The Parabolic Implosion for
$f_0(z) = z + z^{\nu+1} + \mathcal{O}(z^{\nu+2})$

Richard Oudkerk

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Doctor of Philosophy at the University of Warwick

Mathematics Institute
University of Warwick
Coventry CV4 7AL
England

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Declaration

I declare that to the best of my knowledge the material contained in this Thesis is original, except where explicitly stated otherwise.

Summary

In this thesis we examine the bifurcation in behaviour (for the dynamics) which occurs when we perturb the holomorphic germ $f_0(z) = z + z^{\nu+1} + \mathcal{O}(z^{\nu+2})$ defined in a neighbourhood of 0, so that the multiple fixed point at 0 splits into $\nu + 1$ fixed points (counted with multiplicity). The phenomenon observed is called the *parabolic implosion*, since the perturbation will typically cause the filled Julia set (if it is defined) to "implode."

The main tool used for studying this bifurcation is the Fatou coordinates and the associated Écalle cylinders. We show the existence of these for a family of *well behaved* f's close to f_0, and that these depend continuously upon f.

Each well behaved f will have a *gate structure* which gives a qualitative description of the "egg-beater dynamic" for f. Each gate between the fixed points of f will have an associated complex number called the *lifted phase*. (Minus the real part of the lifted phase is a rough measure of how many iterations it takes for an orbit to pass through the gate.) The existence of maps with any desired gate structure and any (sensible) lifted phases is shown. This leads to a simple parameterisation of the well behaved maps.

We are particularly interested in sequences $f_k \to f_0$ where all the lifted phases of the f_k converge to some limits, modulo $\mathbb{Z}$. We show that there is a non-trivial *Lavaurs map* g associated with these limits, which commutes with f_0. The dynamics of f_k are shown to (in some sense) converge to the dynamics of the system $\langle f_0, g \rangle$.

We also prove that for any Lavaurs map g there is a sequence $f_k \to f_0$ so that $f_k^k \to g$ as $k \to +\infty$, uniformly on compact sets.

Notation

Standard notation

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$	the natural numbers, integers, real numbers and complex numbers respectively
$\mathbb{N}_0$	the set of non-negative integers $\{0, 1, 2, \dots\}$
$\overline{\mathbb{C}}$	the Riemann sphere $\mathbb{C} \cup \{\infty\}$
$\mathbb{C}^*$	the set $\mathbb{C} \setminus \{0\}$
$\mathbb{D}$	the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$
D_R	the open disc $\{z \in \mathbb{C} \mid |z| < R\}$ where $R > 0$
$D(a, R)$	the open disc $\{z \in \mathbb{C} \mid |z - a| < R\}$ where $a \in \mathbb{C}$ and $R > 0$
$\overline{X}, \mathring{X}, \partial X$	the closure, interior and boundary respectively of the set X
$\#X$	the number of elements contained in the set X
$A \sqcup B$	the disjoint union of A and B
$A \cong B$	A is conformally isomorphic to B
id or id_X	the identity map on the set X
$[w, z]$	the set $\{(1 - t)w + tz \mid t \in [0, 1]\}$ where $w, z \in \mathbb{C}$
$\mathrm{mult}(f, \sigma)$	the multiplicity of f at σ where σ is a fixed point of f
$\iota(f, \sigma)$	the holomorphic index of f at σ, where σ is a fixed point of f (p. 7)
$\|f\|_K$	the uniform norm $\|f\|_K = \sup_{z \in K} |f(z)|$, where K is a compact subset of $\mathbb{C}$, and f is well defined on K
$\mathcal{O}(g(z))$ (as $z \to 0$)	an arbitrary map $f(z)$ such that $f(z)/g(z)$ is bounded (in some neighbourhood of 0)
$o(g(z))$ (as $z \to 0$)	an arbitrary map $f(z)$ such that $f(z)/g(z) \to 0$ (as $z \to 0$)
$x \gg a$ or $a \ll x$	x is much greater than a
$A := B$ or $B =: A$	A is defined to be B

Symbols used in the main text

$X_s = X_s^f$	the s-time flow for the vector field $\dot{z} = f(z) - z$ (p. 3)
$Y_t = Y_t^f$	the t-time flow for the vector field $\dot{z} = i[f(z) - z]$ (p. 3)
$\Psi_f : U \to \mathbb{C}$	the change of coordinate $\Psi_f(z) := \int_{z_0}^{z} \frac{d\zeta}{f(\zeta) - \zeta}$, where $U \subset \mathbb{C}$ is simply connected and contains no fixed points of f, and $z_0 \in U$ (p. 4)
$\mathcal{H}$	the set of holomorphic maps, together with the compact-open topology (p. 7)
f_0	the holomorphic germ $f_0(z) = z + z^{\nu+1} + \mathcal{O}(z^{\nu+2})$ defined in a neighbourhood of K_0 (p. 7)
K_0	the compact disc $\overline{D_{2r_0}}$, where $r_0 > 0$ is very small (p. 7)
r_0	half the radius of K_0 (p. 7)
$\mathcal{N}_0$	a small neighbourhood of $f_0 \in \mathcal{H}$, made up of holomorphic maps defined in a neighbourhood of K_0 (p. 7)
u_f	given $f \in \mathcal{N}_0$, u_f is a holomorphic map defined in a neighbourhood of K_0, such that $f(z) = z + (z - \sigma_0) \dots (z - \sigma_\nu) u_f(z)$ for some $\sigma_0, \dots, \sigma \in K_0$ (p. 8)
$\mathrm{Comp}^*(\overline{\mathbb{C}})$	the set $\{X \subset \overline{\mathbb{C}} \mid X \neq \varnothing \text{ is compact}\}$ together with the Hausdorff metric (p. 8)
$z_{i,+}, z_{i,-}$	points of distance r_0 away from 0, lying along the ith attracting direction, and ith repelling direction respectively, where $i \in \mathbb{Z}/\nu\mathbb{Z}$ (p. 9)
$\mathcal{WB}$	the maps in $\mathcal{N}_0$ which are well behaved (p. 10)
$\gamma_{i,s,f}$	the "maximal" trajectory for $\dot{z} = i[f(z) - z]$ which satisfies $\gamma_{i,s,f}(0) = z_{i,s}$, where $i \in \mathbb{Z}/\nu\mathbb{Z}$, $s \in \{+, -\}$ and $f \in \mathcal{N}_0$ (p. 9)

Organisation of the paper

This paper comes in three chapters: the Introduction, the Results, and the Proofs (together with an appendix).

It is organised into chapters, sections and subsections. §2 denotes the 2nd chapter, and §3.7.2 denotes the 2nd subsection of the 7th section of the 3rd chapter etc.

Formulae and figures are numbered within chapters, so (3.9) denotes the 9th formula of the 3rd chapter, and Figure 2.4 is the 4th figure of the 2nd chapter.

Statements are numbered within sections, so Corollary 4.7.3 is the 3rd statement of §4.7. By a statement we mean a Theorem, Proposition, Corollary, Lemma, Remark or Definition.

Proofs begin with "**Proof.**" and end with "■."

All the results will be stated in Chapter 2, and the page on which the corresponding proof can be found appears in a box like $\boxed{\text{p. 54}}$ in the margin. All sections in Chapter 2 will have a corresponding section in Chapter 3. So for example, a result in §2.3 will have its proof located in §3.3.

Chapter 1

Introduction

In this paper we examine the bifurcation in behaviour which occurs when we perturb the holomorphic germ

$$f_0(z) = z + z^{\nu+1} + \mathcal{O}(z^{\nu+2})$$

defined in a neighbourhood of 0, so that the multiple fixed point at 0 splits into $\nu+1$ fixed points (counted with multiplicity). The phenomenon observed is called the *parabolic implosion,* since the perturbation will typically cause the filled Julia set (if it is defined) to "implode." See Figure 1.1.

The main tool used for studying this bifurcation is the theory of Écalle cylinders, which was first introduced in [DH]. In this paper we show the existence of Fatou coordinates and Écalle cylinders for a fairly general family f's close to f_0. (These Fatou coordinates will conjugate f to the translation $T(w) := w + 1$ on particular regions.)

The ν incoming and ν outgoing Fatou coordinates for f_0 are quite easy to obtain (see [Mi]). In [La], [Sh1-3], [Do] and [Zi] the "persistence" of the Fatou coordinates (and Écalle cylinders) for certain perturbations is shown in the special case $\nu = 1$. ([DSZ], [EY] and [Wi] contain applications of these Fatou coordinates.)

In this thesis we prove the persistence of the Fatou coordinates in the general case $\nu \geqslant 1$. All the Theorems and Propositions in Chapter 2 are new (although A. Epstein may have some unpublished work on the same problem). Much of the notation used in this thesis follows [Sh2].

We will consider a fairly general family of *well behaved* maps f in a neighbourhood of f_0 and construct Fatou coordinates for these which depend continuously upon f.

Each well behaved f will have a *gate structure* which gives a qualitative description of the "egg-beater dynamic" for f. Each gate between the fixed points of f will have an associated complex number called the *lifted phase.* (Minus the real part of the lifted phase is a rough measure of how many iterations it takes for an orbit to pass through the gate.) The existence of maps with any desired gate structure and any (sensible) lifted phases is shown. This gives us a simple parameterisation of the well behaved maps.

We are particularly interested in sequences $f_k \to f_0$ where all the lifted phases of the f_k converge to some limits, modulo $\mathbb{Z}$. In this case, there is a non-trivial *Lavaurs map g* associated with these limits, which commutes with f_0. The dynamics of f_k are shown to (in some sense) converge to the dynamics of the system $\langle f_0, g \rangle$.

We also prove that for any Lavaurs map g there is a sequence $f_k \to f_0$ so that $f_k^k \to g$.

The Fatou coordinates and Écalle cylinders are constructed using the following Lemma. (See Lemma 3.3.13 in §3.3.5.)

> **Main Lemma** *Let K be a closed Jordan domain and let $f : K \to \mathbb{C}$ be analytic. Suppose that*
>
> 1. *$|f(z) - z| < \frac{1}{10}$ and $|f'(z) - 1| < \frac{1}{10}$ for every $z \in K$;*
> 2. *$\gamma : \mathbb{R} \to \overset{\circ}{K}$ solves $\dot{z} = i[f(z) - z]$ and $\gamma(t) \to \sigma_\pm$ as $t \to \pm\infty$ (where σ_+, σ_- are fixed points of f, which need not be distinct;)*
> 3. *$f(\ell) \subset \overset{\circ}{K}$ and $\ell \cap f(\ell) = \varnothing$, where $\ell := \gamma(\mathbb{R})$, (see Figure 1.2).*

(a)   (b)

(c)   (d)
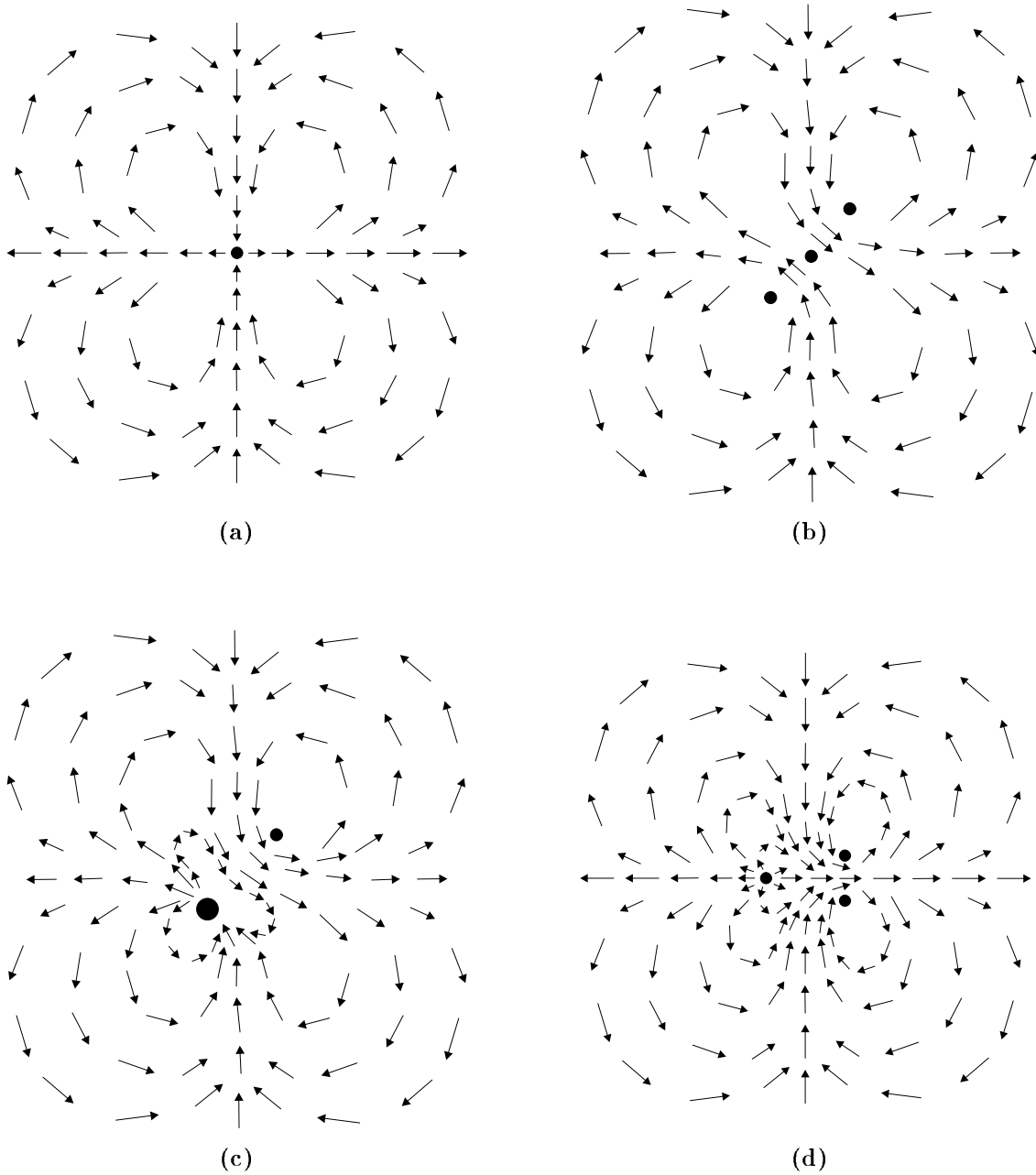
Figure 1.1: *In (a) we show the trajectories for $f_0(z) = z + z^3$, the unperturbed map. (b), (c) and (d) show the dynamics of maps close to f_0. Notice that the left-lower fixed point in (c) is a double fixed point. (a), (b) and (c) are all well behaved maps, and Fatou coordinates can be constructed for each of them. (d) however is not well behaved, although we do deal briefly with this example in the appendix*

*Then we can let S be the closed set bounded by the loop $\ell \cup f(\ell) \cup \{\sigma_+, \sigma_-\}$ and $S' :=$ $S \setminus \{\sigma_+, \sigma_-\}$ (which we call a **fundamental region**).*

There is an analytic, injective map $\Phi : S' \to \mathbb{C}$ such that

$$\Phi(f(z)) = \Phi(z) + 1 \qquad \text{for every } z \in \ell,$$

*and Φ is unique up to addition by a constant. We call Φ a **Fatou coordinate**.*

*We can construct the quotient S'/f by identifying $z \in \ell$ with $f(z) \in f(\ell)$. This is a cylinder with the structure of a Riemann surface. $[z]_f \mapsto [\Phi(z)]_{\mathbb{Z}}$ induces a conformal isomorphism $S'/f \xrightarrow{\cong} \mathbb{C}/\mathbb{Z}$ (where $[z]_f$ denotes the class of $z \in S'$ in S'/f and $[w]_{\mathbb{Z}}$ denotes the class of $w \in \mathbb{C}$ in $\mathbb{C}/\mathbb{Z}$). We call S'/f an **Écalle cylinder**.*
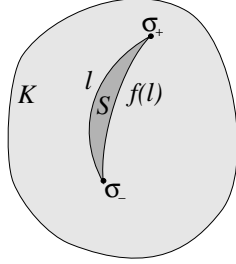
Figure 1.2:

Now in the cases which we will be dealing with, condition **(1.)** will be immediate if we make K a sufficiently small neighbourhood of 0. And if both **(1.)** and **(2.)** are satisfied, then condition **(3.)** will be immediate (although we may need to extend K slightly.)

Therefore much of the work we must do to prove the existence of the Fatou coordinates is aimed at proving that the trajectories for the vector field

$$\dot{z} = i[f(z) - z] \tag{1.1}$$

which pass through certain points will satisfy condition **(2.)**.

If these trajectories "do what they are supposed to" we will say that f is well behaved, and we will be able to apply the Main Lemma above to show the existence of the incoming and outgoing Fatou coordinates for f.

Vector fields and approximate Fatou coordinates

We now take a quick look at the two vector fields $\dot{z} = f(z) - z$ and $\dot{z} = i[f(z) - z]$, and how these are related to the dynamics of f. The flows for these vector fields are used many times in the proofs, and they also lead to a convenient change of coordinate.

Assuming that $f(z) - z$ and $f'(z) - 1$ are both small, we can in some sense "approximate" the discrete time dynamical system associated with f by the continuous time dynamical system associated with the vector field

$$\dot{z} = f(z) - z. \tag{1.2}$$

Orbits of points under f will roughly follow trajectories for (1.2). In fact f is the Euler method map of step length 1 for (1.2). (Compare with [Do, §10].) Notice also that σ is a fixed point of the flow for (1.2) if and only if it is a fixed point for f. (Notice also that trajectories for the vector field (1.2) will never cross one another since the vector field does not depend upon time. The same is true for (1.1).)

