_k)}, \Omega_i) < \frac{1}{k}$, which proves that $i \in I$. $\square$

Lemma 4.8. *If $i \in I$, $i \neq i_0$, then there is a $j \in I$, $j \neq i$ such that $\Omega_j > \Omega_i$.*

Proof of Lemma 4.8. Pick $\delta_0 > 0$ such that

$$\delta_0 < \frac{1}{2} \min_{1 \leq i_1 < i_2 \leq l} d(\Omega_{i_1}, \Omega_{i_2})$$

By assumption there exist $\mu_k \to \infty$ and $s_k \leq 0$ such that $y_{s_k}^{(\mu_k)} \to \Omega_i$. Choose $t_k < s_k$ minimal such that $d(y_{t_k}^{(\mu_k)}, \Omega_i) < \delta_0$. This is possible because $i \neq i_0$. Define $\hat{w}^{(k)}$ by $w_s^{(k)} = y_{s+t_k}^{(\mu_k)}$. By passing to a subsequence we may assume that $\hat{w}^{(k)} \to \hat{w}$ as $k \to \infty$. We claim that $w_s \to \Omega_i$ as $s \to \infty$. To see this we first consider the case when $s_k - t_k \to \infty$. Then $d(w_s^{(k)}, \Omega_i) < \delta_0$ for $0 \leq s \leq s_k - t_k$, so we must have $w_s \to \Omega_i$ as $s \to \infty$. The second case is when $s_k - t_k$ is bounded as $k \to \infty$. By passing to a subsequence we may assume that $s_k - t_k = r \geq 0$ for all k. But then $d(w_r^{(k)}, \Omega_i) \to 0$ as $k \to \infty$ so $w_r \in \Omega_i$. Thus $w_s \to \Omega_i$ is this case too.

Similarly, we have $w_s \to \Omega_j$ as $s \to -\infty$ for some j. Hence $\Omega_j > \Omega_i$ and we have $j \neq i$ by Lemma 4.1. It remains to be seen that $j \in I$. But for each $m \geq 1$ we may choose $u_m < 0$ such that $d(z_{u_m}, \Omega_j) < \frac{1}{m}$. Then we find $\mu_m \to \infty$ such that $d(y_{u_m+t_m}^{(\mu_m)}, \Omega_j) < \frac{1}{m}$. This shows that $j \in I$. $\square$

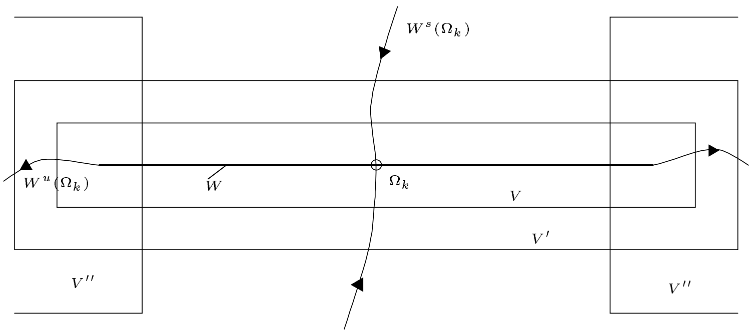


FIGURE 6. Dynamics near Ω_k

We now continue the proof of Lemma 4.6. By Lemma 4.7, Lemma 4.8 and the no-cycle property there exists a chain

$$\Omega_{i_0} > \Omega_{i_1} > \cdots > \Omega_{i_l},$$

such that $x \in W^u(\Omega_{i_l})$. Again by the no-cycle property we must have $i_l \leq i_0 \leq k$ so $x \in \Lambda_k$. This proves that Λ_k is compact. Similarly we may prove that $M - \bigcup_{i \leq k} W^s(\Omega_i) = \bigcup_{i>k} W^u(\Omega_i)$ is compact so $\bigcup_{i \leq k} W^s(\Omega_i)$ is open and it contains Ω_k by the labeling of the basic sets. □

Proof of Proposition 4.5. We construct the sets M_k and choose the fundamental neighborhoods U_k inductively. Compare with Figure 5. First note that Ω_1 is an attracting set, because

$$W^u(\Omega_1) = \bigcup_{1 \leq j \leq l} W^u(\Omega_1) \cap W^s(\Omega_j) = \Omega_1$$

by Lemma 4.1 and the labeling of the Ω_i. Hence, if U_1 is small enough and $M_1 = \overline{U_1}$, then we can find $m \geq 1$ such that $f^m(M_1) \subset U_1$. Note that $\Lambda_1 = \Omega_1 \subset \mathrm{int}(M_1)$. Now suppose that $2 \leq k \leq l$ and that we have an integer $m' \geq 1$ and compact sets $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_{k-1}$ such that $\Lambda_j \subset \mathrm{int}(M_j)$, $f^{m'}(M_j) \subset \mathrm{int}(M_j)$ for $1 \leq j \leq k-1$, and $f^{m'}(M_j - U_j) \subset \mathrm{int}(M_{j-1})$ for $2 \leq j \leq k-1$.

If $x \in W^u(\Omega_k)$, then $x \in W^s(\Omega_i)$ for some $i < k$ by Lemma 4.1 and the labeling of the Ω_i. Given $\epsilon, \delta, \delta', \delta'' > 0$, define the sets W,V,V',V'' as follows. W is the open ϵ-neighborhood of Ω_k in $W^u(\Omega_k)$, V (V') is the closed δ-neighborhood (δ'-neighborhood) of W in M, and V'' is the closed δ''-neighborhood of $W^u(\Omega_k) - (M_{k-1} \cup W)$ in M. See Figure 6.

By the hyperbolicity of f on Ω_k we may choose $m_1 \geq 1$ and $\epsilon > 0$ such that for every $m \geq m_1$ and every $\delta'' > 0$ there exist $\delta' > \delta > 0$ such that $f^m(V') \subset V \cup \mathrm{int}(V'')$.

Now $W^u(\Omega_k) - (W \cup \mathrm{int}(M_{k-1}))$ is compact, so by the induction hypothesis there is an $m_2 \geq m_1$ such that $f^{m_2}(W^u(\Omega_k) - W) \subset \mathrm{int}(M_{k-1})$. Choose δ''

so small that $f^{m_2}(V'') \subset \operatorname{int}(M_{k-1})$ and find $\delta' > \delta > 0$ such that $f^{m_2}(V') \subset V \cup \operatorname{int}(V'')$.

Hence, if we let $m = m'm_2$, $U_k = \operatorname{int}(V')$ and $M_k = M_{k-1} \cup V'' \cup V'$, then M_k is an open neighborhood of Λ_k, $f^m(M_k) \subset \operatorname{int}(M_k)$, and $f^m(M_k - U_k) \subset \operatorname{int}(M_{k-1})$. This completes the induction. □

References

[R] Ruelle, D. *Elements of differentiable dynamics and bifurcation theory.* Academic Press, 1989.

[PS] Pugh, C, Shub, M. Ergodic attractors. *Trans. Amer. Math. Soc.* **312**(1989), 1–54.

[S] Shub, M. *Global stability of dynamical systems.* Springer-Verlag, 1987.

Dept. of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden
E-mail address: mjo@math.kth.se

Paper II

HOLOMORPHIC MOTIONS OF HYPERBOLIC SETS

MATTIAS JONSSON

Abstract. We show how hyperbolic sets for holomorphic families of endomorphisms of a complex Hermitian manifold give rise to holomorphic motions or analytic multifunctions.

0. Introduction

Let M be a complex Hermitian manifold and $\{f_a\}_{a\in\mathbf{D}}$ a holomorphic family of endomorphisms of M, where $\mathbf{D}$ is the unit disk. This means that the map $\mathbf{D}\times M\to M$, defined by $(a,x)\to f_a(x)$ is holomorphic. Suppose that $f=f_0$ has a compact surjectively invariant subset K, i.e. $f(K)=K$. For example, K could be a fixed point or a periodic orbit, but also a more complicated set such as the Julia set of a rational function. We may then ask if K is persistent under the perturbation f_a of the map f. For instance, if K is a fixed point of f, then we ask if f_a has a fixed point K_a near K for a small enough. A sufficient (albeit not necessary) condition for this is that the fixed point K is hyperbolic, meaning that the derivative of f at K has no eigenvalue of modulus one.

There is a natural notion of hyperbolicity for general sets K. Let us first consider the case when the maps f_a are diffeomorphisms. The precise definition can then be found in e.g. [R] and will not be stated here, but it says, loosely, that the tangent bundle over K splits continuously into two invariant subbundles on which the derivative of f is expanding and contracting, respectively.

One basic result in real dynamics is that hyperbolic sets are persistent under perturbations in the map f (see [R]). In our case this means that if a is small enough, then f_a has a hyperbolic set K_a close to K, and there exists a homeomorphism h_a close to the identity conjugating $f|_K$ to $f_a|_{K_a}$.

If K is a hyperbolic fixed point, then it follows from the implicit function theorem that the fixed point K_a of f_a depends holomorphically on a. The natural generalization of this to more general sets K is the notion of a holomorphic motion, the definition of which is given in section 1.

Theorem A. *Let $\{f_a\}_{a\in\mathbf{D}}$ be a holomorphic family of diffeomorphisms of a Hermitian manifold M parameterized by the unit disk $\mathbf{D}$. Suppose that $f=f_0$*

1991 *Mathematics Subject Classification.* Primary: 32H50, Secondary: 58F23,58F15.

Key words and phrases. Holomorphic motions, analytic multifunctions, hyperbolic sets, holomorphic dynamics.

Partially supported by a grant from NFR (Swedish Natural Science Research Council).

has a hyperbolic subset K. Then K moves holomorphically with the parameter a at $a = 0$. More precisely, there exist $r > 0$ and a holomorphic motion $h : \mathbf{D}_r \times K \to M$ such that for each $a \in \mathbf{D}_r$

(1) *$K_a := h(a, K)$ is a hyperbolic subset for f_a.*
(2) *The map $h_a := h(a, \cdot) : K \to K_a$ is a homeomorphism and $f_a \circ h_a = h_a \circ f$.*

Let us now return to the situation of a holomorphic family $\{f_a\}$ of endomorphisms of a Hermitian manifold M. There is a notion of a hyperbolic set K in this setting too [R]. Again, we will not give the precise definition, but let us note that it involves the set $\hat{K} = \{(x_k)_{k\leq 0}; x_k \in K, f(x_k) = x_{k+1}\}$ of backwards orbits in K.

The real theory [R] tells us that for a small enough, f_a has a hyperbolic set K_a close to K and there exists a continuous surjective map $h_a : \hat{K} \to K_a$. Now K and K_a need not be homeomorphic so K does, in general, not move holomorphically with a. Nevertheless, the dependence of K_a on a reflects the complex structure; one way of saying this is that $a \to K_a$ is a strongly analytic multifunction, the definition of which is given in section 1.

Theorem B. *Let $\{f_a\}_{a\in\mathbf{D}}$ be a holomorphic family of endomorphisms of a Hermitian manifold M parameterized by the unit disk $\mathbf{D}$. Suppose that $f = f_0$ has a hyperbolic subset K. Then $\hat{K}$ moves holomorphically with the parameter a at $a = 0$ and $a \to K_a$ is a strongly analytic multifunction. More precisely, there exist $r > 0$ and a continuous map $h : \mathbf{D}_r \times \hat{K} \to M$ such that*

(1) *For each $a \in \mathbf{D}_r$, $K_a := h_a(\hat{K})$ is a hyperbolic set for f_a, where $h_a = h(a, \cdot)$.*
(2) *For each $a \in \mathbf{D}_r$ the map h_a satisfies the relation $f_a \circ h_a = h_a \circ \hat{f}$ and lifts to a homeomorphism $\widehat{h_a} : \widehat{K} \to \widehat{K_a}$, which is just the identity for $a = 0$.*
(3) *The map $h(\cdot, \hat{x}) : \mathbf{D}_r \to M$ is holomorphic for each $\hat{x} \in \hat{K}$.*
(4) *The set $\bigcup_a(\{a\} \times K_a)$ in $\mathbf{D}_r \times M$ is foliated by holomorphic graphs over $\mathbf{D}_r$.*

Sometimes a hyperbolic set K does move holomorphically with the parameter even for endomorphisms. An important situation when this happens is when K is a repellor, meaning that the derivative of f is expanding on the whole tangent bundle over K (see Definition 1.3).

Theorem C. *If $\{f_a\}$ is a holomorphic family of endomorphisms and K is a repellor for $f = f_0$, then K moves holomorphically with a at $a = 0$ in the sense of Theorem A.*

Theorem C applies to show that the Julia set of a rational function moves holomorphically with the parameter on the open set of parameter space consisting of hyperbolic maps. In [MSS], the authors prove that in fact one has a holomorphic motion for a (possible larger) open dense set of parameter space.

1. Definitions

In this section we recall the definitions of holomorphic motions and analytic multifunctions. For notational simplicity we will let these be parameterized by the unit disk.

Definition 1.1. Let $\mathbf{D}$ be the unit disk, M a complex manifold and X a subset of M. Then a holomorphic motion of X parameterized by $\mathbf{D}$ is a continuous map $\phi : \mathbf{D} \times X \to M$ such that:

(1) $\phi(0, \cdot) = \mathrm{id}$.
(2) $\phi(\cdot, x) : \mathbf{D} \to M$ is holomorphic for every $x \in X$.
(3) $\phi(a, \cdot) : X \to M$ is injective for every $a \in \mathbf{D}$.

Holomorphic motions have mostly been studied for subsets of the Riemann sphere. In [MSS] Mañe, Sad and Sullivan proved the celebrated λ-lemma, which says that each map $\phi(a, \cdot)$ is quasiconformal and that the continuity assumption on ϕ is redundant. Later on Słodkowski [S], strengthening previous results, proved that a holomorphic motion of any subset X of $\hat{\mathbf{C}}$ can be extended to a holomorphic motion of the whole Riemann sphere.

In higher-dimensional complex manifolds, such extension and continuity properties do not hold in general. Indeed, it is easy to construct a holomorphic motion of a subset X of $\mathbf{C}^2$, such that all the maps $\phi(a, \cdot)$ are discontinuous for $a \neq 0$. Moreover, the role of quasiconformality is not clear, at least not for arbitrary sets X. Some results on quasiconformality and holomorphic motions in higher dimension can be found in [ABR].

Next we discuss analytic multifunctions. Let M be a complex manifold. Then a multifunction from $\mathbf{D}$ to M is a map K from $\mathbf{D}$ to the set $\mathcal{K}(M)$ of compact subsets of M. K is called continuous (upper semicontinuous) if it is continuous (usc) in the Hausdorff metric on $\mathcal{K}(M)$. Its graph is defined by $\Gamma(K) = \bigcup_{a \in \mathbf{D}} (\{a\} \times K(a))$ and it is easy to see that K is usc iff $\Gamma(K)$ is closed in $\mathbf{D} \times M$.

Definition 1.2. A strongly analytic multifunction is an usc multifunction K such that $\Gamma(K)$ is the union of graphs of holomorphic maps from $\mathbf{D}$ to M.

From the definition it follows that a strongly analytic multifunction K is both continuous and an analytic multifunction in the sense of [A]. The latter statement means that if $D \subset\subset \mathbf{D}$ is open and ψ is plurisubharmonic in a neighborhood of $\Gamma(K|_D)$, then $\phi(\lambda) := \sup\{\psi(\lambda, x); x \in K(\lambda)\}$ is subharmonic on D. Also note that a holomorphic motion can be viewed as a strongly analytic multifunction K such that $\Gamma(K)$ is the union of *disjoint* graphs.

Analytic multifunctions appear naturally in complex dynamics. For example, Baribeau and Ransford [BR] proved that if f_a is a holomorphic family of rational functions, then $a \to J_a^*$ is a analytic multifunction, where J_a^* is the usc regularization of the Julia set J_a of f_a, i.e. the graph $\Gamma(a \to J_a^*)$ is the closure of the graph $\Gamma(a \to J_a)$.

Let us finally give the definition of a repellor as is needed in the statement of Theorem C.

Definition 1.3. Let f be a holomorphic endomorphism of a Hermitian manifold M and K a compact invariant set. Then K is said to be a repellor if there exists $c > 0, \lambda > 1$ such that $|f_*^n v| \geq c\lambda^n |v|$ for all tangent vectors v over K and all $n \geq 1$.

2. Proofs

Proof of Theorem A. From the real theory [R] we know that we may find an $r > 0$ and for all $a \in \mathbf{D}_r$ a continuous map $h_a : K \to M$ such that $K_a := h_a(K)$ is a hyperbolic subset for f_a, $h_a : K \to K_a$ is a homeomorphism and the relation $f_a \circ h_a = h_a \circ f$ holds. Moreover h_0 is the inclusion $K \hookrightarrow M$ and the map $a \to h_a$ is C^∞ as a map from D to the real Banach manifold $C(K, M)$ of continuous functions of K into M. All of this is proved using the Implicit Function Theorem on $C(K, M)$.

We want to prove that the map $a \to h_a(x)$ is holomorphic for all $x \in K$ and depend continuously on x. But the smoothness of $a \to h_a$ implies that $a \to h_a(x)$ is C^∞ and that all derivatives of $h_a(x)$ with respect to a depend continuously on x. Fix $b \in \mathbf{D}_r$ and let μ be the section of the tangent bundle of M over K_b defined by $\mu(h_b(x)) := \frac{\partial}{\partial \bar{a}} h_a(x)|_{a=b}$; this makes sense since h_b is a homeomorphism. Then μ is a continuous, hence bounded, section of TM over the compact set K_b. We want to prove that $\mu \equiv 0$. From the relation $f_a \circ h_a = h_a \circ f$ we easily get $\mu \circ f_b = (f_b)_* \mu$, where $(f_b)_*$ is the derivative of f_b. But then the following lemma tells us that $\mu \equiv 0$, which completes the proof. □

Lemma 2.1. *Let K be a hyperbolic set for an endomorphism f of a Riemannian manifold M and let $(x_i)_{i \in \mathbf{Z}}$ be an orbit in K. Suppose that μ is a bounded section of the tangent bundle over (x_i), i.e. $\mu(x_i) \in T_{x_i}M$, with the property $\mu(x_{i+1}) = f_*(x_i)\mu(x_i)$. Then $\mu(x_i) = 0$ for all i.*

Proof. We prove the lemma in the case when f is a diffeomorphism — the modifications in the endomorphism case are left to the reader. There is a continuous f_*-invariant splitting of the tangent bundle over K into unstable and stable bundles E^u and E^s, respectively, so we may write $\mu = \mu_u + \mu_s$, where μ_u and μ_s are bounded sections over (x_i) of E^u and E^s, respectively. We then have that $\mu_u(x_{i+1}) = f_*(x_i)\mu_u(x_i)$ and $\mu_s(x_{i+1}) = f_*(x_i)\mu_s(x_i)$. Suppose that $\mu_u(x_i) \neq 0$ for some i. Then the expansion along E^u gives that $|\mu_u(x_{i+n})| = |f_*^n(x_i)\mu_u(x_i)| \to \infty$ as $n \to \infty$. This contradicts the assumption that μ_u was bounded. Hence $\mu_u \equiv 0$. In the same way, we see that $\mu_s \equiv 0$ so $\mu \equiv 0$. □

Proof of Theorem B. The proof is very similar to that of Theorem A. The existence of r and h satisfying (1)–(2) follows from the real theory [R]. This time h

is constructed using the Implicit Function Theorem on the real Banach manifold $C(\hat{K}, M)$ of continuous functions from $\hat{K}$ to M. To prove (3) we take b in $\mathbf{D}_r$ and consider the map μ from $\widehat{K_b}$ to TK_b defind by $\mu(\widehat{h_b}(\hat{x})) := \frac{\partial}{\partial \bar{a}} h_a(\hat{x})|_{a=b}$. Then μ is well-defined since $\widehat{h_b}$ is a homeomorphism. Moreover, μ is continuous, hence bounded, and satisfies the relation $\mu \circ \widehat{f_b} = (f_b)_* \mu$. Therefore, if (x_i) is any orbit in K_b, then Lemma 2.1 shows that $\mu((x_i)) = 0$. This proves (3). Finally (4) follows immediately from (3). □

Proof of Theorem C. Let h_a be as in Theorem B. We claim that there exists a homeomorphism $g_a : K \to K_a$ such that $g_a \circ \pi = h_a$, where $\pi : \hat{K} \to K$ is the projection $\pi((x_k)) = x_0$. To see this, take any $x \in K$ and let $\hat{x} = (x_k)$ and $\hat{y} = (y_k)$ be two points in $\hat{K}$ with $\pi(\hat{x}) = \pi(\hat{y}) = x$ (i.e. $x_0 = y_0 = x$). We must show that $h_a(\hat{x}) = h_a(\hat{y})$. Suppose that this is not the case and let $x(a) = h_a(\hat{x})$ and $y(a) = h_a(\hat{y})$. Then for $n \geq 0$ we have

$$d(f_a^n(x(a)), f_a^n(y(a))) \leq d(f_a^n(x(a)), f^n(x)) + d(f_a^n(y(a)), f^n(x)) \leq c(a),$$

where $c(a) \to 0$ as $a \to 0$. Hence the forward orbits of $x(a)$ and $y(a)$ are very close if a is small. Because of the expansion, this is only possible if $x(a) = y(a)$. Therefore, the map $h_a : K \to K_a$ is well-defined. It remains to be seen that $a \to h_a(x)$ is holomorphic for all $x \in K$ but this follows immediately from the fact that the maps $a \to h_a(\hat{x})$ are holomorphic. □

It is also possible to give a direct proof of Theorem C without using Theorem B. Let us sketch how to do this. The idea is to use Sullivan's telescope construction as described in [HO]. For simplicity we assume that the constant c in the definition of a repellor is equal to one; this can be achieved by changing the metric on M slightly (a construction originally due to Mather). Let $U_0(x)$ be the ball of radius $\epsilon > 0$ centered at $x \in K$. The expansion implies that $f^{-1}(U_0(f(x)))$ has a unique component contained in $U_0(x)$ for $x \in K$ if ϵ is small enough. Call this component $U_1(x)$. Inductively we find a nested sequence (telescope) of open sets $\{U_n(x)\}_{n \geq 0}$ for $x \in K$ and the expansion implies that the diameter of $U_n(x)$ is uniformly exponentially small. In particular the intersection $\cap_{n \geq 0} U_n(x)$ (the focus of the telescope) is the single point x. If a is small enough, then we may construct a perturbed telescope $\{U_{n,a}(x)\}_{n \geq 0}$ for $x \in K$ so that $U_{n,a}(x)$ is a connected component of $f_a^{-n}(U_0(f^n(x)))$. We will still have that the diameter of $U_{n,a}(x)$ is uniformly exponentially small, so the focus of the telescope is a well-defined point $h_a(x)$. It is easy to see, using the fact that expansion on K is bounded above, that $h_a(x)$ depends continuously on x — in fact h_a is Hölder continuous. Exchanging the roles of f and f_a we see that (for a small enough) h_a is a homeomorphism, which is bi-Hölder. Define $K_a := h_a(K)$. It is clear from the construction that h_a conjugates f on K to f_a on K_a. Finally, for fixed x, $h_a(x)$ is given as a uniform limit of functions holomorphic in a so $a \to h_a(x)$ is holomorphic. This completes the second proof of Theorem C.

3. EXAMPLES

Our first example concerns polynomial diffeomorphisms of $\mathbf{C}^2$, for which we use [BS] as a reference. We only consider diffeomorphisms which are conjugate to finite compositions of (generalized) Hénon maps.

A polynomial diffeomorphism of $\mathbf{C}^2$ is said to be hyperbolic if it is hyperbolic on its non-wandering set; in this case the non-wandering set consists of a basic set J of unstable dimension one and a finite number of repelling or attracting periodic points.

It follows from Theorem A that if $\{f_a\}_{a\in\mathbf{D}}$ is a holomorphic family of polynomial diffeomorphisms of $\mathbf{C}^2$ and $f = f_0$ is hyperbolic, then J moves holomorphically with a at $a = 0$.

The second example is of a polynomial endomorphism f of $\mathbf{C}^2$, defined by $f(z,w) = (z^2, w^2)$. The non-wandering set Ω of f is the union $\Omega_0 \cup \Omega_1 \cup \Omega_2$, where $\Omega_0 = \{(0,0)\}$, $\Omega_1 = \{|w| = 1, z = 0\} \cup \{|z| = 1, w = 0\}$, $\Omega_2 = \{|z| = |w| = 1\}$. In this case f is hyperbolic on all of Ω and it has unstable dimension i on Ω_i.

We now embed f in a holomorphic family $\{f_a\}$ of endomorphisms of $\mathbf{C}^2$ with $f_0 = f$. It then follows from from Theorem C that the set Ω_2 moves holomorphically with a for a small enough; the same is true for Ω_0. On the other hand, the set Ω_1 does not move holomorphically in general. To see this, consider the component $K = \{|z| = 1, w = 0\}$ of Ω_1. We embed f_0 in the holomorphic family $\{f_a\}$ defined by $f_a(z,w) = (z^2, w^2 + az), |a| < 1/4$. Then the Riemann surface $V_a = \{w^2 = r^2 z\}$ is invariant, where $r = 1/2 - \sqrt{1/4 - a}$ and the branch of the root is chosen so that $\sqrt{1/4} = 1/2$. If we use z as a variable on V_a, then the dynamics on V_a is given by $z \to z^2$. Hence $K_a = \{|z| = 1, w^2 = r^2 z\}$. For $a \neq 0$ this is a fiber bundle over the circle $\{|z| = 1, w = 0\}$ with a two point set as a fiber and it is clear that K_a is not a holomorphic motion of K.

In fact, the discontinuity of K_a in this example is misleadingly simple. If we take $f_a(z,w) = (z^2, w^2 + w/10 + az)$, then one can see that the set K_a, which is a perturbation of the set $K_0 = \{|z| = 1, w = 0\}$ for small $a \neq 0$ is a fiber bundle over the circle $|z| = 1$ with Cantor sets as fibers.

Acknowledgment. This work was partially done when the author was visiting the University of Michigan and he wants to thank the Department of Mathematics for its hospitality.

References

[A] Aupetit, B. Analytic multifunctions and their applications. In Gauthier, P. M., Sabidussi, G., editor, *Complex Potential Theory*, pages 1–74. Kluwer Academic Publishers, 1994.

[ABR] Astala, K., Balogh, Z., Reimann, H. M. Lempert mappings and holomorphic motions in $\mathbf{C}^n$. Preprint.

[BR] Baribeau, L., Ransford,T.J. Meromorphic multifunctions in complex dynamics. *Ergodic Theory Dynam. Systems* **12**(1992), 39–52.

[BS] Bedford, E., Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$: Currents, equilibrium measure and hyperbolicity. *Invent. Math.* **103**(1991), 69–99.

[HO] Hubbard, J.H., Oberste-Vorth, W. Hénon mappings in the complex domain II. In Branner, B., Hjorth, P., editor, *Real and complex dynamical systems*, pages 89–132. Kluwer Academic Publishers, 1995.

[MSS] Mañé, R., Sad, P., Sullivan, D. On the dynamics of rational maps. *Ann. Sci. Éc. Norm. Sup.* **16**(1983), 193–217.

[R] Ruelle, D. *Elements of differentiable dynamics and bifurcation theory.* Academic Press, 1989.

[S] Słodkowski, Z. Holomorphic motions and polynomial hulls. *Proc. Amer. Math. Soc.* **111**(1991), 347–357.

Dept. of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden
E-mail address: mjo@math.kth.se

Paper III

SOME PROPERTIES OF 2-CRITICALLY FINITE HOLOMORPHIC MAPS OF $\mathbf{P}^2$

MATTIAS JONSSON

ABSTRACT. We define 2-critically finite maps of $\mathbf{P}^2$ and show that they have no nontrivial closed backward invariant sets. In particular, their Julia sets J_1, defined as the support of a natural invariant measure, are equal to $\mathbf{P}^2$. We also prove that repelling periodic points are dense for such maps.

0. INTRODUCTION

The present paper deals with the dynamics of 2-critically finite holomorphic maps on $\mathbf{P}^2$. These are natural generalizations of the so called Thurston maps on $\mathbf{P}^1$ which by definition are rational maps all of whose critical points are strictly preperiodic (we do not allow superattracting periodic points). See §1 for precise definitions.

Our first main result is that *if $f : \mathbf{P}^2 \to \mathbf{P}^2$ is 2-critically finite then f has no nontrivial closed backward invariant set.* More precisely, if E is a nonempty closed set with $f^{-1}(E) \subset E$ then $E = \mathbf{P}^2$.

Our second main result is that *repelling periodic points are dense in $\mathbf{P}^2$ for any 2-critically finite $f : \mathbf{P}^2 \to \mathbf{P}^2$.*

We use two major tools for proving the above results. The first one is Kobayashi hyperbolicity. More precisely, we apply two theorems of M. Green [Gr1], [Gr2] which together tell us that the complement in $\mathbf{P}^2$ of certain unions of algebraic curves is complete hyperbolic and hyperbolically embedded. The second tool is a theorem of Ueda [Ue1] which says that families of branches of inverse iterates of f are normal whenever they are defined.

In dimension one, a 1-critically finite map is a Thurston map and both our main results are equivalent to the well-known fact that such maps have empty Fatou sets. Indeed, Thurston [Th] has given a topological classification of them and proved that they admit an expanding metric with singularities at the post-critical points.

The study of 2-critically finite maps of $\mathbf{P}^2$ was initiated by Fornæss and Sibony [FS3], [FS1] (they use a slightly different definition and call their maps strictly critically finite, see Remark 1.10). In [FS1] they proved that, under a technical

---

1991 *Mathematics Subject Classification.* Primary: 32H50; Secondary: 32H20.

Key words and phrases. Holomorphic dynamical systems, critically finite, invariant set, periodic points, Kobayashi hyperbolic.

assumption on the postcritical set, 2-critically finite maps have empty Fatou sets, i.e. that their Julia sets J_0 are all of $\mathbf{P}^2$. Ueda [Ue1] later gave a proof without this technical assumption. It is easy to see that for general holomorphic self-maps of $\mathbf{P}^k$, J_0 is a nonempty closed completely invariant set which contains all the repelling periodic points of f. Hence our main results are stronger than the results in [FS1] and [Ue1].

Furthermore, J_0 is only one possible generalization of the Julia set in one variable. Another natural candidate is the support J_1 of the "Green measure" describing the distribution of preimages of quasi-every point (cf [FS2] and §2). Alternatively, one can consider the set where iterates are not normal even on 1-dimensional subvarieties (cf [FS2]). All these Julia sets share the common property that they are nonempty closed completely invariant sets, so our first main result imply that they are equal to $\mathbf{P}^2$ for 2-critically finite maps.

The results in this paper are restricted to $\mathbf{P}^2$ and the arguments to prove them cannot, in general, be generalized to $\mathbf{P}^k$ with $k \geq 3$. The main reason for this is that when analyzing a k-critically finite map $f : \mathbf{P}^k \to \mathbf{P}^k$ we often need to study the restriction of f to algebraic subvarieties of $\mathbf{P}^k$ of dimension between 1 and $k-1$. This is possible in dimension 2, because then the subvarieties have normalizations that are nonsingular compact Riemann surfaces - in fact they must be Riemann spheres or tori. In $\mathbf{P}^k$ for $k \geq 3$, the situation need not be that simple.

The paper is divided into four sections. In §1 we define what we mean by a k-critically finite holomorphic map of $\mathbf{P}^k$ and analyze the structure of the postcritical set in the case $k = 2$. Then in §2 we prove our first main result, that a 2-critically finite map of $\mathbf{P}^2$ has no nontrivial closed backward invariant set. After that we turn to the study of periodic points and prove that repelling periodic points are dense for a 2-critically finite map of $\mathbf{P}^2$. This is proved in §3. Finally, in §4, we address the question of determining asymptotic distribution of periodic points of a 2-critically finite map of $\mathbf{P}^2$. We show that a certain conjecture about the limits of families of branches of inverse iterates implies that (repelling) periodic points are distributed according to the "Green measure".

1. Critically finite maps and their postcritical sets

Let us fix some notation and definitions which will be used throughout the paper. For general information on holomorphic dynamical systems on $\mathbf{P}^k$, see [FS1], [FS2] and [FS4]. The maps that we call k-critically finite will be generalizations of Thurston maps in one dimension, so let us first define the latter.

Definition 1.1. A rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ is a Thurston map if the postcritical set of f is finite and f has no superattracting periodic points.

Remark 1.2. One often allow Thurston maps to have superattracting periodic points but the above definition is more convenient for our purposes.

If $f : \mathbf{P}^k \to \mathbf{P}^k$ is a holomorphic map, we let C_1 be its critical set, i.e. the set of points p where df_p is non-invertible. Then C_1 is algebraic of codimension 1. We let $D_1 := \cup_{j>0} f^j C_1$ and $E_1 := \cap_{j>0} f^j D_1$. Note that D_1 is the postcritical set of f and that if D_1 is closed then E_1 is the ω-limit set of C_1.

Definition 1.3. We say that f is 1-critically finite if D_1, and hence E_1 are algebraic sets and C_1 and E_1 have no common irreducible component.

Note that D_1 is algebraic iff the sequence of sets $\{f^j C_1\}_j$ is preperiodic. In that case, $f^{l_1+1} D_1 = f^{l_1} D_1$ for some minimal $l_1 \geq 1$ so $E_1 = f^{l_1} D_1$ and C_1, D_1 and E_1 all have codimension 1.

Inductively we define j-critically finite maps of $\mathbf{P}^k, 1 < j \leq k$ as follows:

Definition 1.4. Suppose f is $(j-1)$-critically finite. This means in particular that the set E_{j-1} has been inductively defined as an algebraic set of codimension $j-1$ with no irreducible component contained in C_1. Then $C_j := E_{j-1} \cap C_1 = E_{j-1} \cap C_{j-1}$ is algebraic of codimension j. We say that f is j-critically finite if $D_j := \cup_{i>0} f^i C_j$ is algebraic and the set $E_j := \cap_{i>0} f^i D_j$, which then is necessarily algebraic of codimension j has no irreducible component contained in C_1.

In the case of $\mathbf{P}^2$ the situation is simpler, because of the following lemma by Ueda (Lemma 4.2 in [Ue1]).

Lemma 1.5. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be 1-critically finite. Then the set $D_2 := \cup_{j>0} f^j C_2$ is algebraic of codimension 2 and is contained in sing(D_1). The set $E_2 := \cap_{j>0} f^j D_2$ is algebraic of codimension 2 as well.*

Sketch of proof. Let $q \in D_2$. Then $q = f^j(p)$ for some $p \in C_2 = C_1 \cap E_1$. Since E_1 and C_1 have no common irreducible component, p must be a singular point of $C_1 \cup E_1$ and since $C_1 \cup E_1 \subset f^{-j} D_1$ we must have $p \in \text{sing}(f^{-j} D_1)$. But then by Lemma 2.5 in [Ue1], $q = f^j(p) \in \text{sing}(D_1)$. Thus $D_2 \subset \text{sing}(D_1)$. Now let V be an irreducible component of $C_2 = E_1 \cap C_1$. Since $V \subset \text{sing}(D_1)$, V must have codimension 2. Then for each $j > 0$ $f^j V$ is an irreducible algebraic set of codimension 2 contained in $\text{sing}(D_1)$ so it must coincide with one of the irreducible components of $\text{sing}(D_1)$. This shows that D_2 is contained in the (finite) union of irreducible components of $\text{sing}(D_1)$ and hence is algebraic of codimension 2. The set E_2 is the algebraic of codimension 2 because it is the intersection of the decreasing family of algebraic sets $\{f^j(D_2)\}_{j>0}$. $\square$

Because of Lemma 1.5, $f : \mathbf{P}^2 \to \mathbf{P}^2$ is 2-critically finite iff it is 1-critically finite and the set E_2, which by the lemma is a finite set, contains no critical point.

The following are examples of 2-critically finite maps of $\mathbf{P}^2$:

$$[x:y:z] \to [(-x+y+z)^2 : (x-y+z)^2 : (x+y-z)^2]$$
$$[x:y:z] \to [(x-y+z)^2 : (-x+y+z)^2 : (x+y-z)^2]$$
$$[x:y:z] \to [(x+y-z)^2 : (-x+y+z)^2 : (x-y+z)^2]$$
$$[x:y:z] \to [(x-2y)^2 : (x-2z)^2 : x^2]$$

An example of a k-critically finite map of $\mathbf{P}^k$ is the following:

$$[x_0 : \cdots : x_k] \to [(x_0 - 2x_1)^2 : \cdots : (x_0 - 2x_k)^2 : x_0^2]$$

(The first three examples are due to Ueda [Ue2] and the two last ones to Fornæss and Sibony [FS3].)

Suppose $f : \mathbf{P}^2 \to \mathbf{P}^2$ is 2-critically finite. We analyze the structure of the sets E_1 and E_2 and the restrictions $f|_{E_1}, f|_{E_2}$. Note that neither E_1 nor E_2 change if we replace f by an iterate. Hence, for the time being, we may assume that all irreducible components of E_1 and all points in E_2 are fixed by f.

Let V be an irreducible component of E_1 which is fixed by f. Then V is an irreducible algebraic set of dimension 1. Fornæss and Sibony ([FS2], Proposition 7.5) proved that $f|V$ cannot be injective. Let $\hat{V}$ be the normalization of V (see [Gu]) with projection $\pi : \hat{V} \to V$. Since V has dimension 1, $\hat{V}$ is a compact (nonsingular) Riemann surface, and the map $f|_V$ lifts to a holomorphic map $\hat{f} : \hat{V} \to \hat{V}$

We first prove a result about the critical set of $\hat{f}$.

Lemma 1.6. *Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be any holomorphic map and V an f-invariant irreducible algebraic curve. Let $\hat{V}$ be the normalization of V with projection $\pi : \hat{V} \to V$ and $\hat{f}$ the lifting of $f|_V$ to $\hat{V}$. Then the critical set of $\hat{f}$ is contained in $\pi^{-1}(C_1 \cap V)$.*

Proof. We have that $\hat{p} \in \pi^{-1}(p) \subset \hat{V}$ is critical for $\hat{f}$ if and only if $\hat{f}|_{\hat{U}}$ is non-injective for every neighborhood $\hat{U}$ of $\hat{p}$ in $\hat{V}$ which can only happen if $f|_U$ is non-injective for any neighborhood U of p in V. The latter is only possible if $p \in C_1$, because otherwise f is injective in a neighborhood of p in $\mathbf{P}^2$. $\square$

Now $\hat{V}$ cannot be hyperbolic, for then some iterate of $\hat{f}$ would be the identity, which is impossible since $\hat{f}$ is not injective. So $\hat{V}$ is a torus or the Riemann sphere.

If $\hat{V}$ is a torus then $\hat{f} : \hat{V} \to \hat{V}$ lifts to an endomorphism of $\mathbf{C}$ given by $z \to az + b$. Since $\hat{f}$ is not injective, we must have $|a| > 1$ so $\hat{f}$ is an expanding endomorphism of the torus. In particular all periodic points of $\hat{f}$ are repelling.

If $\hat{V}$ is the Riemann sphere then $\hat{f} : \hat{V} \to \hat{V}$ is a rational map. By Lemma 1.6, its critical points are contained in the set $\pi^{-1}(C_1 \cap V)$. Hence they are contained in the set $\pi^{-1}(C_2 \cap V)$ which is (strictly) preperiodic to the set $\pi^{-1}(E_2 \cap V)$, consisting of noncritical cycles. Therefore, $\hat{f}$ is a Thurston map. In particular, all its periodic points are repelling.

Let us summarize the above discussion in a proposition:

Proposition 1.7. *Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be 2-critically finite. Then the normalization $\hat{V}$ of every irreducible invariant component V of E_1 is biholomorphic to $\mathbf{P}^1$ or a torus. If $\hat{V} \simeq \mathbf{P}^1$ the lifting $\hat{f}$ of the restriction $f|_V$ to $\hat{V}$ is a Thurston map, i.e. its critical points are strictly preperiodic. If $\hat{V}$ is a torus then $\hat{f}$ is an expanding endomorphism. In any case, all periodic points of $\hat{f}$ are repelling.*

Remark 1.8. We only used the fact that V was invariant, so the proposition is also true for all invariant irreducible curves V. A corresponding theorem is true for maps which are only 1-critically finite (so that they may have superattracting periodic points). In this case Lemma 1.5 shows that the sets D_2 and E_2 are finite so the same argument as above proves that the lifting of the restriction $f|_V$ to $\hat{V}$ is an expanding endomorphism of a torus or a critically finite rational map of $\mathbf{P}^1$ (possibly with superattracting periodic points.

We now turn to the set E_2, which is finite. We will call a periodic point p of order r for a holomorphic map $f : \mathbf{P}^k \to \mathbf{P}^k$ *repelling* if all eigenvalues of df_p^r have modulus strictly greater than one.

Proposition 1.9. *If $f : \mathbf{P}^2 \to \mathbf{P}^2$ is 2-critically finite then E_2 is a finite set of repelling periodic points for f.*

Proof. We know that E_2 is a finite forward invariant set, hence it must consist of periodic points. We have to show that these are repelling.

Let $p \in E_2$. By replacing f by an iterate, we may assume that p is fixed. From Lemma 1.5 we know that $p \in \operatorname{sing}(D_1)$. Since p is fixed, all the iterates of the irreducible components of D_1 passing through p still pass through p and all sufficiently high iterates belong to E_1. But p is noncritical, so all iterates of f are local diffeomorphisms at p. This implies that in fact $p \in \operatorname{sing}(E_1)$. Again replacing f by an iterate, we may assume that all the irreducible components of E_1 containing p are invariant under f. In fact, we can, and do, assume that all local branches of E_1 at p are invariant under f.

There are now several possibilities. First suppose that $p \in V$ where V is an irreducible component of E_1 with a cusp singularity at p. As before, let $\hat{V}$ be the normalization of V, $\pi : \hat{V} \to V$ the projection and $\hat{f}$ the lifting to $\hat{V}$ of the restriction $f|_V$. Let $\hat{p} \in \pi^{-1}(p) \subset \hat{V}$ correspond to the cusp at p. Then $\hat{p}$ is a repelling fixed point for $\hat{f}$ according to Proposition 1.7. Choose local coordinates (x, y) around p such that $p = (0,0)$ and the cusp of V is parameterized by $(x,y) = (t^r, t^s + O(|t|^{s+1}))$ where $1 < r < s$ and $r \nmid s$. In the same coordinates we may write $f(x,y) = (ax + by + g(x,y), cx + dy + h(x,y))$ where g and h have no linear terms and $ad - bc \neq 0$ as p is noncritical. Then $f(t^r, t^s + \dots) = (at^r + O(|t|^{r+1}), ct^r + dt^s + h(t^r, t^s + \dots))$. The condition that the cusp is mapped into itself yields that $c = 0$ and that the term $h(t^r, t^s + \dots)$

contains no term of order less than s. Also, since $r \nmid s$,$h(t^r, t^s + \dots)$ contains no term of order s. Thus $f(t^r, t^s) = (at^r + O(|t|^{r+1}), dt^s + O(|t|^{s+1}))$. The invariance of the cusp yields $a^s = d^r$. In particular it holds that $|a|, |d| > 1$, $|a| = |d| = 1$ or $|a|, |d| < 1$. But a, d are the eigenvalues of f at p and $\hat{p}$ is a repelling fixed point for $\hat{f}$, so only the first alternative is possible. Thus p is a repelling fixed point for f.

The next possibility is that two invariant irreducible regular local branches V_1, V_2 of E_1 at p intersect transversally. Then p is a repelling fixed point for $f|_{V_i}$ by Proposition 1.7 so f is expanding in the invariant directions of V_1 and V_2. Thus p is repelling for f.

Finally, consider the case when two invariant irreducible regular local branches V_1, V_2 of E_1 at p intersect tangentially. Choose local coordinates (x, y) such that $p = (0, 0)$ and $V_1 = (y = 0), V_2 = (y = x^r + O(|x|^{r+1}))$ where $r > 1$. As before, we can write $f(x, y) = (ax + by + g(x, y), cx + dy + h(x, y))$ where $ad - bc \neq 0$. Since $f(V_1) \subset V_1$ we must have $c = 0$. Letting $y = x^r + O(|x|^{r+1})$ we get $f(x, x^r + O(|x|^{r+1})) = (ax + O(|x|^2), dx^r + O(|x|^{r+1}))$, so since $f(V_2) \subset V_2$ we must have $a = d$. But d is the derivative of $f|_{V_1}$ at p and p is a repelling fixed point for $f|_{V_1}$ so $|a| = |d| > 1$. Now a, d are the eigenvalues of df_p so p is again a repelling fixed point for f. □

Remark 1.10. Fornæss and Sibony [FS1], [FS3], [FS4] defines a map to be *strictly critically finite* if (translated to our notation) the sets D_1 and E_1 are algebraic and $f^p|_V$ is critically finite for each irreducible component V of E_1 with period p. The last sentence is interpreted as: the lifting of f^p to the normalization of V is critically finite, i.e. its critical points are preperiodic.