Since we will often be working with trajectories for the vector fields (1.2) and (1.1) it makes sense for us to denote the respective flows by $X_t = X_t^f$ and $Y_t = Y_t^f$. More specifically, if $z_0 \in \overset{\circ}{K}_0$ then X_t is determined by

$$\frac{\partial}{\partial t} X_t(z_0) = f(X_t(z_0)) - X_t(z_0) \quad \text{and} \quad X_0(z_0) = z_0,$$

and Y_t is determined by

$$\frac{\partial}{\partial t}Y_t(z_0) = i[f(Y_t(z_0)) - Y_t(z_0)] \quad \text{and} \quad Y_0(z_0) = z_0.$$

These are well defined (at least for small $t \in \mathbb{R}$) and are unique (by Theorem 3.2.1). For a fixed $T \in \mathbb{R}$ the maps X_T and Y_T will be holomorphic on their domains of definition (which may just be the empty set).

Suppose that $U \subset \mathbb{C}$ is a simply connected set containing no fixed points of f, and that f is well defined upon U. Then

$$\Psi_f(z) := \int_{z_0}^{z} \frac{d\zeta}{f(\zeta) - \zeta}$$

Ψ_f is well defined on U if we only allow integration over arcs in U and $z_0 \in U$. (Note that choosing a different $z_0 \in U$ will only have the effect of adding some constant to the map Ψ_f.) Compare with [Sh2, §2.6.1].

The significance of Ψ_f is that it is a "Fatou coordinate for X_1^f," i.e. Ψ_f will conjugate X_1^f to the translation $T(w) := w + 1$. (X_1^f is actually very close to f.) If we let $w = \Psi_f(z)$ then in the w-coordinate the push forwards of the two vector fields $\dot{z} = f(z) - z$ and $\dot{z} = i[f(z) - z]$ become $\dot{w} = 1$ and $\dot{w} = i$ respectively.

This Ψ_f provides us with an *approximate Fatou coordinate* for f. That is to say that $\Psi_f(f(z)) \approx \Psi_f(z) + 1$ when both sides are well defined.
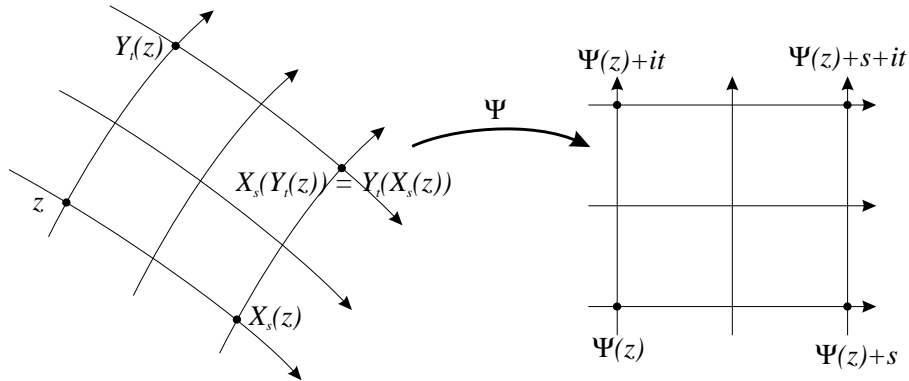
Figure 1.3:

X_s and Y_t commute with each other,

$$X_s \circ Y_t(z) = Y_t \circ X_s(z)$$

for every z, s, t such that both sides are well defined. We also find that $\Psi_f(X_s \circ Y_t(z)) = \Psi_f(z) + (s + it)$. See Figure 1.3.

Trajectories for the vector field $\dot{z} = f(z) - z$ correspond to horizontal lines in the $w = \Psi_f(z)$ coordinate, and trajectories for $\dot{z} = i[f(z) - z]$ correspond to vertical lines.

Ψ_f is basically the change of coordinate used in [Sh1-3] and [Do]. For instance, if $f(z) = z + z^{\nu+1}$ then $w = \Psi_f(z) = -\frac{1}{\nu z^\nu} + \text{const}$. Also if $f(z) = z + z^2 + \varepsilon$ then

$$\Psi_f(z) = \frac{1}{2i\sqrt{\varepsilon}} \log\left(\frac{z - i\sqrt{\varepsilon}}{z + i\sqrt{\varepsilon}}\right) + \frac{\pi}{\sqrt{\varepsilon}}\mathbb{Z} + \text{const}.$$

(See [Do, p.122].)

We will really only be considering the **local** dynamics of maps f close to f_0 (and also in passing the dynamics of the vector fields $\dot{z} = f(z) - z$ and $\dot{z} = i[f(z) - z]$), since we are unsure of the domain of definition of the maps f.

In [DES] a study is given of the **global** dynamics of $\dot{z} = V(z)$ where V is a degree d polynomial. (P. Sentenac gave a seminar based on [DES] in December 1998 at Warwick, but the paper has not been written as yet.) [DES] is not concerned with any discrete dynamical system, and so it has no direct connection with the subject of Fatou coordinates.

However, [DES] does use *"straightening coordinates"* (which correspond to our approximate Fatou coordinates), on each element of a partition of $\mathbb{C}$ to describe the global dynamics for the vector field. This is done for **any** polynomial V, without having to restrict attention to "well behaved" V's.

Each one of our *admissible gate structures* will correspond to one of the *"combinatoric invariants"* in [DES]. Also defined in [DES] are *"integral invariants"* which correspond to our *lifted phases*.

Lemma 3.7.13 (which is used in the proofs) is a special case of a result in [DES]. The proof that we give of Lemma 3.7.13 depends upon the same kind of global study of polynomial vector fields which is carried out in [DES]. (However I do not know whether the proof used in [DES] is quite the same.)

Chapter 2

The Results

2.1 Preliminaries

Given any analytic map, with a multiplier-$e^{2\pi i p/q}$ parabolic fixed point, we can shift the fixed point to the origin so that it is of the form

$$g_0(z) = e^{2\pi i p/q}z + az^{m+1} + \mathcal{O}(z^{m+2}),$$

for some $m \geqslant 1$. g_0^q is then of the form

$$g_0^q(z) = z + bz^{\nu+1} + \mathcal{O}(z^{\nu+2})$$

for a certain $\nu \geqslant 1$, and we are able to conjugate g_0^q (via some map of the form $z \mapsto \alpha z$) to a map $f_0 : \mathcal{D}(f_0) \to \mathbb{C}$, of the form

$$f_0(z) = z + z^{\nu+1}u_{f_0}(z),$$

where $u_{f_0}(0) = 1$ and u_{f_0} is analytic on $\mathcal{D}(f_0)$ (which is a neighbourhood of 0). We will work throughout with f_0 rather than g_0.

> **Definition 2.1.1 (Compact-open topology together with domain of definition)** *For any holomorphic map f defined on a subset of $\overline{\mathbb{C}}$ let $\mathcal{D}(f)$ denote the domain of definition of f. Now set*
>
> $$\mathcal{H} := \left\{ f : \mathcal{D}(f) \to \overline{\mathbb{C}} \;\middle|\; \begin{array}{c} f \text{ is holomorphic and} \\ \partial\big(\mathrm{interior}\,\mathcal{D}(f)\big) = \partial\mathcal{D}(f) \end{array} \right\}$$
>
> *where two functions are considered to be distinct if they have different domains of definition.*
>
> *We can construct a non-Hausdorff[1] topology on $\mathcal{H}$ (which is also defined in [Sh2, §2.5.1]) so that $f_m \to f$ if and only if for every compact set $K \subset \mathring{\mathcal{D}}(f)$ there is an m_0 so that $K \subset \mathring{\mathcal{D}}(f_m)$ for every $m \geqslant m_0$, and $f_m|_K \to f|_K$ uniformly as $m \to +\infty$. Roughly speaking this means the f_m converges to f uniformly on compact sets.*
>
> *In this topology a neighbourhood of $f_1 \in \mathcal{H}$ is any set containing*
>
> $$N(f_1, K, \varepsilon) := \{g \in \mathcal{H} \mid K \subset \mathring{\mathcal{D}}(g), \; \sigma\big(f_1(z), g(z)\big) < \varepsilon \;\forall z \in K\}$$
>
> *for some $\varepsilon > 0$ and a compact set $K \subset \mathring{\mathcal{D}}(f_1)$, where $\sigma(\cdot, \cdot)$ is the spherical metric on $\overline{\mathbb{C}}$.*
>
> *This topology will always be used in this paper. So when we write $f_k \to f$ (for some $f \in \mathcal{H}$) we implicitly mean that convergence is in the compact-open topology.*

We take a very small $r_0 > 0$ so that the closed disc $K_0 = \overline{D_{2r_0}}$ is contained in $\mathring{\mathcal{D}}(f_0)$. This K_0 will remain fixed throughout this paper.

We then take a very small open neighbourhood $\mathcal{N}_0$ of f_0 in the compact-open topology. Assuming this is small enough then $K_0 \subset \mathcal{D}(f)$ for every $f \in \mathcal{N}_0$. In fact we require that r_0 is small enough such that $1/r_0^\nu \gg |\iota(f_0, 0)|$ where $\iota(f_0, 0)$ is the holomorphic index (or dynamical residue) of f_0 at 0, which is defined as follows.

---

[1] This is not Hausdorff since if $f \in \mathcal{H}$ then any extension $\tilde{f}$ of f will lie in an arbitrarily small neighbourhood of f.

Definition 2.1.2 (Holomorphic index $\iota(f, \sigma)$) *We denote by $\iota(f, \sigma)$ the holomorphic index (or dynamical residue)*

$$\iota(f, \sigma) := \frac{1}{2\pi i} \oint_\sigma \frac{dz}{z - f(z)},$$

an integral over an infinitesimal anti-clockwise circle centred on the fixed point σ. This is a conformal invariant. See [Mi, §9].

If σ is a simple fixed point (that is, if it has multiplicity 1) then

$$\iota(f, \sigma) = \frac{1}{1 - f'(\sigma)}.$$

Lemma 2.1.3 (Definition and continuity of $f \mapsto u_f$) *For $f \in \mathcal{N}_0$, with $r + 1$ distinct fixed points inside K_0, let $s_0, \ldots, s_r$ be those fixed points, and let $m_k := \mathrm{mult}(f, s_k)$ be the associated multiplicities for $k = 0, \ldots, r$. Now define*

$$u_f(z) := \frac{f(z) - z}{(z - s_0)^{m_0} \ldots (z - s_r)^{m_r}}$$

on $K_0 \setminus \{s_0, \ldots, s_r\}$. This can be extended analytically to give $u_f : K_0 \to \mathbb{C}$.

The map $f \mapsto u_f$ is continuous on $\mathcal{N}_0$ (with respect to the compact-open topology).

When f_0 is perturbed, we get $\nu + 1$ fixed points counted with multiplicity which are very close to 0 (by Rouché's Theorem). Therefore any $f \in \mathcal{N}_0$ must be of the form

$$f(z) = z + (z - \sigma_0)^{m_0} \cdots (z - \sigma_r)^{m_r} u_f(z)$$

where $\sigma_0, \ldots, \sigma_r$ are close to 0 and $m_0 + \cdots + m_r = \nu + 1$. From Lemma 2.1.3 and the fact that $u_{f_0}(z) \approx 1$ for all $z \in K_0$, we see that $u_f(z) \approx 1$ for all $z \in K_0$ (assuming that $\mathcal{N}_0$ is very small).

Definition 2.1.4 (Hausdorff metric $d_H(\cdot, \cdot)$, and semi-distance $\partial(\cdot, \cdot)$) *We denote by $\partial(\cdot, \cdot)$ the semi-distance*

$$\partial(X, Y) = \sup_{x \in X} \left(\inf_{y \in Y} \sigma(x, y) \right).$$

where $\sigma(\cdot, \cdot)$ is the spherical metric on $\overline{\mathbb{C}}$ and $X, Y \subset \overline{\mathbb{C}}$ are compact. Then the Hausdorff metric is given by

$$d_H(X, Y) = \max \big(\partial(X, Y), \partial(Y, X) \big).$$

With this metric the space

$$\mathrm{Comp}^*(\overline{\mathbb{C}}) := \{ X \subset \overline{\mathbb{C}} \mid X \neq \varnothing \text{ is compact} \}$$

is compact. (See [Do, p. 112].)

Suppose that Λ is a topological space and for each $\lambda \in \Lambda$ there is an $X_\lambda \subset \overline{\mathbb{C}}$ which is compact and non-empty. Then we say that $\lambda \mapsto X_\lambda$ is lower semi-continuous *on Λ if for every $\lambda_0 \in \Lambda$ we have $\partial(X_{\lambda_0}, X_\lambda) \to 0$ as $\lambda \to \lambda_0$.*

Notice that $f \mapsto \overline{\mathcal{D}(f)}$ must be lower semi-continuous on $\mathcal{H}$.

2.2   Fundamental regions and Fatou coordinates for f_0

Definition 2.2.1 (Maximal solution of a vector field) *Let $V : D \to \mathbb{C}$ be holomorphic (where $D \subset \mathbb{C}$) and let $z_0 \in D$. Now suppose that $\gamma : I \to \mathbb{C}$ satisfies*

1. *I is an interval in $\mathbb{R}$ containing 0;*

2. *γ solves $\dot{z} = V(z)$;*

3. *$\gamma(0) = z_0$.*

We say that γ is maximal *if given any other $\tilde{\gamma} : \tilde{I} \to \mathbb{C}$ satisfying (1.), (2.) and (3.) we have $\tilde{I} \subseteq I$.*

(Note that given any $\gamma_1 : I_1 \to \mathbb{C}$ and $\gamma_2 : I_2 \to \mathbb{C}$ satisfying (1.), (2.) and (3.) we must have $\gamma|_{I_1 \cap I_2} = \tilde{\gamma}|_{I_1 \cap I_2}$, by Lemma 3.2.1 below.)

Given $r_0 > 0$ (which was fixed above) we define $z_{k,-} := r_0 e^{2\pi i(k-1)/\nu}$ and $z_{k,+} := e^{\pi i/\nu} z_{k,-}$ for $k \in \mathbb{Z}/\nu\mathbb{Z}$, so that the $z_{k,+}$ are in the attracting directions of f_0, and the $z_{k,-}$ are in the repelling directions. See Figure 2.1.

Lemma 2.2.2 (Fundamental regions for f_0) For $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$ let γ_{i,s,f_0} be the maximal trajectory passing through $z_{i,s}$ for the vector field

$$\dot{z} = i[f_0(z) - z].$$

Then $\gamma_{i,s,f}$ is well defined on $\mathbb{R}$ and $\gamma_{i,s,f_0}(t) \in \overset{\circ}{K}_0$ for all $t \in \mathbb{R}$. Also $\gamma_{i,s,f_0}(t) \to 0$ as $t \to \pm\infty$ for each $i \in \mathbb{Z}/\nu\mathbb{Z}$. None of these paths intersect one another.

In addition, if we define $\ell_{i,s,f_0} := \gamma_{i,s,f_0}(\mathbb{R})$ (for all i, s), then for each i we have that $f(\ell_{i,+,f_0})$ lies "inside the loop $\ell_{i,+,f_0} \cup \{0\}$," and $f(\ell_{i,-,f_0})$ lies "outside the loop $\ell_{i,-,f_0} \cup \{0\}$."

We denote by $S_{i,s,f}$ the closed set bounded by $\ell_{i,s,f_0} \cup f(\ell_{i,s,f_0}) \cup \{0\}$. See Figure 2.2. We also denote by S'_{i,s,f_0} the set $S_{i,s,f_0} \setminus \{0\}$, and we call this the *fundamental region*.
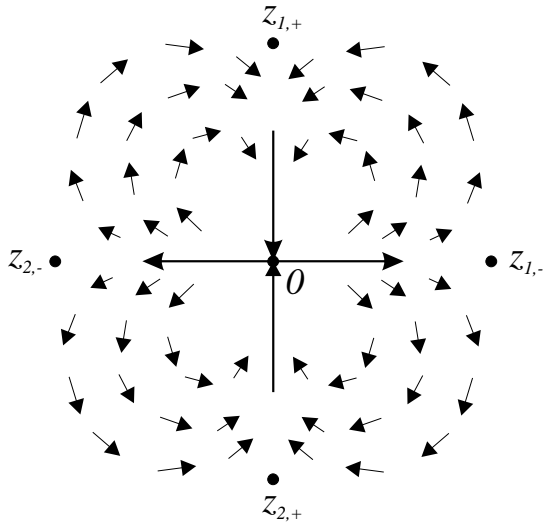
Figure 2.1: *We show the dynamics of $f_0(z) = z + z^3 u_{f_0}(z)$ close to the fixed point 0, along with the attracting and repelling directions.*
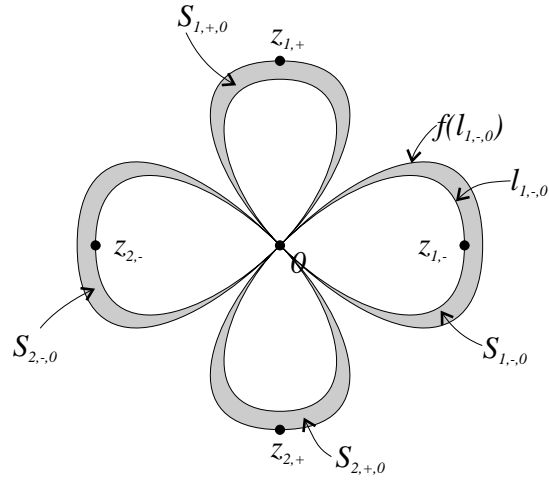
Figure 2.2: *The fundamental regions for $f_0(z) = z + z^3 u_{f_0}(z)$.*

The Main Lemma in the introduction (or Lemma 3.3.13) gives us the existence of the *incoming Fatou coordinates* $\Phi_{i,+,f_0} : S'_{i,+,f_0} \to \mathbb{C}$ and *outgoing Fatou coordinates* $\Phi_{i,-,f_0} : S'_{i,-,f_0} \to \mathbb{C}$ (where $i \in \mathbb{Z}/\nu\mathbb{Z}$). These are unique up to addition by a constant. The existence is proved by a different method in [Mi, Thm. 7.7].