In fact, the last condition is redundant, i.e. a map $f : \mathbf{P}^2 \to \mathbf{P}^2$ is strictly critically finite iff the set D_1 is algebraic. To prove this, suppose $f : \mathbf{P}^2 \to \mathbf{P}^2$ has algebraic postcritical set D_1. If the critical set C_1 has no irreducible component which is periodic then f is 1-critically finite and Remark 1.8 shows that f is in fact strictly critically finite. So, suppose some irreducible component W of the critical set is periodic under f. Replacing f by an iterate, we may assume that W is invariant under f. Let $\hat{W}$ be the normalization of W with projection $\pi : \hat{W} \to W$ and $\hat{f}$ the lifting of $f|_W$ to $\hat{W}$. Then it is not difficult to see that the critical points of $\hat{f}$ must be mapped under π into the set $\operatorname{sing}(C_1) \cap W$. But the proof of Lemma 1.5 shows that the iterates of f map $\operatorname{sing}(C_1)$ into the set $\operatorname{sing}(D_1)$, hence maps the set $\operatorname{sing}(C_1) \cap W$ into $\operatorname{sing}(D_1) \cap W$ which is finite. It follows that the postcritical set of $\hat{f}$ is finite. Now it could of course happen that the critical set C_1 also has some other irreducible component which is strictly preperiodic to a component V of E_1. We can assume that V is invariant under f. But then we can get, in the same way as above, that the lifting of $f|_V$ to the normalization of V has finite postcritical set. Hence f is strictly critically finite even in this case.

2. Backward invariant sets

We recall the definitions of the different Julia sets, $J_0 \supset J_1 \supset \cdots \supset J_{k-1}$ of a holomorphic map $f : \mathbf{P}^k \to \mathbf{P}^k$ of degree $d \geq 2$ (see [FS2], [FS4] for more details).

The map f lifts to a map $F : \mathbf{C}^{k+1} \setminus 0 \to \mathbf{C}^{k+1} \setminus 0$ via the projection $\pi : \mathbf{C}^{k+1} \setminus 0 \to \mathbf{P}^k$ and the Green function $G := \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n\|$ is a continuous plurisubharmonic function on $\mathbf{C}^{k+1} \setminus 0$ with the properties

$$G(tx) = G(x) + \log |t|, \ t \in \mathbf{C}^*$$

and

$$G(F(x)) = d\,G(x).$$

The current dd^cG is then a closed positive current on $\mathbf{C}^{k+1} \setminus 0$ of bidegree $(1,1)$ and there is a unique positive closed current T on $\mathbf{P}^k$ such that $\pi^*T = dd^cG$. We can define $(dd^cG)^j$ inductively as positive closed currents of bidegree (j,j) by:

$$(dd^cG)^j = dd^c(G(dd^cG)^{j-1})$$

for $1 \leq j \leq k$. There exist corresponding positive, closed currents $T^j = T \wedge \cdots \wedge T$, $1 \leq j \leq k$ on $\mathbf{P}^k$ of bidegree (j,j) such that $\pi^*T^j = (dd^cG)^j$; these satisfy the relation $f^*T^j = d^jT^j$, and they have mass $||T^j|| = 1$.

We define $J_j := \mathrm{supp}(T^j)$ and call it the j:th Julia set of f. The Julia sets $J_0 \supset J_1 \supset \cdots \supset J_{k-1}$ are then nonempty closed completely invariant sets. It is known (see [FS2] or [Ue3]) that J_0 is exactly the complement of the largest open set where the family of iterates $\{f^i\}$ is normal.

Fornæss and Sibony [FS1] proved the following theorem (with an additional technical assumption, which was removed by Ueda [Ue1]):

Theorem 2.1. *If $f : \mathbf{P}^2 \to \mathbf{P}^2$ is 2-critically finite then $J_0 = \mathbf{P}^2$.*

We will prove that in fact $J_1 = \mathbf{P}^2$. The only property of the set J_1 that we will use is that it is closed and backward invariant. Indeed, the main result of this section is:

Theorem 2.2. *If $f : \mathbf{P}^2 \to \mathbf{P}^2$ is 2-critically finite then $\mathbf{P}^2$ is the only nonempty closed backward invariant subset of $\mathbf{P}^2$.*

Our result implies Theorem 2.1 but we will use the latter to prove Theorem 2.2. The strategy of proof is the following: Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be 2-critically finite and E be a nonempty closed backward invariant set. First we prove that E must contain all the repelling periodic points of f (Proposition 2.6), hence, by Proposition 1.9, all the points in E_2. Then we prove that the preimages of E_2 are dense in $\mathbf{P}^2$ (Proposition 2.12). From these two results it follows immediately that $E = \mathbf{P}^2$ as was claimed.

We need an elementary but somewhat technical lemma. In favor of brevity we sometimes let the word *preimage* mean *preimage under some iterate of* f. The precise meaning should be clear from the context.

Lemma 2.3. *Let* $f : \mathbf{P}^k \to \mathbf{P}^k$ *be* k*-critically finite. Then any point in* D_1 *has some preimage outside* D_1*. Hence* D_1 *contains no nonempty backward invariant subset.*

Proof. Take any point $x \in D_1$. If $x \notin E_1$ then there is an N_1 such that $f^{-N_1}(x) \not\subseteq D_1$. We only need to choose N_1 such that $f^{N_1}D_1 \subset E_1$. So suppose that $x \in E_i$ for some maximal $i \geq 1$. It suffices to prove that x has some preimage outside E_i, because then we can argue by induction. So suppose $f^{-n}(x) \subset E_i$ $\forall n$. Then there exists an integer $n_1 \geq 1$ and a point $x_1 \in C_i$ such that $f^{n_1}(x_1) = x$. Since $f^{-n_1}(x) \subset E_i$ we must have $x_1 \in E_i \cap C_i = C_{i+1}$. But now we can continue the process and find $n_2, n_3, \dots \geq 1$ and $x_2, x_3, \dots \in C_{i+1}$ such that $f^{n_j}(x_j) = x_{j-1}$. It follows that $x \in f^{n_1+\dots+n_j}C_{i+1}$ for $j = 1, 2, \dots$ and hence $x \in E_{i+1}$ which contradicts the maximality of i. $\square$

Before we state our next result, we recall the following theorem by Ueda [Ue1] that will be used here and in §3.

Theorem 2.4. *Let* $f : \mathbf{P}^k \to \mathbf{P}^k$ *be any holomorphic map of degree at least 2 and* $\{g_\nu\}$ *a family of holomorphic (single-valued) branches of* f^{-j_ν} *defined on an open set* $U \in \mathbf{P}^k$ *(i.e.* $f^{j_\nu} \circ g_\nu = id$ *on* U*). Then the family* $\{g_\nu\}$ *is normal.*

Remark 2.5. Ueda's result in [Ue1] is slightly more general than the statement above but Theorem 2.4 is all that we need. Note that if $f : \mathbf{P}^k \to \mathbf{P}^k$ is k-critically finite (or 1-critically finite) and U is a simply connected open set in $\mathbf{P}^k$ which does not intersect the postcritical set D_1 then *all* branches of f^{-j} are well-defined on U and Theorem 2.4 applies.

Proposition 2.6. *If* $f : \mathbf{P}^k \to \mathbf{P}^k$ *is* k*-critically finite, then any closed backward invariant set* $E \subset \mathbf{P}^k$ *must contain all the repelling periodic points of* f*.*

Proof. Let E be such a set. By Lemma 2.3, there is a point $q \in E \setminus D_1$. Let p be a repelling periodic point of f, say of period m. Then there is a sequence g_j of holomorphic branches of f^{-jm} defined on a neighborhood U of p such that $g_j \to p$ uniformly as $j \to \infty$. Since D_1 is an algebraic set in $\mathbf{P}^k$ of codimension 1, there exist a sequence of simply connected open sets $\{U_i\}_{i=0}^{l}$ in $\mathbf{P}^k$ such that $U_0 = U$, $q \in U_l$, $U_i \cap D_1 = \emptyset$ for $i > 0$, and $U_i \cap U_{i+1} \neq \emptyset$. Then the g_j can be analytically continued along the chain $\{U_i\}$, because the U_i's are simply connected and do not intersect the postcritical set of f (except for U_0). By Theorem 2.4, the family $\{g_j\}$ is normal on each U_i. By successively extracting subsequences if necessary, we get that $g_j \to p$ on U_l so in particular $g_j(q) \to p$. Since q belongs to the closed backward invariant set E, it follows that $p \in E$. $\square$

In dimension 2, we now know from Propositions 1.9 and 2.6 that any closed backward invariant set E must contain the set E_2. Hence, it suffices to show that the preimages of E_2 are dense in $\mathbf{P}^2$. For that we will need a couple of criteria for Kobayashi hyperbolicity. For background on hyperbolicity in complex analysis see [La]. Unless otherwise stated, *hyperbolic* will mean Kobayashi hyperbolic.

Lemma 2.7. *If $V \subset \mathbf{P}^k$ is a irreducible compact (possibly singular) curve and $A \subset V$ is a closed subset containing at least 3 points, then $V \setminus A$ is Brody hyperbolic, i.e. there exists no nonconstant holomorphic map $\mathbf{C} \to V \setminus A$.*

Proof. Suppose $\phi : \mathbf{C} \to V \setminus A$ is holomorphic. Let $\hat{V}$ be the normalization of V, with projection $\pi : \hat{V} \to V$. Since V is one-dimensional, $\hat{V}$ is a compact Riemann surface. The map $\phi : \mathbf{C} \to V$ lifts to a map $\hat{\phi} : \mathbf{C} \to \hat{V}$ such that $\pi \circ \hat{\phi} = \phi$. Now $\hat{A} := \pi^{-1}A$ contains at least three points, so the Riemann surface $\hat{V} \setminus \hat{A}$ is hyperbolic, which means that $\hat{\phi} : \mathbf{C} \to \hat{V} \setminus \hat{A}$ is constant. Thus ϕ is constant and $V \setminus A$ is Brody hyperbolic. □

It is well-known that the complement in $\mathbf{P}^k$ of $2k+1$ lines in general position is hyperbolic. We need a slightly different result:

Proposition 2.8. *Let $X_1, \ldots, X_N$, where $N \geq 2l - 1, l \geq 3$ be distinct irreducible curves in $\mathbf{P}^2$ such that $X_{n_1} \cap \cdots \cap X_{n_l} = \emptyset$ if $1 \leq n_1 < \cdots < n_l \leq N$. Then $\mathbf{P}^2 \setminus \cup_{n=1}^N X_n$ is complete hyperbolic and hyperbolically embedded.*

Proof. The proof is a fairly easy consequence of the following two theorems by M. Green (see [Gr1] and [Gr2]):

Theorem 2.9. *Let D be a finite union of (possibly singular) irreducible hypersurfaces $D_1, \ldots, D_m$ in a compact complex manifold V. Then $V \setminus D$ is complete hyperbolic and hyperbolically embedded in V provided*

1. *there is no nonconstant holomorphic map $\mathbf{C} \to V \setminus D$.*
2. *there is no nonconstant holomorphic map $\mathbf{C} \to D_{i_1} \cap \cdots \cap D_{i_k} \setminus (D_{j_1} \cup \cdots \cup D_{j_l})$ for any choice of distinct indices so that $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} = \{1, \ldots, m\}$*

Theorem 2.10. *Suppose $f : \mathbf{C} \to \mathbf{P}^k$ omits $k+2$ distinct irreducible hypersurfaces. Then $f(\mathbf{C})$ is contained in a compact hypersurface.*

To prove Proposition 2.8, we apply Theorem 2.9 with $D_1, \ldots, D_m$ being the curves $X_1, \ldots, X_N$. We first verify condition 1. Suppose $\phi : \mathbf{C} \to \mathbf{P}^2 \setminus \cup_{n=1}^N X_n$ is holomorphic. Since $N \geq 4$ (in fact $N \geq 5$), Theorem 2.10 shows that $\phi(\mathbf{C})$ must be contained in a compact hypersurface $V \subset \mathbf{P}^2$. Now V must intersect every X_n and the condition that no l of the X_n's intersect at a time implies that V must intersect $\cup_{n=1}^N X_n$ in at least 3 points. But then ϕ is a mapping from $\mathbf{C}$ to $V \setminus \{\text{three points}\}$, so according to Lemma 2.7, ϕ must be constant. Condition 2 is in fact simpler: the only nontrivial case to consider is when $\phi : \mathbf{C} \to X_1 \setminus (X_2 \cup \cdots \cup X_N)$ is a holomorphic map. But the condition on the X_n's imply that the finite set $\cup_{n=2}^N X_1 \cap X_n$ has at least three elements. Hence ϕ is constant according to Lemma 2.7. □

Proposition 2.11. *If $f : \mathbf{P}^2 \to \mathbf{P}^2$ is 2-critically finite and V is any irreducible component of E_1 then there exists an N and a finite collection $X_1, \ldots, X_N$ of irreducible branches of $\cup_{n=0}^{N} f^{-n}V$ such that $\mathbf{P}^2 \setminus \cup_{n=1}^{N} X_n$ is complete hyperbolic and hyperbolically embedded.*

Proof. Let X_1 be any irreducible component of the critical set C_1 preperiodic to V and, inductively, let X_{n+1} be any irreducible component of $f^{-1}X_n$ for $n \geq 1$. We claim that if N is large enough, then $\mathbf{P}^2 \setminus \cup_{n=1}^{N} X_n$ is complete hyperbolic and hyperbolically embedded.

The claim follows from Proposition 2.8 once we prove that there is a number l such that no l of the X_n's intersect at a time. Note that $X_n \cap X_m$ is a finite set if $m \neq n$, because otherwise $X_n = X_m$ which would imply that the irreducible component X_1 of C_1 be periodic, contradicting that f is 1-critically finite. Since f is 2-critically finite, there are numbers l_1, l_2 such that $f^{l_1}C_1 \subset E_1$, and $f^{l_2}C_2 \subset E_2$, $f^{l_2}C_2 \cap C_2 = \emptyset$. We will show that the number $l = l_1 + l_2 + 1$ will do.

For let $l = l_1 + l_2 + 1$ and suppose $p \in X_{n_1} \cap \cdots \cap X_{n_l}$. where $1 \leq n_1 < n_2 < \cdots < n_l \leq N$. Then $f^{n_i}(p) \in C_1,\ 1 \leq i \leq l$. But then $f^{n_i+j}(p) \in E_1, j \geq l_1, 1 \leq i \leq n$. In particular, $f^{n_m}(p) \in E_1$ for $m > l_1$, so that $f^{n_m}(p) \in C_1 \cap E_1 = C_2$ for $m > l_1$. This implies that $f^{n_m+j}(p) \in E_2, j \geq l_2, m > l_1$. In particular $f^{n_l}(p) \in E_2$, so that $f^{n_l}(p) \in C_2 \cap E_2 = \emptyset$, a contradiction. □

Proposition 2.12. *If $f : \mathbf{P}^2 \to \mathbf{P}^2$ is 2-critically finite, then the preimages of any point in E_2 are dense in $\mathbf{P}^2$.*

Proof. Let $p \in E_2$. We show that the preimages of p are dense in some irreducible component V of E_1 and that the inverse images of any irreducible component V of E_1 are dense in $\mathbf{P}^2$.

Let V be any irreducible component of E_1 containing p. Replacing f by an iterate, we may assume that V is invariant under f. Let $\hat{V}$ be the normalization of V and $\hat{f} : \hat{V} \to \hat{V}$ the lifting of $f|_V$ to $\hat{V}$. Then we know from Proposition 1.7 that $\hat{f}$ is (equivalent to) a Thurston map on the Riemann sphere or an expanding endomorphism of a torus. Hence, if $\hat{p} \in \hat{V}$ is any point above p, then the preimages of $\hat{p}$ under $\hat{f}$ are dense in $\hat{V}$. This implies that the preimages of p under f are dense in V.

It remains to be seen that the preimages of V are dense in $\mathbf{P}^2$. Suppose not. Then there is an open ball $U \subset \mathbf{P}^2$ such that the restriction of the family of iterates $\{f^i\}$ to U is a family of holomorphic mappings of U into $\mathbf{P}^2 \setminus \cup_{n=0}^{N} f^{-n}V$ for all N. Now Proposition 2.11 implies that this family is normal, which contradicts the fact that the Fatou set of f is empty (Theorem 2.1). □

We are now able to prove Theorem 2.2:

Proof of Theorem 2.2. Let E be closed and backward invariant. By Propositions 1.9 and 2.6, E must contain all the points in E_2 and hence, by Proposition 2.12, a dense set in $\mathbf{P}^2$. But E is closed so $E = \mathbf{P}^2$. □

Remark 2.13. It is also true that the holomorphic map $f : \mathbf{P}^k \to \mathbf{P}^k$ defined by:

$$[x_0 : \cdots : x_k] \to [(x_0 - 2x_1)^2 : \cdots : (x_0 - 2x_k)^2 : x_0^2]$$

has no nontrivial closed backward invariant subset, hence all its Julia sets are equal to $\mathbf{P}^k$. The proof can be sketched as follows: Suppose E is a nonempty closed backward invariant subset. The set E_1 consists of the hyperplanes $(x_i = x_j)$ for $0 \leq i < j \leq k$ and the set E_l of intersections of l different hyperplanes in E_1. In particular, E_k consists of the single point $[1 : \cdots : 1]$, which is a repelling fixed point. By Proposition 2.6, E must contain the point $[1 : \cdots : 1]$, so it suffices to prove that the preimages of this point are dense in $\mathbf{P}^k$. This is done by showing that the preimages of E_j are dense in E_{j-1} for $j = 1 \ldots k$. The proof of this is very much the same as the proof of Proposition 2.12; since each irreducible component of E_j is a $(k-j)$-plane in $\mathbf{P}^k$ one can apply Theorems 2.9 and 2.10. The details are omitted.

3. Density of periodic points

We now prove that repelling periodic points are dense for a 2-critically finite holomorphic map $f : \mathbf{P}^2 \to \mathbf{P}^2$. The idea is to study sequences of branches of f^{-j}. According to Ueda's result (Theorem 2.4), such sequences always form normal families. If the limit of an appropriate subsequence is constant, the Brouwer fixed point theorem can sometimes be used to assert the existence of a periodic point. Our main objective will therefore be to show that certain such limits must indeed be constant. This leads us to the following definition.

Definition 3.1. Let A be the set of points $a \in \mathbf{P}^2 \setminus D_1$ such that there exists a simply connected open neighborhood $V \subset \mathbf{P}^2 \setminus D_1$ of a and a sequence of branches $\{g_{n_i}\}$ of f^{-n_i} on V such that $g_{n_i} \to a$ uniformly.

Remark 3.2. *A priori*, the set A could be empty. In fact, our main task in this section will be to prove that it is not (Proposition 3.5). The next two results would, however, be true even if A was empty.

Proposition 3.3. *If $a \in A$ then every neighborhood of a contains a repelling periodic point of f.*

Proof. Take V and $\{g_{n_i}\}$ from the definition of A and let V' and V'' be any two sets homeomorphic to an open ball such that $a \in V'' \subset\subset V' \subset V$. Then $g_{n_i} \to a$ uniformly on V' so if i is large then $g_{n_i}(V') \subset V'' \subset\subset V'$. By the Brouwer fixed point theorem, this implies that g_{n_i} has a fixed point p in V'. We claim that this fixed point must be attractive. Indeed, if $K_{V'}$ and $K_{V''}$ are the Kobayashi-Royden metrics on V' and V'' and if $v \in T_p\mathbf{P}^2$ then since g_{n_i} is a biholomorphism from V' to $g_{n_i}(V')$, we have:

$$\begin{aligned} K_{V'}(p,v) &= K_{g_{n_i}(V')}(g_{n_i}(p), g_{n_i *}(v)) \\ &\geq K_{V''}(p, g_{n_i *}(v)) \\ &> K_{V'}(p, g_{n_i *}(v)) \end{aligned}$$

The last inequality follows from the assumption that $V'' \subset\subset V'$. Since p is an attractive fixed point for g_{n_i} it is a repelling fixed point for f^{n_i}. □

Lemma 3.4. *A is backward invariant, i.e. $f^{-1}(A) \subset A$.*

Proof. Suppose $a \in A$ and $f(a') = a$. Let V be a simply connected neighborhood of a in $\mathbf{P}^2 \setminus D$ and $\{g_{n_i}\}$ a family of branches of f^{-n_i} defined on V such that $g_{n_i} \to a$ uniformly on V. Since $a' \in \mathbf{P}^2 \setminus D$, there exists a simply connected open neighborhood $V' \subset \mathbf{P}^2 \setminus D_1$ of a' and analytic continuations of the g_{n_i}'s to V'. By Ueda's result (Theorem 2.4) these continuations form a normal family so a suitable subsequence of them will still converge uniformly to a on V'. Let h be the local inverse of f at a, defined on V, taking a to a'. Then $\{h \circ g_{n_i}\}$ will be a sequence of branches of inverse iterates of f on V', converging uniformly to a'. Hence $a' \in A$. □

Proposition 3.5. *The set A is nonempty.*

Proof. According to Proposition 1.9, the set E_2 consists of repelling periodic points for f. Take any $b \in E_2$. We will prove that a suitable preimage a of b outside D_1 belongs to A. To do this we need to construct a family of branches of inverse iterates converging to a. Using the fact that b is repelling, it is easy to find branches of inverse iterates converging to b. However, $b \in D_1$ and we want the limit to belong to $\mathbf{P}^2 \setminus D_1$ so the idea is to compose the branches converging to b with a branch going from b to its preimage a. The problem with this is that b belongs to the postcritical set D_1, so the latter branch cannot be defined in a whole neighborhood of b. It can, however, be defined on simply connected open sets having b as a boundary point and not intersecting the postcritical set D_1. This means that we must make the images of the branches converging to b stay in such a set. What we have to do is, therefore, to control the dynamics of f and the geometry of D_1 near b. The following technical lemma contains all the information we need.

Lemma 3.6. *Let g be a germ at the origin of a holomorphic map of $\mathbf{C}^2$ with $g(0) = (0)$ and $dg(0)$ having eigenvalues λ_1, λ_2 satisfying $0 < |\lambda_1|, |\lambda_2| < 1$. Let V be a germ of an analytic set of dimension 1 at the origin. Then there exist open subsets $U, U' \subset \mathbf{C}^2$ and a sequence $\{n_i\}$ such that:*

1. *$0 \in \partial U'$ and $U' \cap V = \emptyset$.*
2. *U' is simply connected.*
3. *$g^{n_i}(U) \subset U'$ for i large.*

Assuming Lemma 3.6 is true, we now continue the proof of Proposition 3.5: Take any $a \in f^{-k}\{b\} \setminus D_1$ for some $k > 0$ (the existence of such a and k follows from Lemma 2.3). We claim that $a \in A$. Since b is periodic, say of period m, there exists a locally defined branch h_1 of f^{-m} near b such that $h_1(b) = b$. Since b is repelling for f^m, it is attractive for h_1. We now invoke Lemma 3.6 above. This gives small open sets U and U' and a sequence $\{n_i\}$ such that

1. $0 \in \partial U'$ and $U' \cap D_1 = \emptyset$.
2. U' is simply connected.
3. $h_1^{n_i}(U) \subset U'$ for i large.
4. $h_1^{n_i} \to b$ uniformly on U as $i \to \infty$.

Now note that there exists a j such that $a \in f^j(U)$. Otherwise, the complement in $\mathbf{P}^2$ of the open set $\cup_{j \geq 0} f^j(U)$ would be a nonempty closed backward invariant set, hence equal to $\mathbf{P}^2$ according to Theorem 2.2, and this is clearly impossible. We can therefore find an open simply connected neighborhood V of a in $\mathbf{P}^2 \setminus D_1$ and a branch h_2 of f^{-j} defined on V such that $h_2(V) \subset U$.

Recall that $f^k(a) = b$. Because of properties 1 and 2 above, there exists an branch h_3 of f^{-k} on U' such that $\lim_{x \to b, x \in U'} h_3(x) = a$.

Define $g_i = h_3 \circ h_1^{n_i} \circ h_2$ for i large enough so that 3 holds. Then $\{g_i\}$ is a family of well defined branches of inverse iterates of f defined on V. Since $h_1^{n_i} \to b$ uniformly on U it follows that $g_i \to a$ uniformly on V. Hence $a \in A$ so A is nonempty. $\square$

Proof of Lemma 3.6. We want to linearize the situation locally at the origin. Although this is not always possible (due to resonances among λ_1 and λ_2), we claim that it is sufficient to prove the following lemma:

Lemma 3.7. *Let T be a complex-linear automorphism of $\mathbf{C}^2$ with eigenvalues λ_1, λ_2 satisfying $0 < |\lambda_1|, |\lambda_2| < 1$ and let V be an analytic set of pure dimension one in in the unit ball $B \subset \mathbf{C}^2$ such that $0 \in V$. Then there exist open subsets $U, U' \subset B$ and a sequence $\{n_i\}$ such that:*

1. *$0 \in \partial U'$ and $U' \cap V = \emptyset$.*
2. *U' is simply connected.*
3. *$T^{n_i}(U) \subset U'$ for i large.*

Let us postpone the proof of Lemma 3.7 and instead show how to deduce Lemma 3.6 from it. Suppose $0 < |\lambda_2| \leq |\lambda_1| < 1$. If there are no resonances, i.e. if $\lambda_2 \neq \lambda_1^p$ for $p = 2, 3, \ldots$ then we can linearize the situation locally (cf [Fa]), and the result follows immediately from Lemma 3.7. If $\lambda_2 = \lambda_1^p$ with $p > 1$ minimal, we cannot linearize in general, but after a change of coordinates we can obtain (locally at the origin) $g(x, y) = (\lambda_1 x, \lambda_2 y + a x^p)$ with $a = 0$ or 1 (cf [Fa]). If $a = 0$ we are again in the linear setting, so suppose $a = 1$. Let $\Phi : \mathbf{C}^2 \to \mathbf{C}^2$ be the proper map defined by $(\tilde{x}, \tilde{y}) = \Phi(x, y) = (x^p, y)$. Then the following diagram commutes (locally at the origin):

$$\begin{tikzcd} \mathbf{C}^2 \arrow[r, "g"] \arrow[d, "\Phi"'] & \mathbf{C}^2 \arrow[d, "\Phi"'] \\ \mathbf{C}^2 \arrow[r, "T"] & \mathbf{C}^2 \end{tikzcd}$$

where $T(\tilde{x},\tilde{y}) = (\lambda_2\tilde{x}, \lambda_2\tilde{y} + \tilde{x})$ is linear. Since Φ is proper, it follows from Remmert's theorem (see [Gu]) that $\Phi(V)$ is an analytic set. Let $\tilde{V} = \Phi(V) \cup \{\tilde{x} = 0\}$. Invoking Lemma 3.7 we get open sets $\tilde{U}, \tilde{U'} \subset \mathbf{C}^2$ and a sequence $\{n_i\}$ such that

1. $0 \in \partial\tilde{U'}$ and $\tilde{U'} \cap \tilde{V} = \emptyset$.
2. $\tilde{U'}$ is simply connected.
3. $T^{n_i}(\tilde{U}) \subset \tilde{U'}$ for i large.

Let $U'_1, \dots, U'_p$ be the connected components of $\Phi^{-1}(\tilde{U'})$ and U any connected component of $\Phi^{-1}(\tilde{U})$. Since Φ is an unbranched analytic covering outside $\{x = 0\} \subset \Phi^{-1}(\tilde{V})$, we get that $U'_1, \dots, U'_p$ are disjoint connected simply connected open sets and $\Phi|_{U'_j} : U'_j \to \tilde{U'}$ is a biholomorphism for each j. Since U is connected, the third condition and the commuting diagram above imply that $g^{n_i}(U) \subset U'_j$ for i large, where j may depend on i. Hence, taking a subsequence of $\{n_i\}$ we can obtain $g^{n_i}(U) \subset U'_j$ $\forall i$ for a fixed j. We put $U' = U'_j$. Then we see that:

1. $0 \in \partial U'$ and $U' \cap V = \emptyset$.
2. U' is simply connected.
3. $g^{n_i}(U) \subset U'$ for i large.

Hence Lemma 3.6 follows from Lemma 3.7. $\square$

Proof of Lemma 3.7. After a linear change of coordinates we may assume that the matrix of T takes the form:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}$$

The second form is just a slight variation of the Jordan normal form. Consider these two separately and call them CASE 1 and CASE 2.

Let V' be the analytic set $V \setminus ((x = 0) \cup (y = 0))$. We first claim that if $\delta > 0$ is small then the set

$$A_\delta := \{y; (x,y) \in V', |x|, |y| < \delta, x > 0\} \subset \mathbf{C}$$

is a finite union of real-analytic curves. Note that V' is nonsingular in the punctured ball $B(\delta) \setminus (0,0)$ if δ is small enough. Also, since V' does not contain the lines $x = 0$ and $y = 0$, it is not tangent to those lines anywhere in the same punctured ball (after shrinking δ if necessary). But then the intersection of V' with this ball is a union of graphs of finitely many nonconstant (multi-valued) functions $y = y(x)$ whence the claim easily follows.

CASE 1: By theorem of Dirichlet (see [HW]) we can find a sequence $\{n_i\}$ such that $\arg({\lambda_j}^{n_i}) \to 0$ as $i \to \infty$, $j = 1, 2$. The set A_δ defined above is a union of finitely many real analytic curves for $\delta > 0$ small. In particular, there exists a number $\theta \in (-\pi, \pi)$ such that the sector $\{|\arg(y) - \theta| < 2\epsilon\}$ is disjoint from A_δ for small ϵ. This implies that the set

$$U' := \{(x, y); 0 < |x|, |y| < \epsilon, |\arg(x)| < 2\epsilon, |\arg(y) - \theta| < 2\epsilon\}$$

does not intersect the set V' for small ϵ. But U' does not intersect the lines $x = 0$ and $y = 0$ either so it must be disjoint from V. Note that U' is a product of two simply connected domains, hence it is simply connected. We further define:

$$U := \{(x, y); 0 < |x|, |y| < \epsilon, |\arg(x)| < \epsilon, |\arg(y) - \theta| < \epsilon\}$$

The choice of the sequence $\{n_i\}$ guarantees that $T^{n_i}(U) \subset U'$ for i large. Hence the properties 1-3 all hold.

CASE 2: If we let $(x_n, y_n) = T^n(x, y)$ then $x_n/y_n = x/y + n$; in particular $\arg(x_n/y_n) \to 0$ as $n \to \infty$. We choose our sequence $\{n_i\}$ so that $\arg(\lambda^{n_i}) \to 0$. If $\epsilon > 0$ is small then V has no singularities in the set $\{0 < |x|, |y| < \epsilon\}$. Let L be the set of lines corresponding to tangential directions of the set V at the origin. These lines are well defined even if the origin is a singular point for some irreducible component of V. We now have three cases.

The first case is when L contains no line of the form $(y = 0)$ or $(x = sy)$ with $s \geq 0$. Then we may take

$$U' := \{(x, y); 0 < |x|, |y| < \epsilon, |\arg(x)| < 2\epsilon, |\arg(y)| < 2\epsilon\},$$

$$U := \{(x, y); 0 < |x|, |y| < \epsilon, |\arg(x)| < \epsilon, |\arg(y)| < \epsilon\},$$

and 1-3 are easily verified.

The second case is when L contains some line of the form $(x = sy)$ with $s \geq 0$ but not the line $(y = 0)$. Then we may take

$$U' := \{(x, y); 0 < |x|, |y| < \epsilon, |x| > S|y|, |\arg(x)| < 2\epsilon, |\arg(y)| < 2\epsilon\},$$

$$U := \{(x, y); 0 < |x|, |y| < \epsilon, |x| > S|y|, |\arg(x)| < \epsilon, |\arg(y)| < \epsilon\},$$

if ϵ is small and S large. Indeed, the formula $x_n/y_n = x/y + n$ shows that $T^{n_i}U \subset U'$ for i large and U' does not meet V if S is large enough and ϵ is small. Also, U' is simply connected because it is homeomorphic to the set

$$\{(\theta, \theta') \in \mathbf{R}^2; |\theta|, |\theta'| < 2\epsilon\} \times \{(r, r') \in \mathbf{R}^2; 0 < r, r' < \epsilon, r > Sr'\}$$

which is a product of simply connected domains. Hence 1-3 above hold also in this case.

Finally, we are left with the third case when L contains the line $(y = 0)$ and, perhaps, some lines of the form $(x = sy)$ with $s \geq 0$. Since V is an analytic set, we can then find $\alpha > 0$ such that if $\epsilon > 0$ is small enough, the intersection

$$V \cap \{(x, y); 0 < |x|, |y| < \epsilon, |\arg(x)| < 2\epsilon, |\arg(y)| < 2\epsilon, |x| > S|y|\}$$

is contained in the open set

$$\{(x,y); 0<|x|,|y|<\epsilon, |\arg(x)|<2\epsilon, |\arg(y)|<2\epsilon, |x|>S|y|, |y|<|x|^{1+\alpha}\}.$$

We claim that the sets

$$U' := \{(x,y); 0<|x|,|y|<\epsilon, |\arg(x)|, |\arg(y)|<2\epsilon, |x|>S|y|, |y|>|x|^{1+\alpha}\}$$

and

$$U := \{(x,y); 0<|x|,|y|<\epsilon, |\arg(x)|, |\arg(y)|<\epsilon, |x|>S|y|, |y|>|x|^{1+\alpha}\}$$

will do if ϵ is small enough and S is large enough. It is clear that $U'\cap V=\emptyset$ and that $0\in\partial U'$. Furthermore, U' is homeomorphic to the product

$$\{(\theta,\theta')\in\mathbf{R}^2; |\theta|,|\theta'|<2\epsilon\}\times\{(r,r')\in\mathbf{R}^2; 0<r,r'<\epsilon, r>Sr', r'>r^{1+\alpha}\}$$

and is therefore simply connected. It remains to show that $T^{n_i}(U)\subset U'$ for large i. First note that if $(x,y)\in U$ and ϵ is small then $|\arg(x/y)|<2\epsilon$ so $|x_1/y_1|=|x/y+1|>|x/y|>S$. Using this, we also get

$$\begin{aligned}
\frac{|x_1|^{1+\alpha}}{|y_1|} &= |x_1|^\alpha|\frac{x_1}{y_1}| \\
&= |\lambda|^\alpha|x+y|^\alpha||\frac{x}{y}+1| \\
&\le |\lambda|^\alpha|x+y|^\alpha(|\frac{x}{y}|+1) \\
&\le |\lambda|^\alpha(|x|+|y|)^\alpha(1+\frac{1}{S})|\frac{x}{y}| \\
&= |\lambda|^\alpha|x|^\alpha(1+|\frac{y}{x}|)^\alpha(1+\frac{1}{S})|\frac{x}{y}| \\
&\le |\lambda|^\alpha|x|^\alpha(1+\frac{1}{S})^\alpha(1+\frac{1}{S})|\frac{x}{y}| \\
&\le |\lambda|^\alpha(1+\frac{1}{S})^{1+\alpha}\frac{|x|^{1+\alpha}}{|y|}
\end{aligned}$$

Hence, if S is so large that $|\lambda|^\alpha(1+\frac{1}{S})^{1+\alpha}<1$ and i is large enough, we will have that $T^{n_i}(U)\subset U'$. This means that the properties 1-3 above hold and we are done. □

Theorem 3.8. *Repelling periodic points for f are dense in $\mathbf{P}^2$.*

Proof. It suffices to prove that repelling periodic points are dense in $\mathbf{P}^2\setminus D_1$. Since A is nonempty and backwards invariant, it must be dense in $\mathbf{P}^2$. This follows from Theorem 2.2. Hence if W is any open set in $\mathbf{P}^2\setminus D_1$ we can find $a\in A\cap W$. According to Lemma 3.3, every neighborhood of a contains repelling periodic points of f, hence W contains repelling periodic points. This completes the proof. □

4. Distribution of periodic points

Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be 2-critically finite of degree $d \geq 2$. We know that every sequence $\{g_i\}$ of branches of f^{-n_i} (with $n_i \to \infty$) is necessarily normal (Theorem 2.4). It is natural to ask what the possible limit functions of such sequences are. In section 3 we showed that many such limits must be constant, indeed sufficiently many to prove the density in $\mathbf{P}^2$ of repelling periodic points. The following conjecture is perhaps plausible:

Conjecture 4.1. *All limits of sequences of branches of inverse iterates of f are constant.*

In this section, we show that Conjecture 4.1 implies that the periodic points are distributed according to the "Green measure" $\mu := T \wedge T$ of maximal entropy. Since we know that $\operatorname{supp}(\mu) = J_1 = \mathbf{P}^2$, this is of course stronger than just saying that periodic points are dense in $\mathbf{P}^2$.

We know that f has $(d^{3n} - 1)/(d^n - 1)$ periodic points, counted with multiplicity (cf [FS1]). Define probability measures $\nu_m, \mu_{m,c}$ as:

$$\nu_m = \frac{d^m - 1}{d^{3m} - 1} \sum_{f^m(a)=a} \delta_a$$

$$\mu_{m,c} = \frac{1}{d^{2m}} \sum_{f^m(a)=c} \delta_a$$

Then ν_m and $\mu_{m,c}$ describes the distribution of periodic points of order m and preimages under f^m of a point $c \in \mathbf{P}^2$, respectively. Fornæss and Sibony [FS2] proved that $\mu_{m,c} \to \mu$ weakly as $m \to \infty$ for quasi-every $c \in \mathbf{P}^2$. This is true for any holomorphic $f : \mathbf{P}^2 \to \mathbf{P}^2$. In the 2-critically finite case, using the fact that sequences of branches of inverses of iterates form normal families whenever they are defined, one can prove that $\mu_{m,c} \to \mu$ weakly as $m \to \infty$ for *all* $c \in \mathbf{P}^2$.

Theorem 4.2. *If Conjecture 4.1 holds then periodic points are distributed according to the measure μ, i.e. $\nu_m \to \mu$ weakly as $m \to \infty$.*

The argument given below is similar to the one given by Lyubich in [Ly] in the one-dimensional case (for general rational functions).

Proof. Let W be a simply connected open set in $\mathbf{P}^2$ which does not meet the postcritical set D_1. Take any $c \in W$, $\epsilon > 0$ and find open sets W_1, W_2, W_3 with $W_3 \subset\subset W_2 \subset\subset W_1 \subset\subset W$, $\mu(W \setminus W_3) < \epsilon$. We put $r = \operatorname{dist}(W_2, \partial W_1) > 0$. Since W is simply connected and $W \cap D_1 = \emptyset$, there are d^{2m} well-defined branches $g_i, i = 1, \ldots, d^{2m}$ of f^{-m} defined on W. If Conjecture 4.1 holds, then if m is large enough, $\operatorname{diam}(g_i(W_1)) < r/2$ for $i = 1, \ldots, d^{2m}$. Hence, if $a_i := g_i(c) \in W_2$ for some i then $g_i(W_1) \subset B(a_i, r/2)$ where $B(a_i, r/2)$ is the ball of radius $r/2$ around a_i. But $B(a_i, r) \subset W_1$ so $g_i(B(a_i, r)) \subset B(a_i, r/2) \subset B(a_i, r)$ so by the Brouwer fixed point theorem g_i has a fixed point b_i in $B(a, r) \subset W_1$. Since the sets $g_i(W), i = 1, \dots, d^{2m}$ are all disjoint, the points $b_i, i = 1, \dots, d^{2m}$ are distinct. Note that a fixed point for g_i is a fixed point for f^m. This means that we can associate a fixed point of f^m in W_1 for every preimage under f^m of c in W_2. Hence we have that $\nu_m(W_1) \geq \mu_{m,c}(W_2)$ if m is large enough.

Now suppose $\nu = \lim_{k\to\infty} \nu_{m_k}$ is a weak accumulation point of the measures ν_m. It then follows that

$$\begin{aligned}
\nu(W) &\geq \overline{\lim}_{k\to\infty} \nu_{m_k}(W_1) \\
&\geq \overline{\lim}_{k\to\infty} \mu_{m_k,c}(W_2) \\
&\geq \underline{\lim}_{k\to\infty} \mu_{m_k,c}(W_2) \\
&\geq \mu(W_3) \\
&\geq \mu(W) - \epsilon
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $\nu(W) \geq \mu(W)$. A trivial covering argument yields that $\nu \geq \mu$ outside D_1. Now it follows from the Chern-Levine-Nirenberg inequality (cf [FS2]) that μ puts no mass on pluripolar sets and hence that $\mu(D_1) = 0$. Therefore, $\nu \geq \mu$ everywhere. But ν and μ are probability measures so $\nu = \mu$. This shows that μ is the only accumulation point of the sequence ν_m so we must have that $\nu_m \to \mu$ as $m \to \infty$. □

Remark 4.3. It follows from the above proof that *repelling* periodic points are distributed according to the measure μ. This is because a simple argument using the Kobayashi-Royden metric shows that all the points b_i are attractive for g_i and hence repelling for f^m (compare the proof of Proposition 3.3).

Acknowledgement. The author is grateful to N. Sibony for introducing him into this field and to his advisor M. Benedicks for valuable comments.

References

[Fa] Fatou, P. Substitutions analytiques et équations fonctionelles à deux variables. *Ann. Sci. École Norm. Sup.* 67–142, 1924.

[FS1] Fornæss, J.E., Sibony, N. Complex dynamics in higher dimension I. *Astérisque*, 222:201–231, 1994.

[FS2] Fornaess, J.E., Sibony, N. Complex dynamics in higher dimension II. In Bloom, T. et al, editor, *Modern Methods in Complex Analysis*, number 137 in Annals of Mathematics Studies. Princeton University Press, 1995.

[FS3] Fornæss, J.E., Sibony, N. Critically finite rational maps on $\mathbf{P}^2$. *The Madison Symposium on Complex Analysis*, Contemp. Math., vol. 137, Amer. Math. Soc., Providence, RI, 1992, pp. 245–260.

[FS4] Fornaess, J.E., Sibony, N. Complex dynamics in higher dimension. In Gauthier, P. M., Sabidussi, G., editor, *Complex Potential Theory*, pages 131–186. Kluwer Academic Publishers, 1994.

[Gr1] Green, M. L. Some Picard theorems for holomorphic maps to algebraic varieties. *Amer. J. Math.*, 97:43–75, 1975.

[Gr2] Green, M. L. The hyperbolicity of the complement of $2n + 1$ hyperplanes in general position in $\mathbf{P}^n$ and related results. *Proc. Amer. Math. Soc.* 66:109–113, 1977.

[Gu] Gunning, R. C. *Introduction to holomorphic functions of several variables I-III.* Wadsworth & Brooks/Cole, 1990.

[HW] Hardy, G. H., Wright, E. M.. *An introduction to the theory of numbers*, 4 ed. Oxford, 1960.

[La] Lang, S. *Introduction to complex hyperbolic spaces.* Springer-Verlag, 1987.

[Ly] Lyubich, M. Yu. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dynam. Systems* 3:351–385, 1983.

[Th] Thurston, W. On the combinatorics and dynamics of rational maps. Preprint.

[Ue1] Ueda, T. Critical orbits of holomorphic maps on projective spaces. To appear in *J. Geom. Anal.*

[Ue2] Ueda, T. Complex dynamical systems on projective spaces. Preprint.

[Ue3] Ueda, T. Fatou sets in complex dynamics on projective spaces. *J. Math. Soc. Japan* 46:545–555, 1994.

Dept. of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden
E-mail address: `mjo@math.kth.se`

Paper IV

DYNAMICS OF POLYNOMIAL SKEW PRODUCTS ON $\mathbf{C}^2$: EXPONENTS, CONNECTEDNESS AND EXPANSION

MATTIAS JONSSON

ABSTRACT. We study the dynamics of maps of $\mathbf{C}^2$ of the form $f(z,w) = (p(z), q(z,w))$, where $p(z)$ and $q(z,w)$ are polynomials of degree $d \geq 2$ such that f extends to a holomorphic map of $\mathbf{P}^2$. For such maps we relate expansion and connectedness of Julia sets to behavior of the critical set.

0. INTRODUCTION

Following the successful study of the dynamics on rational maps on the Riemann sphere, a great deal of research has been devoted to complex dynamics in higher dimension, in particular to iterations of holomorphic maps of the complex projective space $\mathbf{P}^k$ [FS1], [HP], [U1]. Despite many results there is still a lack of non-trivial examples whose dynamics can be analyzed in detail. In this paper we begin a study of (polynomial) skew products on $\mathbf{C}^2$; these are maps of the form

$$f(z,w) = (p(z), q(z,w)), \tag{0.1}$$

where p and q are polynomials of the same degree $d \geq 2$ such that f extends to a holomorphic map of $\mathbf{P}^2$.

Another aspect of skew products of the form (0.1) is that they map any vertical line $\{z\} \times \mathbf{C}$ to another vertical line $\{p(z)\} \times \mathbf{C}$ by a polynomial map. Hence the restriction of f^n to $\{z\} \times \mathbf{C}$ can be viewed as a composition of n different polynomial maps of $\mathbf{C}$.

In this paper we study the dynamics of a skew product f on $\mathbf{C}^2$ both as a holomorphic map on $\mathbf{P}^2$ and as a composition of polynomial maps of $\mathbf{C}$. Results on skew products of $\mathbf{C}^2$ have previously been obtained by Heinemann [H1], [H2], but his approach is quite different from the one is this paper. In particular, he works with the one-point compactification $\overline{\mathbf{C}^2}$ of $\mathbf{C}^2$ instead of $\mathbf{P}^2$. At the end of section 6 we review some of his results in the terminology of this paper.