2.3 Fundamental regions, Fatou coordinates and gate structures for f

Given an $i \in \mathbb{Z}/\nu\mathbb{Z}$, $s \in \{+, -\}$ and $f \in \mathcal{N}_0$, we let $\gamma_{i,s,f} : I \to \mathbb{C}$ be the maximal solution of the vector field $\dot{z} = i[f(z) - z]$ (defined on K_0) satisfying $\gamma_{i,s,f}(0) = z_{i,s}$. Also let $\ell_{i,s,f} := \gamma_{i,s,f}(I)$.

The "Continuous dependence of solutions" (see Theorem 3.3.9 below) tells us that if $f \in \mathcal{N}_0$ and $\mathcal{N}_0$ is sufficiently small, then all forward and backward trajectories for the vector field $\dot{z} = i[f(z) - z]$ will enter the open disc $D_{r_0/2}$ (since the same is true for f_0).

More specifically, for any $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$ there are some T_-, T_+ so that $T_- < 0 < T_+$, $\gamma_{i,s,f}(T_-), \gamma_{i,s,f}(T_+) \in D_{r_0/2}$. (This is because the same is true for f_0, and $D_{r_0/2}$ is open.) See Figure 2.3.
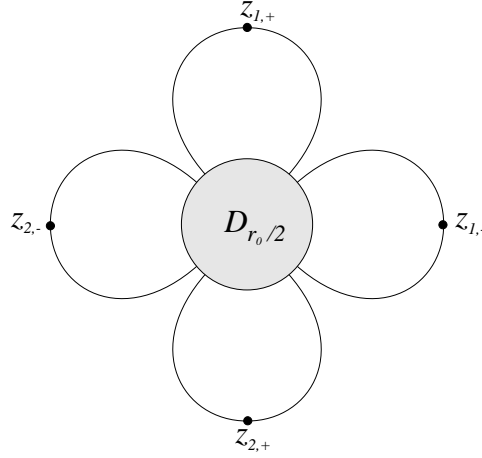
Figure 2.3:

Definition 2.3.1 (Well behaved, $\mathcal{WB}$) *We say that $f \in \mathcal{N}_0$ is well behaved if every forward and backward trajectory for the vector field $\dot{z} = i[f(z) - z]$ passing though the points $z_{i,s}$ stays in $D_{r_0/2}$ once it has entered that disc.*

More specifically, for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$ there are some $t_-, t_+ \in \mathbb{R}$ such that $t_- < 0 < t_+$ and

$$
\begin{aligned}
\gamma_{i,s,f}\big((-\infty, t_-)\big) &\subset D_{r_0/2}, \\
\gamma_{i,s,f}\big([t_-, t_+]\big) &\subset K_0 \setminus D_{r_0/2} \qquad\qquad \text{and} \\
\gamma_{i,s,f}\big((t_+, +\infty)\big) &\subset D_{r_0/2}.
\end{aligned}
$$

We let $\mathcal{WB} := \{f \in \mathcal{N}_0 \mid f \text{ is well behaved}\}$.

The topology of $\mathcal{WB}$ is looked at briefly in §2.9. The reason that we restrict our attention to maps in $\mathcal{WB}$ is that they will have "fundamental regions" with associated "Fatou coordinates" which are fairly easy to construct.

Let $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$. If $\lim_{t \to +\infty} \gamma_{i,s,f}(t)$ exists then we let

$$
\gamma_{i,s,f}(+\infty) := \lim_{t \to +\infty} \gamma_{i,s,f}(t).
$$

And similarly, if $\lim_{t \to -\infty} \gamma_{i,s,f}(t)$ exists then we let

$$
\gamma_{i,s,f}(-\infty) := \lim_{t \to -\infty} \gamma_{i,s,f}(t).
$$

Both $\gamma_{i,s,f}(+\infty)$ and $\gamma_{i,s,f}(-\infty)$ will be fixed points for f (if they exist).

Proposition 2.3.2 *(Combinatorics for well behaved maps) If $f \in \mathcal{WB}$ then the following hold.*

1. *Every trajectory $\gamma_{i,s,f}(t)$ converges to a fixed point (close to 0) as $t \to \pm\infty$.*

2. *For any fixed point σ of f in K_0, there is some $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+,-\}$ such that either $\gamma_{i,s,f}(+\infty) = \sigma$ or $\gamma_{i,s,f}(-\infty) = \sigma$.*

3. *For all $i \in \mathbb{Z}/\nu\mathbb{Z}$ we have $\gamma_{i,-,f}(+\infty) = \gamma_{i,+,f}(+\infty)$ and $\gamma_{i,-,f}(-\infty) = \gamma_{i-1,+,f}(-\infty)$.*

4. *For each $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+,-\}$, either the closure of $\ell_{i,s,f}$ is homeomorphic to a circle (in which case $\gamma_{i,s,f}(+\infty) = \gamma_{i,s,f}(-\infty)$ is a multiple fixed point), or there is a unique $j \in \mathbb{Z}/\nu\mathbb{Z}$ so that the closure of $\ell_{i,s,f} \cup \ell_{j,\bar{s},f}$ is homeomorphic to a circle, where $s \neq \bar{s} \in \{+,-\}$.*

5. *For any $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+,-\}$ we have $\ell_{i,s,f} \cap f(\ell_{i,s,f}) = \varnothing$, and the closure of $\ell_{i,s,f} \cup f(\ell_{i,s,f})$ is a Jordan contour which bounds a closed Jordan domain $S_{i,s,f}$. These $S_{i,s,f}$ (for the various i,s) can only intersect one another at the fixed points (which lie at their end points $\gamma_{i,s,f}(+\infty)$ and $\gamma_{i,s,f}(-\infty)$).*

We set $S'_{i,s,f} := S_{i,s,f} \setminus \{\gamma_{i,s,f}(+\infty), \gamma_{i,s,f}(-\infty)\}$. We call these sets the *fundamental regions* for f.

Notice that if $\sigma = \gamma_{i,s,f}(+\infty)$ (for some $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+,-\}$) then either σ is a multiple fixed point, or $\operatorname{Im} f'(\sigma) > 0$ and "the dynamics of f rotate anti-clockwise around σ." (See Remark 3.3.2 below.) Similarly, if $\sigma = \gamma_{i,s,f}(-\infty)$ then σ is either a multiple fixed point, or $\operatorname{Im} f'(\sigma) < 0$ and "the dynamics of f rotate clockwise around σ."

We can now define the gate structure for an $f \in \mathcal{WB}$.

Definition 2.3.3 *(Gate structure, $\mathsf{gate}(f)$) For an $f \in \mathcal{WB}$ we form the vector $\mathsf{gate}(f) = (\mathsf{gate}_1(f), \ldots, \mathsf{gate}_\nu(f))$ where*

$$\mathsf{gate}_i(f) := \begin{cases} j & \text{if } \overline{\ell_{i,+,f} \cup \ell_{j,-,f}} \text{ is homeomorphic to a circle;} \\ \star & \text{if } \overline{\ell_{i,+,f}} \text{ is homeomorphic to a circle.} \end{cases}$$

This is well defined (by Proposition 2.3.2) and in the particular case of f_0, we get $\mathsf{gate}(f_0) = (\star, \ldots, \star)$.

The ith gate is said to be open *if $\mathsf{gate}_i(f) \neq \star$, and* closed *if $\mathsf{gate}_i(f) = \star$.*

Definition 2.3.4 *(Admissible and $\mathcal{WB}(\mathsf{G})$) Note that although every gate structure has an associated vector $\mathsf{G} \in \{1, \ldots, \nu, \star\}^\nu$, not every such vector corresponds to an admissible gate structure.*

Let us draw a circle and place along it points labelled in anti-clockwise order $(1-)$, $(1+)$, $(2-)$, $(2+)$, $\ldots$, $(\nu-)$, $(\nu+)$. A vector $\mathsf{G} = (\mathsf{G}_1, \ldots, \mathsf{G}_\nu) \in \{1, \ldots, \nu, \star\}^\nu$ is said to be an admissible *gate structure if for each $j \in \mathbb{Z}/\nu\mathbb{Z}$ there is at most one $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that $\mathsf{G}_i = j$, and if we can draw non-intersecting lines on the disc between each pair $(i+)$, $(j-)$ for which $\mathsf{G}_i = j$. See Figure 2.4.*

Let

$$\text{Admissible} := \big\{ \mathsf{G} \in \{1, \ldots, \nu, \star\}^\nu \mid \mathsf{G} \text{ is admissible} \big\} \qquad \text{and}$$
$$\mathcal{WB}(\mathsf{G}) := \{ f \in \mathcal{WB} \mid \mathsf{gate}(f) = \mathsf{G} \}.$$

(Note that when in this paper we write $\mathsf{G} \in$ Admissible, G_k will denote the kth entry of G, where $k = 1, \ldots, \nu$.)

Proposition 2.3.5 *(All gate structures are admissible) If $f \in \mathcal{WB}$ then $\mathsf{gate}(f) \in$ Admissible.*

Remark 2.3.6 *In Figure 1.1 the maps associated with (a), (b) and (c) are all well behaved and have gate structures $(\star, \star)$, $(1, 2)$ and $(1, \star)$ respectively.*
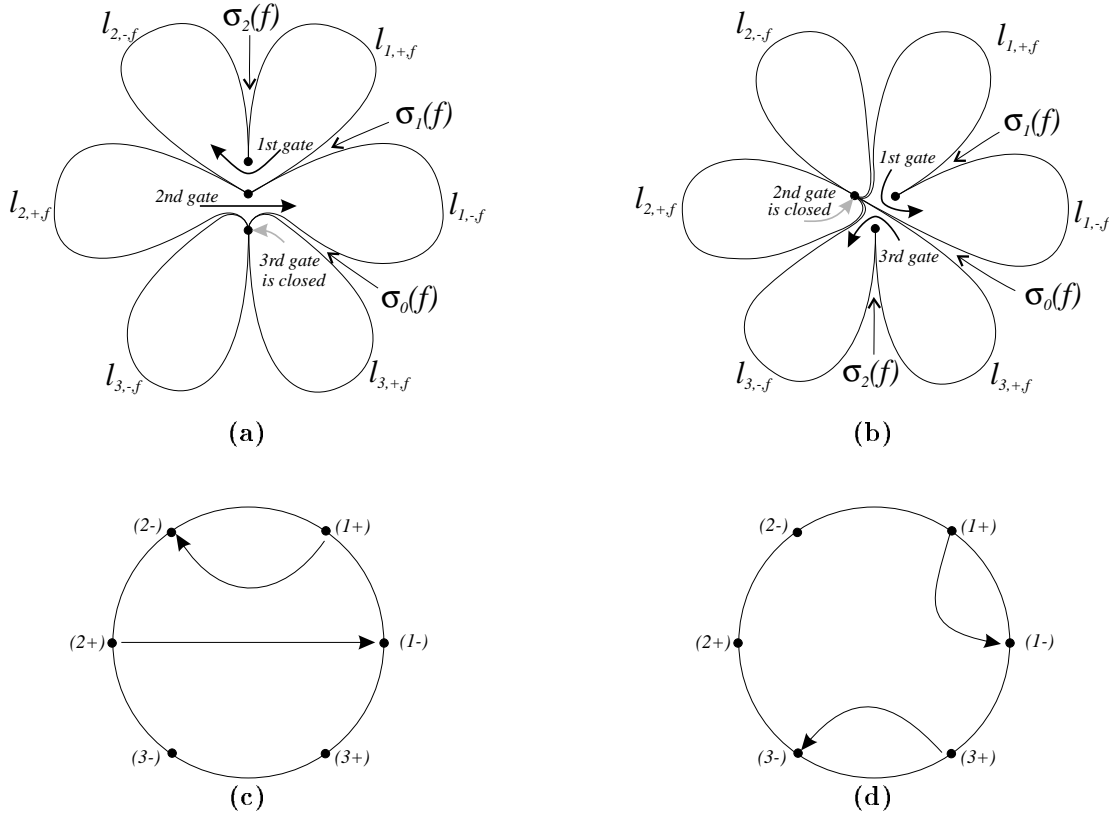
Figure 2.4: *In **(a)** we have the picture of the arrangement of ℓ_{i,s,f_1}'s when $\mathsf{gate}(f_1) = (2,1,\star)$, and the fixed points $\sigma_0(f_1), \sigma_1(f_1), \sigma_2(f_1)$ are labelled. In **(b)** we have the picture for $\mathsf{gate}(f_2) = (1,\star,3)$, again with the fixed points $\sigma_0(f_2), \sigma_1(f_2), \sigma_2(f_2)$ labelled. **(c)** and **(d)** are schematic representations of the gate structures shown in (a) and (b) respectively.*

Remark 2.3.7 *The reason that we choose to restrict our attention to maps in $\mathcal{WB}$ is that for any $f \in \mathcal{WB}$ there is a simple way of constructing our fundamental regions, using the trajectories $\gamma_{i,s,f}$.*

Notice however that for a particular $f \in \mathcal{WB}$ there may be many ways of picking a family of Jordan paths $\{\gamma_{i,s} : \mathbb{R} \to K_0\}_{i,s}$ such that $\gamma_{i,s}(0) = z_{i,s}$ for all i, s and such that Proposition 2.3.2 is still satisfied (where $\gamma_{i,s,f}$ is replaced by $\gamma_{i,s}$ for each i, s). For each such family there will be associated fundamental regions $\{S'_{i,s}\}_{i,s}$ and an associated gate structure $\mathsf{G}' \in \mathsf{Admissible}$ (and perhaps Fatou coordinates defined on the sets $\{S'_{i,s}\}_{i,s}$). However it is quite possible that $\mathsf{G}' \neq \mathsf{G}$.

As a result f may have more than one possible gate structure. In fact it would seem that any $f \in \mathcal{N}_0$ will have at least one possible gate structure (with associated Fatou coordinates). To avoid ambiguity we restrict our attention to well behaved maps, which will have only one "natural" gate structure determined by $\{\gamma_{i,s,f}\}_{i,s}$.

> **Proposition 2.3.8** *(**Numbers of fixed points and open gates**) Suppose that $\mathsf{G} \in \mathsf{Admissible}$ has $r \leqslant \nu$ open gates (i.e. $\#\{i \in \mathbb{Z}/\nu\mathbb{Z} \mid \mathsf{G}_i \neq \star\} = r$). Then every $f \in \mathcal{WB}(\mathsf{G})$ will have exactly $r + 1$ fixed points in K_0.*
>
> *Also for any $i \in \mathbb{Z}/\nu\mathbb{Z}$, $s \in \{+, -\}$ there will be some $m_1, m_2 \in \mathbb{N}$ (dependent on G) such that*
>
> $$\mathrm{mult}\big(f, \gamma_{i,s,f}(+\infty)\big) = m_1 \quad and \quad \mathrm{mult}\big(f, \gamma_{i,s,f}(-\infty)\big) = m_2$$
>
> *for any $f \in \mathcal{WB}(\mathsf{G})$. (That is, the multiplicity of the fixed points $\gamma_{i,s,f}(+\infty)$ and $\gamma_{i,s,f}(-\infty)$ of f will be m_1 and m_2 respectively.)*

Suppose that G has $r \leqslant \nu$ open gates. Then Proposition 2.3.2 parts (2.) and (3.) implies that for every $f \in \mathcal{WB}(\mathsf{G})$ we can label the $r + 1$ distinct fixed points of f using the following algorithm:
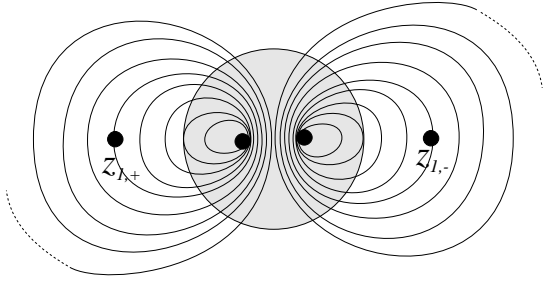
Figure 2.5: *We show the trajectories $\gamma_{1,+,f}$ and $\gamma_{1,-,f}$ through ($z_{1,+}$ and $z_{1,-}$ respectively) for the vector field $\dot{z} = i[f(z) - z]$, for a certain non-well behaved map f close to $f_0(z) = z + z^2$. (The shaded disc is $D_{r_0/2}$.) $\gamma_{1,\pm,f}(t)$ converges to a fixed point as $t \to \mp\infty$, but eventually tries to leave K_0 as $t \to \pm\infty$.*
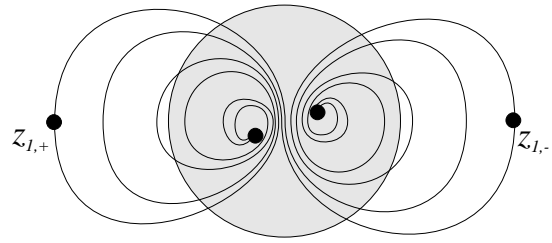
Figure 2.6: *We show the trajectories $\gamma_{1,+,f}$ and $\gamma_{1,-,f}$ through ($z_{1,+}$ and $z_{1,-}$ respectively) for the vector field $\dot{z} = i[f(z) - z]$, for a certain non-well behaved map f close to $f_0(z) = z + z^2$. Although f is not well behaved, $\overline{\gamma_{1,+,f}(\mathbb{R})} \cup \overline{\gamma_{1,-,f}(\mathbb{R})}$ will still be a Jordan contour homeomorphic to a circle (which is what one would find if f were well behaved).*
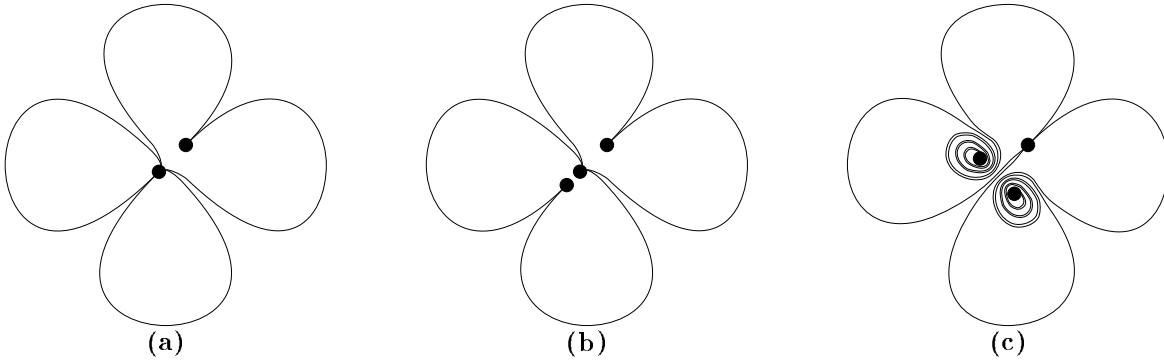
Figure 2.7: *In **(a)** we show the arrangement of the $\ell_{i,s,f}$'s for an $f_1 \in \mathcal{WB}((1, \star))$. It is possible to split apart the double fixed point of f_1 to give some $f_2 \in \mathcal{WB}((1,2))$ close to f_1, as shown in **(b)**. It is also possible to perturb f_1 to give an $f_3 \in \mathcal{WB}((2,1))$ as shown in **(c)**. (It is quite clear that we do not have Hausdorff upper semi-continuity when we perturb f_1 to give an $f \in \mathcal{WB}((2,1))$, and in fact there is a "mini parabolic implosion.")*

(1.) Let $m := 0$ and $i := 1$.

(2.) If $\gamma_{i,-,f}(-\infty)$ has not already been labelled then let $\sigma_m(f) := \gamma_{i,-,f}(-\infty)$ and let $m := m + 1$.

(3.) If $\gamma_{i,-,f}(+\infty)$ has not already been labelled then let $\sigma_m(f) := \gamma_{i,-,f}(+\infty)$ and let $m := m + 1$.

(4.) Let $i := i + 1$.

(5.) If $i < \nu$ then go back to step (2.), or else if $i = \nu$ then stop.

Therefore for any $f \in \mathcal{WB}$ we have $\sigma_0(f) := \gamma_{1,-,f}(-\infty)$. Also, notice that by Proposition 2.3.8 above, $f \mapsto \text{mult}(f, \sigma_m(f))$ is constant on $\mathcal{WB}(\mathsf{G})$ for each $m = 0, \ldots, r$.

Proposition 2.3.9 (*Continuity of the maps $f \mapsto \sigma_k(f)$*) *Fix* $\mathsf{G} \in \mathsf{Admissible}$ *with r open gates and for $f \in \mathcal{WB}(\mathsf{G})$ define $\sigma_0(f), \ldots, \sigma_r(f)$ using the above algorithm.*
Then $f \mapsto \sigma_k(f)$ is continuous on $\mathcal{WB}(\mathsf{G})$ for each $k = 0, \ldots, r$.

Proposition 2.3.10 (*The space of well behaved parameters is open*) *Fix* $\mathsf{G} \in$ $\mathsf{Admissible}$ *with r open gates. Give $\mathbb{C}^{r+1}$ the Euclidean norm, and the space of holomorphic maps $\mathcal{H}$ (as defined in Definition 2.1.1) with the compact-open topology. Then we give $\mathbb{C}^{r+1} \times \mathcal{H}$ the product topology.*
Let the $m_0, \ldots, m_r \in \mathbb{N}$ be those multiplicities such that $m_k = \text{mult}(f, \sigma_k(f))$ for all $f \in \mathcal{WB}(\mathsf{G})$ and $k = 0, \ldots, r$. Now define

$$f_{\mathbf{s},u}(z) = z + (z - s_0)^{m_0} \cdots (z - s_r)^{m_r} u(z)$$

where $\mathbf{s} = (s_0, \ldots, s_r) \in \mathbb{C}^{r+1}$ and $u \in \mathcal{H}$ is a holomorphic map defined in a neighbourhood of K_0. Then

$$\mathcal{P}(\mathsf{G}) := \{(\mathbf{s}, u) \in \mathbb{C}^{r+1} \times \mathcal{H} \mid f_{\mathbf{s},u} \in \mathcal{WB}(\mathsf{G})\}$$

is open in $\mathbb{C}^{r+1} \times \mathcal{H}$.

Whether or not the sets $\mathcal{P}(\mathsf{G})$ are connected (probably) depends upon our choice of $\mathcal{N}_0$. (See §2.9 below.)

Importantly, we have the following Proposition, which actually follows immediately from Corollary 2.7.2 and Proposition 2.4.5 below.

Proposition 2.3.11 (*f_0 is a limit point of $\mathcal{WB}(\mathsf{G})$*) *Given any* $\mathsf{G} \in \mathsf{Admissible}$ *there is a sequence $\{f_k\}_{k \geqslant 0}$ in $\mathcal{WB}(\mathsf{G})$ which converges to f_0 (with respect to the compact-open topology).*

Theorem 2.3.12 (*Existence and continuity of Fatou coordinates*) *Let $f \in \mathcal{WB}$, $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$.*

1. *There exists an analytic univalent map $\Phi_{i,s,f}$ defined in a neighbourhood of $S'_{i,s,f}$, satisfying*

$$\Phi_{i,s,f}(f(z)) = \Phi_{i,s,f}(z) + 1 \qquad \text{if } z \in \ell_{i,s,f}. \tag{2.1}$$

   This is unique up to addition by a constant. (That is, for any two such functions Φ_1, Φ_2, the function $\Phi_1 - \Phi_2$ is a constant.) We call this a Fatou coordinate of f.

2. *The Écalle Cylinder, $\mathcal{C}_{i,s,f} = S'_{i,s,f}/f$ obtained by identifying z and $f(z)$ for all $z \in \ell_{i,s,f}$, is conformally isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$ via $[z]_f \mapsto [\Phi_{i,s,f}(z)]_{\mathbb{Z}}$ (where $[z]_f$ denotes the equivalence class of $z \in S'_{i,s,f}$ in $S'_{i,s,f}/f$, and $[w]_{\mathbb{Z}}$ denotes the equivalence class of $w \in \mathbb{C}$ in $\mathbb{C}/\mathbb{Z}$).*

3. *For each $\mathsf{G} \in \mathsf{Admissible}$, the map $f \mapsto S_{i,s,f}$ is Hausdorff continuous on $\mathcal{WB}(\mathsf{G})$, but only Hausdorff lower semi-continuous on $\mathcal{WB}$. (The compact-open topology is used on $\mathcal{WB}$ and $\mathcal{WB}(\mathsf{G})$.)*

4. *There is a normalisation of the Fatou coordinates such that $f \mapsto (\Phi_{i,s,f} : S'_{i,s,f} \to \mathbb{C})$ is continuous on $\mathcal{WB}$ (using the compact-open topology on both sides).*

We will always use the "*preferred normalisation*" of the Fatou coordinates, which will be introduced later in Theorem 2.4.11. This normalisation satisfies Theorem 2.3.12 part **(4.)**.
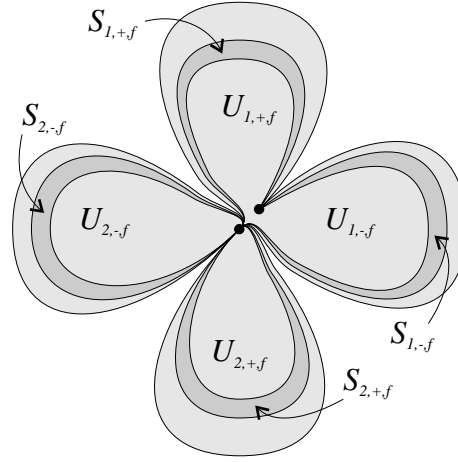
Figure 2.8: *We show typical $U_{i,s,f}$ and $S_{i,s,f}$ when $\mathsf{gate}(f) = (1, \star)$. Notice that $U_{1,+,f} = U_{1,-,f}$.*

Definition 2.3.13 (The sets $U_{i,s,f}$) *Let $f \in \mathcal{WB}$. Then if we have $\mathsf{gate}_i(f) = j \neq \star$ we let $U_{i,+,f} = U_{j,-,f}$ be the open Jordan domains bounded by the closure of $f^{-2}(\ell_{i,+,f}) \cup f^2(\ell_{j,-,f})$.*

If $\mathsf{gate}_i(f) = \star$ then we let $U_{i,+,f}$ be the open set bounded by the closure of $f^{-2}(\ell_{i,+,f})$.

And if $\mathsf{gate}_k(f) \neq j$ for each $k \in \mathbb{Z}/\nu\mathbb{Z}$ then we let $U_{j,-,f}$ be the open set bounded by the closure of $f^2(\ell_{j,-,f})$.

Notice that we will always have $S'_{i,s,f} \subset U_{i,s,f}$, and that for all $i, j \in \mathbb{Z}/\nu\mathbb{Z}$ we have $U_{i,+,f} = U_{j,-,f}$ if and only if $\mathsf{gate}_i(f) = j$. See Figure 2.8.

Proposition 2.3.14 (Extending $\Phi_{i,s,f}$ to $U_{i,s,f}$) *Let $f \in \mathcal{WB}$, $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$.*

1. *We can extend the Fatou coordinate (defined on $S'_{i,s,f}$) to give an analytic map $\Phi_{i,s,f} : U_{i,s,f} \to \mathbb{C}$ satisfying*

$$\Phi_{i,s,f}(f(z)) = \Phi_{i,s,f}(z) + 1 \qquad \text{if } z, f(z) \in U_{i,s,f},$$

and $\Phi_{i,s,f}$ is unique up to addition by a constant. Also we have $U_{i,s,f}/f = S'_{i,s,f}/f = C_{i,s,f}$.

2. *$f \mapsto \left(\Phi_{i,s,f} : U_{i,s,f} \to \mathbb{C} \right)$ and $f \mapsto \overline{U_{i,s,f}}$ are continuous on $\mathcal{WB}(\mathsf{G})$ for each $\mathsf{G} \in \mathsf{Admissible}$. (However neither is continuous on $\mathcal{WB}$, and $f \mapsto \overline{U_{i,s,f}}$ is not even lower semi-continuous.)*

2.4   Lifted phases and ʝ-indices

From now on we use the *"preferred normalisation"* for the Fatou coordinates (which is introduced properly in Theorem 2.4.11 below). Using this normalisation the map $f \mapsto \left(\Phi_{i,s,f} : S'_{i,s,f} \to \mathbb{C} \right)$ is continuous for each i and s. The preferred normalisation is fixed in such a way that the lifted phases defined below can be calculated by the simple formula in Theorem 2.4.11.

Definition 2.4.1 (Lifted phase, $\tilde{\tau}_i(f)$) *Recall that if $f \in \mathcal{WB}(\mathsf{G})$ and $\mathsf{G}_i = j \neq \star$ then $\Phi_{i,+,f}$ and $\Phi_{j,-,f}$ are both defined on $U_{i,+,f} = U_{j,-,f}$ and differ by a constant (since Fatou coordinates are unique up to addition by a constant).*

Therefore the lifted phase for the ith gate

$$\tilde{\tau}_i(f) := \begin{cases} \Phi_{j,-,f} - \Phi_{i,+,f} & \text{if } j = \mathsf{G}_i \neq \star, \\ \infty & \text{if } \mathsf{G}_i = \star, \end{cases}$$

is well defined.

The value of the lifted phase (for a open gate) gives us some idea of what happens to an orbit passing through this gate. For instance, $-\operatorname{Re}\tilde{\tau}_i(f)$ is roughly the number of iterations it takes for a point to pass though the ith gate. (Compare [Do, Prop 17.3].)

Also if $\operatorname{Im}\tilde{\tau}_i(f) \gg 0$ (resp. $\operatorname{Im}\tilde{\tau}_i(f) \ll 0$) then an orbit going through the ith gate tends to be pushed towards the fixed point $\gamma_{i,+,f}(+\infty)$ on the upper side (resp. $\gamma_{i,+,f}(-\infty)$ on the lower side).

More specifically we have the following. (Compare [Sh2, Prop 3.2.2].)

Remark 2.4.2 *Suppose that $f \in \mathcal{WB}(\mathsf{G})$ and $\mathsf{G}_i = j \neq \star$. For each $z \in S'_{i,-,f}$ there is a **unique** positive integer N such that $f^n(z) \in U_{i,+,f}$ for all $n = 0, \ldots, N$ and $f^N(z) \in S'_{j,-,f} \setminus \ell_{j,-,f}$. Also*

$$\Phi_{j,-,f}\big(f^N(z)\big) = \Phi_{i,+,f}(z) + \tilde{\tau}_i(f) + N.$$

Proposition 2.4.3 *(**Continuity of the lifted phase**) For each $i \in \mathbb{Z}/\nu\mathbb{Z}$, $f \mapsto \tilde{\tau}_i(f)$ is continuous as a map from $\mathcal{WB} \to \overline{\mathbb{C}}$.*

Remark 2.4.4 *For any $f \in \mathcal{WB}(\mathsf{G})$ it can be shown (see Lemma 3.7.4 below) that if $\mathsf{G}_i \neq \star$ then*

$$\operatorname{Re}\tilde{\tau}_i(f) < -\frac{1}{2r_0^\nu} \ll 0.$$

Proposition 2.4.5 *(**The size of the ith gate**) If $f \in \mathcal{WB}$ has $\mathsf{G}_i \neq \star$ then the distance between the fixed points $\sigma^u(f) := \gamma_{i,+,f}(+\infty)$ and $\sigma^\ell(f) := \gamma_{i,s,f}(-\infty)$ on either side of the ith gate will be bounded by the inequality*

$$|\sigma^u(f) - \sigma^\ell(f)| \leqslant \frac{const}{|\operatorname{Re}\tilde{\tau}_i(f)|^{1/\nu}}$$

for some constant dependent only upon ν.

Remark 2.4.6 *Proposition 2.4.5 implies that given a sequence $\{f_k\}_k$ in $\mathcal{WB}(\mathsf{G})$ satisfying*

- $\operatorname{Re}\tilde{\tau}_i(f_k) \to -\infty$ as $k \to +\infty$ for each i with $\mathsf{G}_i \neq \star$;

- $\sigma_0(f_k) \to 0$ and $u_{f_k} \to u_{f_0}$ as $k \to +\infty$;

we have $f_k \to f_0$ as $k \to +\infty$.

Note however that we can have a sequence $\{f_k\}_{k \geqslant 1}$ in $\mathcal{WB}(\mathsf{G})$ satisfying $|\tilde{\tau}_i(f_k)| \to +\infty$ for each i such that $\mathsf{G}_i \neq \star$, but such that $f_k \nrightarrow f_0$. (For example, this happens if we have $f \in \mathcal{WB}\big((1,\star)\big)$ and a sequence $\{f_k\}_{k \geqslant 1}$ in $\mathcal{WB}\big((2,1)\big)$ converging to f. See Figure 2.7.)

Now we introduce the *$\jmath$-index* which will be used later in the formulae for the lifted phases in Theorem 2.4.11.

Definition 2.4.7 *(**Definition of the $\jmath$-index, $\jmath(f,\sigma)$**) If $\sigma \in K_0$ is a fixed point of $f \in \mathcal{N}_0$ and $m := \operatorname{mult}(f,\sigma)$ is the multiplicity of σ then we let*

$$\jmath(f,\sigma) := \begin{cases} \frac{-2\pi i}{\log f'(\sigma)} & \text{if } m = 1, \\ 2\pi i \cdot \big[\iota(f,\sigma) - \frac{m}{2}\big] & \text{if } m > 1, \end{cases}$$

where we always take $\operatorname{Im}\log(\cdot) \in (-\pi,\pi]$, and $\iota(\cdot,\cdot)$ is the holomorphic index defined in Definition 2.1.2. We call $\jmath(\cdot,\cdot)$ the $\jmath$-index. (This like $\iota(\cdot,\cdot)$ is a conformal invariant.)

Lemma 2.4.8 *(**Continuity of the $\jmath$-index and holomorphic index**) If $U \subset K_0$ then both the maps*

$$f \mapsto \sum_{\sigma=f(\sigma)\in U} \jmath(f,\sigma) \qquad \text{and} \qquad f \mapsto \sum_{\sigma=f(\sigma)\in U} \iota(f,\sigma)$$

are continuous in a neighbourhood of $f_1 \in \mathcal{N}_0$ if f_1 has no fixed points on ∂U.

Notice that if σ is a simple fixed point of $f \in \mathcal{N}_0$, then in notation of [Sh2] we have $\jmath(f, \sigma) = -\frac{1}{\alpha}$ where $e^{2\pi i \alpha} = f'(\sigma)$ and $\mathrm{Re}\, \alpha \in (-\frac{1}{2}, \frac{1}{2}]$.

The $\jmath$-index behaves in very much the same way as the holomorphic index. In fact, if we have a sequence f_k of maps, each with a simple fixed point $\sigma(f_k)$, such that $f_k'\big(\sigma(f_k)\big) \to 1$ as $k \to +\infty$ then

$$\jmath\big(f_k, \sigma(f_k)\big) = 2\pi i \cdot \big[\iota\big(f_k, \sigma(f_k)\big) - \tfrac{m}{2}\big] + o(1)$$

as $k \to +\infty$ (where $m = \mathrm{mult}(f_k, \sigma(f_k)) = 1$ for all k).