Before describing our results in more detail we introduce some notation. To the map f of $\mathbf{C}^2$ (or $\mathbf{P}^2$) we can associate a Green function G, measuring the rate of escape to infinity, a positive closed current $T = \frac{1}{2\pi} dd^c G$ and an invariant probability measure $\mu = T \wedge T$ (see [FS1], [HP] or [BJ]). The component p of

1991 *Mathematics Subject Classification.* Primary: 32H50, Secondary: 58F23,58F15.

Key words and phrases. Skew products, holomorphic dynamics.

f also has a Green function G_p and an invariant measure $\mu_p = \frac{1}{2\pi} dd^c G_p$. Its Julia set is $J_p = \operatorname{supp}(\mu_p)$. Finally, for each vertical line $\{z\} \times \mathbf{C}$ we can define a Green function R_z, a probability measure μ_z and a Julia set J_z.

In Theorem 2.2 we prove that the measure μ is a skew product of μ_p and μ_z: if φ is a continuous test function, then

$$\int \varphi\, \mu = \int \left(\int \varphi(z, w)\, \mu_z(w) \right) \mu_p(z).$$

This formula provides us with a partial dynamical characterization of the set $J_2 := \operatorname{supp}(\mu)$ (Proposition 3.2).

$$J_2 = \overline{\bigcup_{z \in J_p} \{z\} \times J_z}.$$

Using this formula and a result of Briend [B] we show that J_2 *is the closure of the repelling periodic points of* f. This is in contrast with the example by Hubbard and Papadopol [HP] of a holomorphic map on $\mathbf{P}^2$ with a repelling periodic point outside J_2 (the set J_2 is defined for any holomorphic map of $\mathbf{P}^2$).

For polynomial maps of $\mathbf{C}$ there is an interesting relationship between Lyapunov exponents, critical points, the Green function and connectedness of the Julia set. As we will see, all of this generalizes to skew products on $\mathbf{C}^2$.

To the ergodic measure μ we can associate two Lyapunov exponents λ_1 and λ_2, measuring the average growth of expansion of f^n. In Theorem 2.6 we prove the following integral formulae.

$$\lambda_1 = \log d + \sum_{p'(c)=0} G_p(c),$$

$$\lambda_2 = \log d + \int \left(\sum_{\frac{\partial q}{\partial w}(z,c)=0} G(z, c) \right) \mu_p(z).$$

It follows that $\lambda_1, \lambda_2 \geq \log d$, something which is not generally true for polynomial maps of $\mathbf{C}^2$.

A particularly interesting class of polynomial maps of $\mathbf{C}$ are those with connected Julia sets. We generalize this notion to skew products f on $\mathbf{C}^2$ by saying that f has property C if J_p is connected and J_z is connected for all $z \in J_p$. That property C is a good concept is shown by Theorem 4.10, which asserts that the following statements are equivalent.

(i) f has property C.
(ii) $G_p(c) = 0$ for all critical points c of p and $G(z, c) = 0$ for all $(z, c) \in J_p \times \mathbf{C}$ with $\frac{\partial q}{\partial w}(z, c) = 0$.
(iii) $\lambda_1 = \lambda_2 = \log d$.

Moreover, if f has property C, then J_2 is connected and J_z is connected for all $z \in \mathbf{C}$.

The set of all skew products on $\mathbf{C}^2$ of a given degree $d \geq 2$ can be identified with $\mathbf{C}^N$, $N = N(d)$. Let M_d be the subset of $\mathbf{C}^N$ corresponding to skew products with property C. Then M_d plays the role of the connectedness locus of degree d, a much studied object in the case of polynomial maps of $\mathbf{C}$. One can ask many questions about M_d; in Theorem 5.2 we answer one of them by proving that *M_d is compact.*

The dynamics of a rational map g of $\hat{\mathbf{C}}$ is most easily understood if g is expanding (hyperbolic) on its Julia set J_g. Such rational maps are characterized by the condition that the closure of the postcritical set of g is disjoint from J_g. In Theorem 6.3 we prove the corresponding result for a skew product f on $\mathbf{C}^2$, namely that *f is expanding on J_2 iff the closure of the postcritical set of f is disjoint from J_2.* A consequence of the proof of Theorem 6.3 is that *if f is expanding on J_2, then $z \to J_z$ is continuous on J_p*, which together with Proposition 3.2 implies that

$$J_2 = \bigcup_{z \in J_p} \{z\} \times J_z.$$

This generalizes a result by Heinemann [H2].

The expansion of f on J_2 has consequences for the geometry of the Julia sets J_p, J_z and J_2. In Corollary 6.9 we show that *if f has property C and is expanding on J_2, then J_p, J_2 and J_z for $z \in J_p$ are all connected and locally connected.*

The paper is organized as follows. In section 1 we review some notions from dynamics of polynomial maps of $\mathbf{C}$ and $\mathbf{C}^2$. In section 2 we prove the integral formula for μ (Theorem 2.2) and the formulae for the Lyapunov exponents (Theorem 2.6). The study of the dynamics of f on vertical lines is started in section 3 and continued in section 4, where we introduce and characterize property C (Theorem 4.10). Section 5 is devoted to the connectedness locus M_d and in particular the fact that it is compact (Theorem 5.2). Finally, in section 6, we prove Theorem 6.3 about the relation between expansion on J_2 and the postcritical set and we derive some consequences from it. We also describe some results by Heinemann in our terminology.

1. Polynomial maps of $\mathbf{C}$ and $\mathbf{C}^2$

Polynomial skew products are polynomial maps of $\mathbf{C}^2$ such that the first coordinate is a polynomial map of $\mathbf{C}$. In this section collect some material on polynomial maps of $\mathbf{C}$ from [CG] and on polynomial maps of $\mathbf{C}^2$ from [BJ]. Many of the results from the latter paper are in turn collected from [FS1] and [HP].

Let us start by defining the objects we want to study.

Definition 1.1. A polynomial skew product on $\mathbf{C}^2$ of degree d is a map of the form $f(z, w) = (p(z), q(z, w))$, where p and q are polynomials of degree d and where $p(z) = z^d + O(z^{d-1})$ and $q(z, w) = w^d + O(w^{d-1})$.

For brevity we will often say skew product instead of polynomial skew product. Suppose that $f(z) = (p(z), q(z, w))$ is a skew product. A useful tool in the study of the polynomial map p is the Green function G_p, defined by

$$G_p(z) = \lim_{n\to\infty} d^{-n}\log^+|p^n(z)|.$$

The set $K_p = \{G_p = 0\}$ is called the filled-in Julia set of p. If we view p as a rational map of $\mathbf{C}$, then the Julia set of p is exactly $J_p = \partial K_p$. The asymptotics of G_p at infinity is given by

$$G_p(z) = \log|z| + o(1); \tag{1.1}$$

this implies that K_p and J_p have logarithmic capacity 1. If we let μ_p denote harmonic measure on K_p, then $\mu_p = \frac{1}{2\pi}dd^cG_p$ and $\operatorname{supp}(\mu_p) = J_p$. The measure μ_p is often called the Brolin measure of p and it is the unique invariant measure of maximal entropy for p. We extend G_p to $\mathbf{C}^2$ by $G_p(z,w) = G_p(z)$.

The dynamics of p near infinity is very simple and can be described by introducing a Böttcher coordinate. This is the unique holomorphic function φ_p near infinity such that $\varphi_p(z) = z + O(1)$ as $z \to \infty$, $\varphi_p \circ p = \varphi_p^d$ and $\log|\varphi_p| = G_p$. We can extend φ_p analytically to the set $\{G_p > \max\{G_p(c); p'(c) = 0\}\}$. In particular φ_p extends to $\mathbf{C} - K_p$ iff no critical point is attracted to infinity and this happens iff K_p and J_p are connected. If J_p is connected, then $\psi_p = \varphi_p^{-1}$ maps $\hat{\mathbf{C}} - \bar{\mathbf{D}}$ conformally onto $\hat{\mathbf{C}} - K_p$, extends measurably to S^1 and $(\psi_p)_*(d\theta/2\pi) = \mu_p$. By Carathéodory's theorem ψ_p extends continuously to S^1 iff J_p is locally connected. This happens in particular if p is *expanding* (or hyperbolic) on J_p, i.e. if there exist constants $c > 0$, $\lambda > 1$ such that $|Dp^n(z)| \geq c\lambda^n$ for all $n \geq 1$ and all $z \in J_p$.

Definition 1.1 above implies that f extends to a map of $\mathbf{P}^2$, also denoted by f. If we use homogeneous coordinates $[z : w : t]$ on $\mathbf{P}^2$, where $(z,w) \in \mathbf{C}^2$ is identified with $[z : w : 1] \in \mathbf{P}^2$, then the extension of f is given by

$$f[z : w : t] = [t^d p(z/t) : t^d q(z/t, w/t) : t^d].$$

The line $\Pi := (t = 0)$ at infinity is completely invariant under f and we denote $f|_\Pi$ by f_Π. Note that f_Π is a monic polynomial map of Π: if we use the coordinate $\zeta = w/z$ on Π, then $f_\Pi(\zeta) = q_0(1,\zeta)$, where q_0 is the homogeneous part of q of degree d. The point at infinity for f_Π is the point $[0 : 1 : 0]$. We denote by J_Π and K_Π the Julia set and the filled-in Julia set of f_Π, respectively. Note that the point $[0 : 1 : 0]$ is superattracting for f. Hence its basin of attraction $W^s([0 : 1 : 0])$ is an open subset of the Fatou set of f, i.e. the set of points where $\{f^n\}$ is a locally normal family. The set of points in $\mathbf{P}^2$ which are attracted to J_Π and K_Π are denoted by $W^s(J_\Pi)$ and $W^s(K_\Pi)$, respectively.

Just as in one variable we have a Green function G of f, measuring the rate of escape to Π. It is defined by

$$G(x) = \lim_{n\to\infty} d^{-n}\log^+|f^n(x)|$$

and is continuous, nonnegative, plurisubharmonic and satisfies the fundamental relation $G \circ f = dG$. The definition of G is independent of the norm on $\mathbf{C}^2$. If we fix a norm on $\mathbf{C}^2$, then the asymptotics of G is given by

$$G(z,w) = \log|(z,w)| + \rho_G[z:w:0] + o(1) \tag{1.2}$$

as $(z,w) \to \Pi$, where $[z:w:0]$ is the projection of (z,w) on Π and ρ_G is continuous on Π. The function ρ_G is called the *Robin function* of G; it depends only on the homogeneous part f_0 of degree d of f. By letting $z \in \mathbf{C}$ be a periodic point of p we see from (1.1) and (1.2) that $\rho_G[0:1:0] = 0$ for any skew product f (and any choice of norm on $\mathbf{C}^2$).

All the points in $\mathbf{C}^2$ with bounded orbits form a compact set K, given by $K = \{G = 0\}$. The positive closed current $T := \frac{1}{2\pi} dd^c G$ extends to a positive closed current on $\mathbf{P}^2$, also denoted by T. Much of the importance of T stems from the fact that the set $J_1 := \operatorname{supp}(T)$ is exactly the Julia set of f, i.e. the complement of the Fatou set, the latter set being the set of points where $\{f^n\}$ forms a normal family.

The wedge product $\mu := T \wedge T$ is well-defined and is by definition harmonic measure on K. It has dynamical importance because μ is an invariant measure of maximal entropy $\log d^2$. We denote the support of μ by J_2. The dynamics of f on J_2 are most easily understood if f is expanding on J_2. This means that there exist constants $c > 0$, $\lambda > 1$ such that $|Df^n(x)v| \geq c\lambda^n|v|$ for $n \geq 1$, $x \in J_2$ and $v \in T_x\mathbf{C}^2$.

The critical set plays an important role for the dynamics of a polynomial map of $\mathbf{C}$ and so it does in higher dimension. Let C be the critical set of $f : \mathbf{C}^2 \to \mathbf{C}^2$, i.e the set of points where f is not locally invertible. We write $C = C_1 \cup C_2$, where $C_1 = \{(z,w); p'(z) = 0\}$ and $C_2 = \{(z,w); \partial q/\partial w(z,w) = 0\}$. Note that as a map of $\mathbf{P}^2$ the critical set of f is given by $C_1 \cup C_2 \cup \Pi$.

2. Lyapunov exponents

In this section we describe the measure μ as a skew product (Theorem 2.2). This enables us to prove a formula for the Lyapunov exponents of f with respect to μ (Theorem 2.6).

We start by comparing the Green functions G and G_p.

Lemma 2.1. *$G = G_p$ on J_1.*

Proof. Clearly $G = G_p = 0$ on K and if $G(z,w) > G_p(z,w)$, then (z,w) is in the basin of attraction of $[0:1:0]$, hence in the Fatou set. □

Theorem 2.2. *Let $f(z,w) = (p(z), q(z,w))$ be any skew product on $\mathbf{C}^2$. Then the action of μ on a test function φ is given by*

$$\int \varphi\, \mu = \int \left(\int \varphi(z,w)\, \mu_z(w) \right) \mu_p(z), \tag{2.1}$$

where $\mu_z = \frac{1}{2\pi} dd^c_w G(z,w)$ is the slice of T on the vertical line $\{z\} \times \mathbf{C}$.

Proof. By Lemma 2.1, $G - G_p$ is a continuous function on $\mathbf{C}^2$ which is zero on the support of T. Hence $GT = G_pT$, which implies that

$$\begin{aligned}\mu &= \frac{1}{2\pi} dd^c(GT)\\ &= \frac{1}{2\pi} dd^c(G_pT)\\ &= (\frac{1}{2\pi})^2 dd^c_z G_p \wedge dd^c_w G,\end{aligned}$$

where the last line follows because G_p is independent of w. Applied to a continuous test function φ, this gives exactly (2.1). □

Corollary 2.3. *$(\pi_1)_*\mu = \mu_p$, where $\pi_1(z, w) = z$. In particular, if $E \subset \mathbf{C}$ and $\mu_p(E) = 1$, then $\mu(E \times \mathbf{C}) = 1$.*

The map f has two Lyapunov exponents λ_1, λ_2, with respect to the ergodic invariant measure μ. They can be characterized as follows (see e.g. [Y] for more details). The Lyapunov exponents of f are the two numbers $\lambda_1 \geq \lambda_2$ such that for μ-almost every point $x \in \mathbf{C}^2$ there exists a subspace $E_2(x)$ of $T_x\mathbf{C}^2$ such that

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)v| = \lambda_2 \qquad \forall v \in E_2 - 0,\\ &\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)v| = \lambda_1 \qquad \forall v \in T_x\mathbf{C}^2 - E_2.\end{aligned}$$

For these x we also have

$$\lim_{n\to\infty} \frac{1}{n} \log |\det Df^n(x)| = \Lambda,$$

where $\Lambda = \lambda_1 + \lambda_2$. That all of this is well defined follows from Oseledec's theorem and a point x satisfying the above equations will be called Oseledec generic.

Remark 2.4. If we regard f as a map of $\mathbf{R}^4$, then f has four Lyapunov exponents; these are λ_1, λ_1, λ_2, λ_2.

Similarly, the polynomial map p of $\mathbf{C}$ has a Lyapunov exponent $\lambda(p)$ with respect to the measure μ_p. We have

$$\lim_{n\to\infty} \frac{1}{n} \log |Dp^n(z)| = \lambda(p)$$

for μ_p-a.e. $z \in \mathbf{C}$. Again z will is called Osoledec generic if the above equation holds. It follows from the chain rule and the ergodicity of μ and μ_p that

$$\lambda(p) = \int \log |Dp| \, \mu_p \tag{2.2}$$

$$\Lambda = \int \log |\det Df| \, \mu. \tag{2.3}$$

We are aiming for integral formulae for λ_1 and λ_2. There is an integral formula for $\lambda(p)$, formulated by Przytycki [P].

$$\lambda(p) = \log d + \int G_p \, \mu_{c,p}, \tag{2.4}$$

where $\mu_{c,p}$ is a critical measure defined by

$$\mu_{c,p} = \sum_{p'(c)=0} \delta_c. \tag{2.5}$$

Let $f(z,w) = (p(z), q(z,w))$ be a skew product of degree d. We will write $q_z(w)$ instead of $q(z,w)$. Then q_z is a monic polynomial map of $\mathbf{C}$ of degree d. Let $c_1(z), \ldots, c_{d-1}(z)$ be the critical points of q_z, counted with multiplicity. Define $H = \log |\partial q / \partial w|$. Then $H(z,w) = \log d + \sum_{i=1}^{d-1} \log |w - c_i(z)|$. Define a new critical measure $\mu_{c,q}$ by

$$\mu_{c,q} = (\frac{1}{2\pi})^2 dd^c H \wedge dd^c G_p = (\frac{1}{2\pi})^2 dd^c_w H \wedge dd^c_z G_p. \tag{2.6}$$

This means that if φ is any continuous function on $\mathbf{C}^2$, then

$$\int \varphi \, \mu_{c,q} = \int \left(\sum_{i=1}^{d-1} \varphi(z, c_i(z)) \right) \mu_p(z).$$

We will need the following computational lemma.

Lemma 2.5. *If $z \in J_p$, then*

$$\int H \, \mu_z(w) = \log d + \frac{1}{2\pi} \int G \, dd^c_w H.$$

Proof. The subharmonic function $G(z, \cdot)$ can be reproduced from μ_z by integrating against the kernel $\log |\cdot|$. Hence it follows from the above discussion that

$$\begin{aligned} \int H(z,w) \, \mu_z(w) &= \log d + \int \left(\sum_{i=1}^{d-1} \log |w - c_i(z)| \right) \mu_z(w) \\ &= \log d + \sum_{i=1}^{d-1} G(z, c_i(z)) \\ &= \log d + \frac{1}{2\pi} \int G(z,w) \, dd^c_w H(z,w). \end{aligned}$$

□

Theorem 2.6. *Let $f(z,w) = (p(z), q(z,w))$ be a skew product on $\mathbf{C}^2$ of degree $d \geq 2$. Then the Lyapunov exponents of f with respect to the measure μ are* *given by*

$$\lambda_1 = \log d + \int G_p \, \mu_{c,p}, \tag{2.7}$$

$$\lambda_2 = \log d + \int G \, \mu_{c,q}, \tag{2.8}$$

where the critical measures $\mu_{c,p}$ *and* $\mu_{c,q}$ *are given by* (2.5) *and* (2.6), *respectively.*

Proof. We first compute $\Lambda(f)$ using (2.3).

$$\begin{aligned}\Lambda(f) &= \int \log|\det Df| \, \mu \\ &= \int \log|Dp(z)| \, \mu_p(z) + \int \left(\int H \, \mu_z(w) \right) \mu_p(z) \\ &= \lambda(p) + \log d + \frac{1}{2\pi} \int \left(\int G \, dd^c_w H \right) \mu_p(z) \\ &= \lambda(p) + \log d + \int G \, \mu_{c,q}. \end{aligned} \tag{2.9}$$

The second line follows from Theorem 2.2 and the third line from (2.2) and from Lemma 2.5.

It follows from Corollary 2.3 that there exists a point (z, w) such that (z, w) is Oseledec generic for f and z is Oseledec generic for p, i.e. (2.2) and (2.3) hold. Since $Df(z, w)$ is lower triangular we see that

$$\det Df^n(z, w) = Dp^n(z) \prod_{i=0}^{n-1} \frac{\partial q}{\partial w}(z_i, w_i),$$

where $(z_i, w_i) = f^i(z, w)$, so

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \prod_{i=0}^{n-1} \frac{\partial q}{\partial w}(z_i, w_i) \right| = \Lambda(f) - \lambda(p).$$

Hence, if $v \neq 0$ is a vertical vector, then

$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n(z, w)v| = \Lambda(f) - \lambda(p).$$

It follows that one of the Lyapunov exponents of f, say λ_2, is equal to $\Lambda(f) - \lambda(p)$, so the other one, i.e. λ_1, is equal to $\lambda(p)$. Theorem 2.6 now follows from (2.4) and (2.9). $\square$

Corollary 2.7. $\lambda_1, \lambda_2 \geq \log d$.

Note that it is not true in general for regular polynomial endomorphisms of $\mathbf{C}^2$ of degree d that $\lambda_1, \lambda_2 \geq \log d$. For example, if f is a homogeneous regular polynomial endomorphism of degree 2 such that f_Π is a Lattés example, then one can see that $\lambda_1 = \log 2$ and $\lambda_2 = \frac{\log 2}{2}$.

Corollary 2.8.

(i) $\lambda_1 = \log d$ *iff* $G_p = 0$ *on* C_1.

(ii) $\lambda_2 = \log d$ *iff* $G = 0$ *on* $C_2 \cap (J_p \times \mathbf{C})$.

Proof. This follows from Theorem 2.6 and the continuity of G. □

3. Dynamics on vertical lines

A skew product $f(z,w) = (p(z), q(z,w))$ maps vertical lines to vertical lines. In this section we will study the dynamics of f on these. For a fixed $z \in \mathbf{C}$ we define $q_z(w) = q(z,w)$ and

$$q_z^{(n)} = q_{p^{n-1}(z)} \circ \cdots \circ q_z.$$

Define $R_z(w) := G(z,w) - G_p(z)$. Then R_z is a positive continuous subharmonic function on $\mathbf{C}$ and

$$\begin{aligned} R_z(w) &= \log|(z,w)| + \rho_G[0:1:0] - G_p(z) + o(1) \\ &= \log|w| - G_p(z) + o(1) \end{aligned} \tag{3.1}$$

as $w \to \infty$. This follows from (1.2) and the fact that $\rho_G[0:1:0] = 0$. Let $K_z := \{R_z = 0\}$ and $J_z := \partial K_z$. Then K_z and J_z are compact. The relations $G \circ f = d\,G$ and $G_p \circ p = d\,G_p$ imply that $R_{p(z)} \circ q_z = d\,R_z$. In particular $q_z(K_z) = K_{p(z)}$ and $q_z(J_z) = J_{p(z)}$.

Proposition 3.1. *R_z is the Green function for K_z. Moreover J_z and K_z have logarithmic capacity* $\exp(G_p(z))$.

Proof. We have to show that R_z is harmonic where $R_z > 0$. But $R_z(w) > 0$ iff $(z,w) \in W^s([0:1:0])$ and G is pluriharmonic on the latter set. From (3.1) we read off that the Robin constant of R_z is $-G_p(z)$, so the logarithmic capacity of K_z and J_z is $\exp(G_p(z))$. □

Let $\mu_z := \frac{1}{2\pi} dd^c R_z$ be harmonic measure on K_z. Then $\mathrm{supp}(\mu_z) = J_z$. Note that μ_z is the same measure as in Theorem 2.2. Recall the notation $J_2 := \mathrm{supp}(\mu)$. The measures μ_z vary continuously with z, because G is continuous. Hence $z \to J_z$ is lower semicontinuous in the Hausdorff metric.

Proposition 3.2. *If f is any skew product, then*

$$J_2 = \overline{\bigcup_{z \in J_p} \{z\} \times J_z}. \tag{3.2}$$

Proof. This follows from Theorem 2.2 and the fact that $z \to \mu_z$ is continuous. □

Hubbard and Papadopol [HP] have constructed holomorphic maps of $\mathbf{P}^2$ with repelling periodic points outside J_2. This cannot happen for skew products.

Corollary 3.3. *J_2 is the closure of the repelling periodic points of f.*

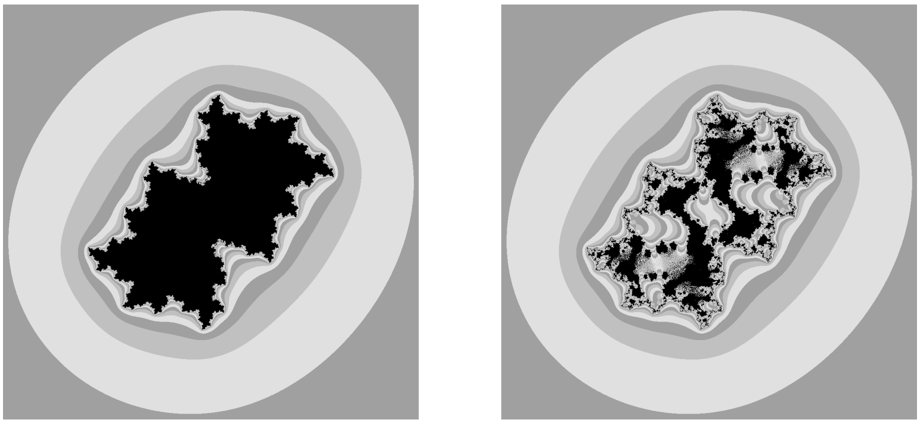


FIGURE 1. Vertical Julia sets.

Proof. Briend [B] has showed that repelling periodic points are dense in J_2 (for a general holomorphic map of $\mathbf{P}^2$), so we only have to show that all repelling periodic points belong to J_2. Now if (z, w) is a repelling periodic point of f, then z is a repelling periodic point of p, say of period k, and w is a repelling periodic point of $q_z^{(k)}$. Hence $z \in J_p$ and $\{f^n\}$ is not normal on $\{z\} \times \mathbf{C}$ at (z, w) so $w \in J_z$. By Proposition 3.2 it follows that $(z, w) \in J_2$. $\square$

The closure in (3.2) could be removed if $z \to J_z$ was upper semicontinuous, hence continuous, but this is not true in general. Consider e.g. the map $f(z, w) = (-z^2, w^2 + \lambda w(2 + z))$, where λ is chosen so that $w \to w^2 + \lambda w$ has a Siegel disk at the origin. Then $0 \notin J_{-1}$ but $0 \in J_z$ for all periodic points $z \neq -1$ of $z \to z^2$, so $z \to J_z$ is discontinuous at $z = -1$. On the other hand, we will prove in section 6 that $z \to J_z$ *is* upper semicontinuous if f is expanding on J_2.

The notation J_z is meant to suggest that it is the Julia set for the iterates of f on the vertical line $z \times \mathbf{C}$. This is indeed the case.

Proposition 3.4. *The family $\{f^n|_{\{z\}\times\mathbf{C}}\}$, viewed as a sequence of mappings of $\{z\} \times \mathbf{C}$ into $\mathbf{P}^2$, is normal exactly on $\{z\} \times (\mathbf{C} - J_z)$.*

Proof. This is a special case of a theorem of Ueda [U3] and is implicitly contained in [FS2]. Compare with the result that $\{f^n\}$ is normal exactly outside $J_1 = \operatorname{supp}(dd^cG)$. $\square$

In Figure 1 we show two vertical Julia sets J_z for the skew product

$$f(z, w) = (z^2, w^2 + 0.21iz + (-0.21 + 0.5i)).$$

The two pictures show $J_{e^{i\theta}}$ for $\theta = 0.7471100934857$ and $\theta = 0.7471101934857$, respectively. It seems that $z \to J_z$ is not continuous on J_p for this map.

We end this section by relating the sets J_z to the sets J_2 and J_1. Ideally we would like to decide whether a point (z, w) belongs to J_2 or J_1 only by checking whether z is in $\operatorname{int}(K_p)$, J_p or $\mathbf{C}-K_p$ and whether w is in $\operatorname{int}(K_z)$, J_z or $\mathbf{C}-K_z$. Such a characterization will not be true in general (compare with the discussion above), but at least we have the following result.

Proposition 3.5. *Let f be any skew product on $\mathbf{C}^2$.*

(i) *If $z \in \mathbf{C}$ and $w \in \mathbf{C} - K_z$, then $(z,w) \notin J_1$.*
(ii) *If $z \in int(K_p)$ and $w \in J_z$, then $(z,w) \in J_1 - J_2$.*
(iii) *If $z \in J_p$ and $w \in J_z$, then $(z,w) \in J_2$.*
(iv) *If $z \in J_p$ and $w \in int(K_z)$, then $(z,w) \in J_1$.*
(v) *If $z \in \mathbf{C} - K_p$ and $w \in J_z$, then $(z,w) \in J_1 - J_2$.*

Proof.

(i) We have already observed that $R_z(w) > 0$ is equivalent to that (z,w) is in the set $W^s([0:1:0])$, which is disjoint from J_1.
(ii) Since R_z is not harmonic near w, G cannot be pluriharmonic near (z,w), so $(z,w) \in J_1$. The fact that $(z,w) \notin J_2$ follows from Proposition 3.2.
(iii) This follows from Proposition 3.2.
(iv) The orbit of (z,w) is bounded, but every neighborhood of (z,w) contains points with unbounded orbits. Hence $\{f^n\}$ is not normal at (z,w), so (z,w) is in J_1.
(v) The proof is the same as for (ii).

□

4. Böttcher coordinates and connectedness of Julia sets

For a polynomial map p of $\mathbf{C}$ of degree $d \geq 2$, the Julia set J_p is connected iff no critical point is attracted to infinity and this happens iff the Lyapunov exponent of p is $\log d$. In this section we analyze when the sets J_z, J_p and J_2 are connected for a skew product. We introduce a condition on skew products, called Property C, meaning that J_p is connected and J_z is connected for all $z \in J_p$. In Theorem 4.10, we show that Property C plays the same role for skew products as does connectedness of the Julia set for a polynomial map of $\mathbf{C}$. The main tool for analyzing the connectedness of J_z are the Böttcher coordinates φ_z, defined near infinity on each vertical line $\{z\} \times \mathbf{C}$ and with similar properties as for polynomial maps of $\mathbf{C}$.

Lemma 4.1. *There exists a positive constant R such that $\{R_z > R\}$ is biholomorphic to a (punctured) disk and R_z is harmonic without critical points on $\{R_z > R\}$ for all $z \in \mathbf{C}$.*

Proof. We have the asymptotic formulas (1.1) and (1.2) for G and G_p. These imply that if R is large enough and $R_z(w) > R$, then (z,w) is close to the point $[0:1:0]$. Hence R_z is harmonic at w and since $\rho_G[0:1:0] = 0$ we have

$$R_z(w) = \log|w| - G_p(z) + g_z(w), \tag{4.1}$$

where the functions g_z are harmonic on $\{R_z > R\}$ and can be made uniformly small by choosing R large enough. Therefore R_z has no critical points on $\{R_z > R\}$. For a fixed z we see from (4.1) that $\{R_z > R'\}$ is biholomorphic to a punctured disk for R' large enough. Since R_z has no critical points on $\{R_z > R\}$, the latter domain is also a punctured disk. □

Proposition 4.2. *For any skew product f there exists a constant $R > 0$ and for any $z \in \mathbf{C}$ there is a unique conformal map φ_z of $\{R_z > R\}$ onto $|\zeta| > e^R$, depending continuously on z, such that*

(i) $\varphi_z(w) = w + o(1)$ *as* $w \to \infty$.
(ii) $\log|\varphi_z| = R_z$.
(iii) $\varphi_{p(z)} \circ q_z = \varphi_z^d$.

We will call φ_z the Böttcher coordinate *of q_z.*

Proof. Let R be as in Lemma 4.1. By (4.1) the function $R_z(w) - \log|w|$ is bounded and harmonic on the punctured disk $\{R_z > R\}$, hence has a harmonic conjugate R_z^* there. We may assume that $R_z^*(\infty) = 0$. Let $\varphi_z = \exp(R_z + iR_z^*)$. Then φ_z is holomorphic on $\{R_z > R\}$ and satisfies (i) and (ii). To see (iii) we note that the relation $R_{p(z)} \circ q_z = d\,R_z$ and (ii) imply that there is a constant c_z of modulus 1 such that $\varphi_{p(z)} \circ q_z = c_z \varphi_z^d$. But q_z is a monic polynomial so by (i) we must have $c_z = 1$. From (ii) it follows that φ_z maps $\{R_z > R\}$ properly onto $\{|\zeta| > e^R\}$, so φ_z is branched covering map. By (i) the sheet number is one near $w = \infty$ so in fact φ_z is a biholomorphism of $\{R_z > R\}$ onto $\{|\zeta| > e^R\}$. Finally recall that R_z depends continuously on z, hence the same is true for φ_z. □

We now try to extend the Böttcher coordinates φ_z analytically, using the relation (iii) above. First note that if $c > 0$, then every connected component of $\{R_z < c\}$ must contain points of J_z. Indeed, otherwise R_z would be harmonic, hence constant by the minimum principle in such a component. We conclude that if $\{R_z < c\}$ has several components, then J_z is disconnected. Since $q_z(J_z) = J_{p(z)}$ we see that if $J_{p(z)}$ is disconnected, then J_z is disconnected.

Let $z \in \mathbf{C}$. First suppose that $R_{p^n(z)}(w) = 0$ for all critical points w of $q_{p^n(z)}$ for all $n \geq 0$. Then we can use the formula $\varphi_{p^{n+1}(z)} \circ q_z = \varphi_{p^n(z)}^d$ to extend the Böttcher coordinates $\varphi_{p^n(z)}$ to all of $\mathbf{C} - K_z$. Thus φ_z maps $\mathbf{C} - K_z$ conformally onto $\mathbf{C} - \bar{\mathbf{D}}$.

Now suppose there is an $n \geq 0$ and a critical point w of $q_{p^n(z)}$ such that $R_{p^n(z)}(w) > 0$. Then we have

$$c = \sup\{d^{-n} R_{p^n(z)}(w); (z, w) \in C_2\} > 0 \tag{4.2}$$

It follows from Lemma 4.1 that the supremum in (4.2) is achieved for some pair (n, w). Then $R_{p^n(z)}$ has a critical point at w, so the set $\{R_{p^n(z)} < d^n c\}$ is disconnected. It follows from the remarks above that $J_{p^i(z)}$ is disconnected for all $i \leq n$. Summing up, we have proved

Proposition 4.3. *Let f be a skew product on $\mathbf{C}^2$ and $z \in \mathbf{C}$. Then J_z is connected iff $R_{p^n(z)}(w) = 0$ for all critical points w of $q_{p^n(z)}$ and all $n \geq 0$. If J_z is connected, then the Böttcher coordinate φ_z extends to a conformal map of $\mathbf{C} - K_z$ onto $\mathbf{C} - \bar{\mathbf{D}}$.*

Corollary 4.4. *Let X be the subset of $z \in X$ for which J_z is disconnected. Then X is open and $p^{-1}(X) \subset X$.*

Proof. This follows from Proposition 4.3 and the continuity of G and G_p, because if $z \in \mathbf{C}$ and $(z, w) \in C_2$, then for every z' close to z there exists a w' close to w such that $(z', w') \in C_2$. □

Remark 4.5. For a polynomial map p of $\mathbf{C}$ it is true that J_p is either connected or has uncountably many components. For the sets J_z, the situation is different. Consider for instance $f(z, w) = (z^2, w^2 + z - 1)$. Then $J_p = \{|z| = 1\}$ and $J_1 = \{|w| = 1\}$. Hence $J_{-1} = q_{-1}^{-1}(J_1) = \{|w^2 - 2| = 1\}$ has exactly two components.

Suppose that J_z is connected for some $z \in \mathbf{C}$. Then φ_z maps $\mathbf{C} - K_z$ conformally onto $\mathbf{C} - \bar{\mathbf{D}}$. Let ψ_z be the inverse of φ_z, mapping $\mathbf{C} - \bar{\mathbf{D}}$ conformally onto $\mathbf{C} - K_z$. Note that if we extend ψ_z to ∞, then ψ_z is the Riemann map onto $\hat{\mathbf{C}} - K_z$. The function $\psi_z(\zeta)/\zeta$ is bounded, so the radial limits $\lim_{r \to 1} \psi_z(re^{i\theta})$ exist for a.e. $\theta \in S^1$ and the extension of ψ_z to S^1 is measurable and maps S^1 into J_z.

Proposition 4.6. *If J_z is connected, then $(\psi_z)_*(d\theta/2\pi) = \mu_z$.*

Proof. This is true because ψ_z is the Riemann map and μ_z is harmonic measure on K_z. □

In general there is no reason why ψ_z should extend continuously to S^1 or, equivalently, why J_z should be locally connected. It is perfectly possible for the set J_z not to be locally connected (consider J_{-1} for $f(z, w) = (-z^2, w^2 + c)$, where the Julia set of $w \to w^2 + c$ is not locally connected). On the other hand we will prove in section 6 that if f is expanding on J_2, then ψ_z *does* extend continuously to S^1 for all $z \in J_p$.

There are several ways of characterizing polynomial maps of $\mathbf{C}$ with connected Julia sets. As we will see, there is a natural generalization of this to skew products on $\mathbf{C}^2$. The relevant property for a skew product will be called property C and it means that J_p is connected and J_z is connected for all $z \in J_p$. We therefore try to understand when a map has this property. Let us start with the connectedness of J_p. The following result is really one-dimensional.

Proposition 4.7. *If f is a skew product on $\mathbf{C}^2$ of degree $d \geq 2$, then the following statements are equivalent.*

(i) $G_p = 0$ *on* C_1.
(ii) J_p *is connected.*

(iii) $\lambda_1 = \log d$.

Proof. The equivalence of (i) and (ii) was already mentioned in section 1 and Corollary 2.8 shows that (iii) is equivalent to (i). $\square$

From Proposition 4.3 we see that the question whether J_z is connected is related to the function $G - G_p$ on C_2. Let $\bar{C}_2$ be the closure of C_2 in $\mathbf{P}^2$ and let

$$\Omega_p = ((\mathbf{C} - K_p) \times \mathbf{C}) \cup (\Pi - [0:1:0]).$$

Then Ω_p is open in $\mathbf{P}^2$.

Lemma 4.8. *The function $G - G_p$ extends to a continuous plurisubharmonic function on Ω_p.*

Proof. $G - G_p$ is clearly continuous and plurisubharmonic on $(\mathbf{C} - K_p) \times \mathbf{C}$. Take any point $[z_0 : w_0 : 0] \in \Pi$ with $z_0 \neq 0$. We recall some facts from [BJ] (in turn adapted from [FS1]). The functions $\tilde{G}(z,w,t) := G(z,w) + \log|t|$ and $\tilde{G_p}(z,w,t) := G_p(z) + \log|t|$, defined for $t \neq 0$, extend to plurisubharmonic functions on $\mathbf{C}^3 - 0$ which are logarithmically homogeneous of degree 1, i.e.

$$\tilde{G}(\lambda x) - \tilde{G}(x) = \tilde{G_p}(\lambda x) - \tilde{G_p}(x) = \log|\lambda|$$

for all $\lambda \in \mathbf{C}^*$. For (z,w) close to $[z_0 : w_0 : 0]$ we therefore have

$$\begin{aligned} G(z,w) - G_p(z,w) &= \tilde{G}(z,w,1) - \tilde{G_p}(z,w,1) \\ &= \tilde{G}(1, w/z, 1/z) - \tilde{G_p}(1, w/z, 1/z). \end{aligned} \tag{4.3}$$

Now G_p is pluriharmonic on $\pi^{-1}(\Omega_p)$, where $\pi : \mathbf{C}^3 - 0 \to \mathbf{P}^2$ is the projection, so the right hand side of (4.3) defines a continuous plurisubharmonic function in a neighborhood of $[z_0 : w_0 : 0]$. This completes the proof. $\square$

Proposition 4.9. *If f is a skew product on $\mathbf{C}^2$ of degree $d \geq 2$, then the following statements are equivalent.*

(i) $G = 0$ *on* $C_2 \cap (J_p \times \mathbf{C})$.
(ii) $G - G_p = 0$ *on* $\bar{C}_2$.
(iii) J_z *is connected for all* $z \in J_p$.
(iv) J_z *is connected for all* $z \in \mathbf{C}$ *and* J_Π *is connected.*
(v) $\lambda_2 = \log d$.

Proof. Clearly (ii) implies (i). Suppose that $G = 0$ on $C_2 \cap (J_p \times \mathbf{C})$. By Lemma 4.8 we can apply the maximum principle to $G - G_p$ on $\bar{C}_2 \cap \Omega_p$ to see that $G - G_p = 0$ there. Similarly we see that $G - G_p = 0$ on $C_2 \cap (\mathrm{int}(K_p) \times \mathbf{C})$, so $G - G_p = 0$ on $\bar{C}_2$. Hence (i) and (ii) are equivalent. Trivially (iv) implies (iii) and from Proposition 4.3 we know that (i) is equivalent to (iii). By Corollary 2.8 (v) is equivalent to (i). We complete the proof by showing that (ii) implies (iv). If $G - G_p = 0$ on C_2, then we know by Proposition 4.3 that J_z is connected for all $z \in \mathbf{C}$. Also, (ii) implies that $\bar{C}_2 \cap \Pi \subset K_\Pi$ so J_Π is connected. Hence (ii) implies (iv) and we are done. $\square$

We now combine Proposition 4.7 and Proposition 4.9.

Theorem 4.10. *If f is a skew product on $\mathbf{C}^2$ of degree $d \geq 2$, then the following statements are equivalent.*

(i) $G_p = 0$ *on* C_1 *and* $G = 0$ *on* $C_2 \cap (J_p \times \mathbf{C})$.
(ii) $G_p = 0$ *on* C_1 *and* $G - G_p = 0$ *on* $\bar{C}_2$.
(iii) J_p *is connected and* J_z *is connected for all* $z \in J_p$.
(iv) J_p *is connected,* J_Π *is connected and* J_z *is connected for all* $z \in \mathbf{C}$.
(v) $\lambda_1 = \lambda_2 = \log d$.

Proof. Everything is a consequence of Proposition 4.7 and Proposition 4.9. □

Definition 4.11. We say that a skew product on $\mathbf{C}^2$ of degree $d \geq 2$ has property C if f satisfies conditions (i)–(v) above.

Proposition 4.12. *If the skew product f has property C, then J_2 is connected.*

Proof. If f has property C, then, by definition, J_p is connected and J_z is connected for all $z \in J_p$. Suppose U is a nonempty closed and open subset of J_2. For $z \in J_p$ define $U_z := \{w \in J_z; (z,w) \in U\}$. Then U_z is closed and open in J_z so for each $z \in J_p$ we have $U_z = \emptyset$ or $U_z = J_z$. Let $A = \{z \in J_p; U_z = J_z\}$. Then A is open, closed and nonempty so $A = J_p$. Hence $U \supset \cup_{z \in J_p} \{z\} \times J_z$ and since U is closed it follows from Proposition 3.2 that $U = J_2$. □

Question 4.13. *Is there a skew product on $\mathbf{C}^2$ such that J_2 is connected but f does not have property C?*

5. Compactness of the connectedness locus

For polynomial maps of $\mathbf{C}$ there is an interesting interplay between dynamical space and parameter space. Of special interest is the connectedness locus, corresponding to polynomials with connected Julia sets.

In this section we define a connectedness locus M_d for skew products on $\mathbf{C}^2$ of degree d and prove that M_d is compact. The proof goes along the same lines as in [BH], where the authors show that the connectedness locus for polynomial maps of $\mathbf{C}$ of degree $d \geq 2$ is compact (indeed, much more is proved in [BH]).

First we have to specify what our parameter space is. A general skew product of degree $d \geq 2$ can be written as

$$f(z,w) = (\sum_{i \leq d} a_i z^i, \sum_{i+j \leq d} b_{i,j} z^i w^j),$$

where $a_d = b_{0,d} = 1$. After a linear change of coordinates we may assume that $a_{d-1} = b_{1,d-1} = 0$. This leaves us with $N = N(d) = (d-1) + ((d+1)(d+2)/2 - 2)$ parameters. We write $f = f_{a,b}$, where a and b is the vector of a_i's and $b_{i,j}$'s, respectively.

Definition 5.1. The connectedness locus M_d is the set of $(a,b) \in \mathbf{C}^N$ such that $f_{a,b}$ has property C.

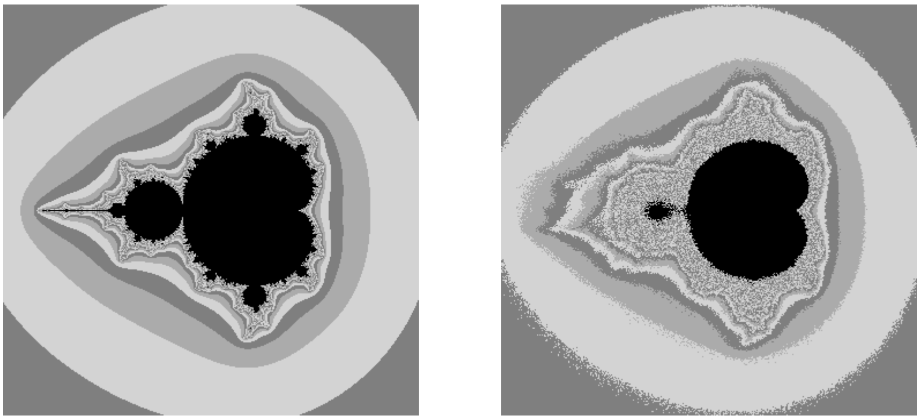


FIGURE 2. Slices of M_2.