Remark 2.4.9 *If σ is a simple fixed point of f, then it is attracting if and only if $\mathrm{Im}\,\jmath(f, \sigma) > 0$ and repelling if and only if $\mathrm{Im}\,\jmath(f, \sigma) < 0$.*

A multiple fixed point σ of f is parabolic-attracting (in the language of [Ep] and [EY, Appendix A]) if and only if $\mathrm{Im}\,\jmath(f, \sigma) > 0$ and parabolic-repelling if and only if $\mathrm{Im}\,\jmath(f, \sigma) < 0$.

Lemma 2.4.8 above implies that if f has a parabolic fixed point σ, then when we perturb f we get at least one attracting or parabolic-attracting fixed point close to σ.

Also if σ is a simple fixed point of f then the dynamics of f around it will "rotate anti-clockwise" (see Remark 3.3.2) if and only if $\mathrm{Re}\,\jmath(f, \sigma) < 0$, and "clockwise" if and only if $\mathrm{Re}\,\jmath(f, \sigma) > 0$.

Definition 2.4.10 (Fix$^u(i, f)$ and Fix$^\ell(i, f)$) *Suppose that $f \in \mathcal{WB}$, $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $\mathsf{gate}_i(f) \neq \star$. Notice that by the definition of f being well behaved, $\overline{D_{r_0/2}} \setminus U_{i,+,f}$ has two components. (See Figure 2.9.) We denote by $Upper(i, f)$ the component which contains $\gamma_{i,+,f}(+\infty)$, and by $Lower(i, f)$ the component which contains $\gamma_{i,+,f}(-\infty)$.*

We can then decompose the set of fixed points $\mathrm{Fix}(f) := \{\sigma \in K_0 \mid f(\sigma) = \sigma\}$ into the disjoint union $\mathrm{Fix}_i^u(f) \sqcup \mathrm{Fix}_i^\ell(f)$ by letting $\mathrm{Fix}^u(i, f) := Upper(i, f) \cap \mathrm{Fix}(f)$ and $\mathrm{Fix}^\ell(i, f) := Lower(i, f) \cap \mathrm{Fix}(f)$.

In some sense, $\mathrm{Fix}^u(i, f)$ is made up of those fixed points in K_0 which are above the ith gate, and $\mathrm{Fix}^\ell(i, f)$ is made up of those below.
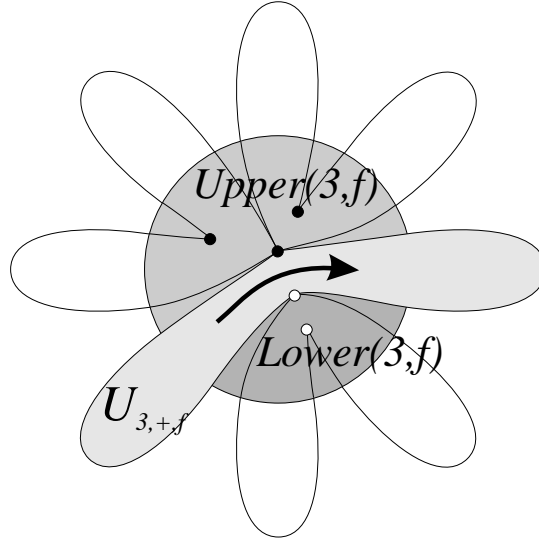
Figure 2.9: *We show the closed set $Upper(3, f)$ which contains the black fixed points which constitute $\mathrm{Fix}^u(3, f)$. We also show $Lower(3, f)$ which contains the white fixed points which constitute $\mathrm{Fix}^\ell(3, f)$.*

Theorem 2.4.11 (Formula for the lifted phases) *Suppose that we have $f \in \mathcal{WB}(\mathsf{G})$. There is a preferred normalisation of the Fatou coordinates (so that Theorem 2.3.12 part (4.) is satisfied) such that if $\mathsf{G}_i \neq \star$ then the lifted phase of the ith gate is given by*

$$\tilde{\tau}_i(f) = \begin{cases} +\sum_{\sigma \in Fix^u(i,f)} \jmath(f, \sigma) & \text{if } \sigma_0(f) \notin Fix^u(i, f); \\ -\sum_{\sigma \in Fix^\ell(i,f)} \jmath(f, \sigma) & \text{if } \sigma_0(f) \notin Fix^\ell(i, f). \end{cases}$$

This *preferred normalisation* will always be used from now on.

Proposition 2.4.12 (Bijections between lifted phases, $\jmath$-indices and holomorphic indices) *Suppose that* $\mathsf{G} \in \mathsf{Admissible}$ *has r open gates, and let $a_1, \ldots, a_r \in \mathbb{Z}/\nu\mathbb{Z}$ be such that $\{a_1, \ldots, a_r\} = \{i \mid \mathsf{G}_i \neq \star\}$ and $a_1 < a_2 < \cdots < a_r$. (Note that for $i, j \in \mathbb{Z}/\nu\mathbb{Z}$ we say $i < j$ if and only if there are $i', j' \in \{1, \ldots, \nu\}$ such that $i = i' + \nu\mathbb{Z}$, $j = j' + \nu\mathbb{Z}$ and $i' < j'$.)*

For $f \in \mathcal{WB}(\mathsf{G})$, let

$$
\begin{aligned}
T(f) &:= \big(\tilde{\tau}_{a_r}(f), \ldots, \tilde{\tau}_{a_r}(f)\big) \in \mathbb{C}^r, \\
P(f) &:= \big(\jmath(f, \sigma_1(f)), \ldots, \jmath(f, \sigma_r(f))\big) \in \mathbb{C}^r \qquad and \\
Q(f) &:= \big(\iota(f, \sigma_1(f)), \ldots, \iota(f, \sigma_r(f))\big) \in \mathbb{C}^r.
\end{aligned}
$$

Then there is an invertible linear map $B = B(\mathsf{G}) : \mathbb{C}^r \to \mathbb{C}^r$ so that $T(f) = B(P(f))$ for all $f \in \mathcal{WB}(\mathsf{G})$. There is also an invertible holomorphic map $M = M(\mathsf{G})$ (defined on a subset of $\mathbb{C}^r$) so that $P(f) = M(Q(f))$ for all $f \in \mathcal{WB}(\mathsf{G})$.

Therefore, if $f_1, f_2 \in \mathcal{WB}(\mathsf{G})$ we have

$$
T(f_1) = T(f_2) \iff P(f_1) = P(f_2) \iff Q(f_1) = Q(f_2).
$$

Recall that $f \mapsto (\Phi_{i,s,f} : S'_{i,s,f} \to \mathbb{C})$ is continuous on $\mathcal{WB}$. In particular it is true that $(\Phi_{i,s,f} : S'_{i,s,f} \to \mathbb{C})$ converges to $(\Phi_{i,s,f_0} : S'_{i,s,f_0} \to \mathbb{C})$ as $f \to f_0$.

However when we extend the Fatou coordinate to $U_{i,s,f}$ we no longer have continuity on all of $\mathcal{WB}$ since $f \mapsto \overline{U_{i,s,f}}$ is not even lower semi-continuous on $\mathcal{WB}$. Therefore the fact that a sequence $\{f_k\}_{k \geqslant 1}$ in $\mathcal{WB}$ converges to f_0 does **not** necessarily imply that $(\Phi_{i,s,f_k} : U_{i,s,f_k} \to \mathbb{C})$ converges to $(\Phi_{i,s,f_0} : U_{i,s,f_0} \to \mathbb{C})$ as $k \to +\infty$ (even when $f_k \in \mathcal{WB}(\mathsf{G})$ for all $k \geqslant 1$). However, we do have the following:

Proposition 2.4.13 (Equivalent convergence criteria) *Suppose that we have a sequence $\{f_k\}$ in $\mathcal{WB}(\mathsf{G})$ converging to f_0. Then the following are equivalent:*

1. *$\operatorname{Re} \tilde{\tau}_i(f_k) \to -\infty$ as $k \to +\infty$ for every $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that $\mathsf{G}_i \neq \star$;*

2. *$S_{i,s,f_k} \to S_{i,s,f_0}$ as $k \to +\infty$ for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$;*

3. *$\overline{U_{i,s,f_k}} \to \begin{cases} \overline{U_{i,+,f_0}} \cup \overline{U_{j,-,f_0}} & \text{if } s = + \text{ and } j = \mathsf{G}_i \neq \star, \\ \overline{U_{i,-,f_0}} \cup \overline{U_{j,+,f_0}} & \text{if } s = - \text{ and } \exists j \text{ s.t. } \mathsf{G}_j = i, \\ \overline{U_{i,s,f_0}} & \text{otherwise} \end{cases}$*
   as $k \to +\infty$ for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$;

4. *$\Phi_{i,s,f_k} : U_{i,s,f_k} \to \mathbb{C}$ converges to $\Phi_{i,s,f_0} : U_{i,s,f_0} \to \mathbb{C}$ as $k \to +\infty$ for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$.*

In fact given any sequence $\{f_k\}_k$ in $\mathcal{WB}(\mathsf{G})$ such that $u_{f_k} \to u_{f_0}$, $\sigma_0(f_k) \to 0$ and the real parts of all the lifted phases (of the open gates) converge to $-\infty$, then we must have $f_k \to f_0$ (by Lemma 3.7.8 below).

2.5 The Lavaurs map g, and $\langle f_0, g \rangle$, $J(f_0, g)$, $K(f_0, g)$

Proposition 2.5.1 (The Partial Lavaurs map) *Fix $f_0(z) = z + z^{\nu+1} u_{f_0}(z)$. There exists $\eta > 0$ such that if*

$$
\tilde{\theta} \in P_\eta = \{w \mid |\arg(w - \eta)| > 3\pi/4\}
$$

then

$$
h_{i,j,\tilde{\theta}} := \Phi^{-1}_{j,-,f_0} \circ T_{\tilde{\theta}} \circ \Phi_{i,+,f_0} : S'_{i,+,f_0} \to K_0
$$

is well defined, injective and analytic, where $T_{\tilde{\theta}}(w) := w + \tilde{\theta}$. (The preferred normalisation is used for the Fatou coordinates.) We call this a partial Lavaurs map and it satisfies

$$
f_0 \circ h_{i,j,\tilde{\theta}} = h_{i,j,\tilde{\theta}+1} = h_{i,j,\tilde{\theta}} \circ f_0 \tag{2.2}
$$

where both sides are well defined.

Now let $\mathsf{G} \in \mathsf{Admissible}$ with and $\mathsf{G}_i \neq \star$. If there is a sequence f_k in $\mathcal{WB}(\mathsf{G})$ and a sequence of integers $N_k \to +\infty$ such that $N_k + \tilde{\tau}_i(f_k) \to \tilde{\theta} \in P_\eta$ as $k \to +\infty$, then

$$
f_k^{N_k} \to h_{i,j,\tilde{\theta}} \tag{2.3}
$$

in the compact-open topology.

The following Corollary follows easily (and no proof will be given).

Corollary 2.5.2 *In the case where f_0 is a rational function (or entire function) for any $\tilde{\theta} \in \mathbb{C}$ it is clear that $\tilde{\theta} - m \in P_\eta$ if $m > 0$ is a large enough integer. Therefore*

$$h_{i,j,\tilde{\theta}} := f^m \circ h_{i,j,\tilde{\theta}-m} : S'_{i,+,f_0} \to \mathbb{C}$$

is well defined. Thus we can use the relation (2.2) to extend $h_{i,j,\tilde{\theta}}$ to the whole of

$$\mathcal{B}(i, f_0) := \bigcup_{k \in \mathbb{Z}} f_0^k(S'_{i,+,f_0}) = \bigcup_{m \geqslant 0} f_0^{-m}(U_{i,+,f_0}),$$

which is the parabolic basin of the ith attracting direction. This extension still satisfies (2.2) and (2.3). However it will not be injective since all pre-images of the critical point(s) of f_0 in $\mathcal{B}(i, f_0)$ will be critical points for $h_{i,j,\tilde{\theta}}$.

Definition 2.5.3 (The Lavaurs map g and "f_k approaches $\langle f_0, g\rangle$") *Let $\mathsf{G} \in$ Admissible, and $(\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)$ be a vector with $\tilde{\theta}_i \in P_\eta$ if $\mathsf{G}_i \neq \star$, and $\tilde{\theta}_i = \infty$ if $\mathsf{G} = \star$. We can define $g = g(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu) : \bigcup_{i \in \mathbb{Z}/\nu\mathbb{Z}} S'_{i,+,f_0} \to \mathbb{C}$ by*

$$g(z) := \begin{cases} h_{i,j,\tilde{\theta}_i}(z) & \text{if } \mathsf{G}_i = j \neq \star; \\ 0 & \text{if } \mathsf{G}_i = \star. \end{cases}$$

We call g the Lavaurs map and define the gate structure of this to be $\mathsf{gate}(g) := \mathsf{G}$. This g will still commute with f_0, and if f_0 is a rational function (or entire function) then it can be extended to the whole parabolic basin $\mathcal{B}(f_0) = \bigcup_{i \in \mathbb{Z}/\nu\mathbb{Z}} \mathcal{B}(i, f_0)$.

We say that the sequence f_k approaches $\langle f_0, g(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)\rangle$ if $f_k \in \mathcal{WB}(\mathsf{G})$ for all k, $f_k \to f_0$ as $k \to +\infty$ and $[\tilde{\tau}_i(f_k)]_\mathbb{Z} \to [\tilde{\theta}_i]_\mathbb{Z}$ as $k \to +\infty$ for every i with $\mathsf{G}_i \neq \star$.

Corollary 2.5.4 (Consequence of f_k approaching $\langle f_0, g\rangle$) *Suppose that we have a* *Lavaurs map $g = g(\mathsf{G}, \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu) : \mathcal{D}(g) \to \mathbb{C}$ associated with f_0. ($\mathcal{D}(g)$ may be the set $\cup_i S'_{i,+,f_0}$ or the domain of an extension still satisfying $f_0 \circ g = g \circ f_0$ where both sides are well defined.)*

If f_k approaches $\langle f_0, g\rangle$ then for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that $\mathsf{gate}_i(g) \neq \star$ there is a sequence of integers $N_k^{(i)}$ such that $N_k^{(i)} + \tilde{\tau}_i(f_k) \to \tilde{\theta}_i$ and

$$f_k^{N_k^{(i)}} \to g \qquad \text{uniformly on compact subsets of } \mathcal{D}(g) \cap U_{i,+,f_0}.$$

Suppose that we have a sequence of rational maps $f_k \to f_0$, where f_0 is also rational.

Given a z in the parabolic basin of 0, the orbit of z under f_0 converges to 0 along some attractive direction. However the orbit of z under f_k (where k is large) approaches the vicinity of 0 roughly along the same "attracting direction" and then leaves the vicinity of 0 along one of the repelling directions.

Suppose that f_0 is a rational map. Suppose that $g(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)$ is a Lavaurs map for f_0. Then

$$G_c := g(\mathsf{G}, \tilde{\theta}_1 + c, \ldots, \tilde{\theta}_\nu + c)$$

is well defined on the whole parabolic basin for any $c \in \mathbb{C}$ (where "$\infty + c = \infty$"). An *admissible pair* of integers is a pair (m, n) satisfying either $m \geqslant 0$, $n \geqslant 0$ or $m > 0$, $l \in \mathbb{Z}$. We order the set of admissible pairs by saying that

$$(m, n) < (m', n') \iff (m < m') \text{ or } (m = m' \text{ and } n < n')$$

For an admissible pair (m, n) we define

$$g^{m,n} := \begin{cases} f_0^n \circ G_0^m & \text{if } n \geqslant 0, \\ G_n \circ G_0^{m-1} & \text{if } n < 0. \end{cases}$$
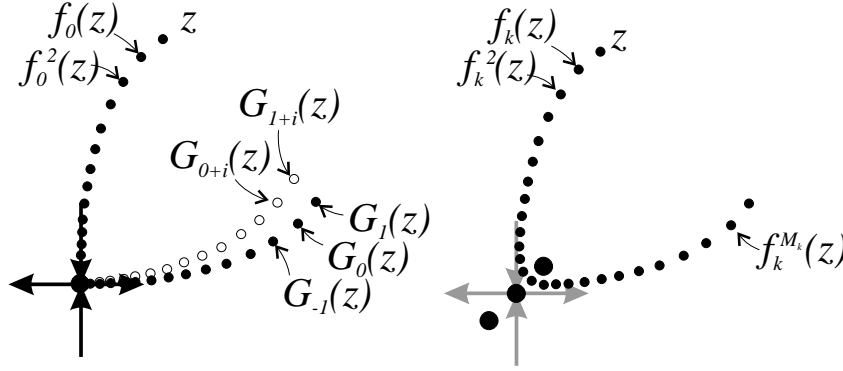
Figure 2.10: *On the left we show the attractive and repelling directions for $f_0(z) = z + z^3$. The black dots show the orbit of a point z under $\langle f_0, g \rangle$, where $\mathsf{gate}(g) = (1, 2)$. (The white dots show the effect on this orbit of adding i to $\tilde\theta_1$.) On the right we show the orbit of z under f_k. Note that $f_k^{M_k}(z) \approx G_0(z)$.*

Compare [DSZ, §2]. (Note that for any particular z and admissible pair (m, n) , $g^{m,n}(z)$ is well defined if and only if $g^a(z) \in \mathcal{B}(f_0)$ for all $a < m$. Also the domain of definition of $g^{m,n}$ is open.)

Then the orbit of a point z under $\langle f_0, g \rangle$ is defined to be

$$\{ g^{m,n}(z) \mid (m, n) \text{ is admissible and } g^{a,b}(z) \in \mathcal{B}(f_0) \text{ for all } (a, b) < (m, n) \}.$$

Note that this orbit only depends upon the class of the values $\tilde\theta_i$ in $\mathbb{C}/\mathbb{Z}$ (for those i with $\mathsf{G}_i \neq \star$).

If f_k approaches $\langle f_0, g \rangle$ then orbits for f_k converge to orbits for $\langle f_0, g \rangle$ in the following sense: if $g^{m,n}(z)$ is well defined then there is a sequence of integers $M_k \to +\infty$ such that $f_k^{M_k}(z) \to g^{m,n}(z)$. See Figure 2.10. (In fact $f_k^{M_k} \to g^{m,n}$ uniformly on some compact neighbourhood of z.)

The Lavaurs map can also be considered as taking "incoming orbits" for f_0 and sending them to "outgoing orbits."

Definition 2.5.5 (The Julia-Lavaurs set and filled Julia-Lavaurs set, $J(f_0, g)$, $K(f_0, g)$) *If f_0 is a polynomial, then we can extend g to the whole parabolic basin of 0. We can then define the filled Julia-Lavaurs set for $\langle f_0, g \rangle$ and Julia-Lavaurs set for $\langle f_0, g \rangle$ as*

$$K(f_0, g) := \{ z \mid g^m(z) \in K(f_0) \text{ for all } m \geqslant 0 \} \qquad \text{and}$$
$$J(f_0, g) := \partial K(f_0, g).$$

respectively, where $J(f_0)$ is the Julia set of f_0 and $K(f_0)$ is the filled Julia set of f_0. Note that $z \in K(f_0, g)$ if and only if the orbit of z under $\langle f_0, g \rangle$ is bounded.

These sets actually only depend on the phases $\theta_i = [\tilde\theta_i]_\mathbb{Z}$ (where $\mathsf{G}_i \neq \star$) of the Lavaurs map. See Figure 2.11.

These $K(f_0, g)$ and $J(f_0, g)$ are related to accumulation points of the sequences $\{K(f_k)\}_k$ and $\{J(f_k)\}_k$ in the Hausdorff metric, when f_k approaches $\langle f_0, g \rangle$. We can prove the following Proposition and Corollary (by generalising proofs in [Do]).

Proposition 2.5.6 (Convergence to $J(f_0, g)$ and $K(f_0, g)$) *Suppose that $f_0(z) = z + z^{\nu+1} + \mathcal{O}(z^{\nu+2})$ is a polynomial of degree d with no indifferent cycles other than 0, and that we have a sequence of degree-d polynomials $\{f_k\}_{k \geqslant 1}$ which approaches $\langle f_0, g \rangle$.*

Then we must have

$$\partial(J(f_0, g), J(f_k)) \to 0$$
$$\partial(K(f_k), K(f_0, g)) \to 0$$

as $k \to +\infty$, where $\partial(\cdot, \cdot)$ is the Hausdorff semi-distance in Definition 2.1.4.

Corollary 2.5.7 *(**Limit points of** $\{J(f_k)\}_k$ **and** $\{K(f_k)\}_k$) Recall from Definition 2.1.4 that $\mathrm{Comp}^*(\overline{\mathbb{C}})$ (the set of compact non-empty subsets of $\overline{\mathbb{C}}$, together with the Hausdorff metric) is compact. Thus if f_0 is a degree-d polynomial and $\{f_k\}_{k \geqslant 1}$ is a sequence of degree-d polynomials which approaches $\langle f_0, g \rangle$ then $\{J(f_k)\}_k$ and $\{K(f_k)\}_k$ must have some respective accumulation points, J^* and K^*.*

If g is non-trivial (that is, if $\mathsf{gate}(g) \neq (\star, \ldots, \star)$) then we then must have

$$J(f_0) \subsetneq J(f_0, g) \subseteq J^* \subseteq K^* \subseteq K(f_0, g) \subsetneq K(f_0).$$

Clearly if $J(f_0, g) = K(f_0, g)$, then we must have $J(f_k) \to J(f_0, g)$ and $K(f_k) \to K(f_0, g)$.

2.6   The return map and renormalised multiplier

Suppose $f \in \mathcal{WB}$ and that $\sigma^u := \gamma_{j,-,f}(+\infty)$ is a simple fixed point of f. Then if $z \in S'_{j,-,f}$ is sufficiently close to σ^u there is some least integer $p = p(z) > 1$ such that $f^i(z) \in K_0$ for every $i = 0, \ldots, p$, and $f^p(z) \in S'_{j,-,f}$. (See Figure 2.13, and see the proof of Lemma 3.4.4 below for the existence of such a p.)

There is an induced map from "the upper end of $\mathcal{C}_{j,-,f}$" back to $\mathcal{C}_{j,-,f}$. Since $\mathcal{C}_{j,-,f}$ is isomorphic to $\mathbb{C}/\mathbb{Z}$, this induces a map $\hat{\mathcal{R}}_f^{(j,u)} : \{[w]_{\mathbb{Z}} \in \mathbb{C}/\mathbb{Z} \mid \mathrm{Im}\, w > M\} \to \mathbb{C}/\mathbb{Z}$ (where $M > 0$ is large), such that $\hat{\mathcal{R}}_f^{(j,u)}([w]_{\mathbb{Z}}) = [\Phi_{j,-,f}(f^p(z))]_{\mathbb{Z}}$ if $w = \Phi_{j,-,f}(z)$ and $z \in S'_{j,-,f}$. (Compare this with $\tilde{\mathcal{R}}_f^{(j,u)}$ which is defined in Lemma 3.4.4 below.)

Now since $\mathbb{C}/\mathbb{Z}$ is isomorphic to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ via $\pi([w]_{\mathbb{Z}}) \mapsto e^{2\pi i w}$, there is an induced analytic map called the *return map* $\mathcal{R}_f^{(j,u)} := \pi \circ \hat{\mathcal{R}}_f^{(j,u)} \circ \pi^{-1} : \{z \in \mathbb{C}^* \mid |z| < e^{-2\pi M}\} \to \mathbb{C}^*$.

In the same way, if $\sigma^\ell = \gamma_{k,-,f}(-\infty)$ is a simple fixed point of f, and if $z \in S'_{k,-,f}$ is sufficiently close to σ^ℓ then there is some least integer $q = q(z) > 1$ such that $f^i(z) \in K_0$ for every $i = 0, \ldots, q$, and $f^q(z) \in S'_{k,-,f}$. There will again be some induced Return map $\mathcal{R}_f^{(k,\ell)} : \{z \in \mathbb{C}^* \mid |z| > e^{2\pi M}\} \to \mathbb{C}^*$ (if $M > 0$ is large enough).

Proposition 2.6.1 *(**Renormalised multipliers**) Suppose that $f \in \mathcal{WB}$. If $\sigma^u := \gamma_{j,-,f}(+\infty)$ is a simple fixed point then $\mathcal{R}_f^{(j,u)}$ (as constructed above) is well defined and extends analytically to 0 so that $\mathcal{R}_f^{(j,u)}(0) = 0$. The multiplier at 0 is $\left(\mathcal{R}_f^{(j,u)}\right)'(0) = e^{2\pi i J(f, \sigma^u)}$.*

Also if $\sigma^\ell := \gamma_{k,-,f}(-\infty)$ is a simple fixed point then $\mathcal{R}_f^{(k,\ell)}$ is well defined and extends analytically to ∞ so that $\mathcal{R}_f^{(k,\ell)}(\infty) = \infty$. The multiplier at ∞ is $\left(\mathcal{R}_f^{(k,\ell)}\right)'(\infty) = e^{2\pi i J(f, \sigma^\ell)}$.

So if $\sigma_m(f) := \gamma_{j,-,f}(+\infty)$ (resp. $\sigma_m(f) := \gamma_{j,-,f}(-\infty)$) is a simple fixed point then we can let $\lambda_m(f) := \left(\mathcal{R}_f^{(j,u)}\right)'(0)$ (resp. $\lambda_m(f) := \left(\mathcal{R}_f^{(j,\ell)}\right)'(\infty)$) without ambiguity. We call these $\lambda_m(f)$ the *renormalised multipliers for f*. They are given by $\lambda_m(f) = e^{2\pi i J(f, \sigma_m(f))}$ (provided that $\sigma_m(f)$ is a simple fixed point).
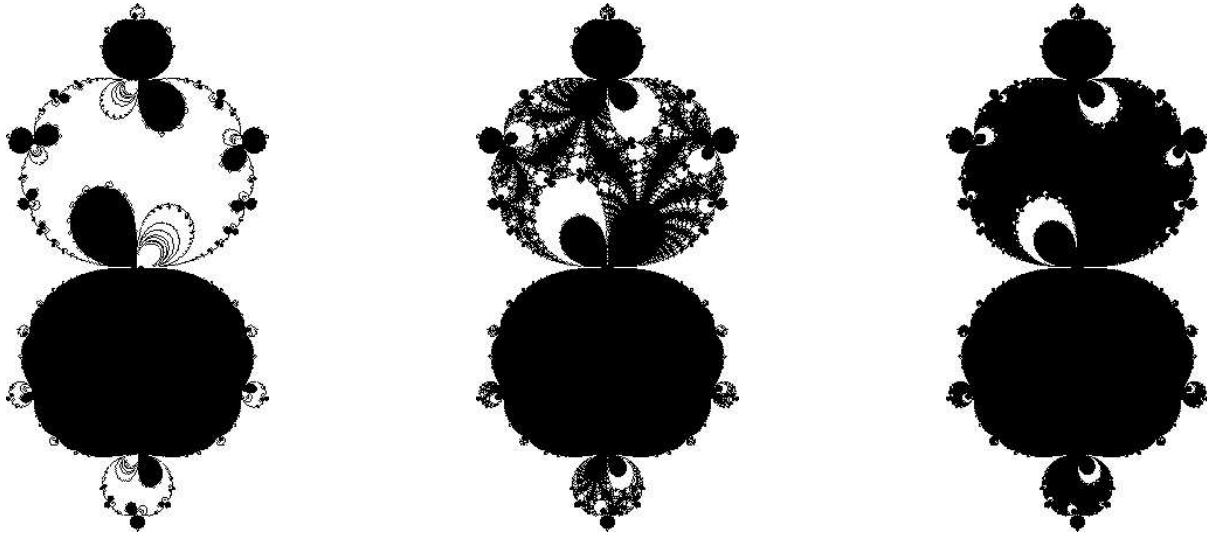
Figure 2.11: *Here are $K(f_0, g)$ for $f_0(z) = z + z^3$ and three choices of g with $\mathsf{gate}(g) = (1, \star)$. The first has a repelling "upper-right" virtual multiplier, with the upper critical point escaping. The second, again has a repelling upper-right virtual multiplier, but the upper critical point tends to a (generalised) attracting period cycle. The third has an attracting upper right virtual multiplier. (See Proposition 2.6.1.) Compare these with the filled Julia sets in Figure 2.12.*

Figure 2.12: *Here are three filled Julia sets $K(f)$, where f is well behaved, and of the form $f(z) = z + z^2(z - \sigma)$. These roughly correspond to the three $K(f_0, g)$ shown above in Figure 2.11. Each has a fixed parabolic basin of roughly the same shape. If g is a Lavaurs map with gate structure $(1, \star)$ and we take a sequence f_k approaching $\langle f_0, g \rangle$, we find that the parabolic basin becomes "pinched" into two pieces as $k \to +\infty$, as we can see in the "limit" filled Julia sets shown in Figure 2.11.*
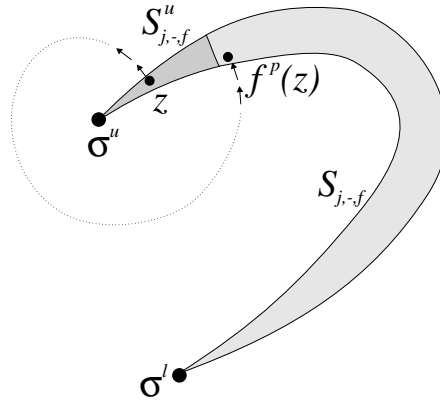
Figure 2.13: There is a small neighbourhood U of σ^u such that for every $z \in S^u_{j,-,f} := S'_{j,-,f} \cap U$ the orbit of that point will circle σ^u and eventually fall in $S_{j,-,f}$ again.

Proposition 2.6.2 *(Limits of return maps and renormalised multipliers)* $\boxed{\text{p. 50}}$

1. *Given a Lavaurs map $g = g(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)$ (associated with f_0) where G has $r \leqslant \nu$ open gates, we can find values $\varphi_m \in \mathbb{C}/\mathbb{Z}$ ($1 \leqslant m \leqslant r$) so that a sequence $f_k \to f_0$ in $\mathcal{WB}(\mathsf{G})$ approaches $\langle f_0, g \rangle$ if and only if $[\jmath(f_k, \sigma_m(f_k))]_{\mathbb{Z}} \to \varphi_m$ as $k \to +\infty$ for each $m = 1, \ldots, r$.*

2. *Suppose that f_k approaches $\langle f_0, g \rangle$ and $\gamma_{j,-,f}(+\infty)$ (resp. $\gamma_{j,-,f}(-\infty)$) is a simple fixed point for all $f \in \mathcal{WB}(\mathsf{G})$. Then there will be a map $\mathcal{R}^{(j,u)}_{\langle f_0, g \rangle}$ defined in a neighbourhood of 0 (resp. $\mathcal{R}^{(j,\ell)}_{\langle f_0, g \rangle}$ defined in a neighbourhood of ∞) so that $\mathcal{R}^{(j,u)}_{f_k} \to \mathcal{R}^{(j,u)}_{\langle f_0, g \rangle}$ (resp. $\mathcal{R}^{(j,\ell)}_{f_k} \to \mathcal{R}^{(j,\ell)}_{\langle f_0, g \rangle}$) as $k \to +\infty$.*

3. *If f_k approaches $\langle f_0, g \rangle$ and $\sigma_m(f)$ is a simple fixed point for all $f \in \mathcal{WB}(\mathsf{G})$ then $\lambda_m(f_k) \to \lambda_m(f_0, g) = e^{2\pi i \varphi_m}$. We call these $\lambda_m(f_0, g)$ the renormalised/virtual multipliers for $\langle f_0, g \rangle$.*

4. *If $\mathsf{gate}(g)$ has no closed gates then the product of all the virtual multipliers for $\langle f_0, g \rangle$ is*

$$\exp\big(2\pi i \jmath(f_0, 0)\big) = \exp\big(-4\pi^2(\iota(f_0, 0) - \tfrac{\nu+1}{2})\big).$$

The above limit maps $\mathcal{R}^{(j,\cdot)}_{\langle f_0, g \rangle}$ can be defined in terms of the *Écalle transformations* (see Definition 3.4.5) and the values $[\tilde{\theta}_1]_{\mathbb{Z}}, \ldots, [\tilde{\theta}_\nu]_{\mathbb{Z}}$ where $g = g(\mathsf{G}, \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)$.

Notice that for a simple fixed point $\sigma_m(f)$, we have $|\lambda_m(f)| < 1 \iff |f'(\sigma_m(f))| < 1$ and $|\lambda_m(f)| > 1 \iff |f'(\sigma_m(f))| > 1$.

Remark 2.6.3 If f_0 is a polynomial and $\langle f_0, g \rangle$ has an "attracting" virtual multiplier $\lambda_m(f_0, g)$ (i.e. $|\lambda_m(f_0, g)| < 1$ for some m), then this will have its own "attracting basin" in $\overset{\circ}{K}(f_0, g)$. (c.f. [La], [Do] and [Zi].)

It can be deduced (using the collapsing trap argument in [Do, §6]) that if f_k approaches $\langle f_0, g \rangle$, and $\overset{\circ}{K}(f_0, g)$ is made up only of pre-images of basins of (super-)attracting (generalised) periodic points or attracting virtual multipliers for $\langle f_0, g \rangle$ then

$$K(f_0, g) = \lim_{k \to +\infty} K(f_k) \qquad \text{and} \qquad J(f_0, g) = \lim_{k \to +\infty} J(f_k).$$

(This is the case for "almost every" polynomial f_0 and associated g.)

No study of the connected components of $\overset{\circ}{K}(f_0, g)$ will be carried out here, but one can compare [La], [Do] and [Zi]. It would seem that the results and proofs contained there can be generalised with suitable adaptions.

2.7   Realising maps with particular gate structures and lifted phases

Recall that for every $f \in \mathcal{N}_0$ there is an analytic map $u_f : K_0 \to \mathbb{C}$ close to u_{f_0} such that f can be written

$$f(z) = z + (z - s_0) \ldots (z - s_\nu) u_f(z),$$

for some $s_0, \ldots, s_\nu \in \mathbb{C}$ close to 0.

We can restrict our attention to a holomorphic family of maps of the form

$$f_{\mathbf{s}}(z) = z + z(z - s_1) \ldots (z - s_\nu) v_{\mathbf{s}}(z) \tag{2.4}$$

where $\mathbf{s} = (s_1, \ldots, s_\nu) \in \mathbb{C}^\nu$ lies in some small neighbourhood $\mathcal{N}(\mathbf{0}) \subset \mathbb{C}^\nu$ of $\mathbf{0} = (0, \ldots, 0)$ and the family $\{v_{\mathbf{s}}\}_{\mathbf{s} \in \mathcal{N}(\mathbf{0})}$ satisfies

- $v_{\mathbf{s}} \in \mathcal{H}$ is defined in a neighbourhood of K_0 for each $\mathbf{s} \in \mathcal{N}(\mathbf{0})$;

- $v_{\mathbf{0}} = u_{f_0}$;

- $\mathbf{s} \mapsto v_{\mathbf{s}}(z)$ is holomorphic for each $z \in K_0$;

- $v_{\mathbf{s}} = v_{\mathbf{s}'}$ if $\mathbf{s}, \mathbf{s}' \in \mathcal{N}(\mathbf{0})$ are permutations of each other.