Theorem 5.2. *M_d is a compact subset of $\mathbf{C}^N$.*

Proof. From (i) in Theorem 4.10 it follows that M_d is closed so it suffices to show that M_d is bounded. We will let A be an unspecified positive constant. Suppose that $f_{a,b}$ has property C. Then J_p is connected so by [BH] we have that $|a_i| \leq A$ for all i. From Theorem 4.10 we know that $G = 0$ on $C_2 \cap (K_p \times \mathbf{C})$ so ψ_z is defined and maps $\hat{\mathbf{C}} - \bar{\mathbf{D}}$ conformally onto $\hat{\mathbf{C}} - K_z$ for all $z \in K_p$. By Koebe's one-quarter theorem we have $K_z \subset \mathbf{D}_A$ for all $z \in K_p$. In particular, the critical points and critical values of q_z are in $\mathbf{D}_A$ for all $z \in K_p$. Therefore, if we write

$$q_z(w) = w^d + c_{d-1}(z)w^{d-1} + \cdots + c_0(z),$$

then $|c_i| \leq A$ on K_p for all i (after increasing A). It follows from the maximality of the Green function that $|c_i(z)| \leq A\exp(dG(z))$ for $z \in \mathbf{C}$. By increasing A and using Cauchy's estimates we get that $|c_i^{(j)}(0)| \leq Aj!$ for all i, j with $i+j \leq d$. Hence $|b_{i,j}| \leq A$ and we are done. □

It would be interesting to know whether M_d is connected. Figure 2 shows two slices of M_2. A general skew product of degree 2 has the form

$$f(z,w) = (z^2 + a_0, w^2 + b_{0.2}z^2 + b_{0,1}z + b_{0,0}).$$

The left hand picture shows the intersection of M_2 with the line $\{a_0 = b_{0,2} = b_{0,0} = 0\}$. This is the usual Mandelbrot set. In the right hand picture the line is $\{b_{0,2} = b_{0,0} = 0, a_0 = 0.15i\}$. It is difficult to see if the intersection of M_2 with this line is connected.

6. EXPANSION OF f ON J_2

It is well known that a rational function h is expanding on its Julia set J_h if and only if the postcritical set does not accumulate on J_h. In Theorem 6.3

we prove the corresponding result for a skew product f, with J_h replaced by $J_2 = \mathrm{supp}(\mu)$.

If f is any skew product on $\mathbf{C}^2$, then we will denote the closure of the post-critical set of f by PC. In other words

$$PC = \overline{\bigcup_{n \geq 1} f^n(C)}$$

We start by proving the following result. The proof follows section 3 in [U2] to some extent.

Proposition 6.1. *Let f be a skew product on $\mathbf{C}^2$ such that $J_2 \cap PC = \emptyset$. Then $z \to J_z$ is upper semicontinuous on J_p.*

Before giving the proof let us introduce some notation. If $z \in J_p$, then PC_z is the set of $w \in \mathbf{C}$ such that $(z, w) \in PC$. Moreover $B(w, \delta)$ denotes the disk in $\mathbf{C}$ centered at w with radius δ.

Proof. Suppose that $J_2 \cap PC = \emptyset$ but $z \to J_z$ is not upper semicontinuous at $z_0 \in J_p$. Then by Proposition 3.2 there exists a point $w_0 \in \mathbf{C} - J_{z_0}$ such that $(z_0, w_0) \in J_2$. Clearly $G(z_0, w_0) = 0$ so $w_0 \in \mathrm{int}(K_{z_0})$. We may find an increasing sequence $\{n_j\}$ such that $R_j := q_{z_0}^{(n_j)}$ is uniformly convergent in a neighborhood of w_0 and $(z_j, w_j) := f^{n_j}(z_0, w_0)$ converges to $(z_\infty, w_\infty) \in J_2$.

By assumption there exists a $\delta > 0$ such that $B(w_j, 2\delta) \cap PC_{z_j} = \emptyset$ for all j. We may assume that $|w_j - w_\infty| < \delta$ for all j. Then $B(w_\infty, \delta) \cap PC_{z_j} = \emptyset$ for all j. Define holomorphic functions $g_j : B(w_\infty, \delta) \to \mathbf{C}$ such that $R_j \circ g_j = \mathrm{id}$ and $g_j(w_j) = w_0$. Then $\{g_j\}$ is a normal family on $B(w_\infty, \delta)$.

Let U be a compact neighborhood of w_0 on which $\{R_j\}$ is uniformly convergent. By decreasing δ and passing to a further subsequence we may assume that $g_j(B(w_\infty, \delta)) \subset U$ for all j. Hence $R_j(U) \supset B(w_\infty, \delta)$.

Define $S_{i,j} := q_{z_i}^{(n_j - n_i)}$ for $1 \leq i < j$. We claim that $S_{i,j} \to \mathrm{id}$ uniformly on $B(w_\infty, \delta)$ as $i \to \infty$. Indeed, if $w \in B(w_\infty, \delta)$, then $w = R_i(\xi_i)$ for some $\xi_i \in U$. Hence $|S_{i,j}(w) - w| = |R_j(\xi_i) - R_i(\xi_i)|$, which is small if i is large.

Let V be the set where $S_{i,j} \to \mathrm{id}$ locally uniformly as $i \to \infty$. More precisely, $w \in V$ iff there exists an $\epsilon > 0$ such that

$$\limsup_{i \to \infty} \sup_{j > i} \sup_{|\xi - w| \leq \epsilon} |S_{i,j}(\xi) - \xi| = 0. \tag{6.1}$$

Then V is open and $B(w_\infty, \delta) \subset V$ by the calculation above.

Pick a large number $R > 0$ such that if $|w| > R$, then $|q_z(w)| \geq 3|w|$ for all $z \in J_p$. We then have $V \subset \bar{\mathbf{D}}_R$. Take any point $x \in \partial V$. We claim that $(z_\infty, x) \in J_2$. To prove this it is, in view of Proposition 3.2, sufficient to find, for each $\gamma > 0$, an increasing sequence $\{i_k\}$ such that $d(w_{i_k}, J_{z_{i_k}}) < \gamma$ for all k. Fix $\gamma > 0$ and pick $\xi \in V$ with $|\xi - x| < \gamma$. It is clear from (6.1) that $\xi \in K_{z_i}$ for all sufficiently large i. On the other hand we claim that we can find sequences $\{i_k\}$,

$\{j_k\}$, $\{\xi_k\}$ with $i_k \to \infty$, $j_k > i_k$ such that $|\xi_k - x| \leq \gamma/2$ and $|S_{i_k,j_k}(\xi_k)| > R$ for all k. If this was not true, then we would have

$$\limsup_{i\to\infty} \sup_{j>i} \sup_{|\xi-x|\leq\gamma/2} |S_{i,j}(\xi)| \leq R. \tag{6.2}$$

Hence any sequence $\{S_{i_k,j_k}\}_{k\geq 0}$ would be normal on $B(x,\gamma/3)$ and by (6.1) must converge to the identity there. It would follow that

$$\limsup_{i\to\infty} \sup_{j>i} \sup_{|\xi-x|\leq\gamma/3} |S_{i,j}(\xi) - \xi| = 0, \tag{6.3}$$

so $B(x,\gamma/3) \subset V$, contradicting our assumption that $x \in \partial V$.

It follows that we may find $i_k \to \infty$, $j_k > i_k$ and $\xi_k \in B(x,\gamma/2)$ such that $|S_{i_k,j_k}(\xi_k)| > R$ for all k. Then $\xi_k \in \mathbf{C} - K_{z_{i_k}}$. We conclude that $d(x, K_{z_{i_k}}) < \gamma$ and $d(x, \mathbf{C} - K_{z_{i_k}}) < \gamma$, so $d(x, J_{z_{i_k}}) < \gamma$ for all large k. This completes the proof of the claim that $(z_\infty, x) \in J_2$.

By our choice of δ all branches of $S_{i,j}^{-1}$ are defined and holomorphic on $B(x,\delta)$ for all i,j.

Pick open sets V_1,V_2 with $w_\infty \in V_2 \subset\subset V_1 \subset\subset V$ and $B(x,\delta/4) \cap V_2 \neq \emptyset$. If i is large enough, then $S_{i,j}$ is close to the identity on V_1 for all $j > i$. In particular $S_{i,j}$ is biholomorphic on V_1 and $S_{i,j}(V_1) \supset V_2$. Thus there exist holomorphic functions $h_{i,j}$ on V_2 such that $h_{i,j}(V_2) \subset V_1$, $S_{i,j} \circ h_{i,j} = \mathrm{id}$ on V_2 and $h_{i,j} \circ S_{i,j} = \mathrm{id}$ on $S_{i,j}^{-1}(V_2) \cap V_1$. By (6.1) we have that

$$\limsup_{i\to\infty} \sup_{j>i} \sup_{\xi\in V_2} |h_{i,j}(\xi) - \xi| = 0. \tag{6.4}$$

Now choose branches $g_{i,j}$ of $S_{i,j}^{-1}$ on $B(x,\delta)$ such that $g_{i,j} = h_{i,j}$ on $B(x,\delta) \cap V_2$ for large i. We claim that

$$\limsup_{i\to\infty} \sup_{j>i} \sup_{|\xi-x|\leq\delta/3} |g_{i,j}(\xi) - \xi| = 0. \tag{6.5}$$

If (6.5) was not true, then we could find $i_k \to \infty$, $j_k > i_k$ and $\xi_k \in B(x,\delta/2)$ such that $|g_{i_k,j_k}(\xi_k) - \xi_k| \geq c > 0$ for all k. But this is a contradiction, because $\{g_{i_k,j_k}\}$ is normal on $B(x,\delta)$ and $g_{i_k,j_k} \to \mathrm{id}$ uniformly on $B(x,\delta/2) \cap V_2$. Therefore (6.5) holds, which implies that $B(x,\delta/4) \subset V$, again contradicting $x \in \partial V$. $\square$

Corollary 6.2. *If $PC \cap J_2 = \emptyset$, then*

$$J_2 = \bigcup_{z\in J_p} J_z.$$

Proof. This follows from Proposition 3.2 and Proposition 6.1. $\square$

We now come to the main result of this section.

Theorem 6.3. *Let f be any skew product on $\mathbf{C}^2$. Then f is expanding on J_2 iff J_2 is disjoint from the closure PC of the postcritical set of f.*

Proof. First assume that f is expanding on J_2. Then $J_2 \cap C = \emptyset$ and we can find a neighborhood U of J_2 such that $f^{-1}(U) \subset U$ and $U \cap C = \emptyset$. It then follows that $U \cap PC = \emptyset$.

Now assume that $PC \cap J_2 = \emptyset$. From Proposition 6.1 and Corollary 6.2 we know that $J_2 = \cup_{z\in J_p} J_z$ and that $z \to J_z$ is continuous on J_p. The key step is to prove the following lemma.

Lemma 6.4. *There exists an $l \geq 1$ such that*

$$\left|\frac{\partial}{\partial w} q_z^{(l)}(w)\right| \geq 2$$

for all $(z, w) \in J_2$.

We postpone the proof of Lemma 6.4 and show instead how to prove that f is expanding on J_2.

The condition $PC \cap J_2 = \emptyset$ implies that the closure of the postcritical set of p is disjoint from J_p and this, in turn, implies that p is expanding on J_p. By Lemma 6.4 we may therefore find an $l \geq 1$ such that

$$\left|\frac{\partial}{\partial w} q_z^{(l)}(w)\right| \geq 2, \qquad \left|\frac{\partial}{\partial z} p^{(l)}(z)\right| \geq 2 \tag{6.6}$$

for all $(z, w) \in J_2$.

Let $(z_i, w_i)_{i\geq 0}$ be any orbit under f^l in J_2, i.e. $f^l(z_i, w_i) = (z_{i+1}, w_{i+1})$ for all $i \geq 0$. Define

$$X_i = \left(Df^l(z_i, w_i)\right)^{-1},$$

for $i \geq 0$ and

$$Y_j = \left(Df^{jl}(z_i, w_i)\right)^{-1} = X_0 \dots X_{j-1}$$

for $j \geq 0$. We write

$$X_i = \begin{pmatrix} a_i & 0 \\ b_i & c_i \end{pmatrix}.$$

Then (6.6) implies that $|a_i|, |c_i| \leq 1/2$ and $|b_i| \leq B$ for all i, where B is independent of i and of the orbit (z_i, w_i).

Similarly we write

$$Y_j = \begin{pmatrix} A_j & 0 \\ B_j & C_j \end{pmatrix}.$$

Then $|A_j|, |C_j| \leq 2^{-j}$. We first prove inductively that $|B_j| \leq B$ for all $j \geq 0$. This is clear for $j = 0$ and for $j \geq 0$ we have $B_{j+1} = a_j B_j + b_j C_j$, so $|B_{j+1}| \leq (|B_j| + B)/2$.

Let k be so large that $\max(1, B) \leq 2^{k-5/2}$. Then $Y_{2k} = Y_k Y_k'$, where

$$Y_k' = X_k \dots X_{2k-1} = \begin{pmatrix} A_i' & 0 \\ B_i' & C_i' \end{pmatrix}.$$

It follows that

$$|A_{2k}| = |A_k A'_k| \le 2^{-2k} \le 2^{-5/2},$$
$$|C_{2k}| = |C_k C'_k| \le 2^{-2k} \le 2^{-5/2}$$

and

$$|B_{2k}| = |B_k A'_k + C_k B'_k| \le 2^{1-k} B \le 2^{-5/2}.$$

From an easy calculation it now follows that the norm of the matrix $Y_{2k} = (Df^{2kl}(z_0, w_0))^{-1}$ is less than $1/2$. Theorem 6.3 follows, because k and l are independent of the point $(z_0, w_0) \in J_2$. $\square$

Proof of Lemma 6.4. Let $R > 0$ be so large that $|q_z(w)| \ge 2|w|$ if $|w| > R$ and $z \in J_p$. Clearly $J_z \subset \bar{\mathbf{D}}_R$ for all $z \in J_p$. Pick $\epsilon \in (0, 1/2)$ so small that $2\epsilon < d(J_2, PC)$.

Let $\mathcal{D}_0$ be the collection of all disks U centered at points $c(U)$, where $c(U) \in J_z$ for some $z \in J_p$. If $z \in J_p$, $n \ge 1$ and $U \in \mathcal{D}_0$ is a disk centered at $c(U) \in J_{p^n(z)}$, then all branches of $(q_z^{(n)})^{-1}$ are holomorphic and univalent on $2U$, where $2U$ denotes the disk of radius 2ϵ centered at $c(U)$. Let $\mathcal{D}_n$ be the collection of all preimages of disks $U \in \mathcal{D}_0$ under all these branches.

Each $U \in \mathcal{D}_n$ has a natural center $c(U)$ which is mapped to the center of the corresponding disk in $\mathcal{D}_0$. Let $r(U) := \sup\{|w - c(U)|; w \in U\}$ be the radius of U. By Koebe distortion we have that

$$B(c(U), r(U)/4) \subset U \subset B(c(U), r(U)), \tag{6.7}$$

for all $U \in \mathcal{D}_n$ and all $n \ge 0$.

To prove the lemma it is sufficient to show that

$$\sup\{r(U); U \in \mathcal{D}_n\} \to 0 \text{ as } n \to \infty, \tag{6.8}$$

because then we can find an $l \ge 1$ such that $r(U) \le \epsilon/2$ for all $U \in \mathcal{D}_l$. By Schwarz's Lemma this implies that

$$\left| \frac{\partial}{\partial w} q_z^{(l)}(w) \right| \ge 2$$

for all $z \in J_p$ and all $w \in J_z$. This inequality and Corollary 6.2 together imply Lemma 6.4.

Hence it suffices to show (6.8) and we argue by contradiction. If (6.8) does not hold, then there exists a $\delta > 0$, a sequence $n_j \to \infty$ and elements $U_j \in \mathcal{D}_{n_j}$ with $r(U_j) \ge 8\delta$ for all j. By (6.7) this implies that there exist $z_j \in J_p$, $w_j \in J_{z_j}$ and euclidean disks $\Delta_j := B(w_j, 2\delta)$ such that

$$q_{z_j}^{(n_j)}(\Delta_j) \subset \mathbf{D}_{R+1} \tag{6.9}$$

for all j. By passing to a subsequence we may assume that (z_j, w_j) converges to $(z_\infty, w_\infty) \in J_2$ and that $\Delta_j \supset \Delta$ for all j, where $\Delta = B(w_\infty, \delta)$. Hence

$$q_{z_j}^{(k)}(\Delta) \subset \mathbf{D}_{R+1}$$

for all $k \leq n_j$ and $j \leq 1$. Now $B(z_\infty, \delta/2) \cap J_{z_j} \neq \emptyset$ for j large enough, so by upper semicontinuity of $z \to J_z$ we have $\Delta \cap J_{z_\infty} \neq \emptyset$. But then there exists an $n \geq 1$ such that

$$q_{z_\infty}^{(n)}(\Delta) \cap (\mathbf{C} - \mathbf{D}_{R+2}) \neq \emptyset.$$

By continuity and by the choice of R we see that

$$q_{z_j}^{(k)}(\Delta) \cap (\mathbf{C} - \mathbf{D}_{R+2}) \neq \emptyset,$$

for $k \geq n$ and j large enough. This contradicts (6.9). □

Corollary 6.5. *If f is expanding on J_2, then $z \to J_z$ is continuous on J_p and*

$$J_2 = \bigcup_{z \in J_p} J_z.$$

Corollary 6.5 generalizes a previous result by Heinemann. To see this, we recall how Heinemann defines the Julia set, which we will denote by J^*. We say that $x \notin J^*$ if there exists a neighborhood U of x and for each $y \in U$ a one-dimensional analytic set $\gamma \ni y$ such that $\{f^n|_\gamma\}$ is normal. Note that Heinemann works with the one-point compactification $\mathbf{C}^2 \cup \{\infty\}$ of $\mathbf{C}^2$. Hence $J^* \subset K$. It is not clear to the author whether $J^* = J_2$ in general, but equality holds under additional assumptions.

Proposition 6.6. *Suppose f is a skew product on $\mathbf{C}^2$ such that f is expanding on J_2 and $q_z^{(n)}$ is expanding on J_z for all attracting periodic points of p (where n is the period of z). Then $J^* = J_2$.*

Sketch of Proof. Let us say that $x \in J'$ if there exists no analytic set containing x on which $\{f^n\}$ is normal. Then J^* is the closure of J'. We claim that $J' = J_2$. To see this, write $x = (z, w)$. If $z \in \mathbf{C} - K_p$, then $x \notin J'$ by the remark above. If $z \in J_p$, then $x \in J'$ iff $w \in J_z$. Finally, if $z \in \operatorname{int}(K_p)$, then $(z, w) \notin J'$. [The assumptions in the proposition imply that $\{(z, w); z \in \operatorname{int}(K_p); w \in J_z\}$ is a union of stable manifolds of saddle points. Using these stable manifolds we see that the above set does not intersect J'. Also, $\{(z, w); z \in \operatorname{int}(K_p); w \in \operatorname{int}(K_z)\} = \operatorname{int}(K)$ and $\{(z, w); z \in \operatorname{int}(K_p); w \notin K_z\} \cap K = \emptyset$.] Hence $J' = J_2$, so $J^* = J_2$. □

In [H1] Heinemann studied so called "Cantor skews". These are quadratic skew products of the type $f(z, w) = (z^2 + c, w^2 + k(z))$, where the Julia set of $z^2 + c$ is disconnected and $\sup_{z \in J_p} |k(z)| < \frac{1}{4}$. Using Theorem 6.3 it is not difficult to see that a Cantor skew satisfies the assumptions in Proposition 4.9. Hence it follows from Corollary 6.5 that $J^* = \bigcup_{z \in J_p} \{z\} \times J_z$. This generalizes

Theorem 3.2 in [H1] (it should be mentioned that Heinemann also proved that the sets J_z, $z \in J_p$ are Jordan curves).

We now turn to some consequences of expansion on J_2. First recall the result by Sibony (see [CG]) that if p is a polynomial map of $\mathbf{C}$, then its Green function G_p is Hölder continuous. Moreover, if p is expanding on J_p, then G_p can be bounded from above and below on $\mathbf{C} - K_p$ in terms of the distance to J_p. The next result generalizes this to skew products.

Proposition 6.7. *If f is expanding on J_2, then there are constants $\alpha_1, \alpha_2 > 0$ and $C_1, C_2 > 0$ such that*

$$C_1 R_z(w)^{\alpha_1} \leq d(w, J_z) \leq C_2 R_z(w)^{\alpha_2} \tag{6.10}$$

for all $(z, w) \in J_p \times \mathbf{C}$ with $0 < G(z, w) \leq 1$. The left hand inequality in (6.10) *is valid without the assumption that f is expanding on J_2.*

Proof. It follows from Corollary 6.5 that the boundary of $(J_p \times \mathbf{C}) \cap K$ in $J_p \times \mathbf{C}$ is J_2. Since f is expanding on J_2 there exists an $l \geq 1$, a neighborhood U of J_2 in $J_p \times \mathbf{C}$ and a constant $A > 0$ such that

$$2d(w, J_z) \leq d(w_l, J_{z_l}) \leq Ad(w, J_z) \tag{6.11}$$

for all $(z, w) \in U$, where $(z_l, w_l) = f^l(z, w)$. Choose α_1, α_2 so that $d^{\alpha_1 l} \geq A$ and $d^{\alpha_2 l} \leq 2$. Let R be so small that $(J_p \times \mathbf{C}) \cap \{0 < G \leq R\} \subset U$ and choose C_1, C_2 so that (6.10) holds on $(J_p \times \mathbf{C}) \cap \{R \leq G \leq 1\}$. It then follows from (6.11) and the fact that $R_{z_l}(w_l) = d\, R_z(w)$ that (6.10) holds on $(J_p \times \mathbf{C}) \cap \{0 < G \leq 1\}$. $\square$

Suppose that J_z is connected for all $z \in J_p$. Then the Böttcher coordinates φ_z are defined on all of $\mathbf{C} - K_z$ and their inverses ψ_z map $\mathbf{C} - \bar{\mathbf{D}}$ onto $\mathbf{C} - K_z$ for all $z \in J_p$. The following Theorem shows that ψ_z extend continuously to S^1 if f is expanding on J_2.

Theorem 6.8. *If f is expanding on J_2 and J_z is connected for all $z \in J_p$, then ψ_z extends Hölder continuously to $\mathbf{C} - \mathbf{D}$ for all $z \in J_p$. More precisely, there exist constants $C > 0$ and $\alpha > 0$ such that*

$$|\psi_z(\zeta) - \psi_z(\zeta')| \leq C|\zeta - \zeta'|^{\alpha} \tag{6.12}$$

in the spherical metric on $\hat{\mathbf{C}}$ for $\zeta, \zeta' \in \hat{\mathbf{C}} - \mathbf{D}$ and $z \in J_p$. The map $(z, \zeta) \to (z, \psi_z(\zeta))$ is continuous on $J_p \times (\mathbf{C} - \mathbf{D})$ and maps $J_p \times S^1$ onto J_2.

Proof. It is sufficient to prove the statement for an iterate of f. Corollary 6.5 implies that the boundary of $(J_p \times \mathbf{C}) \cap K$ in $J_p \times \mathbf{C}$ is J_2. We may therefore choose $R > 0$ such that

$$\left|\frac{\partial q_z}{\partial w}\right| \geq 2 \tag{6.13}$$

if $z \in J_p$ and $0 < R_z(w) \leq R$. Let $\alpha > 0$ be so small that $d^{\alpha} < 2$. We may assume that $d^{\alpha} R^{d-1} \leq 2$. Recall that ψ_z satisfies

$$\psi_z(\zeta) = g(\psi_{p(z)}(\zeta^d)) \tag{6.14}$$

for $|\zeta| > 1$, where g is a suitable branch of $(q_z)^{-1}$. By differentiating (6.14) and using the estimate (6.13) we see that

$$|D\psi_z(\zeta)| \leq \frac{d}{2}|D\psi_{p(z)}(\zeta^d)||\zeta|^{d-1} \tag{6.15}$$

for $1 < |\zeta| \leq e^{R/d}$. Define

$$m(r) = \sup_{z \in J_p} \sup_{|\zeta|=r} |D\psi_z(\zeta)|$$

for $r > 1$. Then there exists a constant $C' < \infty$ such that

$$m(r) \leq C'(r-1)^{\alpha-1} \tag{6.16}$$

for $e^{R/d} \leq r \leq e^R$. Using (6.15) we see inductively that (6.16) holds for $1 < r \leq e^R$. By integrating (6.16) we see that ψ_z extends continuously to $\mathbf{C} - \mathbf{D}$ for all $z \in J_p$ and that (6.12) holds. That $(z, \zeta) \to \psi_z(\zeta)$ is continuous is a consequence of (6.12) and the fact that $(z, \zeta) \to \psi_z(\zeta)$ is continuous on $J_p \times \{|\zeta| = r\}$ for all $r > 1$. Finally $(z, \zeta) \to (z, \psi_z(\zeta))$ maps $J_p \times S^1$ onto J_2 by Proposition 4.6 and Corollary 6.5. □

Corollary 6.9. *If f is expanding on J_2 and has property C, then the sets J_p, J_2 and J_z for $z \in J_p$ are all connected and locally connected.*

Proof. The assumptions imply that p is expanding on J_2, that J_p is connected and that J_z is connected for all $z \in J_p$. Hence J_p is locally connected [CG] and we have continuous surjective maps from S^1 to J_z for $z \in J_p$ and from $J_p \times S^1$ to J_2. Hence the sets J_z for $z \in J_p$ and J_2 are all connected and locally connected. □

Acknowledgment. This work was prepared when the author was visiting the University of Michigan and he thanks the Department of Mathematics for its hospitality. Special thanks go to Eric Bedford. Several results in this paper were obtained jointly with him and his comments and remarks had a very definite influence on the form and contents of the paper.

References

[BJ] Bedford, E., Jonsson, M. Regular polynomial endomorphisms of $\mathbf{C}^N$. Preprint.

[B] Briend, J. Exposants de Liapounoff des endomorphismes holomorphes de $\mathbf{CP}^k$. Preprint.

[BH] Branner, B, Hubbard, J. H. The iteration of cubic polynomials. I. The global topology of parameter space. *Acta Math.* **160**(1988), 143–206.

[CG] Carleson, L., Gamelin, T. W. *Complex Dynamics.* Springer-Verlag, 1993.

[FS1] Fornæss, J.E., Sibony, N. Complex dynamics in higher dimension. In Gauthier, P. M., Sabidussi, G., editor, *Complex Potential Theory*, pages 131–186. Kluwer Academic Publishers, 1994.

[FS2] Fornæss, J.E., Sibony, N. Complex dynamics in higher dimension. II. In Bloom, T. et al, editor, *Modern Methods in Complex Analysis*, number 137 in Annals of Mathematics Studies, pages 135–182. Princeton University Press, 1995.

[H1] Heinemann, S.-M. Julia sets for holomorphic endomorphisms of $\mathbf{C}^n$. *Ergodic Theory Dynam. Systems* **16**(1996), 1275–1296.

[H2] Heinemann, S.-M. *Dynamische Aspekte holomorpher Abbildungen in* $\mathbf{C}^n$. PhD thesis, Universität Göttingen, 1994.

[HP] Hubbard, J.H, Papadopol, P. Superattractive fixed points in $\mathbf{C}^n$. *Indiana Univ. Math. J.* **43**(1994), 321–365.

[P] Przytycki, F. Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map. *Invent. Math.* **80**(1985), 161–179.

[U1] Ueda, T. Fatou sets in complex dynamics on projective spaces. *J. Math. Soc. Japan* **46**(1994), 545–555.

[U2] Ueda, T. Critical orbits of holomorphic maps on projective spaces. *To appear in J. Geom. Anal.*

[U3] Ueda, T. Personal communication.

[Y] Young, L.S. Ergodic theory of differentiable dynamical systems. In Branner, B., Hjorth, P., editor, *Real and complex dynamical systems*, pages 293–336. Kluwer Academic Publishers, 1995.

Dept. of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden
E-mail address: mjo@math.kth.se

Paper V

SUMS OF LYAPUNOV EXPONENTS FOR SOME POLYNOMIAL MAPS OF $\mathbf{C}^2$

MATTIAS JONSSON

Abstract. We give a formula for the sum of the Lyapunov exponents of a nondegenerate polynomial map f of $\mathbf{C}^2$ close to $(z,w) \to (z^d, w^d)$. The formula only involves the behavior of f at infinity. In particular, it follows that the sum only depends on the homogeneous part of f of degree d.

0. Introduction

Every holomorphic map f of $\mathbf{P}^k$ of degree at least two carries a natural invariant mixing measure of maximal entropy (see [FS3]). To this measure μ we can assign Lyapunov exponents, which measure the rate of growth of tangent vectors. It is a result by Briend [B] that the Lyapunov exponents of f are always nonnegative.

In this paper we study the case when f is a polynomial map of $\mathbf{C}^2$ of degree $d \geq 2$, which has an extension to a holomorphic map of $\mathbf{P}^2$. The measure μ above is then the complex equilibrium measure of the compact set K consisting of points with bounded orbits. We assume that f is close to the map $(z,w) \to (z^d, w^d)$. Let $\lambda_1 \geq \lambda_2 \geq 0$ be the Lyapunov exponents of f with respect to μ. The main purpose of this paper is to give a formula for $\lambda_1 + \lambda_2$, which measures the growth of the Jacobian of f^n.

The line Π at infinity is completely invariant under f and the restriction of f to Π is a rational map close to $\zeta \to \zeta^d$. There is a natural invariant measure ν for this rational map and we denote by $\lambda(f|_\Pi)$ the Lyapunov exponent of the restriction with respect to ν. Our main result in this paper is then (cf. Theorem 4.4)

Theorem. *If f is sufficiently close to the map $(z,w) \to (z^d, w^d)$, then the Lyapunov exponents of f satisfy*

$$\lambda_1 + \lambda_2 = \log d + \lambda(f|_\Pi). \tag{0.1}$$

Let us compare formula (0.1) with what is known about Lyapunov exponents in the one-variable setting. If $g(z) = z^d + \dots$ is a polynomial map of $\mathbf{C}$ of degree d, then the harmonic measure of the compact set K consisting of bounded orbits is again the natural invariant measure μ and the Green function of K satisfies

1991 *Mathematics Subject Classification.* Primary: 32H50, Secondary: 58F23.

Key words and phrases. Holomorphic dynamical systems, Lyapunov exponents.

$G = \lim_{n\to\infty} d^{-n} \log^+ |g^n|$. The Lyapunov exponent of g with respect to μ is given by the Brolin-Manning formula

$$\lambda = \log d + \sum_{g'(c)=0} G(c). \tag{0.2}$$

If g is close to the map $z \to z^d$, then all critical points have bounded orbits and the formula reduces to $\lambda = \log d$. In the one-dimensional case the object corresponding to Π is the point at infinity $\{\infty\}$ and it is reasonable to say that the Lyapunov exponent of the restriction of f to this point is zero. With this convention the two formulae (0.1) and (0.2) agree.

In one dimension, Lyapunov exponents provide information on parameter space. For instance, for the quadratic family $p_c(z) = z^2 + c$, the function $c \to \lambda(p_c)$ is proportional to the Green function of the Mandelbrot set. In higher dimension, parameter space is not well understood; one might hope that Lyapunov exponents could be a useful tool in its study.

The proof of Theorem 4.4 consists of two parts. First we prove an integral formula for $\lambda_1+\lambda_2$, which implies that the sum depends only on the homogeneous part of f of degree d. Then we prove Theorem 4.4 in the special case when f is homogeneous of degree d.

To state the integral formula mentioned above we need a few definitions. The Green function G of K is given by $G = \lim_{n\to\infty} d^{-n} \log^+ |f^n|$; this expression measures the superexponential rate of escape to infinity. Close to the line Π at infinity we have the following asymptotic expansion for G (see Proposition 1.1)

$$G(z,w) = \log |(z,w)| + \gamma[z:w] + o(1),$$

where γ is a continuous function on Π and $[z:w]$ denotes the projection of (z,w) on $\Pi \simeq \mathbf{P}^1$. Similarly, if $H := \log|\det Df|$ then we have the asymptotic formula

$$H(z,w) = 2(d-1)\log|(z,w)| + \delta[z:w] + o(1),$$

where δ is continuous on Π outside the critical points of $f|_\Pi$. The integral formula is then (cf. Theorem 3.2)

Theorem. *The Lyapunov exponents of f satisfy the relation*

$$\lambda_1 + \lambda_2 = \int_\Lambda (\delta - 2(d-1)\gamma)\, d\nu. \tag{0.3}$$

It is not too hard to see that the entities γ, δ and ν depend only on the homogeneous part of f of degree d. Hence the same is true for the sum of the Lyapunov exponents as well. One can check that a formula corresponding to (0.3) is valid in the one-dimensional case as well.

The main ingredient in the proof of formula (0.3) above is a geometric description of the current $T = dd^cG$ in the set $\mathbf{P}^2 - K$ consisting of points with unbounded orbits. The description says that T in this set has a global laminar structure, which we now describe. The line Π at infinity is completely invariant under f and the restriction of f to Π is, under our hypotheses, a hyperbolic rational map. The Julia set Λ of the restriction is therefore a hyperbolic set for f and there exists a natural invariant probability measure ν whose support is exactly Λ. In fact, ν is the restriction of T to Π. We prove in Proposition 2.8 that there exists an embedded analytic disk $W^s(p)$ with boundary on $\mathrm{supp}(\mu)$ through each point p of Λ, such that the different disks are pairwise disjoint and f maps $W^s(p)$ onto $W^s(f(p))$. The disk $W^s(p)$ is to be seen as the analytic continuation of the local superstable manifold at $p \in \Lambda$. We then prove that the action of the current T on a test form in the open set $\mathbf{P}^2 - K$ is given by

$$< T, \phi >= \int_\Lambda d\nu(p) \int_{W^s(p)} \phi. \tag{0.4}$$

In fact, the same formula is valid if ϕ is only supposed to be continuous and bounded (with respect to the Fubini-Study metric on $\mathbf{P}^2$) in a neighborhood of $\mathrm{supp}(T|_{\mathbf{P}^2-K})$. It is then not *a priori* clear that the right hand side is well defined. However, we prove in Proposition 2.8 that the disks $W^s(p)$ have uniformly bounded volume in the Fubini-Study metric so the expression in the right hand side does make sense.

Local laminarity results for the current T of more general hyperbolic endomorphisms of $\mathbf{P}^2$ are proven in [FS4]. Equation (0.3) shows that, in our case, the current T has a *global* laminar structure in all of $\mathbf{P}^2 - K$.

In the case of a polynomial map g of $\mathbf{C}$, the first term in the Brolin-Manning formula (0.2) is reminiscent of the behavior of g at infinity, whereas the second term is an integration of the Green function against a critical measure, which in this case is simply the sum of the point masses at the critical points of g.

A similar description has recently been given by Bedford and Smillie [BS2] in the context of polynomial diffeomorphisms of $\mathbf{C}^2$. They establish formulae for the Lyapunov exponents λ^+ and λ^- with respect to the harmonic measure μ of the compact set K consisting of points with bounded forward and backward orbits. Their formulae express the Lyapunov exponents in terms of integration of the Green functions G^+ and G^- against certain critical measures, which, of course, in their situation are much more complicated than a finite sum of Dirac masses.

In a later paper [BJ] we will consider the problem of finding formulae for (the sum of) the Lyapunov exponents for more general polynomial maps than those treated in this article.

This paper is organized as follows. In the first section we review some basic results from iterations of maps of $\mathbf{P}^2$ and prove an asymptotic formula for the Green function in the case of a polynomial map. Then in section 2 we describe the current T in the domain $\mathbf{P}^2 - K$, i.e. we prove formula (0.4). The third section is devoted to the proof of the integral formula (0.3). Finally, in section 4, we study the case of a homogeneous polynomial map of $\mathbf{C}^2$ and prove the main result in the paper, namely Theorem 4.4.

Acknowledgment. The author wants to thank N. Sibony, J.E. Fornæss and E. Bedford for valuable comments on the material in this paper, in particular N. Sibony for outlining the proof of Theorem 2.10. He is also grateful to the University of Michigan for its hospitality. The final thanks goes to the referee for a very careful reading of the paper and for several most useful suggestions and remarks.

1. Basic facts

In this section we review some basic results and definitions from the theory of iterations of holomorphic maps of $\mathbf{P}^2$. Most of the material is lifted from [FS1] and [FS2]. In the end of the section we give the definition of the Lyapunov exponents λ_1, λ_2 in the context of polynomial maps of $\mathbf{C}^2$.

Throughout this paper we let f be a polynomial map of $\mathbf{C}^2$ of degree $d \geq 2$ such that the homogeneous part of f of degree d has no zeros outside the origin. This means precisely that f extends to a holomorphic map of $\mathbf{P}^2$, still denoted by f. The line $\Pi = \mathbf{P}^2 - \mathbf{C}^2$ at infinity is then completely invariant and the restriction of f to Π is a rational map on $\Pi \simeq \mathbf{P}^1$.

We endow $\mathbf{P}^2$ with the Fubini-Study metric and all distances and volumes are measured with respect to this metric unless otherwise stated. Let us note that the Fubini-Study metric is comparable with the Euclidean metric on $\mathbf{C}^2$ on compact subsets of $\mathbf{C}^2$.

The set $\mathcal{H}_d$ of all holomorphic maps of $\mathbf{P}^2$ of degree d is a complex variety (see [FS1]) and the set of polynomial maps of $\mathbf{C}^2$ that extend to $\mathbf{P}^2$ is a complex subvariety, which we denote by $\mathcal{P}_d$. The map $(z,w) \to (z^d, w^d)$ clearly belongs to $\mathcal{P}_d$. From now on, let $\mathcal{V}$ be a (small) neighborhood of this map in $\mathcal{P}_d$.

If $f \in \mathcal{P}_d$ then we define the Green function of f as

$$G(z,w) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z,w)|.$$

Then G is a continuous plurisubharmonic function in all of $\mathbf{C}^2$ with

$$G(z,w) = \log|(z,w)| + O(1) \text{ as } |(z,w)| \to \infty.$$

In fact, G is the pluricomplex Green function with pole at infinity of the compact set $K := \{G = 0\}$ consisting of points in $\mathbf{C}^2$ with bounded forward orbits (cf. [K]). The Green function G satisfies the fundamental relation $G \circ f = dG$.

It is natural to define $G = \infty$ on Π. Later on, we will need more precise information on the asymptotics of G close to Π. For this, it is again useful to regard f as a holomorphic map of $\mathbf{P}^2$. Then f lifts to a map $\tilde{f}$ of $\mathbf{C}^3 - \{0\}$ in an obvious way such that $\pi \circ \tilde{f} = f \circ \pi$, where $\pi : \mathbf{C}^3 - \{0\} \to \mathbf{P}^2$ is the natural projection. If we use homogeneous coordinates $[z : w : t]$ on $\mathbf{P}^2$ where $\mathbf{C}^2 = \mathbf{P}^2 - \Pi$ is identified with the set $t = 1$ and $f(z,w) = (p(z,w), q(z,w))$, then the map $\tilde{f}$ is given by

$$\tilde{f}(z,w,t) = (t^d p(z/t, w/t), t^d q(z/t, w/t), t^d). \tag{1.1}$$

We define the homogeneous Green function for $\tilde{f}$ to be the function

$$\tilde{G}(z,w,t) = \lim_{n\to\infty} \frac{1}{d^n} \log|\tilde{f}^n(z,w,t)|.$$

Then $\tilde{G}$ is continuous and plurisubharmonic on $\mathbf{C}^3 - \{0\}$ and it satisfies

$$\tilde{G}(tx) = \log|t| + \tilde{G}(x) \quad t \in \mathbf{C}^*.$$

The relation between G and $\tilde{G}$ is given by

$$\tilde{G}(z,w,t) = G\left(\frac{z}{t}, \frac{w}{t}\right) + \log|t| \quad t \neq 0.$$

Now suppose that $(z,w) \to p$, where $p \in \Pi$. We may represent p by $p = [z_0 : w_0]$, where $|(z_0,w_0)| = 1$. Then

$$G(z,w) = \log|(z,w)| + \log\frac{|(z,w,1)|}{|(z,w)|} + \tilde{G}\left(\frac{(z,w,1)}{|(z,w,1)|}\right),$$

so since $\tilde{G}$ is continuous we get

$$G(z,w) - \log|(z,w)| \to \tilde{G}(z_0,w_0,0).$$

If we define $\gamma[z:w] := \tilde{G}(z,w,0) - \log|(z,w)|$ then γ is well-defined and continuous on $\Pi \simeq \mathbf{P}^1$ and $G(z,w) = \log|(z,w)| + \gamma[z:w] + o(1)$ as $|(z,w)| \to \infty$. From equation (1.1) and the definition of $\tilde{G}$ above we see that $\tilde{G}(z,w,0)$ depends only on the homogeneous part of degree d of the polynomial map f. Let us summarize all this.

Proposition 1.1. *The Green function G has the following asymptotics at Π*

$$G(z,w) = \log|(z,w)| + \gamma[z:w] + o(1) \quad |(z,w)| \to \infty.$$

Here γ is a continuous function on $\Pi \simeq \mathbf{P}^1$ which only depends on the homogeneous part of f of degree d.

The positive closed $(1,1)$-current $T := dd^cG$ is called the Green current of f and it has an extension as a positive closed current (also denoted by T) to $\mathbf{P}^2$. The support of T is exactly the complement of the largest open subset of $\mathbf{P}^2$ where the family of iterates $\{f^n\}$ is normal. Moreover, if Δ is an analytic disk in $\mathbf{P}^2$ on which $\{f^n\}$ is normal, then G is harmonic on $\Delta - \Pi$.

Since G is continuous and plurisubharmonic, the wedge product $\mu := T \wedge T = (dd^cG)^2$ is a well defined probability measure on $\mathbf{C}^2$. It is known that μ is an invariant mixing measure of maximal entropy for f. The support of μ is the Shilov boundary of the compact set K and μ does not charge pluripolar sets.

We end this section with a brief review of the notion of Lyapunov exponents in the present context. For an introduction to Lyapunov exponents in general we refer to [Y]. By Oseledec's Theorem there are two numbers $\lambda_1 \geq \lambda_2$, called the Lyapunov exponents of f, such that for μ-almost every $x \in \mathbf{C}^2$ there exists a complex subspace $E_2(x)$ of $\mathbf{C}^2$ of dimension 1 with the property that

$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)v| = \lambda_2 \quad \text{if} \quad v \in E_2(x), v \neq 0,$$
$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)v| = \lambda_1 \quad \text{if} \quad v \notin E_2(x), v \neq 0$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log |\det Df^n(x)| = \lambda_1 + \lambda_2.$$

Although we are interested in the sum of the Lyapunov exponents, we will not use the last formula directly. Instead, we will work with the following well known formula, which follows e.g. from the Ergodic Theorem

$$\lambda_1 + \lambda_2 = \int \log |\det Df| \, d\mu.$$

Let us finally note that Briend [B] has proved that the Lyapunov exponents of a general holomorphic map of $\mathbf{P}^k, k \geq 1$ are nonnegative. In particular, $\lambda_1 \geq \lambda_2 \geq 0$.

2. Geometric description of $T|_{\mathbf{P}^2 - K}$

In this section we will prove that the restriction of the current T to the domain $\mathbf{P}^2 - K$ has a laminar structure arising from superstable manifolds at the line at infinity. We start by studying these manifolds.

Recall that the restriction of f to the line Π at infinity is a hyperbolic rational map. We denote its Julia set by Λ and notice that it is a hyperbolic set for f as a map of $\mathbf{P}^2$. From the general theory of hyperbolic dynamical systems (cf. [R]) we know that there exists a local superstable manifold through each point of Λ. To state this precisely, let Π_ε be the neighborhood of Π defined by $\Pi_\varepsilon = \{G > \log 1/\varepsilon\}$.

Lemma 2.1. *If we define the sets $W_0^s(p)$ for $p \in \Lambda$ as*

$$W_0^s(p) = \{x \in \Pi_\epsilon ; d(f^n x, f^n p) \to 0 \text{ as } n \to \infty\},$$

then for ε small enough the following holds.