It is clear that $f_{\mathbf{s}}$ does not depend on the ordering of $\mathbf{s}$. Then let

$$\mathcal{F} := \{f_{\mathbf{s}}\}_{\mathbf{s} \in \mathcal{N}(\mathbf{0})}.$$

This holomorphic family $\mathcal{F}$ is assumed to be fixed throughout the rest of this section.[2]

Now let H_ξ be the left half-plane $\{w \mid \operatorname{Re} w < -\xi\}$ where $\xi \gg 0$. Given $\mathsf{G} \in \mathsf{Admissible}$ we can let

$$\mathbf{H}(\mathsf{G}, \xi) := M_1 \times \cdots \times M_\nu \subset \overline{\mathbb{C}}^\nu \qquad \text{where} \qquad M_i := \begin{cases} H_\xi & \text{if } \mathsf{G}_i \neq \star, \\ \{\infty\} & \text{if } \mathsf{G}_i = \star. \end{cases}$$

Theorem 2.7.1 (Injectivity of $\mathcal{T}$) *If $\mathcal{T} : \mathcal{WB}(\mathsf{G}) \cap \mathcal{F} \to \overline{\mathbb{C}}^\nu$ is defined as*

$$\mathcal{T}(f) := \big(\tilde{\tau}_1(f), \ldots, \tilde{\tau}_\nu(f)\big)$$

then $\mathcal{T}$ is injective, and there exists $\xi_1, \xi_2 > 0$ such that

$$\mathbf{H}(\mathsf{G}, \xi_1) \subset \mathcal{T}\big(\mathcal{WB}(\mathsf{G}) \cap \mathcal{F}\big) \subset \mathbf{H}(\mathsf{G}, \xi_2).$$

In fact if $\mathcal{N}_0$ is chosen suitably then ξ_1 and ξ_2 can be chosen so that $\xi_1/\xi_2 \approx 1$. These will both then be roughly $(2^\nu - 1)\frac{1}{\nu r_0^\nu}$.

Corollary 2.7.2 (Existence of $f(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)$)

1. *For all $(\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu) \in \mathbf{H}(\mathsf{G}, \xi_1)$ there exists a unique $f = f(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu) \in \mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ such that*
$$\mathcal{T}(f) = (\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu).$$

2. *The map $(\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu, z) \mapsto f(z)$ is holomorphic as a map from $\mathbf{H}(\mathsf{G}, \xi_1) \times K_0 \to \mathbb{C}$.*

Corollary 2.7.3 (Every $\langle f_0, g \rangle$ is approached by some sequence $\{f_k\}$) *Suppose that f_0 has an associated Lavaurs map $g = g(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu) : \mathcal{D}(g) \to \mathbb{C}$.*

---

[2] Alternatively, if $w_\varepsilon(z)$ depends analytically upon (ε, z) close to $(0, 0) \in \mathbb{C}^{\nu+1} \times \mathbb{C}$, and $w_0(0) = 1$, then we could consider the family $\mathcal{G}$ of maps of the form

$$h_\varepsilon(z) := z + (\varepsilon_1 z + \cdots + \varepsilon_\nu z^\nu) + z^{\nu+1} + w_\varepsilon(z)$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\nu)$. $\mathcal{G}$ can be "rewritten" so that it is in the same form as the family $\mathcal{F}$ and vice versa, although no proof is given here.

Then we can find a sequence $\{f_k\}_{k \geqslant 1}$ *in* $\mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ *approaching* $\langle f_0, g \rangle$, *and such that* $f_k^k \to g$ *as* $k \to +\infty$.

Note that the above Corollary is much stronger than simply saying that there is a sequence f_k approaching $\langle f_0, g \rangle$. This is because (see Corollary 2.5.4) "f_k approaches $\langle f_0, g \rangle$" only implies the existence of sequences of integers $N_k^{(i)} \to +\infty$ for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that $f_k^{N_k^{(i)}} \to g$ uniformly on compact subsets of $\mathcal{D}(g) \cap U_{i,+,f_0}$.

Theorem 2.7.4 (Simultaneous orbit correspondence) *Suppose that for all* $i, j \in \mathbb{Z}/\nu\mathbb{Z}$ *we have compact sets* $X_i \subset U_{i,+,f_0}$ *and* $Y_j \subset U_{j,-,f_0}$, *and that* $a_i : \mathcal{N}_0 \to X_i$ *and* $b_j : \mathcal{N}_0 \to Y_j$ *are continuous maps for each* $i \in \mathbb{Z}/\nu\mathbb{Z}$.

For a large enough k_0 *there is a **unique** sequence* $\{f_k\}_{k \geqslant k_0}$ *in* $\mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ *such that*

    1. $f_k^m(a_i(f_k)) \in U_{i,+,f_k}$ for all $m = 0, \ldots, k$ and $i \in \mathbb{Z}/\nu\mathbb{Z}$;

    2. $f_k^k(a_i(f_k)) = b_j(f_k)$ for each $k \geqslant k_0$ and $i, j \in \mathbb{Z}/\nu\mathbb{Z}$ with $\mathsf{G}_i = j$;

and we have $f_k \to f_0$.

Remark 2.7.5 *If* f_0 *is globally defined and we have the situation in Theorem 2.7.4, there will be a Lavaurs map* g *which is well defined upon the whole of the parabolic basin (of 0 for* f_0*) such that*

$$f_k^k \to g$$

in the compact-open topology. However, if f_0 *is not globally defined then we can only be sure that the "Lavaurs map"* g *is well defined in some neighbourhood of* $\{a_1(f_0), \ldots, a_\nu(f_0)\}$.

2.8 Parameterisation of the well behaved maps

Theorem 2.8.1 (Injectivity of $\mathcal{T}^\#$) *If* $\mathcal{T}^\# : \mathcal{WB}(\mathsf{G}) \to \overline{\mathbb{C}}^\nu \times \mathbb{C} \times \mathcal{H}$ *is defined as*

$$\mathcal{T}^\#(f) := \left(\tilde{\tau}_1(f), \ldots, \tilde{\tau}_\nu(f), \sigma_0(f), u_f \right)$$

then $\mathcal{T}^\#$ *is injective.*

Also, there exists some large $\xi_1, \xi_2 > 0$, *some small* $\delta_1, \delta_2 > 0$ *and some small neighbourhoods* $\mathcal{N}_1(u_{f_0}), \mathcal{N}_2(u_{f_0})$ *of* u_{f_0} *such that*

$$\mathbf{H}(\mathsf{G}, \xi_1) \times D_{\delta_1} \times \mathcal{N}_1(u_{f_0}) \subset \mathcal{T}^\#\left(\mathcal{WB}(\mathsf{G}) \right) \subset \mathbf{H}(\mathsf{G}, \xi_2) \times D_{\delta_2} \times \mathcal{N}_2(u_{f_0}).$$

This Theorem implies that $\mathcal{WB}(\mathsf{G})$ is parameterised by the set $\mathcal{T}^\#\left(\mathcal{WB}(\mathsf{G}) \right)$.

Corollary 2.8.2 (Existence of $f(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu; \sigma_0; u)$)

    1. For all $(\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu, \sigma_0, u) \in \mathbf{H}(\mathsf{G}, \xi_1) \times D_{\delta_1} \times \mathcal{N}_1(u_{f_0})$ there exists a unique $f = f(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu; \sigma_0, u) \in \mathcal{WB}(\mathsf{G})$ such that

$$\mathcal{T}^\#(f) = (\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu, \sigma_0, u).$$

    2. The map $(\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu, \sigma_0, u) \mapsto f(z)$ is continuous as a map from

$$\mathbf{H}(\mathsf{G}, \xi_1) \times D_{\delta_1} \times \mathcal{N}_1(u_{f_0}) \to \mathcal{H}.$$

We can then make the definitions

$$\mathcal{WB}^*(\mathsf{G}) := \{f \in \mathcal{WB}(\mathsf{G}) \mid \mathcal{T}^\#(f) \in \mathbf{H}(\mathsf{G}, \xi_1) \times D_{\delta_1} \times \mathcal{N}_1(u_{f_0})\}$$

and

$$\mathcal{WB}^* := \bigcup_{\mathsf{G}} \mathcal{WB}^*(\mathsf{G}).$$

By Theorem 2.8.1 we really do not loose anything if we consider the space of maps $\mathcal{WB}^*$ instead of $\mathcal{WB}$. All the results up to now will still hold if we replace "$\mathcal{WB}$" and "$\mathcal{WB}(\mathsf{G})$" by "$\mathcal{WB}^*$" and "$\mathcal{WB}^*(\mathsf{G})$."

The topology of each $\mathcal{WB}^*(\mathsf{G})$ is very easy to understand since it is basically the same as that for $\mathbf{H}(\mathsf{G}, \xi_1) \times D_{\delta_1} \times \mathcal{N}_1(u_{f_0})$.

2.9   Additional comments

The topology of $\mathcal{WB}$ and $\mathcal{WB}^*$

First we consider the topology of $\mathcal{WB}^*$. The following statements hold (although we give no proofs).

1. $\mathcal{WB}^*$ and $\mathcal{WB}^* \cap \mathcal{F}$ are neither open nor closed.

2. If $\mathsf{G} \in \mathsf{Admissible}$ has no closed gates then $\mathcal{WB}^*(\mathsf{G})$ and $\mathcal{WB}^*(\mathsf{G}) \cap \mathcal{F}$ are open (in $\mathcal{N}_0$ and $\mathcal{N}_0 \cap \mathcal{F}$ respectively).

3. If $\mathsf{G} \in \mathsf{Admissible}$ has one or more closed gates then $\mathcal{WB}^*(\mathsf{G}) \subset \partial\mathcal{WB}^*$ and $\mathcal{WB}^*(\mathsf{G}) \cap \mathcal{F} \subset \partial\big(\mathcal{WB}^* \cap \mathcal{F}\big)$.

4. Suppose that $\mathsf{G}, \mathsf{G}' \in \mathsf{Admissible}$ are distinct and that $\mathsf{G}'_i = \mathsf{G}_i$ for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that $\mathsf{G}_i \neq \star$. Then $\mathcal{WB}^*(\mathsf{G}) \subset \partial\mathcal{WB}^*(\mathsf{G}')$ and $\mathcal{WB}^*(\mathsf{G}) \cap \mathcal{F} \subset \partial\big(\mathcal{WB}^*(\mathsf{G}') \cap \mathcal{F}\big)$.

5. $\overline{\mathcal{WB}^*} \subset \overset{\circ}{\mathcal{N}}_0$ and $\overline{\mathcal{WB}^* \cap \mathcal{F}} \subset \overset{\circ}{\mathcal{N}}_0$.

6. $\mathcal{WB}^*(\mathsf{G})$ and $\mathcal{WB}^*(\mathsf{G}) \cap \mathcal{F}$ are path connected and simply connected.

Statements (1.)–(4.) certainly hold if we replace $\mathcal{WB}^*$ by $\mathcal{WB}$. When we consider (5.) we find that although $\overline{\mathcal{WB} \cap \mathcal{F}} \subset \overset{\circ}{\mathcal{N}}_0$, we have $\overline{\mathcal{WB}^*} \not\subset \overset{\circ}{\mathcal{N}}_0$. However if $\mathcal{N}_0$ is suitably chosen then $f \in \partial\mathcal{WB} \cap \partial\mathcal{N}_0$ will imply that u_f is "relatively far away from u_{f_0}." It seems likely that (6.) is also true for $\mathcal{WB}$.

A much strengthened version of part (4.) is the following. Suppose that $\mathsf{G}, \mathsf{G}' \in \mathsf{Admissible}$ are distinct and that $\mathsf{G}'_i = \mathsf{G}_i$ for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that $\mathsf{G}_i \neq \star$. If a sequence $\big\{\big(\tilde{\theta}_1^{(k)}, \ldots, \tilde{\theta}_\nu^{(k)}\big)\big\}_k$ in $\mathbf{H}(\mathsf{G}', \xi_1)$ converges to $(\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu) \in \mathbf{H}(\mathsf{G}, \xi_1)$ then

$$f\big(\mathsf{G}'; \tilde{\theta}_1^{(k)}, \ldots, \tilde{\theta}_\nu^{(k)}\big) \to f(\mathsf{G}'; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)$$

as $k \to +\infty$.

If $f \in \partial\mathcal{WB} \cap \overset{\circ}{\mathcal{N}}_0$ then either $\mathsf{G} := \mathsf{gate}(f)$ has a closed gate, or there is some $i \in \mathbb{Z}/\nu\mathbb{Z}$, $s \in \{+, -\}$ such that $\gamma_{i,s,f}$ intersects and is tangent to the circle $\partial D_{r_0/2}$ at some point.

It is a little unfortunate that $\mathcal{WB}$ depends upon the choice of r_0. By decreasing (resp. increasing) the size of r_0 we would effectively shrink (resp. enlarge) the set $\mathcal{WB}$.

Remark 2.9.1 *It would be nice to take r_0 as large as possible. The condition for f to be well behaved is really a condition on the vector field $\dot{z} = i[f(z) - z]$. If f is a $\nu + 1$ degree polynomial, then the vector field is globally defined and in some sense it is possible to use $r_0 = +\infty$ and $z_{i,s} := \infty$ for each i, s. (This is basically what is done in [DES] and the proof of Lemma 3.7.13.)*

The non-well behaved cases for "$r_0 = +\infty$" will then be those polynomial vector fields which have one or more "homoclinic links." Such cases are degenerate (and belong to a set with "small dimension"). Note however that if f is "well behaved for $r_0 = +\infty$" then this does not imply the existence of fundamental regions for f.

Examples in the parameter space

Let us consider the family of maps $f_c(z) = z^2 + c$ close to $f_{1/4}(z) = z^2 + \frac{1}{4}$. Then the set

$$C_\eta = \{c \mid \text{if } f_c(w) = w \text{ then } |\operatorname{Im}\iota(f_c, w)| > \eta\}$$

is a cardioid. By Corollary 3.7.5 and Lemma 3.7.4 below there must be some $\eta_-, \eta_+ > 0$ such that

$$C_{\eta_+} \subset \{c \mid f_c \in \mathcal{WB}\} \subset C_{\eta_-}.$$

See Figure 2.14.

As another example, if we consider the maps $h_\sigma(z) = z + z^2(z - \sigma)$ where σ is close to 0, then

$$C'_\eta = \{\sigma \mid \text{if } h_\sigma(w) = w \text{ then } |\operatorname{Im}\iota(h_\sigma, w)| > \eta\}$$

forms a four-leaved-clover. Once again, we find that there are some η_-, η_+ such that

$$C'_{\eta_+} \subset \left\{\sigma \mid h_\sigma \in \mathcal{WB}(\mathsf{G}) \text{ where } \mathsf{G} \in \{(1,\star), (2,\star), (\star,2), (\star,1)\}\right\} \subset C'_{\eta_-}.$$

See Figure 2.15. Each of the "leaves" corresponds to one of the gate structures $(1,\star)$, $(2,\star)$, $(\star,2)$ and $(\star,1)$.
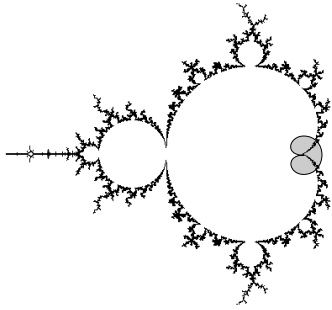
Figure 2.14: *The boundary of the Mandlebrot set is shown, and the shaded region shaped like a cardioid is the set of parameters c corresponding to well behaved maps f_c.*
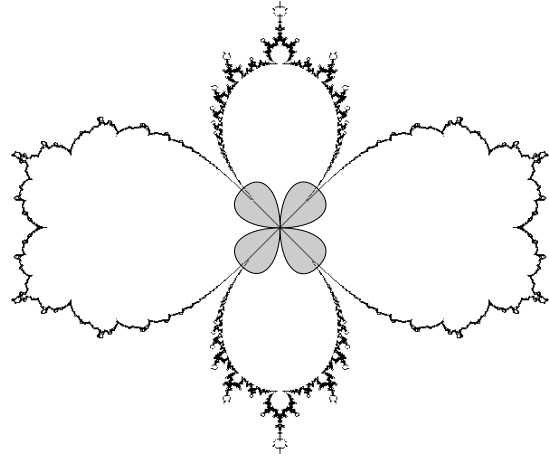
Figure 2.15: *The boundary of the connectedness locus is shown, and the shaded region shaped like a four-leaved-clover is the set of parameters σ corresponding to well behaved maps h_σ.*

As a final example, suppose that $f(z) := z + z(z - s_1)\ldots(z - s_\nu)$, and that $f'(s_k) \in D(i\varepsilon, 2\varepsilon)$ for $k = 1, \ldots, \nu$, for some small ε. Then $\operatorname{Im}\iota(f, s_k) = \operatorname{Im}\frac{1}{1 - f'(s_k)} > \frac{1}{2\varepsilon}$ for all k. Also note that $\iota(f, 0) + \iota(f, s_1) + \cdots + \iota(f, s_\nu) = 0$.

Corollary 3.7.5 implies that $f \in \mathcal{WB}$. In fact, since $s_1, \ldots, s_\nu$ are all sinks and 0 is a source for $\dot{z} = i[f(z) - z]$, only one gate structure is possible, and this will be $(1, 2, \ldots, \nu)$. Thus $f \in \mathcal{WB}\big((1, 2, \ldots, \nu)\big)$.

Unfinished business

Throughout this paper we have been considering a holomorphic germ f_0 with a multiple fixed point, and maps close to this.