1. *Each set $W_0^s(p)$ is an embedded disk intersecting Π only at p and transverse to Π at p.*
2. *The disks $W_0^s(p)$ and $W^s(q)$ are disjoint if $p \neq q$ and $W^s(p)$ depends continuously on p.*
3. *The restriction of G to $W_0^s(p) - \{p\}$ is a harmonic function without critical points and has a logarithmic pole at p.*
4. *$f(W_0^s(p))$ is compactly contained in $W_0^s(f(p))$.*

Proof. The Stable Manifold Theorem [R] provides us with local stable manifolds of the form

$$W^s_{\text{loc}}(p) = \{x \in \mathbf{P}^2; d(f^n x, f^n p) < \varepsilon' \quad \forall n \geq 0\}$$

for $\varepsilon' > 0$ small enough; these are pairwise disjoint embedded disks, varying continuously with p and if $x \in W^s_{\text{loc}}(p)$, then $d(f^n x, f^n p) \to 0$ superexponentially fast as $n \to \infty$. The disks $W^s_{\text{loc}}(p)$ are all transverse to Π because the unstable direction of the hyperbolic splitting over Λ is along Π. We may therefore parameterize a neighborhood $Z(p)$ of p in $W^s_{\text{loc}}(p)$ for $p \in \Lambda$ as

$$(z, w) = \left(\frac{z_0}{s}\left(1 + \phi_p(s)\right), \frac{w_0}{s}\left(1 + \psi_p(s)\right)\right) \quad s \in \mathbf{D}_{\varepsilon''}, \tag{2.1}$$

where $|(z_0, w_0)| = 1$ and $[z_0 : w_0] = p$ and ϕ_p, ψ_p are bounded analytic functions depending continuously on p with $\phi_p(0) = \psi_p(0) = 0$. If we choose ε'' small enough, then the $Z(p)$ are embedded disks. Using Proposition 1.1 we see that on $Z(p)$

$$G(z(s), w(s)) = -\log|s| + g_p(s), \tag{2.2}$$

where $g_p(s)$ is uniformly bounded and $g_p(0) = \gamma(p)$. Now the family f^n restricted to $W^s_{\text{loc}}(p)$ is normal, so G restricted to $W^s_{\text{loc}}(p) - \{p\}$ is harmonic. It follows that the g_p's are uniformly bounded harmonic functions on $\mathbf{D}_{\varepsilon''}$ and (2.2) then yields that G is has no critical points on $Z(p) - \{p\}$ if ε'' is small enough. Therefore, the intersection of $Z(p)$ with Π_ε is an embedded disk for all p if ε is small enough. This proves (1), (2) and (3). Finally (4) follows from the equation $G \circ f = dG$. □

Next we prove a few easy properties of the dynamics of f in the set $\mathbf{P}^2 - K$.

Lemma 2.2. *The hyperbolic set Λ has local product structure.*

Proof. The local unstable manifold of a (history of a) point $p \in \Lambda$ can be identified with a neighborhood of p in Π. Therefore, if $p, q \in \Lambda$ are two nearby points, then the intersection of $W^s_0(p)$ and the unstable manifold of (any history of) q is exactly the point q, which belongs to Λ. □

Corollary 2.3. *Let $W^s(\Lambda)$ be the stable set of Λ, i.e.*

$$W^s(\Lambda) = \{x \in \mathbf{P}^2; d(f^n(x), \Lambda) \to 0 \text{ as } n \to \infty\}.$$

Then, if $\varepsilon > 0$ is small enough, we have $\bigcup_{p \in \Lambda} W^s_0(p) = W^s(\Lambda) \cap \Pi_\varepsilon$.

Proof. This follows from the local product structure (see [R]). □

Corollary 2.4. *$W^s(\Lambda)$ has empty interior.*

Proof. It is sufficient to prove that $W^s(\Lambda)$ has no interior near Π. Indeed, $W^s(\Lambda)$ is completely invariant under f and every compact subset of $W^s(\Lambda)$ is iterated into any neighborhood of Π. Let Σ be a complex line close to Π and meeting every disk $W^s_0(p)$ transversely and let $\chi : \Sigma \cap W^s(\Lambda) \to \Lambda$ be the holonomy map defined by $\chi(x) = p$ if $x \in W_0^s(p)$. Then χ is a homeomorphism so since Λ has empty interior, $\Sigma \cap W^s(\Lambda)$ has empty interior. This is true for all lines sufficiently close to the line Π so we conclude that $W^s(\Lambda)$ has no interior near Π, which completes the proof. $\square$

Lemma 2.5. *The set $W^s(\Lambda)$ is contained in the Kobayashi hyperbolic open set*

$$U = \{[z : w : t] \in \mathbf{P}^2; |(z,w)| > \frac{1}{2}|t|, \frac{1}{10}|w| < |z| < 10|w|\}.$$

Moreover, $W^s(\Lambda)$ only intersects the critical set of f in Λ.

Proof. Recall that f is close to the map $(z,w) \to (z^d, w^d)$. Hence Λ is close to the circle $\{|z| = |w|, t = 0\}$ which is contained in U, so it follows from Corollary 2.3 that $W^s(\Lambda) \cap \Pi_\varepsilon \subset U$ for ε small enough. But it is easy to check that $f^{-1}(U) \subset U$, so the invariance of $W^s(\Lambda)$ implies that $W^s(\Lambda) \subset U$. The set U is biholomorphic to an open subset of the bidisk $\{|z_1| < 10, |z_2| < 2\}$ and is therefore Kobayashi hyperbolic. Finally, a perturbation argument yields that U intersects the critical set of f only in Π. This completes the proof. $\square$

Lemma 2.6. *The set $W^s(\Lambda)$ is equal to the support of $T|_{\mathbf{P}^2 - K}$.*

Proof. Any point in $\mathbf{P}^2 - K$ is attracted to the line Π at infinity and the restriction of f to Π is a hyperbolic rational map. Let x be any point in $\mathbf{P}^2 - K$. If $x \notin W^s(\Lambda)$, then the orbit of x will converge to an attracting cycle in Π, as will the orbit of points sufficiently close to x. It follows that $\{f^n\}$ is normal in a neighborhood of x so $x \notin \operatorname{supp}(T)$. On the other hand, $W^s(\Lambda)$ has empty interior by Corollary 2.4, so if $x \in W^s(\Lambda)$ then every neighborhood of x will contain points whose orbits converge to an attracting cycle in Π, whereas the orbit of x itself is attracted to Λ. Therefore $\{f^n\}$ is not normal in any neighborhood of x and $x \in \operatorname{supp}(T)$. $\square$

Corollary 2.7. *The support of the current $T|_{\mathbf{P}^2 - K}$ only intersects the critical set of f in Λ.*

Proof. This follows immediately from Lemma 2.5 and Lemma 2.6. $\square$

Now we want to extend the disks $W_0^s(p)$ away from Π_ε by pulling them back by f^{-1}. Recall that $f(W_0^s(p))$ is compactly contained in $W_0^s(f(p))$ for all $p \in \Lambda$. Let us define a sequence of disks

$$W_0^s(p) \subset\subset W_1^s(p) \subset\subset W_2^s(p) \subset\subset \dots$$

for every $p \in \Lambda$ as follows. First, we let $W_1^s(p)$ be the connected component of $f^{-1}(W_0^s(f(p)))$ containing $W_0^s(p)$. Note that $f^{-1}(W_0^s(f(p)))$ is contained in $W^s(\Lambda)$ and therefore, by Lemma 2.5, meets the critical set of f only at Π. Hence f is a local biholomorphism at every point of $f^{-1}(W_0^s(f(p))) - \Pi$ so the latter set is an embedded manifold. In fact $f^{-1}(W_0^s(f(p)))$ is an embedded manifold, because its intersection with Π_ε is the union of the disks $W_0^s(q)$ for $f(q) = f(p)$.

We claim that $f^{-1}(W_0^s(f(p)))$ has d different components, each containing exactly one of the disks $W_0^s(q)$ with $f(q) = f(p)$. If this was not true, then there would exist two different preimages q_1 and q_2 of $f(p)$ and a curve Γ inside $f^{-1}(W_0^s(f(p)))$ joining q_1 and q_2. The curve $f(\Gamma)$ would then be homotopic rel. p to the constant curve at p and if $f(\Gamma)$ and the deformed curves were sufficiently nice at p, then the homotopy would lift to a homotopy rel. $\{q_1, q_2\}$ of Γ to a constant curve. This is impossible. It follows that $W_1^s(p)$ is an embedded manifold, which is furthermore a branched cover of degree d of the disk $W_0^s(f(p))$, branched only at p. Hence $W_1^s(p)$ is an embedded disk containing $W_0^s(p)$ as a relatively compact subset and intersecting Π only at p. From the equation $G \circ f = dG$ and Lemma 2.1 it follows that G has no critical points on $W_1^s(p)$ for $p \in \Lambda$. It is also clear that $W_1^s(p) \cap W_1^s(q) = \emptyset$ if $p \neq q$.

We now repeat the procedure and inductively construct, for each $p \in \Lambda$, a sequence of embedded disks $\{W_n^s(p)\}_{n \geq 0}$ with the following properties.

(1) $W_n^s(p)$ is compactly contained in $W_{n+1}^s(p)$ and f maps $W_{n+1}^s(p)$ onto $W_n^s(f(p))$ as a branched covering of degree d, branched only at p.
(2) G is harmonic on $W_n^s(p) - \{p\}$ and has no critical points there.
(3) The disks $W_n^s(p)$ and $W_n^s(q)$ are disjoint if $p \neq q$.

Let $W^s(p)$ be the increasing union of all the $W_n^s(p)$ over $n \geq 0$. Note that $W^s(p)$ is not the superstable manifold at p in the usual sense but rather the connected component containing p of the superstable manifold at p. We arrive at the following.

Proposition 2.8. *If $\mathcal{V}$ is sufficiently small, then for any $f \in \mathcal{V}$ the following properties hold*

(1) *$W^s(p)$ is an embedded disk for all $p \in \Lambda$ and $W^s(p) \cap W^s(q) = \emptyset$ if $p \neq q$.*
(2) *G is harmonic without critical points on $W^s(p) - \{p\}$ for all $p \in \Lambda$.*
(3) *If $W^s(\Lambda)$ is defined as in Corollary 2.3, then*

$$\operatorname{supp}(T|_{\mathbf{P}^2 - K}) = W^s(\Lambda) = \bigcup_{p \in \Lambda} W^s(p).$$

(4) *We have that $\overline{W^s(\Lambda)} \cap K \subset \operatorname{supp}(\mu)$. In particular, the boundary of the disk $W^s(p)$ is contained in $\operatorname{supp}(\mu)$ for all $p \in \Lambda$.*
(5) *The volumes $|W^s(p)|$ of $W^s(p)$ in the Fubini-Study metric on $\mathbf{P}^2$ are uniformly bounded.*

Proof.

(1) We have that $W^s(p)$, being the increasing union of embedded disks, is biholomorphic to either $\mathbf{C}$ or $\mathbf{D}$. To see that $W^s(p)$ is a disk we note that $W^s(p)$ is contained in the Kobayashi hyperbolic set U defined in Lemma 2.5. If $p \neq q$, then $W^s(p)$ and $W^s(q)$ are disjoint, because $W_n^s(p)$ and $W_n^s(q)$ are disjoint for all $n \geq 0$.
(2) This is clear since G is harmonic without critical points on $W_n^s(p)$ for $n \geq 0$.

(3) We know from Lemma 2.6 that the sets $\operatorname{supp}(T|_{\mathbf{P}^2-K})$ and $W^s(\Lambda)$ are equal so we only need to show that the latter set is equal to $\bigcup_{p\in\Lambda} W^s(p)$. But these two sets are completely invariant under f and the orbit of a point in any one of them approaches the line Π at infinity. Hence the equality to be proved follows from Corollary 2.3.

(4) We will make heavy use of the assumption that f is close to the map $f_0(z,w) = (z^d, w^d)$, which is Axiom A and satisfies the No-Cycles condition. Let us briefly verify this. The non-wandering set of f_0 has seven components Ω_i^0, $i = 1,\dots,7$. Here $\Omega_1^0 = \{[0:0:1]\}$, $\Omega_2^0 = \{[0:1:0]\}$ and $\Omega_3^0 = \{[1:0:0]\}$ are attracting fixed points, $\Omega_4^0 = \{|z| = |w|, t = 0\}$, $\Omega_5^0 = \{|t| = |z|, w = 0\}$ and $\Omega_6^0 = \{|w| = |t|, z = 0\}$ have unstable index 1 and $\Omega_7^0 = \{|z| = |w| = |t|\}$ has unstable index 2. It is easy to see that f_0 is transitive on Ω_i^0 and that periodic points are dense in Ω_i^0 for all i. Hence f_0 is Axiom A (i.e. hyperbolic on its non-wandering set and with periodic points dense there) and its basic sets are exactly Ω_i^0. We write $\Omega_i^0 \prec \Omega_j^0$ if there exists an orbit $(x_k)_{k\in\mathbf{Z}}$ under f_0 such that $x_k \to \Omega_i^0$ as $k \to -\infty$ and $x_k \to \Omega_j^0$ as $k \to \infty$. It is then easy to verify that f_0 satisfies the No-Cycles condition, i.e. there is no nontrivial sequence $i_0, i_1, \dots, i_k = i_0$ such that $\Omega_{i_0}^0 \prec \Omega_{i_1}^0 \prec \cdots \prec \Omega_{i_k}^0$.

Hence, by Smale's Ω-stability theorem (cf. [R]), we can assume that the perturbed map f is Axiom A and that the basic sets of f are close (in the Hausdorff metric) to those of f_0. In particular, f has three basic sets Ω_5, Ω_6 and Ω_7 inside ∂K; these are perturbations of Ω_5^0,Ω_6^0 and Ω_7^0, respectively. We know that $\operatorname{supp}(\mu)$ is completely invariant and that $f|_{\operatorname{supp}(\mu)}$ is expanding and topologically transitive [FS2]. Hence $\operatorname{supp}(\mu)$ is a basic set so in fact $\Omega_7 = \operatorname{supp}(\mu)$.

Now suppose that there is a point $x \in (\overline{W^s(\Lambda)} \cap K) - \operatorname{supp}(\mu)$. By definition, the orbit of x must converge to the non-wandering set of f and since the orbit of x is contained in ∂K and $\operatorname{supp}(\mu)$ is repelling, the orbit must converge to $\Omega_5 \cup \Omega_6$. But the orbit is contained in the closed set $\overline{W^s(\Lambda)}$ so we conclude that the sets $\overline{W^s(\Lambda)}$ and $\Omega_5 \cup \Omega_6$ have a nonempty intersection. This is a contradiction, because $\overline{W^s(\Lambda)}$ is contained in the set U defined in Lemma 2.5 and Ω_i is close to Ω_i^0 for all i.

(5) It is clear that $|W_0^s(p)|$ is uniformly bounded if ε is small enough. Let $A_n(p)$ for $n \in \mathbf{Z}$ be the annuli defined by $A_n(p) := W^s(p) \cap \{d^n < G < d^{n+1}\}$. For n sufficiently large we have $A_n(p) \subset W_0^s(p)$ for $p \in \Lambda$. Also, for each n, the volumes $|A_n(p)|$ are uniformly bounded, because if m is sufficiently large, then $f^m(A_n(p)) \subset W_0^s(p)$ for $p \in \Lambda$. It is therefore sufficient to prove that there exist constants $C < \infty$ and $\lambda > 1$ such that for n sufficiently large negative and all $p \in \Lambda$ we have $|A_n(p)| \le C\lambda^n$. Since the sets $|A_n(p)|$ are uniformly far away from the line Π at infinity for n large negative, it suffices to prove the same estimate in the Euclidean metric on $\mathbf{C}^2$.

For the unperturbed map $f_0(z,w) = (z^d, w^d)$ we have $|Df_0(x)v| = d|v|$ for all $x \in \Omega_7^0$ and all $v \in \mathbf{C}^2$. If $\mathcal{V}$ is small enough and $f \in \mathcal{V}$ then by continuity we will have $|Df(x)v| \geq \frac{3d}{4}|v|$ for all x in a small neighborhood of supp(μ) and all $v \in \mathbf{C}^2$. In particular, it follows from (4) that the last estimate will hold for $x \in A_n(p)$ for all sufficiently large negative n and all $p \in \Lambda$.

Now $f : A_{n-1}(p) \to A_n(f(p))$ is a covering map of order d. Hence, for n large negative and $p \in \Lambda$ we have

$$\begin{aligned} |A_n(f(p))|_{\mathrm{eucl}} &= \frac{1}{d}\int_{A_{n-1}(p)} |Df|_{A_{n-1}(p)}|^2 \\ &\geq \frac{1}{d}(\frac{3d}{4})^2 |A_{n-1}(p)|_{\mathrm{eucl}} \\ &\geq \frac{9}{8}|A_{n-1}(p)|_{\mathrm{eucl}}. \end{aligned}$$

It follows that $|A_n(p)|_{\mathrm{eucl}} \leq C(\frac{9}{8})^n$ for n sufficiently large negative, which completes the proof.

□

We have shown that the union of the disks $W^s(p)$ is equal to the support of the current T in the open set $\mathbf{P}^2 - K$. Our next objective is to describe the action of T in this set on test forms in terms of integration over the $W^s(p)$.

Let us first note that the slice of T on the invariant line Π at infinity is a measure ν which satisfies $f^*\nu = d\nu$. In fact ν is the unique invariant measure of maximal entropy for the restriction of f to Π(cf. [L], [FLM], [HP] or [FS2]). The support of ν is exactly the hyperbolic set Λ.

We may try to define a current S on $\mathbf{P}^2 - K$ by declaring

$$< S, \phi >:= \int_\Lambda d\nu(p) \int_{W^s(p)} \phi, \tag{2.3}$$

for a smooth form ϕ with compact support in $\mathbf{P}^2 - K$. Since the $W^s(p)$ are embedded disks with uniformly bounded volume, this makes sense and defines S as a positive closed $(1,1)$-current. In fact, formula (2.3) defines $< S, \phi >$ for bounded (in the metric on $\mathbf{P}^2$) continuous $(1,1)$-forms in a neighborhood of $W^s(\Lambda)$. The change of variables formula implies that $f^*[W^s(p)] = \sum_{f(q)=p}[W^s(q)]$. Together with the fact that $f^*\nu = d\nu$ this shows that $f^*S = dS$.

We arrive at the main theorem in this section.

Theorem 2.9. *We have $T = S$ on $\mathbf{P}^2 - K$. More precisely, if ϕ is a bounded continuous $(1,1)$-form in a neighborhood of supp$(T|_{\mathbf{P}^2-K})$, then*

$$< T|_{\mathbf{P}^2-K}, \phi >= \int_\Lambda d\nu(p) \int_{W^s(p)} \phi. \tag{2.4}$$

We will prove that Theorem 2.9 follows from the following weaker statement. The author is grateful to N. Sibony for the idea of its proof.

Theorem 2.10. *Equation* (2.4) *holds if ϕ is continuous with compact support in a small neighborhood of the line Π at infinity.*

Proof of Theorem 2.9. We assume that Theorem 2.10 has been proven and let ϕ be a bounded continuous $(1,1)$-form in a neighborhood of $\operatorname{supp}(T|_{\mathbf{P}^2-K})$. We may assume that ϕ is defined, continuous and bounded in all of $\mathbf{P}^2 - K$. Let $\{\psi_n\}_{n\geq 0}$ be a sequence of continuous functions with $0 \leq \psi_n \leq 1$, $\psi_n = 1$ on $\{G > 2/n\}$ and $\psi_n = 0$ on $\{G < 1/n\}$. Then

$$< T|_{\mathbf{P}^2-K}, \psi_n\phi > \to < T|_{\mathbf{P}^2-K}, \phi >$$

and

$$< S, \psi_n\phi > \to < S, \phi >$$

as $n \to \infty$ because S and T are positive currents with finite mass. Hence we may assume that ϕ is continuous with compact support in $\mathbf{P}^2 - K$. Now let V be a neighborhood of Π on which the statement in Theorem 2.10 is true. We may assume that the support of ϕ does not intersect Π, because otherwise we just write $\phi = \phi_1 + \phi_2$ where $\operatorname{supp}(\phi_1) \subset V$ and where $\operatorname{supp}(\phi_2) \cap \Pi = \emptyset$.

There exists an $n \geq 0$ such that $f^n(\operatorname{supp}(\phi))$ is contained in $V - \Pi$. The support of S and $T|_{\mathbf{P}^2-K}$ only meets the critical set of f^n at Λ (Corollary 2.7), so after multiplying ϕ with a suitable cut-off function we may assume that f^n is a local biholomorphism in a neighborhood of every point of $\operatorname{supp}(\phi)$. But then $(f^n)_*\phi$ is a well-defined continuous form with support in V so using the properties $f^*T = dT$ and $f^*S = dS$ we obtain

$$\begin{aligned} < T, \phi > &= \frac{1}{d^n} < T, (f^n)_*\phi > \\ &= \frac{1}{d^n} < S, (f^n)_*\phi > \\ &= < S, \phi > . \end{aligned}$$

□

Proof of Theorem 2.10. Recall that f is close to the map $(z,w) \to (z^d, w^d)$. Let $\chi : W^s(\Lambda) \to \Lambda$ be the holonomy map defined by following the leaves of the stable foliation, i.e. $\chi(x) = p$ if $x \in W^s(p)$. It is clear that χ commutes with f.

There is an open set of lines such that every line Σ in this open set is very close to the line Π, intersects every leaf of $W^s(\Lambda)$ transversely and in exactly one point and that the intersection point is in Π_ε.

For such a Σ the restriction of χ to $\Sigma \cap W^s(\Lambda)$ is a homeomorphism onto Λ which is close to the canonical projection π on Π. The slice $T|_\Sigma$ is a measure on Σ supported on $\Sigma \cap W^s(\Lambda)$. The key observation is the following.

Lemma 2.11. *The relation $\chi_*(T|_\Sigma) = \nu$ holds. In other words, the measure ν is a transversal measure of the foliation $\{W^s(p)\}_{p\in\Lambda}$ of $W^s(\Lambda)$.*

Theorem 2.10 follows from Lemma 2.11. Indeed, Lemma 2.11 holds for an open set of lines Σ, so we can use slicing theory for currents to prove Theorem 2.10. See [BS1] for details on this kind of argument. $\square$

Proof of Lemma 2.11. The idea of the proof is to use the property $f^*T = dT$ together with the fact that $f^n(\Sigma)$ approaches Π very fast as $n \to \infty$.

Take any continuous function ϕ on $\mathbf{P}^2$ such that $\phi \circ \chi = \phi$ on $\Sigma \cap W^s(\Lambda)$. It suffices to show that

$$\int_\Sigma \phi(T|_\Sigma) = \int_\Pi \phi \, d\nu.$$

We may use coordinates $\zeta = z/w$ and $t = 1/w$ in a neighborhood of Λ. Note that $\pi(\zeta, t) = \zeta$ in these coordinates. Cover Λ by a finite union of bidisks $B_i = D_i \times \mathbf{D}_\eta$, where D_i are small disks in Π centered at points of Λ and $\eta > 0$ is small. We may assume that each B_i intersects the closure of the postcritical set of f only at Π and that $\Sigma \cap W^s(\Lambda)$ is contained in the union of the B_i's. We may also assume that $\Sigma \cap B_i \cap \Pi = \emptyset$ for all i. Because of the hyperbolicity (in particular the "Lambda Lemma") we have that for all i and all $n \geq 0$ the set $f^n\Sigma \cap B_i$ has finitely many components, each of which is a graph over D_i, uniformly close to D_i as $n \to \infty$. Since these components are disks which do not intersect the closure of the postcritical set of f, we see that $f^{-n}(f^n\Sigma \cap B_i) \cap \Sigma$ consists of d^n distinct disks $\widetilde{E_{ij}}, j = 1, \ldots, d^n$. Let $\widetilde{D_{ij}} = f^n(\widetilde{E_{ij}})$ and $\widetilde{g_{ij}}$ be the single-valued branch of f^{-n} defined on $\widetilde{D_{ij}}$ with values in $\widetilde{E_{ij}}$. Note that if f^n is not injective on $\Sigma \cap W^s(\Lambda)$, then the disks $\widetilde{D_{ij}}$ will not all be different.

Let $D_{ij} = \pi(\widetilde{D_{ij}}) = D_i$, g_{ij} be the single-valued branch of $(f|_\Pi)^{-n}$ such that $g_{ij} \circ \chi = \chi \circ \widetilde{g_{ij}}$ on $\widetilde{D_{ij}}$ and let $E_{ij} = g_{ij}(D_{ij})$. Then $\{\widetilde{E_{ij}}\}_{i,j}$ and $\{E_{ij}\}_{i,j}$ are covers of $\Sigma \cap W^s(\Lambda)$ and Λ, respectively. Choose a C^1 partition of unity $\{\rho_i\}$ subordinate to the cover $\{D_i\}$ of Λ and let $\{\rho_{ij}\}$ and let $\{\widetilde{\rho_{ij}}\}$ be the partitions of unity subordinate to the previous two covers defined by the properties $\rho_i = \rho_{ij} \circ g_{ij}$ and $\rho_i \circ \pi = \widetilde{\rho_{ij}} \circ \widetilde{g_{ij}}$, respectively. We then have

$$\begin{aligned}
\int_\Sigma \phi(T|_\Sigma) &= \sum_{i,j} \int_{\widetilde{E_{ij}}} (\widetilde{\rho_{ij}}\phi) \left(T|_{\widetilde{E_{ij}}}\right) \\
&= d^{-n} \sum_{i,j} \int_{\widetilde{E_{ij}}} (\widetilde{\rho_{ij}}\phi) \, (\widetilde{g_{ij}})_* \left(T|_{\widetilde{D_{ij}}}\right) \\
&= d^{-n} \sum_{i,j} \int_{\widetilde{D_{ij}}} ((\widetilde{\rho_{ij}}\phi) \circ \widetilde{g_{ij}}) \left(T|_{\widetilde{D_{ij}}}\right) \\
&= d^{-n} \sum_{i,j} \int_{\widetilde{D_{ij}}} (\rho_i \circ \pi) \, (\phi \circ \widetilde{g_{ij}}) \left(T|_{\widetilde{D_{ij}}}\right) \\
&= d^{-n} \sum_{i,j} \int_{D_{ij}} \rho_i \left(\phi \circ g_{ij} \circ \chi \circ \pi^{-1}\right) \pi_* \left(T|_{\widetilde{D_{ij}}}\right).
\end{aligned}$$

Here the second line follows from the property $f^*T = dT$, and lines three and five from the change of variables formula. A similar computation shows that

$$\int_\Pi \phi\, d\nu = d^{-n} \sum_{i,j} \int_{D_{ij}} \rho_i \left(\phi \circ g_{ij}\right) \left(T|_{D_{ij}}\right).$$

Now, as $n \to \infty$, $\widetilde{D_{ij}}$ is uniformly close to D_{ij} and $\{g_{ij}\}$ is equicontinuous so $\rho_i(\phi \circ g_{ij} \circ \chi \circ \pi^{-1})$ is uniformly close to $\rho_i(\phi \circ g_{ij})$. Since the slice measure $T|_S$ depends continuously on S this implies that

$$\left| \int_{D_{ij}} \rho_i(\phi \circ g_{ij} \circ \chi \circ \pi^{-1})\, \pi_*(T|_{\widetilde{D_{ij}}}) - \int_{D_{ij}} \rho_i(\phi \circ g_{ij})\, (T|_{D_{ij}}) \right|$$

is uniformly small as $n \to \infty$. Hence it follows that

$$\left| \int_\Sigma \phi(T|_\Sigma) - \int_\Pi \phi\, d\nu \right|$$

is arbitrarily small, which completes the proof. □

3. THE INTEGRAL FORMULA

Having described the current T in the domain $\mathbf{P}^2 - K$ as a laminar current, we now proceed to obtain an integral formula for the sum $\lambda_1 + \lambda_2$ of the the Lyapunov exponents of f. Perhaps the most striking about this formula is that it only depends on the homogeneous part of f of degree d.

Let us recall the following asymptotic expansion from Proposition 1.1.

$$G(z, w) = \log |(z, w)| + \gamma[z : w] + o(1),$$

where $[z : w]$ is the projection of (z, w) on $\Pi \simeq \mathbf{P}^1$. If $H = \log|\det Df|$, then we have a similar formula:

$$H(z, w) = 2(d - 1) \log |(z, w)| + \delta[z : w] + o(1).$$

Note that γ and δ depend only on the homogeneous part of f of degree d. This is easy to see for δ and was shown in Proposition 1.1 for γ. As before, let ν be the measure on $\Lambda \subset \Pi$ defined by $\nu = T|_\Pi$. We will need the following result, which is a fairly straightforward application of Green's formula.

Lemma 3.1. *Let M be a Riemann surface, p a point on M, ξ a coordinate on M with $\xi(p) = 0$ and u, v harmonic functions on $M - \{p\}$ with*

$$u(\xi) = c_u \log |\xi|^{-1} + \rho_u + o(1),$$
$$v(\xi) = c_v \log |\xi|^{-1} + \rho_v + o(1)$$

as $\xi \to 0$, where c_u,c_v,ρ_u, and ρ_v are constants. Then, if Γ is a positively oriented simple closed C^1-curve homotopic to a circle $|\xi| = \eta$, we have

$$\int_\Gamma (u d^c v - v d^c u) = c_u \rho_v - c_v \rho_u.$$

Let us now state and prove the integral formula for the sum of the Lyapunov exponents of f.

Theorem 3.2. *If $\mathcal{V}$ is sufficiently small and $f \in \mathcal{V}$, then*

$$\lambda_1 + \lambda_2 = \int_\Lambda (\delta - 2(d-1)\gamma)d\nu.$$

Proof. Let χ be a smooth function on $\mathbf{C}^2$ with $0 \le \chi \le 1$, $\chi = 1$ on $\mathbf{P}^2 - \Pi_{\varepsilon/2}$ and $\chi = 0$ on $\Pi_{\varepsilon/3}$. Then

$$\begin{aligned}\lambda_1 + \lambda_2 &= \int H\, d\mu \\ &= \int (\chi H)\, d\mu \\ &= \int (\chi H)\, dd^c G \wedge T \\ &= \int G\, dd^c(\chi H) \wedge T \\ &= \int_{\mathbf{P}^2 - K} G\, dd^c(\chi H) \wedge T.\end{aligned}$$

Here the first line is well-known and follows e.g. from the Ergodic Theorem. The second line holds since $\chi = 1$ in a neighborhood of $\operatorname{supp}(\mu)$ and the fourth line since the potential G of T is continuous. Finally, the last line is true since the current T is of order zero and $G = 0$ on K.

Now H is pluriharmonic outside the critical set of f and the latter set does not intersect the support of $\operatorname{supp}(T|_{\mathbf{P}^2-K})$ outside Π. Hence $dd^c(\chi H)$ is smooth and bounded in a neighborhood of $\operatorname{supp}(T|_{\mathbf{P}^2-K})$ so Theorem 2.9 yields

$$\begin{aligned}\lambda_1 + \lambda_2 &= \int_{\mathbf{P}^2 - K} G\, dd^c(\chi H) \wedge T \\ &= \int_\Lambda d\nu(p) \int_{W^s(p)} G\, dd^c(\chi H) \\ &= \int_\Lambda d\nu(p) \int_{W^s_0(p)} G\, dd^c(\chi H) \\ &= \int_\Lambda d\nu(p) \int_{\partial W^s_0(p)} (G d^c H - H d^c G).\end{aligned}$$

Here the third line follows because χH is harmonic on $W^s(p) \cap (\mathbf{P}^2 - \Pi_{\varepsilon/2})$ and the fourth line from an integration by parts. If we apply Lemma 3.1 (with the coordinate ξ given by (2.1)), then we get

$$\lambda_1 + \lambda_2 = \int_\Lambda (\delta - 2(d-1)\gamma)d\nu,$$

and the proof is complete. □

Corollary 3.3. *If $\mathcal{V}$ is sufficiently small and $f \in \mathcal{V}$, then $\lambda_1 + \lambda_2$ depends only on the homogeneous part of f of degree d.*

Proof. This is clear since the measure ν and the functions δ and γ in the statement of Theorem 2.9 depend only on the homogeneous part of degree d of f. □

4. HOMOGENEOUS POLYNOMIALS

From Corollary 3.3 we know that, for the maps we are considering, the sum of the Lyapunov exponents depends only on the homogeneous part of f of maximal degree. This motivates a further study of Lyapunov exponents for homogeneous polynomial maps of $\mathbf{C}^2$ and will lead us to the main result of the paper.

Suppose f is a nondegenerate homogeneous map of $\mathbf{C}^2$ of degree d, i.e. $f(z,w) = (p(z,w), q(z,w))$ with p and q homogeneous of degree d, and $f^{-1}(0) = \{0\}$. We may then define a rational map $\hat{f} : \mathbf{P}^1 \to \mathbf{P}^1$ in a natural way by letting $\hat{f}[z : w] = [p(z,w) : q(z,w)]$. Note that $\hat{f}$ can be identified with the restriction of f to Π.

Let $\nu = T|_\Pi$ be the unique invariant measure of maximal entropy for $\hat{f}$. We first prove a result about the relation between μ and ν. There is a natural projection $\pi : \mathbf{C}^2 - \{0\} \to \mathbf{P}^1$ such that $\pi(z,w) = [z : w]$. We know that $\operatorname{supp}(\mu)$, being the Shilov boundary of K, is contained in ∂K. Since μ does not charge pluripolar sets, this shows that ∂K has positive capacity. Furthermore, since f is homogeneous, the intersection of ∂K with a complex line of the form $\pi^{-1}(p)$ is a circle which we denote by S_p. Let μ_p be the Lebesgue measure on S_p, normalized so that $\mu_p(S_p) = 1$. We have

Proposition 4.1. *If ϕ is a continuous function on $\mathbf{C}^2$, then*

$$\int \phi \, d\mu = \int_{\mathbf{P}^1} d\nu(q) \int_{S_q} \phi \, d\mu_q. \tag{4.1}$$

Proof. We will use the following two results (see [FS2]). First, if $p \in \mathbf{P}^1$ with at most two exceptions, then $((\hat{f}^n)^* \delta_p)/d^n \to \nu$ weakly as $n \to \infty$. Second, for any $x \in \mathbf{C}^2$ outside a set of capacity zero we have $((f^n)^* \delta_x)/d^{2n} \to \mu$ weakly as $n \to \infty$. Therefore, we may find an $x \in \mathbf{C}^2$ such that the above convergence results are true for $p = \pi(x)$ and x, respectively. We may assume that x is in the set ∂K, because we noticed above that the latter set has positive capacity. Also, we may assume that no preimage of x is in the critical set of f, because the set of x without this property has capacity zero. For any $n \geq 1$, the preimages of x are then grouped into d^n groups, where each group consists of d^n points $\{x_1^{(q)}, \dots, x_{d^n}^{(q)}\}$ equidistributed on the circle S_q, and where q runs through the d^n preimages of $p = \pi(x)$ under $\hat{f}$. Hence, if ϕ is a continuous function on $\mathbf{C}^2$, then

$$< \frac{1}{d^{2n}} (f^n)^* \delta_x, \phi >= \frac{1}{d^n} \sum_{\hat{f}^n(q)=p} \frac{1}{d^n} \sum_{i=1}^{d^n} \phi(x_i^{(q)}). \tag{4.2}$$

Here the left hand side converges to $< \mu, \phi >$ by the choice of x, so we only need to check that the right hand side of (4.2) converges to the right hand side of (4.1). Let $\hat{\phi}$ be the continuous function on $\mathbf{P}^1$ defined by $\hat{\phi}(q) =< \mu_q, \phi >$. Then we get

$$\begin{aligned}
&\left| \frac{1}{d^n} \sum_{\hat{f}^n(q)=p} \frac{1}{d^n} \sum_{i=1}^{d^n} \phi(x_i^{(q)}) - \int_{\mathbf{P}^1} d\nu(q) \int_{S_q} \phi \, d\mu_q \right| \\
&\leq \frac{1}{d^n} \sum_{\hat{f}^n(q)=p} \left| \frac{1}{d^n} \sum_{i=1}^{d^n} \phi(x_i^{(q)}) - \hat{\phi}(q) \right| + \left| \frac{1}{d^n} \sum_{\hat{f}^n(q)=p} \hat{\phi}(q) - \int_{\mathbf{P}^1} \hat{\phi}(q) \, d\nu(q) \right| \\
&\leq \sup_q \left| \frac{1}{d^n} \sum_{i=1}^{d^n} \phi(x_i^{(q)}) - \hat{\phi}(q) \right| + \left| \frac{1}{d^n} \sum_{\hat{f}^n(q)=p} \hat{\phi}(q) - \int_{\mathbf{P}^1} \hat{\phi}(q) \, d\nu(q) \right| .
\end{aligned}$$

Here the first term tends to zero because ϕ is uniformly continuous on ∂K and the second term tends to zero because of the choice of $p = \pi(x)$ and the continuity of $\hat{\phi}$ on $\mathbf{P}^1$. ◻

Corollary 4.2. *We have $\pi_*\mu = \nu$. In particular, $\mu(E) = 1$ implies $\nu(\pi(E)) = 1$*

Proof. This follows immediately from Proposition 4.1. ◻

We next want to find the relation between the Lyapunov exponent λ of $\hat{f}$ with respect to the measure ν and the sum $\lambda_1 + \lambda_2$ of the Lyapunov exponents of f with respect to μ. The answer is the following

Theorem 4.3. *If $f : \mathbf{C}^2 \to \mathbf{C}^2$ is a nondegenerate homogeneous polynomial map of degree d, then, with the notation above, $\lambda_1 + \lambda_2 = \lambda + \log d$.*

Proof. Let (z, w) be coordinates on $\mathbf{C}^2$, $[z : w]$ homogeneous coordinates on $\mathbf{P}^1$ and $\zeta := z/w$. The map $\hat{f}$ then looks like $\hat{f}(\zeta) = p(\zeta, 1)/q(\zeta, 1)$. The Jacobian of f is given by $\det Df = p_z q_w - q_z p_w$. If (z, w) are such that $w \neq 0$ and $q(z, w) \neq 0$ then the norm of the derivative of $\hat{f}$ at the point $\zeta = z/w$ in the Euclidean metric on $\mathbf{C} \simeq \{w \neq 0\}$ is given by

$$\begin{aligned}
&\left| \frac{1}{q(\zeta,1)^2} (p_z(\zeta,1) q(\zeta,1) - q_z(\zeta,1) p(\zeta,1)) \right| \\
&\qquad\qquad = \left| \frac{w}{q(z,w)^2} (p_z(z,w) q(z,w) - q_z(z,w) p(z,w)) \right| ,
\end{aligned}$$

where we have used the homogeneity of p, q, p_z, q_z. Therefore, the norm of the derivative of $\hat{f}$ in the spherical metric on $\mathbf{P}^1$ is

$$\begin{aligned}\left|\hat{f}'(\zeta)\right| &= \frac{1+|\zeta|^2}{1+|\hat{f}(\zeta)|^2}\left|\frac{w}{q(z,w)^2}(p_z(z,w)q(z,w)-q_z(z,w)p(z,w))\right| \\ &= \frac{|z|^2+|w|^2}{|p(z,w)|^2+|q(z,w)|^2}\left|\frac{1}{w}(p_z(z,w)q(z,w)-q_z(z,w)p(z,w))\right| \\ &= \frac{1}{d}\frac{|z|^2+|w|^2}{|p(z,w)|^2+|q(z,w)|^2}\left|p_z(z,w)q_w(z,w)-q_z(z,w)p_w(z,w)\right|.\end{aligned}$$

The last line follows from the relations $zp_z + wp_w = dp$ and $zq_z + wq_w = dq$, which are true since p and q are homogeneous of degree d. By continuity we get that the expression in the last line is the norm of the derivative of $\hat{f}$ in the spherical metric at the point ζ for any $\zeta \in \mathbf{P}^1$. We may now iterate this. If $(z, w) \in \mathbf{C}^2$, $(z_i, w_i) = f^i(z, w)$ and $\zeta_i = \hat{f}^i(\zeta) = z_i/w_i$ for $i \geq 0$, then

$$\begin{aligned}\left|(\hat{f}^n)'(\zeta)\right| &= \prod_{i=0}^{n-1}\left|\hat{f}'(\zeta_i)\right| \\ &= \prod_{i=0}^{n-1}\frac{1}{d}\frac{|z_i|^2+|w_i|^2}{|z_{i+1}|^2+|w_{i+1}|^2}\left|\det Df(z_i,w_i)\right| \\ &= \frac{1}{d^n}\frac{|(z,w)|^2}{|(z_n,w_n)|^2}\left|\det Df^n(z,w)\right|,\end{aligned}$$

so we arrive at

$$\frac{1}{n}\log|(\hat{f}^n)'(\zeta)| + \log d = \frac{2}{n}\log\frac{|(z,w)|}{|(z_n,w_n)|} + \frac{1}{n}\log|\det Df^n(z,w)|. \tag{4.3}$$

Now $\operatorname{supp}(\mu)$ is a compact subset of $\mathbf{C}^2 - \{0\}$. Therefore, if $(z, w) \in \operatorname{supp}(\mu)$, then $\log(|(z, w)|/|(z_n, w_n)|)$ is bounded so for μ-almost all $(z, w) \in \operatorname{supp}(\mu)$, the right hand side in (4.3) tends to $\lambda_1 + \lambda_2$ whereas, by Corollary 4.2, the left hand side tends to $\lambda + \log d$ as $n \to \infty$. This completes the proof. □

We are now in position to prove the main result of this paper.

Theorem 4.4. *If $\mathcal{V}$ is small enough and $f \in \mathcal{V}$, then*

$$\lambda_1 + \lambda_2 = \log d + \lambda(f|_\Pi).$$

Proof. This follows from Theorem 4.3 and Corollary 3.3. □

As a special case we consider mappings of the type $f(z, w) = (p(z, w), w^d)$, where p is a homogeneous polynomial of degree d such that $p(z, 0) = 0$ only if $z = 0$. Then the corresponding rational map $\hat{f}$ is the polynomial map $\hat{f}(\zeta) = p(\zeta, 1)$. If now $\mathcal{V}$ is small and $f \in \mathcal{V}$, then $\hat{f}$ is close to the map $\zeta \to \zeta^d$ and the Lyapunov exponent is $\log d$ by the Brolin-Manning formula (0.2). By Theorem 4.4 we therefore get

Corollary 4.5. *If $\mathcal{V}$ is sufficiently small and $f \in \mathcal{V}$ is a map of the form $f(z,w) = (p(z,w), q(z,w))$ where the homogeneous part of $q(z,w)$ of degree d is the monomial w^d, then the Lyapunov exponents of f satisfy $\lambda_1 + \lambda_2 = 2\log d$.*

References

[B] Briend, J. Exposants de Liapounoff et points périodiques d'endomorphismes holomorphes de $\mathbf{CP}^k$. *C. R. Acad. Sci. Paris Sér. I Math.* 323:805–808, 1996.

[BJ] Bedford, E., Jonsson, M. On dynamics of regular polynomial endomorphisms of $\mathbf{C}^N$. In preparation.

[BS1] Bedford, E., Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$: currents, equilibrium measure and hyperbolicity. *Invent. Math.*, 103:69–99, 1991.

[BS2] Bedford, E., Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$ V: critical points and Lyapunov exponents. To appear in *J. Geom. Anal.*.

[FLM] Freire, A., Lopez, A., Mañé, R. An invariant measure for rational maps. *Bol. Soc. Bras. Mat.*, 6:45–62, 1983.

[FS1] Fornæss, J.E., Sibony, N. Complex dynamics in higher dimension. I. *Astérisque*, 222:201–231, 1994.

[FS2] Fornæss, J.E., Sibony, N. Complex dynamics in higher dimension. II. In Bloom, T. et al, editor, *Modern Methods in Complex Analysis*, number 137 in Annals of Mathematics Studies. Princeton University Press, 1995.

[FS3] Fornæss, J.E., Sibony, N. Complex dynamics in higher dimension. In Gauthier, P. M., Sabidussi, G., editor, *Complex Potential Theory*, pages 131–186. Kluwer Academic Publishers, 1994.

[FS4] Fornæss, J.E., Sibony, N. Hyperbolic maps on $\mathbf{P}^2$. Preprint.

[HP] Hubbard, J.H., Papadopol, P. Superattractive fixed points in $\mathbf{C}^n$. *Indiana Univ. Math. J.*, 43:321–365, 1994.

[K] Klimek, M. *Pluripotential theory.* Number 6 in London Mathematical Society Monographs, New Series. Oxford Science Publications, 1991.

[L] Lyubich, M. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dynam. Systems*, 3:351–385, 1983.

[R] Ruelle, D. *Elements of differentiable dynamics and bifurcation theory.* Academic Press, 1989.

[Y] Young, L.S. Ergodic theory of differentiable dynamical systems. In Branner, B., Hjorth, P., editor, *Real and complex dynamical systems*, pages 293–336. Kluwer Academic Publishers, 1995.

DEPT. OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, 100 44 STOCKHOLM, SWEDEN
E-mail address: mjo@math.kth.se

Paper VI

REGULAR POLYNOMIAL ENDOMORPHISMS OF $\mathbf{C}^k$

ERIC BEDFORD AND MATTIAS JONSSON

0. Introduction

We consider a polynomial mapping $f : \mathbf{C}^k \to \mathbf{C}^k$, $k > 1$ as a dynamical system. We let $\mathbf{P}^k$ denote complex projective space and view $\mathbf{C}^k$ as an affine coordinate chart. Thus $\Pi := \mathbf{P}^k - \mathbf{C}^k$ is isomorphic to $\mathbf{P}^{k-1}$ and will be considered as the hyperplane at infinity. We study mappings f of degree $d \geq 2$ which have a continuous (and thus holomorphic) extension to $\mathbf{P}^k$. It follows that the hyperplane Π is completely invariant, i.e. $\Pi = f(\Pi) = f^{-1}(\Pi)$. We let f_Π denote the induced dynamical system at infinity. Π is (super)-attracting in the normal direction, so the basin A of points which are attracted to Π in forward time is an open set containing Π.

To study the dynamics of f, we follow the approach introduced in [HP] and developed more generally and systematically in [FS1-3]. Namely, there is an invariant current T, and the exterior powers $T^l := T \wedge \cdots \wedge T$, $1 \leq l \leq k$, are well defined, positive, closed currents of bidimension $(k-l, k-l)$. The supports $J_l := \mathrm{supp}(T^l)$ serve as a family of intermediate Julia sets. In this paper we will be concerned with the measures $\mu := T^k$ and $\mu_\Pi := T_\Pi^{k-1}$ (corresponding to f_Π). In favor of brevity we denote their supports by J and J_Π, respectively. Equally important will be the restriction of T^{k-1} to A, which will be written $T^{k-1} \llcorner A$.

In the study of the dynamics of a one-dimensional polynomial $p : \mathbf{C} \to \mathbf{C}$, a special role is played by the point at infinity. There is a conformal mapping φ, the Böttcher coordinate, which is defined in a neighborhood of infinity, and which conjugates p to the canonical model $\zeta \to \zeta^d$ near infinity. If the filled Julia set K is connected, then φ in fact extends to a conformal equivalence $\varphi : \mathbf{C} - K \to \mathbf{C} - \bar{\mathbf{D}}$. This leads naturally to the study of J in terms of external rays, a powerful tool developed by Douady and Hubbard [DH]. The point at infinity, being completely invariant, can also play the role of the pole for a Green function for the set $\mathbf{C} - K$; this serves as the starting point for the use of potential-theoretic methods in the study of polynomial mappings, as was introduced by Brolin [B] and further developed by Sibony (see [CG]) and Tortrat [T].

In our study of polynomial mappings of $\mathbf{C}^k$, we will use the function G, defined in (1.1), which measures the superexponential rate at which orbits approach Π. If we set $K := \mathbf{P}^k - A \subset \mathbf{C}^k$, then G is the pluricomplex Green function for K with logarithmic pole along Π. We will replace the point at infinity in the one-dimensional case by J_Π, the Julia set at infinity. And we will replace the one-dimensional set $\mathbf{C} - K$ by the set $W^s(J_\Pi)$, consisting of points which are attracted to J_Π in forward time. The main objective of this paper is to study $T^{k-1} \llcorner A$. We show that, if f_Π is (uniformly) expanding on J_Π, the support of $T^{k-1} \llcorner A$ is $W^s(J_\Pi)$, and we show how $T^{k-1} \llcorner A$ provides a connection between μ_Π and μ.

Define a small neighborhood A_0 of Π by $A_0 := \{G > \log \frac{1}{\epsilon}\}$. Results in hyperbolic dynamics imply that $W^s(J_\Pi) \cap A_0$ is equal to the union of complex disks $W_0^s(a)$, each of which is properly embedded in A_0. It follows that $W^s(J_\Pi)$ has the structure of a Riemann surface lamination outside the set $\bigcup_{n \geq 0} f^{-n}(\mathcal{C})$, where $\mathcal{C}$ is the hypersurface of critical points of f in $\mathbf{C}^k$, i.e. the set where f is not locally invertible. We denote this lamination by $\mathcal{W}^s(J_\Pi)$.

Our first use of $\mathcal{W}^s(J_\Pi)$ is to obtain a laminar structure for $T^{k-1} \llcorner A$. In general the leaves of $\mathcal{W}^s(J_\Pi)$ can be dense in $W^s(J_\Pi)$. However, we show that there is a (small) closed subset S of A such that $W^s(J_\Pi) - S$ is a union of Riemann surfaces $\{W_a : a \in J_\Pi\}$, and each W_a has the structure of a complex disk which is properly embedded in $A - S$. For μ_Π-almost every $a \in J_\Pi$, the disk W_a defines a current of integration $[W_a]$ with finite mass, and $T^{k-1} \llcorner A$ is a laminar current of the form $\int [W_a]\, \mu_\Pi(a)$.

Second, we investigate the dynamics of f on $W^s(J_\Pi)$. Let f_h be the homogeneous part of f of degree d, and let G_h be the homogeneous Green function for f_h. The sets $W_h^s(J_\Pi)$, $W_{h,0}^s(J_\Pi)$ and $W_{h,0}^s(a)$ are defined in the same way as $W^s(J_\Pi)$, $W_0^s(J_\Pi)$ and $W_0^s(a)$, but using f_h instead of f. We show that the restriction of f to $W_0^s(J_\Pi)$ is conjugate to the restriction of f_h to $W_{h,0}^s(J_\Pi)$. Moreover, this conjugation can be extended as follows. There is a closed subset S_h of $W_h^s(J_\Pi)$, such that restriction of f to $W^s(J_\Pi) - S$ is conjugate to $W_h^s(J_\Pi) - S_h$. The set S_h is a union of rays of the form $\{z = as; 1 \leq s \leq r\}$, where $a \in \mathbf{C}^k$, $G_h(a) = 0$ and $r > 1$.

Third, we use the stable lamination $\mathcal{W}^s(J_\Pi)$ to construct a family of curves $\mathcal{E}$ which play a role analogous to that played by the external rays for a polynomial mapping of $\mathbf{C}$. An external ray $\gamma \in \mathcal{E}$ corresponds to the image of a radial line in $\mathcal{H}$ under the conjugacy mentioned in the previous paragraph. (Equivalently, external rays are lines where the harmonic conjugate of the restriction $G|_{W_a}$ is locally constant.) We may identify $\mathcal{E}$ with $J_\Pi \times S^1$, and we consider the measure $\mu_\Pi \otimes \frac{d\theta}{2\pi}$ on $\mathcal{E}$. For $\mu_\Pi \otimes \frac{d\theta}{2\pi}$-almost every point $(a, \theta) \in \mathcal{E}$, the corresponding ray $\gamma(a, \theta)$ has a well defined endpoint $e(a, \theta) \in J$. Further,

$$e_*(\mu_\Pi \otimes \frac{d\theta}{2\pi}) = \mu.$$

Fourth, we consider Lyapunov exponents. As in [BS2] we find it useful to work with a measure μ_c on the set of critical points; this measure is defined by $\mu_c := [\mathcal{C}] \wedge (T^{k-1} \llcorner A)$. If ν is an ergodic measure, then by general results of smooth ergodic theory, there are Lyapunov exponents $\lambda_j(\nu)$, $1 \leq j \leq k$. We consider the quantity $\Lambda(\nu) = \sum_j \lambda_j(\nu)$, which gives the time average of the infinitesimal rate of volume growth, $\lim_{n\to\infty} \frac{1}{n} \log|\det Df^n(x)|$, of f at ν-almost every point x. By [FS3] the measures μ and μ_Π are ergodic. Using the laminar structure of $T^{k-1} \llcorner A$, we show that the values of Λ for μ and μ_Π are related as follows:

$$\Lambda(\mu) = \Lambda(\mu_\Pi) + \log d + \int G\mu_c. \tag{5.1}$$

This generalizes the formula for polynomials in dimension one, as formulated by Przytycky [Pr], as well as the 2-dimensional formula of Jonsson [J1].

Finally, we restrict ourselves to $k = 2$ and the case where $\mu_c = 0$, i.e., where there are no critical points in the sense that $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$. It follows that $S = \emptyset$, so each disk W_a is properly embedded in A. An interesting question is whether the endpoint map e defined above is continuous. Examples from one-dimensional dynamics show that this is not always true. However, we show that if f satisfies a suitable hyperbolicity condition (‡), then e maps $\mathcal{E}$ Hölder continuously onto J. Conditions which together imply (‡) are: (1) f is Axiom A, (2) all repelling periodic points belong to J, and (3) $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$.

The difficult part in proving that e is continuous is to show that the boundary of any disk W_a accumulates only at J. To do this we must show that there are no saddle connections; i.e., there can be no intersection between $W^s(J_\Pi)$ and $W^u(S_1)$, where S_1 is the part of the nonwandering set in $\mathbf{C}^2$ which is hyperbolic of index one. One consequence of our result is that J is given as a topological quotient of the space $J_\Pi \times \partial\mathbf{D}$. Our hope is that this will provide a starting point for a more detailed study of the topology of J.

Let us note that related results have been obtained by other authors. Corollary 4.4 and Theorem 6.5 have been obtained independently by G. Peng [Pe]. Hubbard and Papadopol [HP] have considered a problem which is in some sense dual to what we have described above, and which was influential in motivating our approach. In particular, they considered the case of a superattracting fixed point at the origin and interpreted $\mathbf{P}^{k-1}$ as the fiber of the blow-up of $\mathbf{C}^k$ over the origin. The dynamics of regular polynomial endomorphisms were also studied by Heinemann [H], who focused on the behavior of f on K rather than the behavior at infinity. Finally, Fornæss and Sibony [FS4] considered hyperbolic maps on $\mathbf{P}^2$; their results apply to provide different proofs of Theorem 4.1 and Corollary 4.7.

The organization of the paper is as follows. In the first two sections we recall results on Green functions and invariant currents for general regular polynomial endomorphisms of $\mathbf{C}^k$. Then, in Section 3 we start assuming that f_Π is expanding on J_Π and show that $W^s_0(J_\Pi)$ is the union of local stable disks $W^s_0(a)$. The laminar structure of $T^{k-1} \llcorner A$ is discussed in Section 4 and used in Section 5 to prove (5.1). We define external rays in Section 6 and describe how they provide a link between the measures μ and μ_Π. In the same section we discuss

Böttcher coordinates, which provide a model for the dynamics of f on $W^s(J_\Pi)$. Finally, in the last two sections, we focus on regular polynomial endomorphisms of $\mathbf{C}^2$ satisfying suitable hyperbolicity conditions; in particular we introduce condition (‡). Different hyperbolicity conditions are discussed in Section 7 and in Section 8 we prove that if f satisfies (‡), then the endpoint map e maps $\mathcal{E}$ Hölder continuously onto J. For the convenience of the reader we have gathered some basic facts on hyperbolicity for endomorphisms in Appendix A.

List of notation

f	regular polynomial endomorphism of $\mathbf{C}^k$ of degree d.
f_h	homogeneous part of f of degree d.
Π	hyperplane at infinity.
f_Π	restriction of f to Π.
π	projection of $\mathbf{C}^k - \{0\}$ on Π or $\mathbf{C}^{k+1} - \{0\}$ on $\mathbf{P}^k$.
A	basin of Π for f.
A_h	basin of Π for f_h.
K	complement of A.
G	Green function for f.
G_h	homogeneous Green function for f_h.
ρ_G	Robin function for G.
T	invariant current for f.
T_h	invariant current for f_h.
T_Π	invariant current for f_Π.
μ	T^k.
μ_Π	T_Π^{k-1}.
J	support of μ.
J_Π	support of μ_Π.
$W^s(J_\Pi)$	stable set of J_Π for f.
$W^s_h(J_\Pi)$	stable set of J_Π for f_h.
A_0	subset of A where $G > \log\frac{1}{\epsilon}$.
$A_{h,0}$	subset of A_h where $G_h > \log\frac{1}{\epsilon}$.
A_n	$f^{-n}(A_0)$.
$W^s_0(J_\Pi)$	$W^s(J_\Pi) \cap A_0$.
$W^s_{h,0}(J_\Pi)$	$W^s_h(J_\Pi) \cap A_{h,0}$.
$W^s_0(a)$	local stable disk for f at $a \in J_\Pi$.
$W^s_{h,0}(a)$	local stable disk for f_h at a.

$W^s(a)$	global stable manifold of a.
$\Lambda(\mu)$	sum of Lyapunov exponents of f with respect to μ.
$\Lambda(\mu_\Pi)$	sum of Lyapunov exponents of f_Π with respect to μ_Π.
$\mathcal{C}$	critical set of f.
μ_c	critical measure.
$\mathcal{W}^s(J_\Pi)$	stable lamination for f.
$\mathcal{C}_{-\infty}$	$\bigcup_{n\geq 0} f^{-n}(\mathcal{C})$.
S	union of "bad" gradient lines in $\mathcal{W}^s(J_\Pi)$.
S_h	union of gradient lines in $W^s_h(J_\Pi)$.
$\tilde{W}^s(J_\Pi)$	$W^s(J_\Pi) - S$.
W_a	component of $W^s(a) - S$ containing a.
$\mathcal{E}$	set of external rays.
S_1	union of basic sets of unstable index 1.
S_2	union of basic sets of unstable index 2.
$W^u(J)$	backwards attracting basin for J.
$W^u(S_1)$	unstable set of S_1.
$W^s_\delta(p)$	local stable manifold at p.
$W^u_\delta(\hat{q})$	local unstable manifold at $\hat{q}$.
$W^s(p)$	global stable manifold at p.
$W^u_\delta(\hat{q})$	global unstable manifold at $\hat{q}$.

1. Regular polynomial endomorphisms and their Green functions

In the following two sections we summarize several basic results that we will use. These may be found in [HP], [FS1-3], and [U]. We recommend the unified treatment in [FS3]. Throughout this paper, we will let f be a regular polynomial endomorphism of $\mathbf{C}^k$ of degree $d \geq 2$. This means that the components of f are polynomials of degree d and that the homogeneous part f_h of degree d of f satisfies $f_h^{-1}(0) = \{0\}$. Alternatively, f is regular if and only if $\liminf |f(z)|/|z|^d > 0$ as $|z| \to \infty$.

We will use the compactification $\mathbf{P}^k$ of $\mathbf{C}^k$, i.e. we let $z = (z_1, \ldots, z_k)$ denote (inhomogeneous) coordinates on $\mathbf{C}^k$, and $[z : t] = [z_1 : \ldots : z_k : t]$ denote homogeneous coordinates on $\mathbf{P}^k$, and we choose the embedding of $\mathbf{C}^k$ in $\mathbf{P}^k$ given by $z \mapsto [z : 1]$. In the same notation, let $\Pi = \{t = 0\}$ be the hyperplane at infinity of $\mathbf{P}^k$. Then Π may be identified with $\mathbf{P}^{k-1}$ using homogeneous coordinates $[z] = [z_1 : \ldots : z_k]$. We equip $\mathbf{P}^k$ with the Fubini-Study metric and measure distances and volumes in that metric unless otherwise stated.

A regular polynomial endomorphism f extends to an endomorphism of $\mathbf{P}^k$, still denoted by f, by the formula $f[z : t] = [t^d f(z/t) : t^d]$. In fact, a holomorphic endomorphism of $\mathbf{P}^k$ has a completely invariant hyperplane exactly when it may be identified with a regular polynomial endomorphism of $\mathbf{C}^k$.

There is a projection $\pi : \mathbf{C}^k - \{0\} \to \mathbf{P}^{k-1}$ given by $\pi(z) = [z]$. It is clear that the extension of π to $\mathbf{P}^k - \{[0:1]\}$ given by $\pi[z:t] = [z]$ is holomorphic and that the restriction of π to Π is the identity, with the identification $\Pi \simeq \mathbf{P}^{k-1}$ above.

The hyperplane Π is completely invariant under f. In fact, the set of regular polynomial endomorphisms of $\mathbf{C}^k$ can easily be identified with the set of holomorphic endomorphisms $\mathbf{P}^k$ having a completely invariant hyperplane. Under the identification $\Pi \simeq \mathbf{P}^{k-1}$ the restriction of f to Π is a holomorphic endomorphism of $\mathbf{P}^{k-1}$, which in homogeneous coordinates is given by $[z] \to [f_h(z)]$. When precision is needed, we will denote the map f on $\mathbf{C}^k$, $\mathbf{P}^k$ and Π by $f_{\mathbf{C}^k}$, $f_{\mathbf{P}^k}$ and f_Π, respectively.

We let K be the compact set of points in $\mathbf{C}^k$ with bounded forward orbits and define $A := \mathbf{P}^k - K$. The function

$$G(z) = \lim_{n\to\infty} d^{-n} \log^+ |f^n(z)| \tag{1.1}$$

gives the (super-exponential) rate at which the orbit of $z \in \mathbf{C}^k$ approaches Π. This is continuous and plurisubharmonic on $\mathbf{C}^k$ and coincides with the pluricomplex Green function of K. We will therefore also call G the Green function of f. The homogeneous Green function for the homogeneous part f_h of f of maximal degree d is defined in an analogous way, namely as

$$G_h(z) = \lim_{n\to\infty} d^{-n} \log |f_h^n(z)|. \tag{1.2}$$

The functions G and G_h are continuous on $\mathbf{C}^k$ and $\mathbf{C}^k - \{0\}$, respectively. We use log instead of $\log^+$ so that G_h is logarithmically homogeneous.

It will also be useful to define maps $\tilde{f}$ and $\tilde{f}_h$ on $\mathbf{C}^{k+1}$ by the formulae

$$\tilde{f}(z,t) = (t^d f(z/t), t^d)$$
$$\tilde{f}_h(z,t) = (t^d f_h(z/t), t^d).$$

The homogeneous Green functions for $\tilde{f}$ and $\tilde{f}_h$ are given by

$$\tilde{G}(z,t) = \lim_{n\to\infty} d^{-n} \log |\tilde{f}^n(z,t)|$$
$$\tilde{G}_h(z,t) = \lim_{n\to\infty} d^{-n} \log |\tilde{f}_h^n(z,t)|.$$

for $(z,t) \in \mathbf{C}^{k+1} - \{0\}$.

The projection $\pi : \mathbf{C}^{k+1} - \{0\} \to \mathbf{P}^k$ given by $\pi(z,t) = [z:t]$ semiconjugates $\tilde{f}$ to f, and $\tilde{f}_h$ to f_h, i.e. $f \circ \pi = \pi \circ \tilde{f}$ and $f_h \circ \pi = \pi \circ \tilde{f}_h$. We have used the letter π for two different projections but it should be clear in each case which one we are refering to.

From the definitions we easily prove the following result.

Lemma 1.1. *The Green functions satisfy the following relations (whenever they are defined)*

$$\begin{aligned} G(f(z)) &= d\cdot G(z), \\ G_h(f_h(z)) &= d\cdot G_h(z), \\ \tilde{G}(\tilde{f}(z,t)) &= d\cdot \tilde{G}(z,t), \\ \tilde{G_h}(\tilde{f_h}(z,t)) &= d\cdot \tilde{G_h}(z,t), \\ \tilde{G}(z,t) &= G(z/t)+\log|t|, \\ \tilde{G_h}(z,t) &= G_h(z) = \tilde{G}(z,0). \end{aligned}$$

It is easy to see that $G(z)$ and $G_h(z)$ behave like $\log|z|+O(1)$ as $|z|\to\infty$. Later on we will need the following more precise result.

Lemma 1.2. *The asymptotics of G and G_h at Π are given by*

$$\begin{aligned} G_h(z) &= \log|z|+\rho_G[z] \\ G(z) &= \log|z|+\rho_G[z]+o(1), \end{aligned}$$

where $[x]$ is the projection of $\mathbf{C}^k-\{0\}$ on Π defined above and ρ_G is continuous on Π.

Remark. ρ_G is the Robin function for G (cf. [BT]).

Proof. Since G_h is homogeneous we have

$$G_h(z)=\log|z|+G_h(z/|z|).$$

Here the second term is continuous in z and depends only on the projection $[z]$ of z on Π. Hence there exists a continuous function ρ_G on Π such that $G_h(z/|z|)=\rho_G[z]$. This proves the first formula. To prove the second we use Lemma 1.1 and write

$$\begin{aligned} G(z) &= \tilde{G}(z,1) \\ &= \log|z|+\tilde{G}(z/|z|,0)+(\tilde{G}(z/|z|,1/|z|)-\tilde{G}(z/|z|,0)) \\ &= \log|z|+\rho_G[z]+o(1), \end{aligned}$$

where the last line follows from the continuity of $\tilde{G}$ on $\mathbf{C}^{k+1}-\{0\}$. □

2. Invariant Currents

Using Green functions we may define invariant currents; see [FS3] for a more general discussion of these. The purpose of this section is to recall some definitions and to see how the different invariant currents are related.

To begin, we have positive closed currents $T_{\mathbf{C}^k}$ and $T_{h,\mathbf{C}^k}$ on $\mathbf{C}^k$ defined by $T_{\mathbf{C}^k}=\frac{1}{2\pi}dd^cG$ and $T_{h,\mathbf{C}^k}=\frac{1}{2\pi}dd^cG_h$. We also have positive closed currents

$T_{\mathbf{P}^k}$ and $T_{h,\mathbf{P}^k}$ on $\mathbf{P}^k$ defined by $\pi^*(T_{\mathbf{P}^k}) = \frac{1}{2\pi} dd^c \tilde{G}$ and $\pi^*(T_{h,\mathbf{P}^k}) = \frac{1}{2\pi} dd^c \tilde{G}_h$, where $\pi : \mathbf{C}^{k+1} - \{0\} \to \mathbf{P}^k$ is the projection. The last two currents can be calculated explicitly as follows. Let $U \subset \mathbf{P}^k$ be an open set and $s : U \to \mathbf{C}^{k+1}$ a holomorphic section of the line bundle $\pi : \mathbf{C}^{k+1} - \{0\} \to \mathbf{P}^k$. Then $T \llcorner U = \frac{1}{2\pi} dd^c(\tilde{G} \circ s)$ and $T_h \llcorner U = \frac{1}{2\pi} dd^c(\tilde{G}_h \circ s)$, where $T \llcorner U$ denotes the restriction of T to U. Finally, there is a unique, positive, closed current T_Π on Π with the property that $\pi^*(T_\Pi) = T_{h,\mathbf{C}^k}$ with the projection $\pi : \mathbf{C}^k - \{0\} \to \Pi$. To explain this last assertion, we note that, a priori, $\pi^*(T_\Pi)$ is defined only as a current on $\mathbf{C}^k - \{0\}$. However, it is a positive current, so we may extend it to $\mathbf{P}^k$ by assigning it to have mass zero on $\{0\}$. Correspondingly, on the right hand side of the equation, since G_h is continuous on $\mathbf{C}^k - \{0\}$, it follows that $T_{h,\mathbf{C}^k}$ puts no mass on the origin $\{0\}$. Finally, let us note that the current T_Π may be constructed explicitly, using a section of the bundle $\pi : \mathbf{C}^k - \{0\} \to \Pi$.

We remark that if U_k is the open subset of $\mathbf{P}^k$ where $z_k \neq 0$, then

$$[z_1 : \ldots : z_k : t] \to (z_1/z_k, \ldots, z_{k-1}/z_k, 1, t/z_k)$$

is a section of the bundle $\pi : \mathbf{C}^{k+1} - \{0\} \to \mathbf{P}^k$ on U_k, and

$$[z_1 : \ldots : z_k] \to (z_1/z_k, \ldots, z_{k-1}/z_k, 1)$$

is a section of $\pi : \mathbf{C}^k - \{0\} \to \Pi$ on $U_k \cap \Pi$. Similar sections can be defined on the sets $U_j = \{z_j \neq 0\}$ and $U_j \cap \Pi$ for $j = 1, \ldots, k-1$. We also note that $[z : 1] \mapsto (z, 1)$ is a section of $\pi : \mathbf{C}^{k+1} - \{0\} \to \mathbf{P}^k$ over $\{t \neq 0\}$.

The currents above are related, as will be shown in the following lemma. We recall that if $S = \frac{1}{2\pi} dd^c u$ is a positive closed current of bidegree $(1,1)$ on a complex manifold with continuous potential u, and M is a complex submanifold, then the slice $S|_M$ is well-defined and is equal to the current on M defined by $S|_M = \frac{1}{2\pi}(dd^c)|_M(u|_M)$.

Lemma 2.1. *The invariant currents defined above are related as follows.*

$$\begin{aligned} T_{\mathbf{C}^k} &= T_{\mathbf{P}^k} \llcorner \mathbf{C}^k \\ T_{h,\mathbf{C}^k} &= T_{h,\mathbf{P}^k} \llcorner \mathbf{C}^k \\ T_\Pi &= (T_{\mathbf{P}^k})|_\Pi = (T_{h,\mathbf{P}^k})|_\Pi. \end{aligned}$$

Proof. These statements follow easily from the remarks above and Lemma 1.1. To prove the first relation we observe that

$$\begin{aligned} T_{\mathbf{P}^k} \llcorner \mathbf{C}^k &= \frac{1}{2\pi} dd^c \tilde{G}(z,1) \\ &= \frac{1}{2\pi} dd^c G(z) \\ &= T_{\mathbf{C}^k}. \end{aligned}$$

The second relation is proved in the same way. As for the third, we have

$$\begin{aligned}(T_{\mathbf{P}^k}|_\Pi)\llcorner(z_k \neq 0) &= \frac{1}{2\pi}dd^c\tilde{G}(z_1/z_k,\ldots,z_{k-1}/z_k,1,0)\\ &= \frac{1}{2\pi}dd^cG_h(z_1/z_k,\ldots,z_{k-1}/z_k,1)\\ &= T_\Pi\llcorner(z_k \neq 0).\end{aligned}$$

Permuting the variables, we obtain $(T_{\mathbf{P}^k}|_\Pi)\llcorner(z_j \neq 0) = T_\Pi\llcorner(z_j \neq 0)$ for $1 \leq j \leq k-1$, so $T_{\mathbf{P}^k}|_\Pi = T_\Pi$. The proof that $T_{h,\mathbf{P}^k}|_\Pi = T_\Pi$ is identical. □

In view of Lemma 2.1 we may simplify our notation and use only the currents $T = T_{\mathbf{P}^k}$ and $T_h = T_{h,\mathbf{P}^k}$ on $\mathbf{P}^k$, restricting them to $\mathbf{C}^k$ or taking the slice on Π whenever needed.

Although the maps f and f_h are not submersions, they define pullbacks of the invariant currents T and T_h.

Lemma 2.2. *The following relations hold*

$$\begin{aligned}f^*T &= d\cdot T,\\ f_h^*T_h &= d\cdot T_h,\\ f_\Pi^*(T_\Pi) &= d\cdot T_\Pi.\end{aligned}$$

Since the currents T and T_h have continuous local potentials, we may define T^j and T_h^j for $1 \leq j \leq k$; these are positive closed currents of bidegree (j,j) which satisfy $f^*(T^j) = d^jT^j$ and $f_h^*(T_h^j) = d^jT_h^j$. Most important for us will be the currents T^{k-1}, T_h^{k-1}, of bidimension (1,1), and $\mu := T^k$ and $\mu_\Pi := (T|_\Pi)^{k-1}$, of bidimension (0,0). Note that μ and μ_Π are represented by probability measures on $\mathbf{C}^k$ and Π, respectively. We will denote their supports by $J = \mathrm{supp}(\mu)$ and $J_\Pi = \mathrm{supp}(\mu_\Pi)$.

Remark. In the notation of [HP] the latter two sets would be called J_k and $J_{\Pi,k-1}$, respectively. We use J and J_Π for brevity, as we will not be using the other intermediate Julia sets.

Proposition 2.3. *The following formula holds on* $\mathbf{P}^k$.

$$T_h^{k-1} = \int \left[\pi^{-1}(a)\right]\,\mu_\Pi(a). \tag{2.1}$$

Proof. We know that $\pi^*(T_\Pi) = T_h\llcorner(\mathbf{C}^k - \{0\})$. Thus $T_h^{k-1}\llcorner(\mathbf{C}^k - \{0\}) = \pi^*(T_\Pi^{k-1}) = \pi^*(\mu_\Pi)$. Hence, by the definition of π^* as integration over the fibers of π, we have

$$T_h^{k-1}\llcorner(\mathbf{C}^k - \{0\}) = \int \left[\pi^{-1}(a) - \{a\}\right]\,\mu_\Pi(a).$$

Now we note that since T_h has continuous local potentials on $\mathbf{P}^k - \{[0:1]\}$, the current T_h^{k-1} puts no mass at $\{0\}$ or at Π. Hence T_h^{k-1} coincides with the trivial extension of $T_h^{k-1} \llcorner (\mathbf{C}^k - \{0\})$ to $\mathbf{P}^k$. Further, since $\{a\}$ is a set of measure zero with respect to $[\pi^{-1}(a)]$, it follows that $[\pi^{-1}(a)]$ and $[\pi^{-1}(a) - \{a\}]$ define the same current on $\mathbf{P}^k$. Thus the equation above yields (2.1). $\square$

Let $\mathcal{C}_{\mathbf{C}^k}$, $\mathcal{C}_{\mathbf{P}^k}$, and $\mathcal{C}_\Pi$ be the critical sets of $f_{\mathbf{C}^k}$, $f_{\mathbf{P}^k}$, and f_Π, respectively. Thus we have

$$\mathcal{C}_{\mathbf{P}^k} = \mathcal{C}_{\mathbf{C}^k} \cup \Pi$$

$$\mathcal{C}_\Pi = \overline{\mathcal{C}_{\mathbf{C}^k}} \cap \Pi.$$

3. Local stable disks near J_Π

Everything said so far is true for all regular polynomial endomorphisms of $\mathbf{C}^k$. We now want to understand the dynamics of f on the stable set of J_Π, i.e.

$$W^s(J_\Pi) = \{x \in \mathbf{P}^k; d(f^n(x), J_\Pi) \to 0\}.$$

To do this successfully, we impose restrictions on the dynamics of f. Namely, we assume that f_Π is (uniformly) expanding on J_Π. This means that there exist constants $c > 0$ and $\lambda > 1$ such that

$$|Df_x^n v| \geq c\lambda^n |v| \quad x \in J_\Pi,\ v \in T_x\Pi,\ n \geq 1. \tag{3.1}$$

If f is expanding on J_Π and $a \in J_\Pi$, then the tangent space $T_a\mathbf{P}^k$ splits into a direct sum $E^u(a) \oplus E^s(a)$, where $E^u(a) = T_a\Pi$ and $E^s(a)$ is the eigenspace of Df_a associated with the zero eigenvalue. We clearly have $Df_a(E^{u/s}(a)) \subset E^{u/s}(f_\Pi(a))$, and $E^{u/s}(a)$ depends continuously on a. Therefore, with the definition given in Appendix A, f_Π is hyperbolic on J_Π.

The expansion of f_Π on J_Π will allow us to understand the structure of $W^s(J_\Pi)$. In this section we will restrict our attention to a small neighborhood of Π, so let $A_0 := \{G > \log \frac{1}{\epsilon}\}$ and $W_0^s(J_\Pi) := W^s(J_\Pi) \cap A_0$.

The stable manifold theorem asserts that there is a local stable manifold at each point of J_Π. The global stable manifold $W^s(a)$ of $a \in J_\Pi$ is by definition the set of points $x \in \mathbf{P}^k$ such that $d(f^n(x), f^n(a)) \to 0$ as $n \to \infty$. We now define the local stable disk $W_0^s(a)$ at $a \in J_\Pi$ to be the connected component of $W^s(a) \cap A_0$ containing a. By a *complex disk* we will mean the image of a holomorphic injective immersion of the unit disk into $\mathbf{P}^k$.

Theorem 3.1. *If ϵ is small enough, then $W_0^s(a)$ is a complex disk which is properly embedded in A_0 for all $a \in J_\Pi$. Moreover, $W_0^s(a)$ depends continuously on a.*

It is possible to deduce Theorem 3.1 from the stable manifold theorem. We will, however, give a direct and fairly detailed proof, using the graph transform method, because we need some of the constructions in Section 6.

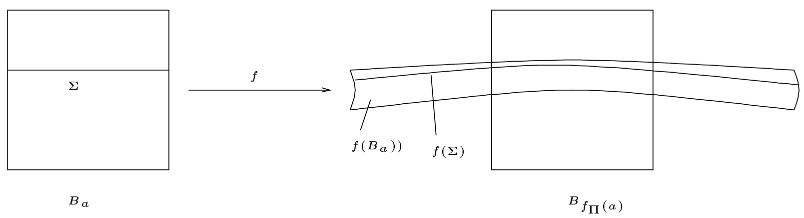
Proof. Let us embed f in a holomorphic one-parameter family f_τ, $|\tau| < 2$, defined by $f_\tau = f + \tau(f - f_h)$. We do this only because we need it in Section 6. Note that $f_0 = f_h$ is homogeneous. To avoid cumbersome notation we write f instead of f_τ. Our first task is to define good coordinate charts at the points in J_Π. Pick $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_k)$ with $\pi(\tilde{a}) = a$ and $|\tilde{a}| = 1$. Permute coordinates so that $|\tilde{a}_k| = \max_{1\le j\le k} |\tilde{a}_j|$. Let $\zeta = (\zeta_1, \dots, \zeta_{k-1})$, where $\zeta_j = z_j/z_k - \tilde{a}_j/\tilde{a}_k$ and let $t = 1/z_k$. We denote the ball $|\zeta| < \delta_1$ by $U_a = U_a(\delta_1)$, the disk $|t| < \delta_2$ by $V_a = V_a(\delta_2)$ and the polycylinder $U_a \times V_a$ by $B_a = B_a(\delta) = B_a(\delta_1, \delta_2)$ for $\delta_1 > \delta_2 > 0$. Note that Π corresponds to $\{t = 0\}$ and the line $\pi^{-1}(a)$ to $\{\zeta = 0\}$. Also, the Euclidean metric on B_a and the Fubini-Study metric on $\mathbf{P}^k$ differ by at most a multiplicative constant $C > 0$. The expansion of f_Π on J_Π implies that if $a, b \in J_\Pi$ and $a \neq b$, then there is an $n \ge 0$ such that $d(f_\Pi^n(a), f_\Pi^n(b)) > 3C\delta_1$. After replacing f by an iterate we may assume that (3.1) holds with $n = 1$, $c = 1$ and $\lambda = 3C$.

Let us introduce some more terminology. A vertical disk in B_a is a disk of the form $\{\zeta = \text{const}\}$ and a vertical-like disk is the graph of a holomorphic map $U_a \to V_a$. Similarly we define horizontal and horizontal-like disks (although, strictly speaking, these are not disks if $k > 2$).

By choosing $1 \gg \delta_1 \gg \delta_2 > 0$ we get that for all $a \in J_\Pi$ and for all $f = f_\tau$ with $|\tau| < 2$:

(1) $f(B_a) \cap B_{f_\Pi(a)} \subset U_{f_\Pi(a)} \times V_{f_\Pi(a)}(\delta_2/2)$.
(2) $f^{-1}(B_{f_\Pi(a)}) \cap B_a \subset U_a(\delta_1/2) \times V_a$.
(3) If Σ is a horizontal disk in B_a, then $f(\Sigma) \cap B_{f_\Pi(a)}$ is a horizontal-like disk in $B_{f_\Pi(a)}$ and the restriction of f to $\Sigma \cap f^{-1}(B_{f_\Pi(a)})$ is a biholomorphism.
(4) The critical set $\mathcal{C} = \mathcal{C}_{\mathbf{C}^k}$ of f does not meet B_a.
(5) If Σ' is a vertical-like disk in B_a on which G is harmonic, then $G|_{\Sigma'}$ has no critical points.

Here (1)–(3) follows from (3.1), (4) follows from the fact that $\mathcal{C}_\Pi \cap J_\Pi = \emptyset$ and (5) is a consequence of Lemma 1.2. Conditions (1)–(3) are illustrated in the picture below.



To produce stable manifolds we have to iterate backwards. We claim that

(6) If Σ' is a vertical-like disk in $B_{f_\Pi(a)}$, then $f^{-1}(\Sigma') \cap B_a$ is a vertical-like disk in B_a.

To see this, note that $f^{-1}(\Sigma') \cap B_a$ is an analytic set in B_a. Let Σ be a horizontal disk in B_a. We claim that $f(\Sigma)$ intersects Σ' in exactly one point.

Indeed, by (3) we may write $f(\Sigma)\cap B_{f_\Pi(a)}=\{t=g(\zeta)\}$ and $\Sigma'=\{\zeta=h(t)\}$, where g and h are holomorphic. Hence the intersection between these two sets is the unique fixed point of the holomorphic map $g\circ h: U_{f_\Pi(a)}(\delta_1)\to U_{f_\Pi(a)}(\delta_2/2)$. By (3) it follows that Σ intersects $f^{-1}(\Sigma')\cap B_a$ in exactly one point. This proves that the latter set is a vertical-like disk.

Now define $B_a^n = B_a\cap f^{-1}(B_{f_\Pi(a)})\cap\ldots\cap f^{-n}(B_{f_\Pi^n(a)})$ for $n\geq 0$ and $B_a^\infty=\cap_{n\geq 0}B_a^n$. The latter set is the local stable manifold of a, i.e. it consists of the points tracking a in positive time. Using the Kobayashi metric on U_a, it follows from (2) and (3) that there is a constant $\kappa>0$ such that the diameter of $\Sigma\cap B_a^n$ is less that $\kappa 2^{-n}$ for every horizontal disk Σ in B_a and all $a\in J_\Pi$. We claim that B_a^∞ is a vertical-like disk. Indeed, the estimate above implies that $\Sigma\cap B_a^\infty$ consists of at most one point for every Σ. On the other hand, repeated applications of (6) show that the set $\gamma_n(a)$, defined inductively by $\gamma_0(a)=\{0\}\times V_a$ and $\gamma_n(a)=f^{-1}(\gamma_{n-1}(f_\Pi(a)))\cap B_a$, is a vertical-like disk in B_a^n. Hence $\gamma_n(a)$ converges to a vertical-like disk. By the remark above, this disk must be exactly B_a^∞.

The sets B_a^∞ are pairwise disjoint, because if $a\neq b$, then there exists an $n\geq 0$ such that $f_\Pi^n(b)\notin B_{f_\Pi^n(a)}(3\delta)$. Hence $B^\infty_{f_\Pi^n(a)}\cap B^\infty_{f_\Pi^n(b)}=\emptyset$, so B_a^∞ and B_b^∞ are disjoint.

Note that if $\delta_2'<\delta_2$ and $\delta_1'=\delta_1$, then $B_a^\infty(\delta')$ is the restriction to $V_a(\delta_2')$ of the vertical-like disk defining $B_a^\infty(\delta)$. We next show that the disks B_a^∞ depend continuously on a. Let $\delta_2'<\delta_2$ and M be larger than the Lipschitz constant for all f_τ on $\mathbf{P}^k$. Assume that b is close to a and choose n maximal so that $M^nC^3|b-a|<(\delta_1^2+\delta_2^2)^{1/2}-(\delta_1^2+\delta_2'^2)^{1/2}$. Then B_b^∞ is contained in B_a^n, and the latter set intersects every horizontal disk in a set of diameter at most $\kappa 2^{-n}$. Hence B_a^∞ depends continuously on a.

Let $\epsilon>0$ be so small that if $x\in W_0^s(J_\Pi)$, then $x\in B_a$ for some a. By the definition of B_a^∞ we see that if $x\in A_0\cap W^s(a)$, then $f^n(x)\in B^\infty_{f_\Pi^n(a)}$ for large n. Thus $x\in B_b^\infty$ for some $b\in J_\Pi$ with $f_\Pi^n(b)=f_\Pi^n(a)$. Since the sets B_a^∞ are disjoint, it follows that the connected component $W_0^s(a)$ of $W^s(a)\cap A_0$ containing a is the subset of B_a^∞ containing a. Hence (5) implies that $W_0^s(a)$ is a complex disk, properly embedded in A_0 and depending continuously on a. □

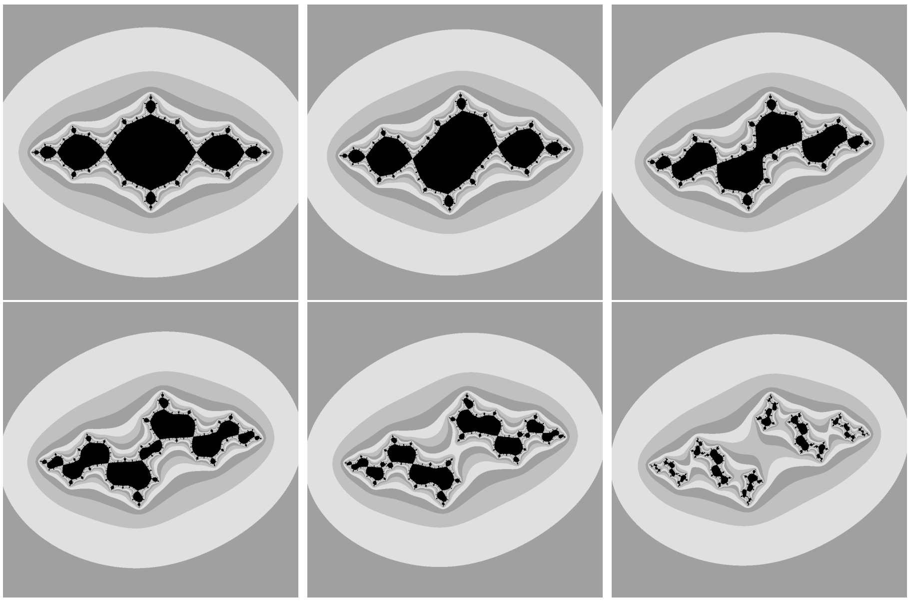
Remark. We note that the proof shows that $W^s(a)\cap A_0$ is the union of the local stable disks $W_0^s(b)$, where $f_\Pi^n(a)=f_\Pi^n(b)$ for some $n\geq 0$.

Proposition 3.2. *For $\epsilon>0$ small enough we have*

$$W_0^s(J_\Pi)=\bigcup_{a\in J_\Pi}W_0^s(a).$$

Proof. This can be proved by showing that the inverse limit space $\widehat{J_\Pi}$ has local product structure (see Proposition A.6), but we will give a direct proof. The inclusion "$\supset$" is trivial, so suppose that $x\in W_0^s(J_\Pi)$. After replacing f by an iterate we may assume that (3.1) holds with $n = c = 1$ and $\lambda = 3$. Let $M \geq 1$ be larger than the Lipschitz constant for f on $\mathbf{P}^k$. Let $\eta > 0$ be so small that if $a \in J_\Pi$,then all branches of f_Π^{-1} are single-valued on the ball $B(f_\Pi(a), 4M\eta)$ in Π and the branch mapping $f_\Pi(a)$ to a maps $B(f_\Pi(a), 4M\eta)$ into the ball $B(a, 2M\eta)$. Now let $x \in W_0^s(J_\Pi)$. Let n be so large that $d(f^{n+j}(x), J_\Pi) < \eta$ for $j \geq 0$ and pick points $a_j \in J_\Pi$ such that $d(f^{n+j}(x), a_j) < \eta$ for $j \geq 0$. Then $(a_j)_{j\geq 0}$ is an $2M\eta$-pseudoorbit in J_Π, i.e. $d(f_\Pi(a_j), a_{j+1}) < 2M\eta$. Let g_j be the branch of f_Π^{-1} on $B(f_\Pi(a_j), 4M\eta)$ mapping $f_\Pi(a_j)$ to a_j. Then $g_j(a_{j+1}) \in B(f_\Pi(a_{j-1}), 4M\eta)$ so the point $b^{(j)} := G_h \circ \ldots \circ g_j(a_{j+1})$ is well-defined. Moreover $d(f_\Pi^i(b^{(j)}), a_i) < 2M\eta$ for $0 \leq i \leq j$. Letting $j \to \infty$ and using the compactness of J_Π we find a point $b \in J_\Pi$ such that $d(f_\Pi^i(b), a_i) < 3M\eta$ for all $i \geq 0$. Hence $d(f^{n+i}(x), f_\Pi^i(b)) < 4M\eta$ for all $i \geq 0$. Assume that $4CM\eta < \delta$, with C and δ from the proof of Theorem 3.1. It follows that $f^n(x) \in W^s(b)$, so $x \in W^s(c)$, where $c \in J_\Pi$ is a point with $f_\Pi^n(c) = b$. By the remark following the proof of Theorem 3.1, this gives the corollary. $\square$

The pictures below show slices of $W^s(J_\Pi)$ by complex lines $\{z = c\}$ for the map $f(z, w) = (z^2 - 0.1, w^2 - z^2 + 0.2z - 0.5i)$. In the coordinate $\zeta = w/z$, we have $f_\Pi(\zeta) = \zeta^2 - 1$. The first picture is the Julia set of f_Π. By Proposition 3.2, the slices above converge (suitably scaled) to this picture as $c \to \infty$. The remaining five pictures show the slices by the lines $\{z = 2\}$, $\{z = 1.3\}$, $\{z = 1.2\}$, $\{z = 1.15\}$, $\{z = 1.1\}$.



4. Structure of T^{k-1} on the Basin of Π

In this section we use the local stable disks to analyze the structure of T^{k-1}

on the basin A of Π. Proposition 2.3 gives us a hint on what to expect: if f is homogeneous, then the local stable disks $W_0^s(a)$ lie inside the lines $\pi^{-1}(a)$ and $T^{k-1} \llcorner A$ is the average with respect to the measure μ_Π of the currents of integration over $\pi^{-1}(a) \cap A$.

In the non-homogeneous case the situation will be similar, but more complicated. We start by proving the laminar formula (4.1) for $T^{k-1} \llcorner A_0$ in terms of the local stable disks $W_0^s(a)$. This induces a formula for for T^{k-1} on each compact subset of A. In Section 6 we will go further and produce a laminar formula for T^{k-1} on all of A.

Finally we will deduce some dynamical consequences of the laminar structure of T^{k-1}. Namely, the support of $T^{k-1} \llcorner A$ is exactly the set $W^s(J_\Pi)$ and each global stable manifold $W^s(a)$ is dense in $W^s(J_\Pi)$.

Theorem 4.1. *If f_Π is expanding on J_Π and ϵ is small enough, then*

$$T^{k-1} \llcorner A_0 = \int [W_0^s(a)] \, \mu_\Pi(a). \tag{4.1}$$

Proof. It follows from Proposition 2.3 that

$$\begin{aligned} &\frac{1}{d^{n(k-1)}} (f^n)^* \left(T_h^{k-1} \llcorner A_0\right) \llcorner A_0 \\ &\qquad = \frac{1}{d^{n(k-1)}} (f^n)^* \left(\int \left[\pi^{-1}(a) \cap A_0\right] \mu_\Pi(a)\right) \llcorner A_0, \end{aligned}$$

for all $n \geq 0$. We claim that the left hand side tends to $T^{k-1} \llcorner A_0$ and the right hand side tends to $\int [W_0^s(a)] \, \mu_\Pi(a)$ as $n \to \infty$.

To prove the first part of the claim, it suffices to show that $d^{-n} \tilde{G}_h \circ \tilde{f}^n \to \tilde{G}$ uniformly on compact subsets of $\pi^{-1}(A_0)$. Now the function $H := \tilde{G}_h - \tilde{G}$ is continuous on $\mathbf{C}^{k+1} - \{0\}$ and satisfies $H(\alpha z, \alpha t) = H(z,t)$ for $\alpha \neq 0$. Further, $H = 0$ on $\pi^{-1}(\Pi)$ according to Lemma 1.1 and $\tilde{f}^n(\pi^{-1}(A_0)) \to \pi^{-1}(\Pi)$ as $n \to \infty$. Hence

$$\begin{aligned} \frac{1}{d^n} \tilde{G}_h \circ \tilde{f}^n - \tilde{G} &= \frac{1}{d^n} \left(\tilde{G}_h \circ \tilde{f}^n - \tilde{G} \circ \tilde{f}^n\right) \\ &= \frac{1}{d^n} H \circ \tilde{f}^n \\ &= o\left(\frac{1}{d^n}\right). \end{aligned}$$

As for the second part of the claim, we calculate

$$\begin{aligned} &\frac{1}{d^{n(k-1)}} (f^n)^* \left(\int \left[\pi^{-1}(a) \cap A_0\right] \, \mu_\Pi(a)\right) \llcorner A_0 \\ &\qquad = \int \frac{1}{d^{n(k-1)}} \left[f^{-n}\left(\pi^{-1}(a) \cap A_0\right) \cap A_0\right] \, \mu_\Pi(a). \end{aligned}$$

From the proof of Theorem 3.1 we know that $f^{-n}(\pi^{-1}(a) \cap A_0) \cap A_0$ is a union of $d^{n(k-1)}$ disjoint complex disks $\gamma_n(b)$, where b runs through the preimages of a under f^n. Hence we get

$$\begin{aligned}\int \frac{1}{d^{n(k-1)}} \left[f^{-n}\left(\pi^{-1}(a) \cap A_0\right) \cap A_0\right] \mu_\Pi(a) &= \int \frac{1}{d^{n(k-1)}} \sum_{f^n(b)=a} [\gamma_n(b)]\ \mu_\Pi(a) \\ &= \int [\gamma_n(a)]\ \frac{1}{d^{n(k-1)}}((f^n)^*\ \mu_\Pi)(a) \\ &= \int [\gamma_n(a)]\ \mu_\Pi(a).\end{aligned}$$

Moreover, from the same proof it follows that $\gamma_n(a)$ converges to the local stable disk $W_0^s(a)$. Moreover, the volumes of $\gamma_n(a)$ are uniformly bounded, so by bounded convergence the last line above converges to $\int [W_0^s(a)]\, \mu_\Pi(a)$ as $n \to \infty$, completing the proof. ☐

Theorem 4.1 allows us to describe the support of $T^{k-1} \llcorner A$ in dynamical terms.

Corollary 4.2. *If f_Π is expanding on J_Π, then* $\operatorname{supp}(T^{k-1} \llcorner A) = W^s(J_\Pi)$.

Proof. It follows from Theorem 4.1 and Proposition 3.2 that the support of $T^{k-1} \llcorner A_0$ is equal to $W^s(J_\Pi) \cap A_0$. This proves the corollary, because the sets $\operatorname{supp}(T^{k-1} \llcorner A)$ and $W^s(J_\Pi)$ are both completely invariant and any compact subset of either of them is mapped by some iterate of f into A_0. ☐

We would like to have a formula similar to (4.1) on all of A. One idea is to try to extend the complex disks $W_0^s(a)$ to closed complex varieties in A, using the fact that $f(W_0^s(a))$ is compactly contained in $W_0^s(a)$. With this in mind we define $A_n := f^{-n}(A_0) = \{G > d^{-n} \log \frac{1}{\epsilon}\}$ and

$$W_n^s(a) := f^{-n}\left(W_0^s\left(f^n(a)\right)\right),$$

for $a \in J_\Pi$ and $n \geq 0$. Then $W_n^s(a)$ is a (possibly disconnected) complex subvariety of A_n. Note that $W_n^s(a_1) = W_n^s(a_2)$ as soon as $f^n(a_1) = f^n(a_2)$, so $W_n^s(a)$ will contain $d^{n(k-1)}$ different local disks. Thus the union over $n \geq 0$ of $W_n^s(a)$ will not be a complex subvariety of A. One might try to get around this problem by taking the irreducible component containing a of $W_n^s(a)$ and hope that the union over $n \geq 0$ of these components would be a complex subvariety of A. However, this union may contain infinitely many local stable disks (see the end of this section for an example).

We will return to the problem of finding a global laminar structure for $T^{k-1} \llcorner A$ in Section 6. At any rate we can now present a formula on each compact subset of A.

Corollary 4.4. *If f_Π is expanding on J_Π and ϵ is small enough, then for every $n \geq 0$ we have*

$$T^{k-1} \llcorner A_n = \int \frac{1}{d^{n(k-1)}} [W_n^s(a)] \; \mu_\Pi(a).$$

Proof. This is an easy consequence of Theorem 4.1. Indeed,

$$\begin{aligned}
T^{k-1} \llcorner A_n &= \frac{1}{d^{n(k-1)}} (f^n)^* \left(T^{k-1} \llcorner A_0\right) \\
&= \frac{1}{d^{n(k-1)}} (f^n)^* \left(\int [W_0^s(a)] \; \mu_\Pi(a) \right) \\
&= \frac{1}{d^{n(k-1)}} \int \left[f^{-n} \left(W_0^s(a) \right) \right] \; \mu_\Pi(a) \\
&= \frac{1}{d^{n(k-1)}} \int \left[f^{-n} \left(W_0^s \left(f^n(a) \right) \right) \right] \; \mu_\Pi(a) \\
&= \int \frac{1}{d^{n(k-1)}} [W_n^s(a)] \; \mu_\Pi(a).
\end{aligned}$$

□

As indicated above, the sets $W_n^s(a)$ will be very large for large n. This is made precise by the following result.

Proposition 4.5. *If f_Π is expanding on J_Π, then for every $a \in J_\Pi$ we have*

$$\frac{1}{d^{n(k-1)}} [W_n^s(a)] \to T^{k-1} \llcorner A$$

as $n \to \infty$.

To prove Proposition 4.5 we need an auxiliary result.

Lemma 4.6. *Given $a \in J_\Pi$ and $n \geq 0$, define*

$$\mu_{\Pi,n,a} := \frac{1}{d^{n(k-1)}} \sum_{f_\Pi^n(b)=a} \delta_b = \frac{1}{d^{n(k-1)}} (f_\Pi^n)^* \delta_a.$$

Then $\mu_{\Pi,n,a} \to \mu_\Pi$ as $n \to \infty$ for every $a \in J_\Pi$.

Proof of Lemma 4.6. It is a general result of Fornæss and Sibony [FS3, Lemma 8.3] that $\mu_{\Pi,n,a} \to \mu_\Pi$ for μ_Π-a.e. $a \in J_\Pi$. In the presence of expansion, however, we can say more. Take any $a \in J_\Pi$ and a small ball U in Π containing a. We may assume that U does not intersect the postcritical set of f_Π. Then, for $n \geq 0$, $f^{-n}(U)$ is a union of $d^{n(k-1)}$ disjoint open sets, the diameter of which tend to zero uniformly as $n \to \infty$. Take any point $a' \in U$ for which $\mu_{\Pi,n,a'} \to \mu_\Pi$ as $n \to \infty$. It then follows that $\mu_{\Pi,n,a} \to \mu_\Pi$ as well. □

Proof of Proposition 4.5. For any $m \geq 0$ and $n \geq m$ we have

$$\begin{aligned}\frac{1}{d^{n(k-1)}}\left[W_n^s(a)\cap A_m\right] &= \frac{1}{d^{n(k-1)}}\left[f^{-m}\left(f^{-(n-m)}\left(W_0^s(a)\right)\cap A_0\right)\right]\\ &= \frac{1}{d^{n(k-1)}}\left[f^{-m}\left(\bigcup_{f^{n-m}(b)=a} W_0^s(b)\right)\right]\\ &= \frac{1}{d^{m(k-1)}}\left(f^m\right)^*\left(\int\left[W_0^s(b)\right]\mu_{\Pi,n-m,a}(b)\right),\end{aligned}$$

which by Lemma 4.6 converges to

$$\frac{1}{d^{m(k-1)}}\left(f^m\right)^*\left(T \llcorner A_0\right) = T \llcorner A_m$$

as $n \to \infty$. This completes the proof. □

Recall the notation $W^s(a)$ for the global stable manifold of a. Note that $W^s(a)$ is the increasing union of $W_n^s(a)$ over all $n \geq 0$.

Corollary 4.7. *If f_Π is expanding on J_Π, then $W^s(a)$ is dense in $W^s(J_\Pi)$ for all $a \in J_\Pi$.*

Proof. This is clear in view of Proposition 4.5. □

We give the following example to show that the global stable manifolds $W^s(a)$ may be quite complicated.

Example. Consider $f(z,w) = (z^2+c, w^2)$, where $c \in \mathbf{C}$ is outside the Mandelbrot set. We have $f_\Pi(\zeta) = \zeta^2$. The line $\{w = 0\}$ is completely invariant and does not intersect $W^s(J_\Pi)$. Define new coordinates on $\mathbf{C}\times\mathbf{C}^*$ by $(u,v) = \Phi(z,w) = (z/w, 1/w)$. Then Φ conjugates f to the homogeneous map $g(u,v) = (u^2+cv^2, v^2)$. Let G be the Green function for f, G_g be the homogeneous Green function for g, and let G_c be the Green function for the one-dimensional polynomial $\zeta \to \zeta^2 + c$. Then $G(z,w) = \max(G_c(z), \log|w|)$ and $G_g(u,v) = -\log|w| + G_g(z,1) = -\log|w| + G_c(z)$. Now $W^s(J_\Pi) = \{(z,w); G_c(z) = \log|w| > 0\}$ so $\Phi(W^s(J_\Pi)) = X_g$, where $X_g := \{(u,v); G_g(u,v) = 0, \pi(u,v) \notin K_c\}$. Here K_c is the filled Julia set for the polynomial map $\zeta \to \zeta^2 + c$. Hubbard and Papadopol [HP, Proposition 8.4] showed that X_g is a Riemann surface lamination, all of whose leaves are dense in X_g. Hence the same is true for $W^s(J_\Pi)$. In fact, the leaves of the Riemann surface foliation are exactly the global stable manifolds $W^s(a)$, so these are dense in $W^s(J_\Pi)$. Further evidence for the complicated structure of $W^s(a)$ was given by Barrett [B] who showed that the Corona Problem fails on the leaves of X_g. Hence it fails on $W^s(a)$ for $a \in J_\Pi$.

5. Lyapunov exponents.

In this section we prove a formula for the sum $\Lambda(f)$ of the Lyapunov exponents of a regular polynomial endomorphism f of $\mathbf{C}^k$. The only assumption we make is that f_Π is expanding on J_Π so that the current $T^{k-1} \llcorner A$ has the laminar structure given by Theorem 4.1. A special case of (5.1) below was proved in [J1].

Let us recall the notion of Lyapunov exponents. For more details we refer to [Y]. The sum of the Lyapunov exponents of f with respect to μ is the number $\Lambda(f)$ defined by the property that

$$\lim_{n\to\infty} \frac{1}{n} \log |\det Df^n(x)| = \Lambda(f),$$

for μ-a.e. $x \in \mathbf{P}^k$. That this is well-defined is part of the statement of Oseledec's Theorem. Hence $\Lambda(f)$ measures average volume growth of the map f^n at μ-a.e. point. The individual Lyapunov exponents measure the average growth of the derivative of f^n in different directions; we will not give the precise definition since we do not need it.

Our formula for $\Lambda(f)$ will involve the integral of the Green function against a critical measure so we begin by defining the latter measure as

$$\mu_c := \frac{1}{2\pi} dd^c H \wedge (T^{k-1} \llcorner A) = [\mathcal{C}] \wedge (T^{k-1} \llcorner A),$$

where $H = \log |\det Df|$.

Then μ_c is a well-defined positive measure because T has a continuous potential, and so the mass of $dd^c H \wedge T^{k-1}$ is finite. For later reference we note that the asymptotics of H at Π are given by

$$H(z) = k(d-1)\log|z| + \rho_H[z] + o(1),$$

where $[z]$ is the projection of z on Π and ρ_H is the Robin function of H. It is easy to see that ρ_H is continuous on $\Pi - \mathcal{C}_\Pi$ and depends only on the homogeneous part f_h of f of degree d.

We will need the following application of Green's formula. The proof is left to the reader.

Lemma 5.1. *Let M be a Riemann surface, a a point on M, ξ a coordinate on M with $\xi(a) = 0$ and u, v harmonic functions on $M - \{a\}$ with*

$$u(\xi) = c_u \log|\xi|^{-1} + \rho_u + o(1),$$
$$v(\xi) = c_v \log|\xi|^{-1} + \rho_v + o(1),$$

as $\xi \to 0$, where c_u,c_v,ρ_u, and ρ_v are constants. Then, if $\delta > 0$ is so small that $\bar{\mathbf{D}}_\delta \subset \xi(M)$, we have

$$\int_{|\xi|=\delta} (u d^c v - v d^c u) = c_u \rho_v - c_v \rho_u.$$

Theorem 5.2. *If f is a regular polynomial endomorphism of $\mathbf{C}^k$ with f_Π expanding on J_Π, then*

$$\Lambda(f) = \log d + \Lambda(f_\Pi) + \int G\,\mu_c. \tag{5.1}$$

Proof. From the ergodic theorem we have

$$\Lambda(f) = \int H\,\mu.$$

Fix a large number R and let χ be a test function supported in $\{G < 3R\}$ that satisfies $0 \le \chi \le 1$ and $\chi = 1$ in a neighborhood of $\{G \le 2R\}$. Then χH is continuous and compactly supported, and the integral above is equal to

$$\begin{aligned}\int \chi H (dd^cG)^k &= \int G dd^c(\chi H) \wedge (dd^cG)^{k-1}\\ &= \int \chi_1 G dd^c(\chi H) \wedge (dd^cG)^{k-1}\\ &+ \int \chi_2 G dd^c(\chi H) \wedge (dd^cG)^{k-1},\end{aligned}$$

where χ_1 is a test function supported on $\{G < 2R\}$ which satisfies $0 \le \chi_1 \le 1$ and $\chi = 1$ in a neighborhood of $\{G \le R\}$ and where $\chi_2 = 1 - \chi_1$.

We may assume that R is so large that μ_c is supported in $\{G < R\}$. Then, since $\chi = 1$ in a neighborhood of $\mathrm{supp}(\chi_1)$, the first term above is, by definition of the critical measure, equal to

$$\int \chi_1 G\,\mu_c = \int G\,\mu_c.$$

As for the second term, Theorem 4.1 tells us that for $R \ge \frac{1}{2}\log\frac{1}{\epsilon}$ it can be written as

$$\begin{aligned}\int \mu_\Pi(a) \int_{W_0^s(a)} \chi_2 G\,dd^c(\chi H) &= \int \mu_\Pi(a) \int_{W_0^s(a)} G\,dd^c(\chi H)\\ &= \int \mu_\Pi(a) \int_{\partial W_0^s(a)} (G d^c H - H d^c G)\\ &= \int (\rho_H - k(d-1)\rho_G)\,\mu_\Pi(a).\end{aligned}$$

The first equality follows because $\chi_1 H = H$ is harmonic on $W_0^s(a) \cap \{G \le 2R\}$ and χ_2 is identically one outside the same set. The second line is an integration by parts, and the last equality is a consequence of Lemma 5.1.

It remains to evaluate the last integral. To do so we first note that the Robin functions ρ_G and ρ_H depend only on the homogeneous part of degree d of f. We may therefore assume that $f = f_h$ is homogeneous and make use of the following result.

Lemma 5.3. *Let f be any homogeneous regular polynomial endomorphism of $\mathbf{C}^k$, and let $|\det(Df)|$ and $|\det(Df_\Pi)|$ be the Jacobians of f and f_Π in the Euclidean metric on $\mathbf{C}^k$ and the Fubini-Study metric on Π, respectively. Then*

$$|\det(Df)(z)| = d \cdot \left(\frac{|f(z)|}{|z|}\right)^k |\det(Df_\Pi)[z]|.$$

Proof of Lemma 5.3. Pick any $z_0 \in \mathbf{C}^k - \{0\}$. After pre- and post-composing with dilations and unitary maps, we may assume that $f(z_0) = z_0 = (0, \dots, 0, 1)$. Since z_0 and $[z_0]$ are now fixed points, the choices of metrics are irrelevant when computing the Jacobians. We use local coordinates (ξ, s) on $\mathbf{P}^k$ and ξ on Π, where $\xi_i = z_i/z_k$ for $1 \le i \le k-1$ and $s = t/z_k$. In these coordinates,

$$\begin{aligned} f(\xi, s) &= (f_1(\xi,1)/f_k(\xi,1), \dots, f_{k-1}(\xi,1)/f_k(\xi,1), s^d/f_k(\xi,1)), \\ f_\Pi(\xi) &= (f_1(\xi,1)/f_k(\xi,1), \dots, f_{k-1}(\xi,1)/f_k(\xi,1)). \end{aligned}$$

Since the first $k-1$ coordinates in $f(\xi, s)$ do not depend on s, we see that

$$\det Df(\xi, s)|_{(\xi,s)=(0,1)} = d \cdot \det Df_\Pi(\xi)|_{\xi=0},$$

which completes the proof. □

We continue the proof of Theorem 5.2. Let

$$h[z] = \frac{|f(z)|}{|z|^d}.$$

This is a well-defined continuous function on Π and from the equations $G(z) = \log|z| + \rho_G[z]$ and $G \circ f = dG$ we get

$$\log h = d\,\rho_G - \rho_G \circ f_\Pi, \tag{5.2}$$

so by invariance of μ_Π

$$\int \log h\, \mu_\Pi = (d-1) \int \rho_G\, \mu_\Pi.$$

On the other hand, Lemma 5.3 shows that

$$\rho_H = \log d + \log|\det Df_\Pi| + k \log h, \tag{5.3}$$

so by (5.2) and (5.3) we arrive at

$$\begin{aligned} \int (\rho_H - k(d-1)\rho_G)\mu_\Pi &= \log d + \int \log|\det Df_\Pi|\, \mu_\Pi \\ &= \log d + \Lambda(f_\Pi), \end{aligned}$$

which completes the proof of Theorem 5.2. □

6. External Rays and Böttcher Coordinates

In this section we will do three things. First, we will continue the work in Section 4 and give a laminar formula for T^{k-1} on all of A. Second, we will show how to define external rays and a measure on the set of these. Every ray starts at J_Π, and almost every ray lands at a point of J. The family of rays gives the connection between the measure μ_Π and the measure μ. Finally, we will give variations on the idea of giving a Böttcher coordinate for the restriction of f to $W^s(J_\Pi)$.

We start by discussing the laminar structure of $W^s(J_\Pi)$. Recall from Section 3 that we may choose $\epsilon > 0$ so that $W_0^s(J_\Pi) = W^s(J_\Pi) \cap A_0$ is the disjoint union of local stable disks $W_0^s(a)$, each of which is a complex disk properly embedded in A_0. It follows that $W_0^s(J_\Pi)$ is a Riemann surface lamination. Now the iterates of f are local biholomorphisms outside the set $\mathcal{C}_{-\infty} := \bigcup_{n\geq 0} f^{-n}(\mathcal{C})$. The expansion of f_Π on J_Π implies that $\mathcal{C}_{-\infty} \cap W^s(J_\Pi)$ is closed and nowhere dense in $W^s(J_\Pi)$. Thus $W^s(J_\Pi) - \mathcal{C}_{-\infty}$ is also a lamination, which we denote by $\mathcal{W}^s(J_\Pi)$. If $a \in J_\Pi$, then $W^s(a) - \mathcal{C}_{-\infty}$ is a disjoint union of leaves of $\mathcal{W}^s(J_\Pi)$. We know from Corollary 4.7 that $W^s(a)$ is dense in $W^s(J_\Pi)$. The precise structure of $W^s(a)$ depends on if, and how, $W^s(J_\Pi)$ intersects $\mathcal{C}$. At the end of section 4 we gave an example where each $W^s(a)$ was a connected Riemann surface. On the other hand, if $W^s(J_\Pi)$ does not intersect the critical set, then we will see below that $W^s(J_\Pi)$ is the disjoint union of complex disks, each of which is properly embedded in A.

Next we define a set S to be removed from $W^s(J_\Pi)$. Each point $x \in W^s(J_\Pi) - \mathcal{C}_{-\infty}$ has the following properties: (1) x is contained in a unique leaf L_x of $\mathcal{W}^s(J_\Pi)$, (2) L_x is nonsingular at x, and (3) the gradient of $G|_{L_x}$ is nonvanishing at x. Thus x is contained in a unique gradient line of $G|_{\mathcal{W}^s(J_\Pi)}$. We let $\tilde{W}^s(J_\Pi)$ denote the set of points $x \in W^s(J_\Pi) - \mathcal{C}_{-\infty}$ for which the gradient line γ_x starting at x and moving in the direction of increasing G has an unlimited continuation, i.e. $\lim_{s\in\gamma_x} G(s) = +\infty$. Denote $W^s(J_\Pi) - \tilde{W}^s(J_\Pi)$ by S. Heuristically speaking, S consists of all points x for which the gradient line encounters $\mathcal{C}_{-\infty}$ If $x \in \tilde{W}^s(J_\Pi)$, then there is an $n \geq 0$ and a neighborhood U of x such that f^n maps U biholomorphically into A_0. Thus $\{G \geq r\} \cap S$ is compact for $r > 0$. Similarly, the set $A - S$ is open. For $a \in J_\Pi$, we write W_a for the connected component of $\tilde{W}^s(J_\Pi)$ containing a. By the construction of $\tilde{W}^s(J_\Pi)$, there is a well-defined gradient flow inside each set W_a, following the lines of increasing values of G.

Theorem 6.1. *For each $a \in J_\Pi$, W_a is simply connected, and W_a is a properly embedded disk in $A - S$. For μ_Π a.e. a, W_a has finite area as a subset of $\mathbf{P}^k$, and we have the laminar formula*

$$T^{k-1} \llcorner A = \int [W_a]\, \mu_\Pi(a). \tag{6.1}$$

Proof. If γ is a closed curve in W_a, then we may apply the gradient flow to γ until the image lies inside the disk $W_0^s(a)$. Since the image of γ is contractible inside this disk, the original curve γ was contractible. For $r > 0$, we observe that $W_a \cap \{G > r\}$ is a relatively compact subset of $W^s(a)$. Thus $W_a \cap \{G > r\}$ has finite area, and it follows that the area of W_a is locally finite inside A, so the current of integration $[W_a]$ is well defined. By Corollary 4.4 and by the fact that the sets $W_a \cap A_n$ are (modulo a set of zero area) just a finite subdivision of $W^s_n(a)$, we have that

$$\int ([W_a] \llcorner A_n) \mu_\Pi(a) = T^{k-1} \llcorner A_n,$$

and thus (6.1) holds. Hence

$$\int [W_a] \, \mu_\Pi(a) \le T^{k-1}.$$

Since the current T^{k-1} has finite mass, it follows that $\int [W_a] \, \mu_\Pi(a)$ has finite mass, so that for μ_Π almost every a, W_a has finite area. □

For each $a \in J_\Pi$ there is a harmonic conjugate function G^*_a for $G|_{W_a}$ in the sense that

$$\varphi_a := e^{G+iG^*_a}$$

is an analytic function. The choice of G^*_a is unique up to the choice of an additive real constant. Note that G^*_a is constant along the gradient lines on W_a. We consider domains H_a of the form

$$H_a = \hat{\mathbf{C}} - (\bar{\mathbf{D}} \cup \bigcup_j R_j), \tag{6.2}$$

where R_j is a ray of the form $(e^{i\theta_j}, r_j e^{i\theta_j}]$, and for each $\epsilon > 0$ there are only finitely many j for which $r_j > 1 + \epsilon$.

Lemma 6.2. *For each $a \in J_\Pi$, there is a domain H_a of the form (6.2) such that $\varphi_a : W_a \to H_a$ is a conformal equivalence. If $\psi_a := (\varphi_a)^{-1} : H_a \to W_a$, and if W_a has finite area, then the radial limits $\lim_{r\to 1^+} \psi_a(re^{i\theta})$ exist for a.e. θ.*

Proof. The first two assertions were proved above. It remains to show that radial limits exist almost everywhere. We work in affine coordinates in $\mathbf{C}^k \subset \mathbf{P}^k$. Let $\tilde{a} \in \mathbf{C}^k$ denote a point with $|\tilde{a}| = 1$ such that $\pi(\tilde{a}) = a \in \mathbf{P}^{k-1}$. Thus we may write $\psi_a(\zeta) = \zeta^{-1}\tilde{a} + h_a(\zeta)$, where h_a is analytic on H_a. Away from the hyperplane at infinity, the Euclidean metric on $\mathbf{C}^k$ is equivalent to the Fubini-Study metric on $\mathbf{P}^k$. The condition that W_a has finite area in $\mathbf{P}^k$ is equivalent to $\int_{H_a} |\nabla h_a|^2 < \infty$. It follows that

$$\int_0^1 |\nabla h_a(re^{i\theta})|^2 \, rdr = \int_0^1 |\frac{\partial h_a(re^{i\theta})}{\partial r}|^2 \, rdr < \infty$$

for almost every θ. Thus radial limits exist for these values of θ. □

Let us define the set $\mathcal{E}$ of external rays as the set of gradient lines in $\tilde{W}^s(J_\Pi)$. For each $a \in J_\Pi$, the unit tangent directions in the tangent space $T_a W_a$ at a give a natural parametrization of the set of external rays which lie in W_a. Thus the measure $\mu_\Pi \otimes \frac{d\theta}{2\pi}$ is defined on $\mathcal{E}$. By Lemma 6.2, we may define an endpoint map $e : \mathcal{E} \to \partial K$ for $\mu_\Pi \otimes \frac{d\theta}{2\pi}$-a.e. ray. For every ray $\gamma \in \mathcal{E}$ and $r > 0$, we let $e_r(\gamma) = \gamma \cap \{G = r\}$. Thus $\lim_{r\to 0^+} e_r = e$ holds for almost every ray $\gamma \in \mathcal{E}$.

Theorem 6.3. $e_*(\mu_\Pi \otimes \frac{d\theta}{2\pi}) = \mu$.

Proof. Let us first fix $a \in J_\Pi$ and consider the mapping e_r restricted to a manifold W_a; it is well defined except possibly at a finite number of points. For $r > 0$, the measure defined by the restriction of $\frac{1}{2\pi} d^c G$ to $W_a \cap \{G = r\}$ is the image, under the gradient flow, of the measure $\frac{d\theta}{2\pi}$. We note that this measure is the same as $\frac{1}{2\pi} d(d^c G \llcorner \{G > r\})$. Thus, continuing to restrict to W_a, we have $e_*(\frac{d\theta}{2\pi}) = \frac{1}{2\pi} dd^c \max(G, r)$.

Let us next consider the current

$$\nu_r := \frac{1}{2\pi} d^c G \wedge T^{k-1} \llcorner \{G > r\}.$$

Note that $d(\nu_r) = (\frac{1}{2\pi} dd^c \max(G, r))^k$. By the laminar structure of $T^{k-1} \llcorner A$ and by the properties of the restriction of e_r to W_a, we have that

$$(e_r)_*(\mu_\Pi \otimes \frac{d\theta}{2\pi}) = (\frac{1}{2\pi} dd^c \max(G, r))^k. \tag{6.3}$$

This is taken by using the result for each a and integrating with respect to μ_Π. Now since $\lim_{r\to 0^+} e_r = e$ almost everywhere, we have that the left hand side of (6.3) converges to $e_*(\mu_\Pi \otimes \frac{d\theta}{2\pi})$ as r decreases to 0. Next, since $\max(G, r)$ decreases to G, it follows that the right hand side of (6.3) converges to $(\frac{1}{2\pi} dd^c G)^k = \mu$ as r decreases to 0. This completes the proof. ◻

Now we proceed to give some interpretations of the Böttcher coordinate. The idea is to describe the dynamics on $W_0^s(J_\Pi)$ or $W^s(J_\Pi)$ in simple terms, just as the ordinary Böttcher coordinate conjugates a polynomial mapping of $\mathbf{C}$ to $\zeta \to \zeta^d$. First we set

$$\mathcal{H} = \bigcup_{a \in J_\Pi} \{a\} \times H_a, \tag{6.4}$$

and we let $\Psi : \mathcal{H} \to \tilde{W}^s(J_\Pi)$ denote the mapping such that $\Psi(a, \cdot) = \psi_a$. Note that $\mathcal{H}$ and Ψ depend on the choice of additive real constants in G_a^*. In any case, Ψ is a Böttcher coordinate in the following sense.

Proposition 6.4. *For each $a \in J_\Pi$, there is a $\nu_a \in S^1$ such that*

$$f(\Psi(a, \zeta)) = \Psi(f_\Pi(a), \nu_a \zeta^d). \tag{6.5}$$

Proof. By construction we have $\log |\varphi_a| = G$ on W_a. From the formula $G \circ f = d{\cdot}G$ we get $|\varphi_{f_\Pi(a)} \circ f| = |\varphi_a|^d$, so there exists $\nu_a \in S^1$ such that $\varphi_{f_\Pi(a)} \circ f = \nu_a \varphi_a^d$. This is equivalent to (6.5). ◻

Proposition 6.4 gives a Böttcher coordinate on all of $\tilde{W}^s(J_\Pi)$ but has the drawback that the constants ν_a depend on the choice of G^*_a. Moreover, it is not a priori clear that the set $\mathcal{H}$ is open in $J_\Pi \times (\hat{\mathbf{C}}-\bar{\mathbf{D}})$ or that Ψ is continuous. These problems are eliminated in the next version of the Böttcher coordinate. The idea is that, as in one dimension, we will conjugate f to its homogeneous part f_h of maximal degree. We work locally near Π and use the notations $W^s_{h,0}(J_\Pi)$ and $W^s_{h,0}(a)$ for the sets defined in the same way as $W^s_0(J_\Pi)$ and $W^s_0(a)$ but using f_h instead of f. The following result is similar to Theorem 9.3 in [HP].

Theorem 6.5. *If f_Π is expanding on J_Π, then for small $\epsilon > 0$ there is a conjugacy Θ between $(f_h, W^s_{h,0}(J_\Pi))$ and $(f, W^s_0(J_\Pi))$. Furthermore, Θ maps $W^s_{h,0}(a)$ conformally onto $W^s_0(a)$ for all $a \in J_\Pi$.*

Proof. The idea is to define the conjugacy as $\lim_{n\to\infty} f^{-n} \circ f^n_h$. We resume the notation from the proof of Theorem 3.1. It is here that we use the embedding of f in the one-parameter family $f_\tau = f_h + \tau(f - f_h)$. Define $W^s_{\tau,0}(a)$ and $W^s_{\tau,0}(J_\Pi)$ just as $W^s_0(a)$ and $W^s_0(J_\Pi)$ but using f_τ instead of f. By the proof of Theorem 3.1 we may assume that are $W^s_{\tau,0}(a)$ pairwise disjoint complex disks whose union is exactly $W^s_{\tau,0}(J_\Pi)$ for $|\tau| < 2$. We have that $f^{-n}_\tau(f^n_h(W^s_{h,0}(a))) \cap B^n_{a,\tau}$ is contained in a vertical-like disk in B_a. By the construction of B_a there are d^n locally defined branches of f^{-n}_τ mapping $f^n_h(W^s_{h,0}(a) - \{a\})$ into $B^n_{a,\tau}$. These branches depend holomorphically on τ. Let $\psi_{a,\tau,n}$ be the branch obtained by analytic continuation of $\psi_{a,0,n} = \mathrm{id}$. Then $\psi_{a,\tau,n}$ is well-defined on $W^s_{h,0}(a)$, depends continuously on a and holomorphically on τ. We may pass to the limit and define $\psi_{a,\tau} = \lim_{n\to\infty} \psi_{a,\tau,n}$. Now $\psi_{a,\tau}$ maps level sets of G_h to the corresponding level sets of G_τ and $\psi_{a,\tau}$ maps $W^s_{h,0}(a)$ into $B^\infty_{a,\tau}$. Hence Hurwitz's theorem implies that $\psi_{a,\tau}$ is a biholomorphism of $W^s_{h,0}(a)$ onto $W^s_{\tau,0}(a)$. Moreover, $\psi_{a,\tau}$ depends continuously on a and holomorphically on τ. If we define $\Theta_\tau : W^s_{h,0}(J_\Pi) \to W^s_{\tau,0}(J_\Pi)$ by $\Theta_\tau(x) = \psi_{a,\tau}(x)$ for $x \in W^s_{h,0}(a)$, then Θ_τ is a homeomorphism for each τ. We claim that Θ_τ conjugates f_h to f. To see this, fix $a \in J_\Pi$ and note that the two mappings $g_\tau := f_\tau \circ \psi_{a,\tau}$ and $h_\tau := \psi_{f_\Pi(a),\tau} \circ f_h$ both map $W^s_{h,0}(a)$ onto $W^s_{\tau,0}(f_\Pi(a))$ as branched coverings, depend holomorphically on τ and satisfy $G_\tau \circ g_\tau = G_\tau \circ h_\tau = d \cdot G_h$. Hence there exists $\nu_\tau \in \mathbf{C}$, depending holomorphically on τ, with $|\nu_\tau| = 1$ and $g_\tau(\nu_\tau x) = h_\tau(x)$. Thus ν_τ is constant so since $G_h = h_0$ we must have $g_\tau = h_\tau$ for all τ. This completes the proof. $\square$

In fact, we may extend the conjugacy in Theorem 6.5 as follows. Namely, $W^s_{h,0}(a)$ is a complex disk in $\mathbf{P}^k$ of the form $\{z = \zeta\tilde{a}; |\zeta| > r_a\}$, where $\pi(\tilde{a}) = a$, $|\tilde{a}| = 1$ and $r_a > 0$. By using the coordinate ζ on $W^s_{h,0}(a)$ we may therefore identify the restriction of Θ to $W^s_{h,0}(a)$ with a parameterization ψ_a as above. This, and Proposition 6.4 imply the following result.

Corollary 6.6. *If f_Π is expanding on J_Π, then there is a closed subset S_h of $W^s_h(J_\Pi)$ and a conjugacy Θ between $(f_h, W^s_h(J_\Pi) - S_h)$ and $(f, W^s(J_\Pi) - S)$. The set S_h is a union of rays of the form $\{z = as; 1 \leq s \leq r\}$, where $a \in \mathbf{C}^k$,*

$G_h(a) = 0$ and $r > 1$. Further, Θ maps $(\pi^{-1}(a) \cap A_h) - S_h$ conformally onto W_a for all $a \in J_\Pi$.

Using Corollary 6.6 we can make a more precise choice of the conjugacy Ψ in Proposition 6.4, at least in dimension $k = 2$. Namely, if $k = 2$, then the restriction of the tautological line bundle $\pi : \mathbf{C}^2 - \{0\} \to \mathbf{P}^1$ to $\pi^{-1}(\mathcal{C}_\Pi)$ is trivial. Further, the set $W^s_{h,0}(J_\Pi)$ is a topological disk bundle over J_Π whose fibers are subsets of the fibers of the bundle π. Also, $\mathcal{C}_\Pi \cap J_\Pi = \emptyset$, since f_Π is expanding on J_Π, so there exists a homeomorphism $\Psi_0 : J_\Pi \times (\hat{\mathbf{C}} - \bar{\mathbf{D}}) \to W^s_h(J_\Pi)$ such that $\Psi_0(a, \infty) = a$ and Ψ_0 maps the disk $\{a\} \times (\hat{\mathbf{C}} - \bar{\mathbf{D}})$ conformally onto $\pi^{-1}(a) \cap A_h$. Let $\mathcal{H} = \Psi_0^{-1}(W^s_h(J_\Pi) - S_h)$ and $\Psi = \Theta \circ \Psi_0$, where S_h and Θ are as in Corollary 6.6. Then $\mathcal{H}$ is a domain of the type (6.4) and Ψ is a homeomorphism of $\mathcal{H}$ onto $W^s(J_\Pi) - S$. Further,

$$f \circ \Psi(a, \zeta) = \Psi(f_\Pi(a), \nu_a \zeta^d),$$

where $\nu_a \in S^1$ depends continuously on a. Hence we have.

Theorem 6.7. *If $k = 2$ and f_Π is expanding on J_Π, then there is an open set $\mathcal{H}$ in $J_\Pi \times (\hat{\mathbf{C}} - \bar{\mathbf{D}})$ of the type (6.4) and a homeomorphism $\Psi : \mathcal{H} \to \tilde{W}^s(J_\Pi)$ satisfying (6.5) with ν_a depending continuously on a.*

We give an example to show that we cannot always obtain $\nu_a = 1$ in Theorem 6.7. Let $f(z, w) = (w^2, z^2)$. Then J_Π is a circle on Π and we can use coordinates $\zeta = z/w, t = 1/w$ in a neighborhood of J_Π. In these coordinates f is given by $(\zeta, t) \to (1/\zeta^2, t^2/\zeta^2)$, J_Π by $|\zeta| = 1, t = 0$, and the Green function by $G = -\log|t|$. Hence f_Π is expanding on J_Π and $\mathcal{H} = J_\Pi \times (\hat{\mathbf{C}} - \bar{\mathbf{D}})$. It follows that any Böttcher coordinate is of the form $\varphi_a(\zeta, t) = c(a)/t$, where c is a self-map of the circle S^1. The numbers ν_a associated with φ_a are given by $\nu_a = c(1/a^2)a^2/c(a)^2$, so $\nu_a = 1$ if and only if $c(a)^2 = c(1/a^2)a^2$. The corresponding map Ψ is continuous on $J_\Pi \times (\hat{\mathbf{C}} - \bar{\mathbf{D}})$ if and only if c is continuous. If this is the case, then c lifts to a continuous map $C : \mathbf{R} \to \mathbf{R}$ such that $C(s + 1) = C(s) + n$ for some integer n and the equation $c(a)^2 = c(1/a^2)a^2$ to $2C(a) = C(-2a) + 2a + m$ for some m. Replacing a by $a + 1$ and subtracting the old equation yields $2n = -2n + 2$, which is impossible.

We end this section by discussing the particularly interesting case when the stable set of J_Π does not meet the critical set, i.e. $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$. Then $S = \emptyset$, each stable disk W_a is properly embedded in A and $H_a = \hat{\mathbf{C}} - \bar{\mathbf{D}}$ for all a. In this case we may strengthen Theorem 6.3 as follows.

Theorem 6.8. *If f_Π is expanding on J_Π and $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$, then for μ_Π almost every a and any choice of harmonic conjugate G^*_a, the embedding $\psi_a : \hat{\mathbf{C}} - \bar{\mathbf{D}} \to \mathbf{P}^k - J$ is proper.*

Proof. We have observed that if $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$, then W_a is a topological disk properly embedded in A. By Theorem 6.3, the radial limits $\lim_{r \to 1} \psi_a(re^{i\theta})$ exist and belong to J for $d\theta$ almost every θ and μ_Π almost every a. Further, for μ_Π almost every a, W_a has finite area as a subset of $\mathbf{P}^k$. It follows from a Theorem of Alexander [A] (see also [Ro]) that $\psi_a : \hat{\mathbf{C}} - \bar{\mathbf{D}} \to \mathbf{P}^k - J$ is proper. $\square$

In dimension $k = 2$ we have a homeomorphism $\Psi : J_\Pi \times (\hat{\mathbf{C}} - \bar{\mathbf{D}}) \to W^s(J_\Pi)$ satisfying the relation (6.5). Theorem 6.8 says that, in a measure theoretic sense, most of the disks W_a have their boundaries in J, but that does not imply that Ψ extends continuously to $J_\Pi \times S^1$. We will return to this problem in Section 8.

7. Axiom A and strong hyperbolicity in dimension 2.

So far we have worked with fairly general regular polynomial endomorphisms of $\mathbf{C}^k$, only assuming that f_Π is expanding on J_Π. This allowed us to understand the dynamics in the set $W^s(J_\Pi)$. Thus, in dimension 2 we have a complete description of the dynamics in A, because all points in $A - W^s(J_\Pi)$ are in the basin of an attracting periodic point in Π. However, the condition that f_Π is expanding on J_Π does not rule out the possibility of very complicated dynamics of f on K, even in dimension 2. For example, if p and q are any two monic polynomial maps of $\mathbf{C}$ of degree $d \geq 2$ and $f(z, w) = (p(z), q(w))$, then the map f_Π is always given by $f_\Pi(\zeta) = \zeta^d$, which is expanding on J_Π. Hence a regular polynomial endomorphism of $\mathbf{C}^2$ with f_Π expanding on J_Π can have as complicated dynamics as any polynomial map of $\mathbf{C}$. Moreover, Gavosto [G] has shown that a holomorphic endomorphism of $\mathbf{P}^2$ can have infinitely many attracting basins and her examples are in fact regular polynomial endomorphisms f of $\mathbf{C}^2$ with f_Π expanding on J_Π.

In the next section we will need further assumptions on the dynamics on f in order to prove that the external rays land *continuously* on J, something which allows us to describe J as a topological quotient of $J_\Pi \times S^1$. Here we digress and discuss hyperbolicity for regular polynomial endomorphisms. We restrict our attention to dimension 2.

The literature on hyperbolic dynamics is vast, but most expositions consider only diffeomorphisms. A regular polynomial endomorphism of $\mathbf{C}^2$ of degree $d \geq 2$ is not invertible, so the theory becomes different. A general treatment of the dynamics of possibly non-invertible maps can be found in [Ru], but it is scarce with details. For the convenience of the reader we give the definitions and results we need in Appendix A. More details can be found in [J2]. We also refer to [FS4], where the authors study hyperbolic endomorphisms of $\mathbf{P}^2$.

Suppose that f is a regular polynomial endomorphism of $\mathbf{C}^2$; as usual we regard f as a holomorphic map of $\mathbf{P}^2$. Since f is not injective, we will often have to work with histories of points instead of the points themselves. Precisely, a history of a point $x \in \mathbf{P}^2$ is a sequence $(x_i)_{i \leq 0}$ of points in $\mathbf{P}^2$ such that $x_0 = x$ and $f(x_i) = x_{i+1}$ for all $i < 0$. We will use the notation $\hat{x}$ for a history (x_i).

Let L be a compact subset of $\mathbf{P}^2$ with $f(L) = L$. We refer to Appendix A for a definition of what it means for f to be hyperbolic on L. Let us only recall that the definition involves the set $\hat{L}$ of histories in L. There is a natural projection

$\pi : \hat{L} \to L$ such that $\pi(\hat{x}) = x_0$. We say that L has unstable index i if the stable bundle E^s has constant dimension $2-i$ on L. If L has unstable index 2, then f is said to be expanding on L (see Appendix A for an alternative definition). If f is hyperbolic on L, then to every point in $x \in L$ and every history $\hat{x} \in \hat{L}$ there is an associated local stable and unstable manifold respectively, defined by

$$W^s_\delta(x) = \{y \in \mathbf{P}^2; d(f^i(y), f^i(x)) < \delta \ \forall i \geq 0\}$$
$$W^u_\delta(\hat{x}) = \{y \in \mathbf{P}^2; \exists \hat{y} \in \widehat{\mathbf{P}^2}, \pi(\hat{y}) = y, d(y_i, x_i) < \delta \ \forall i \leq 0\},$$

for small $\delta > 0$. Then $W^s_\delta(x)$ and $W^u_\delta(\hat{x})$ are complex submanifolds of $\mathbf{P}^2$. If f is expanding on L, then the local stable manifolds are empty and the local unstable manifold at $\hat{x}$ is a neighborhood of x_0 in $\mathbf{P}^2$.

We also define global stable manifolds by declaring

$$W^s(x) = \{y \in \mathbf{P}^2; d(f^i(y), f^i(x)) \to 0 \text{ as } i \to \infty\}$$
$$W^u(\hat{x}) = \{y \in \mathbf{P}^2; \exists \hat{y} \in \widehat{\mathbf{P}^2}, \pi(\hat{y}) = y, d(y_i, x_i) \to 0 \text{ as } i \to -\infty\}.$$

Note that if $n \geq 0$, $y \in L$ and $f^n(y) = f^n(x)$, then $W^s(x)$ contains $W^s_\delta(y)$. Hence the global stable manifolds are in general large and quite complicated objects (compare with Corollary 4.7; see also [FS4]). The unstable manifolds will also be fairly complicated in general; we will have more to say about this in Section 8. All of this should be contrasted with the case of polynomial automorphims of $\mathbf{C}^2$, where the global stable and unstable manifolds are immersed copies of $\mathbf{C}$ [BS].

We next discuss Axiom A regular polynomial endomorphisms of $\mathbf{C}^2$. Recall that a point $x \in \mathbf{P}^2$ is wandering if for every neighborhood V of x there exists an $n \geq 1$ such that $f^n(V) \cap V \neq \emptyset$. The non-wandering set Ω of f is the set of all non-wandering points; it is a compact set. A regular polynomial endomorphism f of $\mathbf{C}^2$ is said to be Axiom A if the periodic points of f are dense in Ω and f is hyperbolic on Ω. If f is Axiom A, then Smale's spectral decomposition theorem (Theorem A.9) asserts that Ω can be written in a unique way as a finite union of disjoint compact sets Ω_j, called basic sets, such that $f(\Omega_j) = \Omega_j$ and $f|_{\Omega_j}$ is transitive, i.e. has a dense orbit. Thus each basic set has a well-defined unstable index.

Let us investigate what the possible basic sets are for a Axiom A regular polynomial endomorphism f of $\mathbf{C}^2$. To do this, we first observe that the four sets Π, $\mathbf{C}^2 - K$, $\mathrm{int}(K)$ and ∂K are all completely invariant and see what basic sets each one of them may contain.

To begin with, it is clear that $\Omega(f) \cap \Pi = \Omega(f_\Pi)$. Now f_Π is a rational map and from one-dimensional dynamics we know that f_Π is Axiom A if and only if f_Π is expanding on J_Π (see [Mi]). Hence, if f is Axiom A, then the basic sets in Π are J_Π, which is of unstable index 1, and a finite union of attracting periodic points, all of whose unstable index is zero.

All the points in the open set $\mathbf{C}^2 - K$ are attracted to Π so $(\mathbf{C}^2 - K) \cap \Omega$ is empty. It is clear that $\{f^n\}$ is normal on the interior of K, so if f is Axiom

A, then the only basic sets in $\mathrm{int}(K)$ are attracting periodic points, all of whose unstable index are zero.

The boundary of K contains the most complicated dynamics. Clearly, no basic sets in ∂K can have unstable index 0. Let S_2 and S_1 be the union of the basic sets in ∂K of index 2 and 1, respectively. We note that S_1 can be empty, as in the example $f(z,w) = (z^2+c, w^2+c)$, with c outside the Mandelbrot set. On the other hand, J is a basic set of unstable index 2 (see [FS2]), so $J \subset S_2$. The question arises whether this inclusion is ever strict or, equivalently, whether f can have repelling periodic points outside J. Hubbard and Papadopol [HP] have in fact given an example of a regular polynomial endomorphism of $\mathbf{C}^2$ with a repelling periodic point outside J but is seems difficult to check whether their map can be made Axiom A.

We say that f is *strongly hyperbolic* if it is Axiom A and $f^{-1}(S_2) = S_2$. This is slightly weaker than the definition of strong hyperbolicity in [FS4]. As mentioned above, it is not completely clear whether strong hyperbolicity is equivalent to Axiom A, but we do have the following.

Lemma 7.1. *Let f be an Axiom A regular polynomial endomorphism of $\mathbf{C}^2$. Then f is strongly hyperbolic if and only if $S_2 = J$, i.e. if all repelling periodic points are contained in J.*

Remark. A proof is given in [FS4]. We give it here for the convenience of the reader.

Proof. The "only if" part is trivial since $f^{-1}(J) = J$, so suppose that f is Axiom A and $f^{-1}(S_2) = S_2$ but $S_2 \neq J$. Let N be an open neighborhood of J such that $f^{-1}(N) \subset N$ and $\cap_{n\geq 0} f^{-n}(N) = J$. Then $N - J$ contains only wandering points, so $S_2 - J$ is at a positive distance from J and is therefore a completely invariant compact set. Let N' be an open neighborhood of $S_2 - J$ disjoint from J with $f^{-1}(N') \subset N'$. Then N' has positive capacity and if $x \in N'$ then $(f^n)^*\delta_x/d^{2n}$ cannot converge to μ as $n \to \infty$. This contradicts Lemma 8.3 in [FS3]. □

Let f be a strongly hyperbolic regular polynomial endomorphism of $\mathbf{C}^2$. It follows from Corollary A.10 and the above discussion that any history of a point $\mathbf{C}^2$ which is not an attracting periodic point must converge to either J or S_1. We define the unstable set of J to be the set of points in $\mathbf{C}^2$ all of whose histories converge to J, i.e.

$$W^u(J) = \{x \in \mathbf{C}^2; (\hat{x} \in \widehat{\mathbf{C}^2}, \pi(\hat{x}) = x) \Rightarrow x_i \to J \text{ as } i \to -\infty\}.$$

We note that this definition differs from the one in [FS4], where $W^u(J)$ is defined as the set of points having *at least* one history converging to J. On the other hand we define the unstable set of S_1 as

$$W^u(S_1) = \{x \in \mathbf{C}^2; \exists \hat{x} \in \widehat{\mathbf{C}^2}, \pi(\hat{x}) = x, x_i \to S_1 \text{ as } i \to -\infty\}.$$

Let N be a neighborhood of J in $\mathbf{C}^2$ as in the proof of Lemma 7.1. Clearly $N \subset W^u(J)$ and every point in $\mathbf{C}^2$ which is not an attracting periodic point is contained in precisely one of the sets $W^u(J)$ and $W^u(S_1)$.

Lemma 7.2. *If $x \in W^u(J)$, then there exists an $n \geq 0$ such that $f^{-n}(x) \subset N$. In particular, $W^u(J)$ is open in $\mathbf{C}^2$ and $W^u(S_1)$ is closed in $\mathbf{C}^2$ except possibly at some of the attracting periodic points.*

Proof. Let Z be the set of points y in $\mathbf{C}^2$ such that for all $n \geq 0$, there is a point in $f^{-n}(y)$ outside N. It is clear that if $y \in Z$, then y has at least one preimage in Z, so every point $y \in Z$ has a whole history inside Z. Such a history cannot converge to J so it follows that $Z \cap W^u(J) = \emptyset$, which completes the proof. □

In the next section we will assume that f satisfies a slightly different hyperbolicity criterion, which we now discuss.

Definition 7.3. *A regular polynomial endomorphism f of $\mathbf{C}^2$ satisfies condition (†) if the following four properties hold:*
(†1) f_Π is expanding on J_Π.
(†2) f is expanding on J.
(†3) The nonwandering set of f in ∂K consists of J and a hyperbolic set S_1 of unstable index 1 with $f(S_1) = S_1$.
(†4) $W^u(S_1) = \bigcup_{\hat{x}\in\hat{S}_1} W^u(\hat{x})$.

Condition (†) is exactly the hyperbolicity assumption that we need for the proof of the main result in Section 8 (Theorem 8.2). It is a weaker condition than strong hyperbolicity:

Proposition 7.4. *Let f be a regular polynomial endomorphism of $\mathbf{C}^2$. If f is strongly hyperbolic, then f satisfies condition (†).*

Proof. Suppose that f is strongly hyperbolic. From the above discussion we know that f satisfies conditions (†1), (†2) and (†3). Finally (†4) follows from Corollary A.10. □

8. Landing of disks.

In this section we consider a regular polynomial endomorphism f of $\mathbf{C}^2$ with f_Π expanding on J_Π and $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$. We know from Section 6 that $W^s(J_\Pi)$ is laminated by complex disks W_a, $a \in J_\Pi$, each of which is properly embedded in A. Moreover, there exists a homeomorphism $\Psi : J_\Pi \times (\hat{\mathbf{C}} - \bar{\mathbf{D}}) \to W^s(J_\Pi)$, whose restrictions $\psi_a = \Psi(a, \cdot)$ maps $\hat{\mathbf{C}} - \bar{\mathbf{D}}$ conformally onto W_a. We have $G(\Psi(a,\zeta)) = \log|\zeta|$ and $f(\Psi(a,\zeta)) = \Psi(f_\Pi(a), \nu_a\zeta^d)$, where $\nu_a \in S^1$ depends continuously on a.

It is a natural question to ask whether the disks W_a land on J, i.e. if Ψ extends continuously to $J_\Pi \times S^1$. Without any further assumptions, this need not be the case. Indeed, if z^2+c is a quadratic polynomial map of $\mathbf{C}$ with connected, but not locally connected Julia set, then results from one-dimensional dynamics [CG] imply that $f(z, w) = (z^2 + c, w^2 + c)$ is a counterexample.

This shows that in order for Ψ to extend it is necessary to impose additional conditions on the dynamics of f. In dimension one, a sufficient (although not necessary) one is for the map to be expanding on its Julia set (and that the critical points have bounded orbits).

In this section we give conditions on the dynamics of f which will ensure that the map Ψ does extend continuously to $J_\Pi \times S^1$. The reason for working in dimension 2 is that in the proof we will consider unstable manifolds, view these as Riemann surfaces, and use the uniformization theorem. This strategy would fail dismally in dimension $k > 2$.

Definition 8.1. *We say that a regular polynomial endomorphism f of $\mathbf{C}^2$ satisfies condition (‡) if f satisfies condition (†) defined in Section 7 and $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$, i.e. if the following five properties hold:*

(‡1) f_Π is expanding on J_Π.
(‡2) f is expanding on J.
(‡3) The nonwandering set of f in ∂K consists of J and a hyperbolic set S_1 of unstable index 1 with $f(S_1) = S_1$.
(‡4) We have $W^u(S_1) = \bigcup_{\hat{x} \in \hat{S}_1} W^u(\hat{x})$.
(‡5) $W^s(J_\Pi) \cap \mathcal{C} = \emptyset$.

It follows from Proposition 7.4 that if f is strongly hyperbolic and satisfies (‡5), then f satisfies (‡). It is proved in [J1] that perturbations of the map $f(z, w) = (z^d, w^d)$ satisfy (‡).

We say that a stable disk W_a *lands* on J if ψ_a extends continuously to $\hat{\mathbf{C}} - \mathbf{D}$ and $\psi_a(S^1) \subset J$. It is our goal to prove the following result.

Theorem 8.2. *If the regular polynomial endomorphism f of $\mathbf{C}^2$ satisfies condition (‡), then all the stable disks W_a land on J. More precisely, there exist constants $C < \infty$ and $\alpha > 0$ such that*

$$d(\psi_a(\zeta), \psi_a(\zeta')) \leq C d(\zeta, \zeta')^\alpha, \tag{8.1}$$

for all $a \in J_\Pi$ and all $\zeta, \zeta' \in \hat{\mathbf{C}} - \mathbf{D}$. Furthermore, Ψ extends to a continuous map of $J_\Pi \times (\hat{\mathbf{C}} - \mathbf{D})$ into $W^s(J_\Pi) \cup J$.

The main difficulty in proving Theorem 8.2 is to show that ∂W_a accumulates only at J for all a. To do this, we must show that there are no saddle connections between S_1 and J_Π, i.e. that there is no complete orbit $(x_i)_{i \in \mathbf{Z}}$ such that $x_i \to S_1$ as $i \to -\infty$ and $x_i \to J_\Pi$ as $i \to \infty$.

Lemma 8.3. *If f satisfies condition (‡), then $W^s(J_\Pi) \cap W^u(S_1) = \emptyset$.*

We postpone the proof of Lemma 8.3 for the moment and show instead how it implies Theorem 8.2.

Proof of Theorem 8.2. The expansion of f on J implies that there exists a neighborhood N of J with $f^{-1}(N) \subset N$, $\lambda > 1$ and a metric equivalent to the Euclidean metric such that $|Df(x)v| \geq \lambda |v|$ for all $x \in N$ and all $v \in T_x\mathbf{C}^2$ with respect to this metric. By Lemma 7.2 and Lemma 8.3 we know that the set $W^s(J_\Pi) \cap \{1 \leq G \leq d\}$ is a compact subset of the open set $W^u(J)$ so by pulling back by f we see that there exists an $R > 1$ such that $\Psi(J_\Pi \times (\bar{\mathbf{D}}_R - \bar{\mathbf{D}})) \subset N$. Let $\alpha > 0$ be so small that $d^\alpha < \lambda$. We may assume that $R^{d-1}d^\alpha < \lambda$. Recall that ψ_a satisfies

$$\psi_a(\zeta) = g\left(\psi_{f_\Pi(a)}\left(\nu_a \zeta^d\right)\right), \tag{8.2}$$

for $|\zeta| > 1$, where g is a suitable, locally well-defined branch of f^{-1} and $|\nu_a| = 1$. By differentiating (8.2) and using the estimates above we get, for $1 < |\zeta| < R^{d^{-1}}$,

$$|D\psi_a(\zeta)| \leq \lambda^{-1}\left|D\psi_{f_\Pi(a)}\left(\nu_a\zeta^d\right)\right| d|\zeta|^{d-1}. \tag{8.3}$$

Define

$$m(r) = \sup_{a \in J_\Pi} \sup_{|\zeta| = r} |D\psi_a(\zeta)|,$$

for $1 < r \leq R$. Then there exists a constant $C' < \infty$ such that

$$m(r) \leq C'(r-1)^{\alpha - 1}, \tag{8.4}$$

for $R^{d^{-1}} \leq r \leq R$. Using the estimate (8.3) we prove inductively that (8.4) holds for $1 < r \leq R$. Integrating (8.4) we see that ψ_a extends continuously to $\hat{\mathbf{C}} - \mathbf{D}$ and that (8.1) holds. The continuity of Ψ on $J_\Pi \times (\hat{\mathbf{C}} - \mathbf{D})$ follows from (8.1) and the fact that the restriction of Ψ to $J_\Pi \times (\hat{\mathbf{C}} - \mathbf{D}_r)$ is continuous for each $r > 1$. □

We now turn to the proof of Lemma 8.3 and proceed in a number of steps. First we show that there is a dichotomy for the stable disks W_a expressed by the following lemma. The dichotomy will be used on several occasions.

Lemma 8.4. *Let W_a be the stable disk of a point $a \in J_\Pi$. Then either $W_a \cap W^u(S_1) = \emptyset$ or there exists a point $\hat{x} \in \hat{S}_1$ such that $W_a^* \subset W^u(\hat{x})$, where $W_a^* = W_a - \{a\}$.*

The key observation in proving the dichotomy is the following.

Lemma 8.5. *If U is a simply connected open subset of a punctured stable disk W_a^*, then all branches of $f^{-i}|_U$ for all $i > 0$ are well-defined and holomorphic on U and they form a normal family there.*

Proof. That the branches are well-defined follows from condition (‡5). If V is relatively compact in U then all branches of f^{-i} on V map V into a fixed compact subset of $\mathbf{C}^2$. Thus they form a normal family on U. □

Proof of Lemma 8.4. Suppose that $y \in W_a \cap W^u(S_1)$. Then by condition (‡4) there exists a point $\hat{x} \in \hat{S}_1$ such that $y \in W^u(\hat{x})$, i.e. y has a history $\hat{y}$ such that $d(y_i, x_i) \to 0$ as $i \to -\infty$. Let U be any simply connected open subset of W_a^* containing y and let g_i be the unique sequence of branches of $f^{-i}|_U$ such that $g_i(y) = y_i$. Then $\{g_i\}$ is equicontinuous by Lemma 8.3, so there is a small neighborhood V of y in U such that the maximal distance from $g_i(V)$ to x_i is uniformly small as $i \to \infty$. Hence $V \subset W^u(\hat{x})$ and, by normality of $\{g_i\}$, $U \subset W^u(\hat{x})$. Since U was arbitrary it follows that $W_a^* \subset W^u(\hat{x})$. □

Corollary 8.6. *Let J'_Π be the set $a \in J_\Pi$ such that $W_a^* \subset W^u(S_1)$. Then J'_Π is closed, $f_\Pi(J'_\Pi) = J'_\Pi$ and $J'_\Pi \neq J_\Pi$.*

Proof. If $a \notin J'_\Pi$, then $W_a^* \cap W^u(S_1) = \emptyset$ by Lemma 8.4. Hence $W_a \cap \{G = 1\}$ is a compact subset of the open set $W^u(J)$ so by continuity there is an open neighborhood X of a in J_Π such that $W_b \cap \{G = 1\} \subset W^u(J)$ for all $b \in X$. By Lemma 8.4 it follows that $X \cap J'_\Pi = \emptyset$ and we conclude that $J_\Pi - J'_\Pi$ is open. That $f_\Pi(J'_\Pi) = J'_\Pi$ follows from the fact that $f(W^u(S_1)) = W^u(S_1)$.

Finally suppose $J'_\Pi = J_\Pi$. Then $W^s(J_\Pi) \subset J_\Pi \cup W^u(S_1)$, so $W^s(J_\Pi)$ does not intersect $W^u(J)$. This contradicts Theorem 6.3, because $W^u(J)$ contains a neighborhood of $J = \mathrm{supp}(\mu)$. □

Recall that we say that a stable disk W_b lands on J if ψ_b extends continuously to S^1 and $\psi_b(S^1) \subset J$.

Lemma 8.7. *There exists a dense set of $b \in J_\Pi$ such that W_b lands on J.*

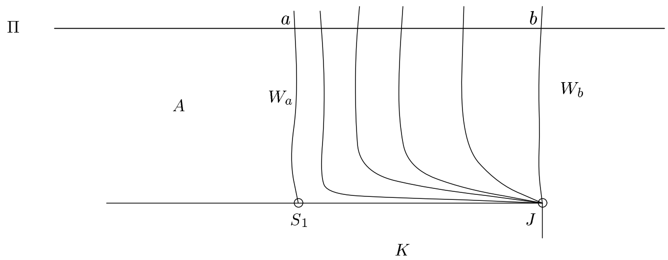
Proof. Since periodic points are dense in J_Π and $J_\Pi - J'_\Pi$ is open and nonempty, we can find a periodic point $b' \in J_\Pi - J'_\Pi$, say of period n. Furthermore, f is expanding on J, so there exists a neighborhood N of J and $\lambda > 1$ with $f^{-1}(N) \subset N$ and

$$|Df^n(y)v| \geq \lambda |v|, \tag{8.5}$$

for all $y \in N$ and all tangent vectors v (we may have to increase n). Now the annulus $\psi_{b'}(\bar{\mathbf{D}}_2 - \mathbf{D}_{2^{d-n}})$ in $W_{b'}$ is a compact subset of $W^u(J)$, so the inverse images under sufficiently high iterates of f of points in this annulus will be in N. In particular, since b' is periodic, it follows that there exists an $R > 1$ such that $\psi_{b'}(\bar{\mathbf{D}}_R - \bar{\mathbf{D}}) \subset N$. Then, using the estimate (8.5) above, we may prove that $\psi_{b'}$ extends to a Hölder continuous map of $\hat{\mathbf{C}} - \mathbf{D}$, mapping S^1 into J. The proof is very similar to the proof of the first part of Theorem 8.2 and is therefore omitted.

We conclude that $W_{b'}$ lands on J and so does W_b for all preimages b of b' under iterates of f. Such preimages are dense in J_Π. □

The picture below illustrates the effect of a saddle connection. Here W_a^* is in the unstable set of S_1 whereas W_b lands on J. The stable disks in the middle are of the form W_{b_n}, where b_n are preimages of b converging to a. Note that the disks W_{b_n} are very "bent" for large n. The idea in the proofs below is to show that this is impossible.



It follows from Lemma 8.4 that for each $a \in J'_\Pi$ there exists a (not necessarily unique) history $\widehat{p_a}$ in S_1 such that $W_a^* \subset W^u(\widehat{p_a})$. In general, an unstable manifold $W^u(\hat{q})$ of a history $\hat{q}$ in S_1 is a complicated object, but, as we will see, the information that $W_a^* \subset W^u(\widehat{p_a})$ implies that $W^u(\widehat{p_a})$ is in fact an algebraic subvariety of $\mathbf{C}^2$. Recall that the image of a holomorphic map of a compact Riemann surface into $\mathbf{P}^2$ is an algebraic variety. The authors thank Jeff Diller for useful conversations on the proof of the following result.

Lemma 8.8. *If $J'_\Pi \neq \emptyset$, then there exists an $a \in J'_\Pi$ such that $W^u(\widehat{p_a})$ is an algebraic subvariety of $\mathbf{C}^2$.*

Proof. Take any point $a \in J'_\Pi$ and a complete orbit $(a_i)_{i \in \mathbf{Z}}$ with $a_0 = a$. Let $(p_i)_{i \in \mathbf{Z}}$ be a complete orbit in S_1 such that $W_{a_i}^* \subset W^u((p_{i+j})_{j \le 0})$ for all i. We write $\widehat{p_i}$ for the history $(p_{i+j})_{j \le 0}$. If $\delta > 0$ is small enough, then the local unstable manifolds $W_\delta^u(\widehat{p_i})$ are complex disks for all i and there exist biholomorphisms $\eta_i : \mathbf{D}_{\delta_i} \to W_\delta^u(\widehat{p_i})$ with $|D\eta_i(0)| = 1$ and complex numbers $\lambda_i \neq 0$ such that

$$\eta_i(\lambda_{i-1}\zeta) = f(\eta_{i-1}(\zeta)), \tag{8.6}$$

for all i and all $|\zeta| < \delta_{i-1}$. Since f is hyperbolic on S_1, the numbers δ_i are uniformly bounded from below and $\lambda_{i-n} \cdots \lambda_{i-1} \to \infty$ as $n \to \infty$ for all i, so (8.6) allows us to extend η_i to maps of $\mathbf{C}$ into $W^u(\widehat{p_i})$ by defining

$$\eta_i(\lambda_{i-n} \cdots \lambda_{i-1}\zeta) = f^n(\eta_{i-n}(\zeta)),$$

for $n \ge 0$.

The maps η_i are surjective by the definition of $W^u(\widehat{p_i})$ but they need not be injective. However, the global unstable manifolds $W^u(\widehat{p_i})$ have a natural structure as abstract Riemann surfaces given by the maps η_i. More precisely, for each i we define a Riemann surface X_i as the quotient $\mathbf{C}/\sim$, where $z \sim w$ if there are open sets $U \ni z$ and $V \ni w$ such that $\eta_i(U) = \eta_i(V)$. Then the map η_i factors as $\eta_i = \eta'_i \circ \pi_i$, where $\pi_i : \mathbf{C} \to X_i$ is surjective, $\eta'_i : X_i \to \mathbf{C}^2$ is locally injective and the set of points $(z, w) \in X_i \times X_i$ with $z \neq w$ and $\eta'_i(z) = \eta'_i(w)$ is discrete. We will be sloppy and make no distinction between the unstable manifold $W^u(\widehat{p}_i)$ and the Riemann surface X_i. Hence we will sometimes view $W^u(\widehat{p}_i)$ as a subset of $\mathbf{C}^2$ and sometimes as an abstract Riemann surface. The precise meaning should be clear from the context.

Now the Riemann surface $W^u(\widehat{p}_i)$ cannot be hyperbolic, because η_i maps $\mathbf{C}$ into it so $W^u(\widehat{p}_i)$ is biholomorphic to $\mathbf{C}^*$, $\mathbf{C}$ or $\mathbf{P}^1$. The last case cannot occur, because then $W^u(\widehat{p}_i)$ would be an algebraic subvariety of $\mathbf{P}^2$, which is impossible. Hence $W^u(\widehat{p}_i)$ is biholomorphic to $\mathbf{C}^*$ or $\mathbf{C}$ for all i.

Write W_i instead of W_{a_i} and note that $W^u(\widehat{p}_i)$ has an open subset biholomorphic to W_i^*. Let Σ_i be the Riemann surface obtained from $W^u(\widehat{p}_i)$ by filling in the hole at a_i. Then Σ_i is biholomorphic to $\mathbf{C}$ or $\mathbf{P}^1$ for all i. If Σ_i is biholomorphic to $\mathbf{P}^1$ for some i, then Σ_i is an algebraic subvariety of $\mathbf{P}^2$ (in fact a line) and we are done, so assume that Σ_i is biholomorphic to $\mathbf{C}$ for all i.

Suppose that $(\Sigma_i - W_i) \cap W^s(J_\Pi) \neq \emptyset$ for some i. Then $(\Sigma_i - W_i) \cap W_b \neq \emptyset$ for some $b \in J_\Pi$, $b \neq a_i$. By the dichotomy given in Lemma 8.4 we then have that $W_b^* \subset (\Sigma_i - W_i)$ so by filling in the hole at b we see that the closure of Σ_i in $\mathbf{P}^2$ is an algebraic subvariety of $\mathbf{P}^2$, which implies that $W^u(\widehat{p}_i)$ is algebraic in this case too.

Let us now suppose that Σ_i is biholomorphic to $\mathbf{C}$ and that $(\Sigma_i - W_i) \cap W^s(J_\Pi) = \emptyset$ for all i. Then there exist unique biholomorphisms $\chi_i : \mathbf{C} \to \Sigma_i$ such that $\chi_i(0) = a_i$ and $\chi_i(\zeta) = \psi_i(1/\zeta) + o(1)$ as $\zeta \to 0$, where we write ψ_i instead of ψ_{a_i}. Note that f induces holomorphic maps of Σ_i onto Σ_{i+1}. Hence we may define entire maps h_i by $\chi_i \circ h_i = f \circ \chi_{i-1}$ for all i. The restriction of f to W_{i-1} is a branched covering of W_i of degree d, branched only at a_i. This implies that $h_i(\zeta) = \zeta^d \exp(u_i(\zeta))$ where u_i is entire. Moreover, the condition $(\Sigma_i - W_i) \cap W^s(J_\Pi) = \emptyset$ implies that the inverse image of W_i in Σ_{i-1} is exactly W_{i-1}. Therefore $\limsup |h_i(\zeta)| > 0$ as $|\zeta| \to \infty$ and this is only possible if u_i is constant. Hence we may write $h_i(\zeta) = c_i \zeta^d$ for some constants $c_i \neq 0$.

We claim that $\chi_i(\zeta) = \psi_i(1/\zeta)$ on $\mathbf{D}$ for all i. To see this, let $g_i = G \circ \chi_i$. Then, for each i, $g_i \geq 0$ is continuous and subharmonic on $\mathbf{C}^*$ and harmonic in a punctured neighborhood of the origin. Recall that $G(\psi_i(1/\zeta)) = -\log|\zeta|$ for $|\zeta| < 1$. Hence it follows from the definition of χ_i that $g_i(\zeta) = -\log|\zeta| + o(1)$ as $\zeta \to 0$. Now the equation $G \circ f = d\,G$ translates into $g_i \circ h_i = d\,g_{i-1}$, i.e. $g_i(c_i \zeta^d) = d\,g_{i-1}(\zeta)$. Iterating this we see that $g_i(\zeta)$ depends only on $|\zeta|$. Since g_i is harmonic in a punctured neighborhood of the origin it follows that $g_i(\zeta) = A_i \log|\zeta| + B_i$ for some constants A_i,B_i. But then the asymptotic formula above shows that $A_i = -1$ and $B_i = 0$, i.e. $g_i(\zeta) = -\log|\zeta|$ near the origin. We conclude from the equation $g_i \circ h_i = d\,g_{i-1}$ that $|c_i| = 1$. Write φ_i for the Böttcher coordinate on W_i, i.e. the inverse of ψ_i. Then the function $(\varphi_i(\chi_i(\zeta)))^{-1} = \zeta + O(\zeta^2)$ is holomorphic near the origin and

$$\log|(\varphi_i(\chi_i(\zeta)))^{-1}| = -G(\chi_i(\zeta)) = -g_i(\zeta) = \log|\zeta|.$$

Thus $(\varphi_i(\chi_i(\zeta)))^{-1} = \zeta$ near the origin, i.e. $\chi_i(\zeta) = \psi_i(1/\zeta)$ near the origin so the latter identity must hold on all of $\mathbf{D}$.

The equation $g_i(\zeta) = G(\psi_i(1/\zeta))$ on $|\zeta| < 1$ implies that $g_i = 0$ on $|\zeta| = 1$ for all i. We saw above that $g_i(\zeta)$ depends only on $|\zeta|$, so for each i either $g_i = 0$ on $|\zeta| \geq 1$ or there exists an $R_i \geq 1$ such that $g_i > 0$ for $|\zeta| > R_i$.

If $g_i = 0$ for $|\zeta| \geq 1$, then χ_i maps $\mathbf{C} - \bar{\mathbf{D}}$ into the bounded set K and must therefore extend to a holomorphic map of $\mathbf{P}^1$ into $\mathbf{P}^2$. Hence $W^u(\widehat{p}_i)$ is algebraic.

If $g_i > 0$ for $|\zeta| > R_i$, then χ_i maps $|\zeta| > R_i$ into $\mathbf{C}^2 - K$, and by our previous assumption, the image does not intersect $W^s(J_\Pi) = \operatorname{supp}(T \llcorner A)$, so g_i is harmonic on $|\zeta| > R_i$. Hence there exist constants $A_i > 0$ and B_i such that $g_i(\zeta) = A_i \log|\zeta| + B_i$ for $|\zeta| > R_i$. Since $G(x) = \log|x| + O(1)$ as $x \to \Pi$, this implies that $|\chi_i(\zeta)| \leq C|\zeta|^{A_i}$ as $\zeta \to \infty$, so again χ_i extends to a holomorphic map of $\mathbf{P}^1$ into $\mathbf{P}^2$. Hence $W^u(\widehat{p}_i)$ is algebraic, which completes the proof of Lemma 8.8. □

We are now in position to prove Lemma 8.3.

Proof of Lemma 8.3. Suppose that $W^s(J_\Pi) \cap W^u(S_1) \neq \emptyset$. Then $J'_\Pi \neq \emptyset$ so Lemma 8.8 shows that there exist $a \in J_\Pi$, a history $\widehat{p}$ in S_1 and an irreducible polynomial $P(z, w)$ such that $W^*_a \subset W^u(\widehat{p}) = \{P = 0\}$. Clearly $W^u(\widehat{p}) \cap J = \emptyset$ so there exists an $\epsilon > 0$ such that $|P| \geq 2\epsilon$ on J.

By Lemma 8.7 there is a dense set of b's such that W_b lands on J. If we choose b close enough to a, then by continuity W_b will intersect the open set $|P| < \epsilon$, so every component U of $\{\zeta \in \mathbf{D}^*; |P(\psi_b(\zeta))| < \epsilon\}$ is relatively compact in $\mathbf{D}^*$. Then P is a holomorphic function without zeros on U, so $-\log|P|$ is harmonic on U. But $|P| < \epsilon$ on U and $|P| = \epsilon$ on ∂U, contradicting the maximum principle for $-\log|P|$ on U. This completes the proof of Lemma 8.3. □

Theorem 8.2 allows us to describe J as a topological quotient of $J_\Pi \times S^1$.

Corollary 8.9. *If f satisfies condition (‡), then the restriction of ψ to $J_\Pi \times S^1$ maps $J_\Pi \times S^1$ continuously onto J.*

Proof. It follows from Theorem 8.2 that the restriction of Ψ to $J_\Pi \times S^1$ maps $J_\Pi \times S^1$ continuously into J. On the other hand, the push-forward of the measure $\mu_\Pi \otimes \frac{d\theta}{2\pi}$ under this map is the measure μ according to Theorem 6.3, so the map must be surjective. □

Corollary 8.10. *If f satisfies condition (‡) and J_Π is connected, then J is connected. If J_Π is also locally connected, then so is J.* □

Proof. If J_Π is connected (and locally connected) then $J_\Pi \times S^1$ is connected (and locally connected) so the statement to be proved follows from Corollary 8.9. □

Appendix A. Hyperbolicity for endomorphisms.

In this appendix we present some basic results on hyperbolicity for smooth endomorphisms. More details can be found in [J2]. Our main references are [Ru] and [PS], see also [FS4]. No proofs are given in this appendix; they can be found in the above references.

Let f be a C^∞ endomorphism of a finite-dimensional Riemannian manifold M. Let L be a compact subset of M with $f(L) = L$ and define

$$\hat{L} = \{(x_i)_{i\le 0}; x_i \in L, f(x_i) = x_{i+1}\}.$$

Then $\hat{L}$ is a closed subset of $L^{\mathbf{N}}$, hence compact. We will often use the notation $\hat{x}$ for a point $(x_i)_{i\le 0}$ in $\hat{L}$. The restriction $f|_L$ lifts to a homeomorphism $\hat{f}$ of $\hat{L}$ given by $\hat{f}((x_i)) = (x_{i+1})$. There is a natural projection π from $\hat{L}$ to L sending $\hat{x}$ to x_0 and the pullback under π of the restriction to L of the tangent bundle of M is a bundle on $\hat{L}$ which we call the tangent bundle $T_{\hat{L}}$. Explicitly, a point in $T_{\hat{L}}$ is of the form $(\hat{x}, v)$ where $\hat{x} \in \hat{L}$ and v is a tangent vector in $T_{x_0}M$. The derivative Df lifts to a map $D\hat{f}$ of $T_{\hat{L}}$ in a natural way.

Now f is said to be *hyperbolic* (or prehyperbolic) on L if there exists a continuous splitting $T_{\hat{L}} = E^u \oplus E^s$ which is invariant under $D\hat{f}$ and such that $D\hat{f}$ is expanding on E^u and contracting on E^s. More precisely, $D\hat{f}(E^{u/s}) \subset E^{u/s}$ and there are constants $c > 0$ and $\lambda > 1$ such that for all $n \ge 1$

$$\begin{aligned} |D\hat{f}^n(v)| &\ge c\lambda^n|v| \quad v \in E^u \\ |D\hat{f}^n(v)| &\le c^{-1}\lambda^{-n}|v| \quad v \in E^s. \end{aligned}$$

Remark. It is possible to make a smooth change of metric in a neighborhood of L and obtain $c = 1$ in the equation above.

Note that whereas the fiber of the unstable bundle E^u at a point $\hat{x} \in \hat{L}$ depends on the whole history $\hat{x}$ of x_0, the fiber of E^s at $\hat{x}$ depends only on the point x_0. Hence the dimension of the fiber of E^u at a point $\hat{x}$ depends only on x_0, so the dimensions of the fibers of the bundles E^u and E^s are locally constant.

As a special case of the above we say that f is expanding on L if the bundle E^s is trivial. This means that there exist constants $c > 0$ and $\lambda > 1$ such that $|D\hat{f}^n(x)v| \ge c\lambda^n|v|$ for all $x \in L$, $v \in T_xM$ and all $n \ge 1$.

Perhaps the most fundamental basic result in hyperbolic dynamics is the stable manifold theorem. For each point p in L and each history $\hat{q}$ in $\hat{L}$, we define local stable and unstable manifolds by

$$\begin{aligned} W^s_\delta(p) &= \{y \in M; d(f^i(y), f^i(p)) < \delta \ \forall i \ge 0\} \\ W^u_\delta(\hat{q}) &= \{y \in M; \exists \hat{y}, \pi(\hat{y}) = y, d(y_i, q_i) < \delta \ \forall i \le 0\}, \end{aligned}$$

for small $\delta > 0$.

The following theorem asserts that the local (un)stable manifolds are indeed nice objects. For a proof see [Ru] or [PS] ([Ru] contains an outline of a proof, whereas [PS] proves a more general theorem).

Theorem A.1 (Stable Manifold Theorem). *If δ is small enough, then*

(i) For all $p \in L$ and all $\hat{q} \in \hat{L}$, $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ are embedded C^∞ submanifolds of M tangent to $E^s(p)$ and $E^u(\hat{q})$ at p and q_0, respectively.

(ii) $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ depend continuously on p and $\hat{q}$, respectively.

(iii) If $x \in W^s_\delta(p)$, then $d(f^n(x), f^n(p)) \to 0$ exponentially fast as $n \to \infty$. Similarly, every point x in $W^u(\hat{q})$ has a unique history $\hat{x}$ such that $x_j \in W^u(\hat{f}^j(\hat{q}))$ for all $j \leq 0$ and $d(x_j, q_j) \to 0$ exponentially fast as $j \to -\infty$.

If δ is small enough, then by continuity $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ are almost flat, i.e. C^1 close to the tangents at p and q_0, respectively for all $p \in L$ and all $q \in \hat{L}$. Therefore $W^s_\delta(p)$ and $W^u_\delta(q)$ intersect in at most one point.

Definition A.2. *We say that L has local product structure if δ can be chosen so that $W^s_\delta(p) \cap W^s_\delta(\hat{q}) \subset L$ for all p and $\hat{q}$.*

If L has local product structure, $p \in L$, $\hat{q} \in \hat{L}$ and if p,q_0 are sufficiently close, then $W^s_\delta(p)$ and $W^u_\delta(\hat{q})$ intersect in exactly one point $x \in L$ and x has a history $\hat{x}$ such that $x_j \in W^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \leq 0$. It is not a priori clear that $\hat{x} \in \hat{L}$, i.e. that $x_j \in L$ for all $j \leq 0$. We therefore make another definition.

Definition A.3. *We say that $\hat{L}$ has local product structure if δ can be chosen so that if the intersection $W^s_\delta(p) \cap W^s_\delta(\hat{q})$ is nonempty, then it consists of a unique point $x \in L$ and the unique history $\hat{x}$ of x with $x_j \in W^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \leq 0$ is contained in $\hat{L}$.*

Definition A.4. *Let $\eta > 0$. An η-pseudoorbit in M is a sequence $(x_i)_{[t_1,t_2]}$, where $-\infty \leq t_1 < t_2 \leq \infty$, such that $d(f(x_i), x_{i+1}) < \delta$ for $t_1 \leq i < t_2$. An η-pseudoorbit $(x_i)_{[t_1,t_2]}$ is ϵ-shadowed by an orbit $(y_i)_{[t_1,t_2]}$ if $d(y_i, x_i) < \epsilon$ for all $i \in [t_1, t_2]$.*

For proofs of the remaining results in this appendix see [J2].

Theorem A.5 (Shadowing Lemma). *Suppose that $\hat{L}$ has local product structure. Then for each $\epsilon > 0$ there exists an $\eta > 0$ such that every η-pseudoorbit in L can be ϵ-shadowed by an orbit in L.*

Using shadowing we control the orbits of f staying near L in positive or negative time.

Proposition A.6 (Fundamental Neighborhood). *Let L be a hyperbolic set for a map f. Assume that $\hat{L}$ has local product structure. Then L has a neighborhood U in M such that*

(i) If $x \in U$ and $f^j(x) \in U$ for all $j \geq 0$, then $x \in W^s_\delta(p)$ for some $p \in L$.

(ii) If $x \in U$ and x has a history $\hat{x}$ with $x_i \in U$ for all $i \leq 0$, then $x \in W^u_\delta(\hat{q})$ for some $\hat{q} \in \hat{L}$.

(iii) If $(x_i)_{i \in \mathbf{Z}}$ is a complete orbit in U then $x_i \in L$ for all i.

Next we consider Axiom A endomorphisms. A point $x \in M$ is *wandering* if it has a neighborhood V such that $f^n(V) \cap V = \emptyset$ for all $n \geq 1$; otherwise it is called *non-wandering*. The *non-wandering set* Ω of f is the set of all non-wandering points; it is a closed set.

Definition A.7. *f is said to be Axiom A if its non-wandering set satisfies*
(i) Ω *is compact.*
(ii) Periodic points are dense in Ω.
(iii) f *is hyperbolic on* Ω.

Remark. If Ω satisfies (i) and (ii), then $f(\Omega) = \Omega$, so (iii) makes sense. Also, if f is Axiom A, then periodic points (under $\hat{f}$) are dense in $\hat{\Omega}$.

The next proposition shows that the preceding results apply to open Axiom A endomorphisms.

Proposition A.8. *If f is Axiom A and open, then $\hat{\Omega}$ has local product structure.*

Theorem A.9 (Spectral decomposition). *If f is an open Axiom A endomorphism, then Ω can be written in a unique way as a disjoint union $\Omega = \cup_{i=1}^{l} \Omega_i$, where each Ω_i is compact, satisfies $f(\Omega_i) = \Omega_i$ and f is transitive on Ω_i. The sets Ω_i are called the basic sets of f. Morover, each Ω_i can be further decomposed into a finite disjoint union $\Omega_i = \cup_{1 \le j \le n_i} \Omega_{i,j}$, where $\Omega_{i,j}$ is compact, $f(\Omega_{i,j}) = \Omega_{i,j+1}$ ($\Omega_{i,n_i+1} = \Omega_{i,1}$) and f^{n_i} is mixing on each $\Omega_{i,j}$.*

Our final result in this appendix describes forward and backward orbits for an Axiom A endomorphism.

Corollary A.10. *Assume that f is Axiom A and M is compact.*
(i) If $x \in M$, then there is a unique basic set Ω_i such that $f^j(x) \to \Omega_i$ as $j \to \infty$. Moreover, there is a (not necessarily unique) $p \in \Omega_i$ such that $d(f^j(x), f^j(p)) \to 0$ as $j \to \infty$.
(ii) If $\hat{x} \in \hat{M}$, then there is a unique basic set Ω_i such that $x_j \to \Omega_i$ as $j \to -\infty$. Moreover, there is a (not necessarily unique) $\hat{q} \in \widehat{\Omega_i}$ such that $d(x_j, q_j) \to 0$ as $j \to -\infty$.

References

[A] H. Alexander, Gromov's method and Bennequin's problem. Invent. Math. 125, 135–148 (1996).

[B] D. Barrett, Holomorphic extension from boundaries with concentrated Levi form, Indiana U. Math. J., 44 (1995), 1075–1087.

[BS1] E. Bedford, J. Smillie, Polynomial diffeomorphims of $\mathbf{C}^2$: currents, equilibrium measure and hyperbolicity. Invent. Math. 103, 69–99 (1991).

[BS2] E. Bedford, J. Smillie, Polynomial diffeomorphims of $\mathbf{C}^2$ V: Critical points and Lyaounov exponents. To appear in J. Geom. Anal.

[BT] E. Bedford, B. A. Taylor, Plurisubharmonic functions with logarithmic singularities. Ann. Inst. Fourier (Grenoble) 38, 133-171 (1988).

[CG] L. Carleson, T. W. Gamelin, Complex dynamics. Springer-Verlag (1993).

[DH] A. Douady, J. H. Hubbard, Étude dynamique des polynômes complexes I–II. Publications Mathématiques d'Orsay, 1984 and 1985.

[FS1] J. E. Fornæss, N. Sibony, Complex dynamics in higher dimension I. Astérisque 222, 201–231 (1994).

[FS2] J. E. Fornæss, N. Sibony, Complex dynamics in higher dimension II. In Annals of Mathematics Studies 137, 135–182. Princeton university press, 1995.

[FS3] J. E. Fornæss, N. Sibony, Complex dynamics in higher dimension, In P. M. Gauthier, G. Sabidussi, editors, Complex potential theory, pages 131–186. Kluwer Academic Publishers, 1994.

[FS4] J. E. Fornæss, N. Sibony, Hyperbolic maps on $\mathbf{P}^2$. Preprint.

[G] E. A. Gavosto, Attracting basins in $\mathbf{P}^2$. To appear in J. Geom. Anal.

[H] S.-M. Heinemann, Julia sets for holomorphic endomorphisms of $\mathbf{C}^n$. Ergodic Theory Dynam. Systems 16, 1275–1296 (1996).

[HP] J. H. Hubbard, P. Papadopol, Superattractive fixed points in $\mathbf{C}^n$. Indiana Univ. Math. J. 43, 321–365 (1994).

[J1] M. Jonsson, Sums of Lyapunov exponents for some polynomial maps of $\mathbf{C}^2$. Preprint.

[J2] M. Jonsson, Hyperbolic theory for endomorphisms. Preprint.

[Mi] J. Milnor, Dynamics in one complex variable: introductory lectures. Preprint SUNY StonyBrook (1990).

[Pe] G. Peng. On the dynamics of non-degenerate polynomial endomorphisms of $\mathbf{C}^2$. PhD thesis.

[Pr] F. Przytycki, Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map. Invent. Math. 80, 161–179 (1985).

[PS] C. Pugh, M. Shub, Ergodic attractors. Trans. Amer. Math. Soc. 312, 1–54 (1989).

[Ro] J.-P. Rosay, A remark on the paper by H. Alexander on Bennequin's problem. Invent. Math. 126, 625–627 (1996).

[Ru] D. Ruelle, Elements of differentiable dynamics and bifurcation theory. Academic Press, 1989.

[T] P. Tortrat, Aspects potentialistes de l'itération des polynômes. Séminaire de Théorie du Potentiel. Lecture Notes in Math 1235, 195–209 (1987).

[U] T. Ueda, Fatou sets in complex dynamics on projective spaces. J. Math. Soc. Japan 46, 545–555 (1994).

[Y] L.-S. Young, Ergodic theory of differentiable dynamical systems. In Branner, B., Hjorth, P., editors, Real and complex dynamical systems, pages 293–336, Kluwer Academic Publishers (1995).