A related problem is to consider an f_0 which has a parabolic cycle of period k, whose multiplier is a qth root of unity. Then (after conjugating by some affine map) either $f_0^{kq} = \text{id}$ or f_0^{kq} is of the form

$$f_0^{kq}(z) = z + z^{\nu q + 1} + \mathcal{O}(z^{\nu q + 2}).$$

In the case where $k = 1$, $\nu = 1$ and $q \geqslant 1$, the persistence of Fatou coordinates for f close to f_0 is dealt with in [Sh1, §7].

For general k, ν, q the results in this paper generalise easily, except for those in §2.7 and §2.8. (We simply have to say that an f close to f_0 is well behaved if and only if f^{kq} is well behaved when treated as a perturbation of f_0^{kq}.)

The results from §2.7 and §2.8 do not have straightforward generalisations—for instance not every admissible gate structure vector (of length νq) can be realised by some f^{kq}. However, it does seem probable that well behaved maps close to f_0 can be parameterised in terms of the lifted phases (and some extra information).

Chapter 3

The Proofs

3.1 Preliminaries

Proof of Lemma 2.1.3 on page 8 **(Definition and continuity of $f \mapsto u_f$)** u_f is clearly analytic upon $K_0 \setminus \{\sigma_0, \ldots, \sigma_\nu\}$. It is quite easy to show that u_f is also analytic at the fixed points. (For instance one can consider the Taylor expansion of $f(z) - z$ at s_k, and note that $(z - s_k)^{m_k}$ is a factor.)

Now assume for contradiction that there is a sequence $f_k \rightarrow f_0$ and $z \in \mathcal{D}(f_0) \setminus \{0\}$ such that $|u_{f_k}(z) - u_{f_0}(z)| > \varepsilon$ for all k and some $\varepsilon > 0$. It would then follow quite easily that $|f_k(z) - f_0(z)| > |z|^{\nu+1} \varepsilon/2$ for sufficiently large k, a contradiction. Thus $u_{f_k}(z) \rightarrow u_{f_0}(z)$ for every $z \neq 0$. The fact that $u_f \rightarrow u_{f_0}$ in the compact-open topology follows quickly. ■

3.2 Fundamental regions and Fatou coordinates for f_0

Proof of Lemma 2.2.2 on page 9 **(Fundamental regions for f_0)** First of all, we need to know that these solution curves actually do exist and are unique. Any analytic function with bounded derivative must be Lipschitz by the Mean Value Theorem. So we can use the following proved in [BR]:

> **Theorem 3.2.1** *(Existence and uniqueness of solution curves) Suppose that the function $Z(z)$ is defined and continuous in the closed domain $|z - z_0| \leqslant K$ and satisfies the Lipschitz condition there. Let $M = \sup |Z(z)|$ in this domain. Then the differential equation*
>
> $$\dot{z} = Z(z)$$
>
> *has a unique solution satisfying $z(t_0) = z_0$ and defined on the interval $|t - t_0| \leqslant K/M$.*

Remember also that for any autonomous differential equation as we have here (i.e. $\dot{z}$ depends on position but not time) the trajectories will never intersect, unless they coincide everywhere.

Let $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$. On the set $P_{i,s} := \{z \in K_0 \setminus \{0\} \mid |\arg(z/z_{i,s})| < 3\pi/4\nu\}$ we make the change of coordinate

$$w = I(z) := -\frac{1}{\nu z^\nu},$$

and in this coordinate we get an $F_{i,s}$ with $F_{i,s} := I \circ f_0 \circ I|_{P_{i,s}}^{-1}$,

$$
\begin{aligned}
F_{i,s}(w) = I(f_0(z)) &= -\frac{1}{\nu z^\nu}\left(1 + z^\nu + \mathcal{O}(z^{\nu+1})\right)^{-\nu} \\
&= w\left(1 - \frac{1}{\nu w} + \mathcal{O}(w^{-1-1/\nu})\right)^{-\nu} \\
&= w\left(1 + \frac{1}{w} + \mathcal{O}(w^{-1-1/\nu})\right) \\
&= w + 1 + \mathcal{O}(w^{-1/\nu}) \qquad \text{as } w \rightarrow \infty.
\end{aligned}
$$

If define $\Gamma_{i,s,f_0} = I \circ \gamma_{i,s,f_0}$ then we will have

$$\Gamma'_{i,s,f_0}(t) = I'(\gamma_{i,s,f_0}(t))\gamma'_{i,s,f_0}(t) = \frac{i[f_0(z) - z]}{z^{\nu+1}} = iu_{f_0}(z) \approx i,$$

29

where $z = \gamma_{i,s,f_0}(t)$.

So $\Gamma_{i,\pm,f_0}(t)$ will be an almost vertical line in the w-coordinate, passing through $w_{i,\pm} := I(z_{i,\pm}) = \pm 1/\nu r_0^\nu \in \mathbb{R}$. Thus $\Gamma_{i,\pm,f_0}(t) \to \infty$ as $t \to \pm\infty$, implying that $\gamma_{i,\pm,f_0}(t) \to 0$ as $t \to \pm\infty$ (since $w = \infty$ corresponds to $z = 0$).

Since $\Gamma = \Gamma_{i,\pm,f_0}$ is an "almost vertical line" through $\pm 1/\nu r_0^\nu$, $|\Gamma(t)|$ is always large and we must have $F_{i,s}(w) \approx w + 1$. This implies that $\Gamma(\mathbb{R})$ cannot intersect $F_{i,s}(\Gamma(\mathbb{R}))$, which in turn implies that $\gamma(\mathbb{R})$ does not intersect $f_0(\gamma(\mathbb{R}))$.

Since $z \to 0$ when $w \to \infty$, the $\ell_{i,\pm,f_0}$ must be loops with their ends at 0.   ■

3.3   Fundamental regions, Fatou coordinates and gate structures for f

Before giving a proof of Proposition 2.3.2 we have to do some ground work.

3.3.1   Sinks and sources

Lemma 3.3.1 *Suppose that σ is a fixed point of the analytic map $f : K_0 \to \mathbb{C}$ and there is an $\varepsilon > 0$ for which $\varepsilon < \arg[f'(\sigma) - 1] < \pi - \varepsilon$. Then the following hold.*

1. *In a sufficiently small disc, B, centred on σ, the dynamics of f rotate anti-clockwise around σ—that is, for every $z \in B$*

$$\tfrac{\varepsilon}{2} < \arg \frac{f(z) - z}{z - \sigma} < \pi - \tfrac{\varepsilon}{2}.$$

2. *If $\gamma : [0, +\infty) \to \mathbb{C}$ is an analytic path satisfying $\dot{z} = i[f(z) - z]$ and there is some $t_1 \geqslant 0$ so that $\gamma(t_1) \in B$ then $\gamma(t) \in B$ for every $t > t_1$ and $\gamma(t) \to \sigma$ exponentially as $t \to +\infty$.*
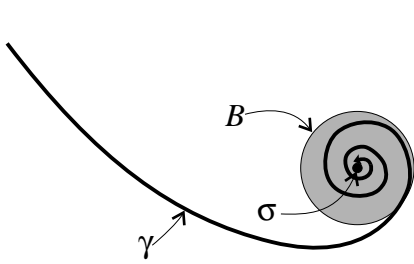
Figure 3.1:
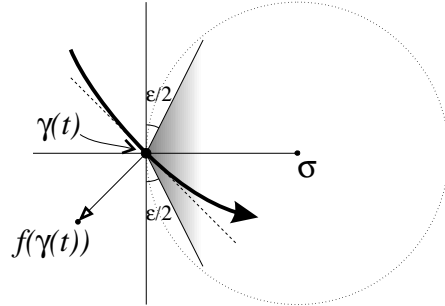
Figure 3.2:

Proof.   We use the fact that if B is very small we have

$$\frac{f(z) - z}{z - \sigma} = \frac{[f(z) - z] - [f(\sigma) - \sigma]}{z - \sigma} \approx \frac{d}{dz}[f(z) - z]\Big|_{z=\sigma} = f'(\sigma) - 1,$$

so we can say that for $z \in B$

$$\tfrac{\varepsilon}{2} < \arg \frac{f(z) - z}{z - \sigma} < \pi - \tfrac{\varepsilon}{2}.$$

Thus when $\gamma(t) \in B$ (see Figure 3.2)

$$-\tfrac{\pi}{2} + \tfrac{\varepsilon}{2} < \arg \frac{\gamma'(t)}{\sigma - \gamma(t)} < \tfrac{\pi}{2} - \tfrac{\varepsilon}{2}$$

and

$$\frac{d}{dt}|\gamma(t) - \sigma| < -|\gamma'(t)| \sin \tfrac{\varepsilon}{2} < -\text{const}|\gamma(t) - \sigma|,$$

for some small const > 0 depending on the size of B. Therefore $|\gamma(t) - \sigma| \to 0$, and the decay is exponential.

It is therefore clear that 1. and 2. are satisfied. ∎

Remark 3.3.2 *We can see that if σ is a fixed point, then $\operatorname{Im} f'(\sigma) > 0$ implies that the dynamics of f will rotate anti-clockwise around σ. Similarly $\operatorname{Im} f'(\sigma) < 0$ implies that the dynamics of f will rotate clockwise around σ.*

Notice also that if σ is a simple fixed point then (see Figure 3.2) we have $\operatorname{Im} f'(\sigma) > 0 \iff \operatorname{Im} \iota(f, \sigma) > 0$, and $\operatorname{Im} f'(\sigma) < 0 \iff \operatorname{Im} \iota(f, \sigma) < 0$.

Also we can see from Figure 3.2 that $\operatorname{Re} f'(\sigma) > 0$ implies that trajectories for $\dot{z} = i[f(z) - z]$ will spiral anti-clockwise around σ, and if $\operatorname{Re} f'(\sigma) < 0$ then the trajectories for $\dot{z} = i[f(z) - z]$ will spiral clockwise.

As an immediate corollary we have

> **Corollary 3.3.3** *If we have $f \in \mathcal{N}_0$, then each fixed point $\sigma \in K_0$ with $\operatorname{Im} f'(\sigma) > 0$ will be a sink of the vector field $\dot{z} = i[f(z) - z]$, and any fixed point σ' with $\operatorname{Im} f'(\sigma') < 0$ will be a source.*
>
> *In fact, these will be the only sinks or sources in K_0.*

3.3.2 Limit cycles

We start with the following Lemma, which gives a good description of the flows $\dot{z} = f(z) - z$ and $\dot{z} = i[f(z) - z]$ on $K_0 \setminus D_{r_0/4}$.

> **Lemma 3.3.4** *Let $f \in \mathcal{N}_0$ and consider the change of coordinate $w = I(z) := -1/\nu z^\nu$ on*
>
> $$R_{i,\pm} = \{z \in K_0 \setminus D_{r_0/4} \mid |\arg(z/z_{i,\pm})| < 3\pi/4\nu\}.$$
>
> *Then the push forwards of $\dot{z} = f(z) - z$ and $\dot{z} = i[f(z) - z]$ on $R_{i,\pm}$ will be*
>
> $$\dot{w} = \frac{f(z) - z}{z^{\nu+1}} \approx 1 \qquad \text{and} \qquad \dot{w} = \frac{i[f(z) - z]}{z^{\nu+1}} \approx i$$
>
> *respectively on*
>
> $$Q_\pm := I(R_{i,\pm}) = \{w \mid |I(r_0/4)| \leqslant |w| \leqslant |I(2r_0)|, \; |\arg \pm w| < 3\pi/4\}.$$
>
> *Therefore trajectories for $\dot{z} = f(z) - z$ (resp. $\dot{z} = i[f(z) - z]$) will be mapped by I to "almost horizontal lines" (resp. "almost vertical lines"). (See Figure 3.3.)*
>
> *Note that the union of all the $R_{i,s}$ forms the annulus $K_0 \setminus D_{r_0/4}$.*

Proof. The calculations are fairly straightforward. ∎

The Poincaré-Bendixson Theorem says that any solution curve of an autonomous differential equation which stays within a compact subset of the plane for all time must either converge to a fixed point or accumulate along a *limit cycle*. By a limit cycle we mean a periodic solution of the vector field, the trail of which forms a closed loop.

> **Lemma 3.3.5 (No limit cycles for $f \in \mathcal{WB}$)** *Suppose that $f \in \mathcal{N}_0$ has a limit cycle L_0 for $\dot{z} = i[f(z) - z]$ which is contained in the interior of K_0.*
>
> *Then every point sufficiently close to L_0 also lies on a limit cycle, which has the same period. Thus if a trajectory γ for $\dot{z} = i[f(z) - z]$ accumulates along a limit cycle then γ is itself a limit cycle.*

Proof. We will suppose that $L_0(t) := Y_t(z_0)$ is a limit cycle with period p (that is there is a minimum $p > 0$ such that $L_0(p) = L_0(0)$). Since the flows X_s and Y_t commute we will have $Y_p(X_s(z_0)) = X_s(Y_p(z_0)) = X_s(z_0)$, implying that $X_s(z_0)$ is also a point on a limit cycle of period p (where $z_0 \in L_0(\mathbb{R})$ and $s > 0$).

It is easy to show that for any z sufficiently close to L_0 there will be some small $s \in \mathbb{R}$ such that $X_s(z) \in L_0$, so the Lemma is proved. ∎
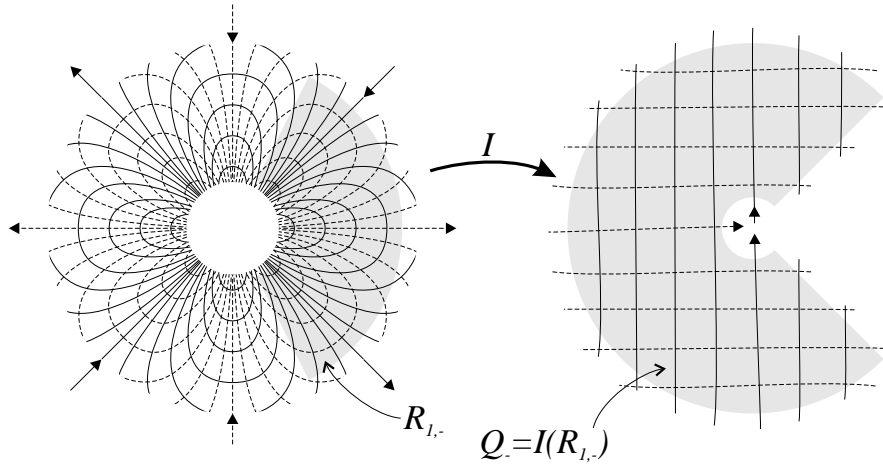
Figure 3.3: *On the left we show the trajectories for the vector fields $\dot{z} = i[f(z) - z]$ and $\dot{z} = f(z) - z$ outside a disc centred upon 0. (f is close to $z \mapsto z + z^3$.) The trajectories for $\dot{z} = i[f(z) - z]$ are shown by solid lines, and those for $\dot{z} = f(z) - z$ are shown by dotted lines. On the right we show the image under $I(z) := -\frac{1}{\nu z^\nu}$ of those trajectories which intersect $R_{1,-}$.*

Remark 3.3.6 *For $f \in \mathcal{N}_0$, any limit cycle $L_0 \subset K_0$ will wind around exactly one simple fixed point σ. The multiplier will lie on the real line: $f'(\sigma) \in \mathbb{R} \setminus \{1\}$. Every point "inside" L_0 other than σ will also lie on a limit cycle with the same period.*

We can also show that every point close enough to a simple fixed point σ satisfying $f'(\sigma) \in (0, +\infty) \setminus \{1\}$ will belong to a limit cycle of period $-2\pi\iota(f, \sigma) = 2\pi/[f'(\sigma) - 1]$.

3.3.3   Combinatorics: Proofs of Props 2.3.2 and 2.3.8

Now we are ready to give the proof of the first Proposition.

Proof of Proposition 2.3.2 on page 10 (Combinatorics for well behaved maps) We know that since f is well behaved, every singular point for the vector field $\dot{z} = i[f(z) - z]$ is a fixed point for f.

Therefore the Poincaré-Bendixson Theorem and Lemma 3.3.5 ensures that if f is well behaved then every forward and backward trajectory through the $z_{i,\pm}$ will converge to a fixed point in $\overline{D_{r_0/2}}$. So **(1.)** is proved.

We now cover every sink (for $\dot{z} = i[f(z) - z]$) with a small "black disc," so that any forward trajectory which enters that sink will stay inside for all time, and converge to the associated sink. A multiplicity $r + 1$ fixed point of f will also be a multiplicity $r + 1$ fixed point of the analytic map Y_1, and each of the r attracting directions for the map Y_1 can be covered by an open "black petal" (see [Mi, §7]) which is forward invariant under Y_1. Any forward orbit which enters one of these petals will converge to the associated multiple fixed point. The multiple fixed points and the sources we mark with a single "white" point. See Figure 3.4.

Importantly, we have marked out exactly $\nu + 1$ black and white "objects," since there are $\nu + 1$ fixed points counted with multiplicity. We let "$D := D_{r_0/2} \setminus \{$the black objects$\}$." Notice that D is connected.

We know that no forward trajectory will converge to one of the white objects, and that any trajectory which converges to a multiple fixed point will eventually enter and stay in one of the black petals. Therefore, for each i, the line $\gamma_{i,+,f}(\mathbb{R})$ enters a black object.

We want to know how many components there are of

$$D' := D \setminus \bigcup_{i \in \mathbb{Z}/\nu\mathbb{Z}} \gamma_{i,+,f}([0, +\infty)).$$

Let s denote the total number of black objects which are converged to by at least one of these forward trajectories.
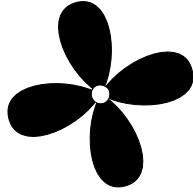
Figure 3.4: *For a multiplicity 5 fixed point, we cover the fixed point itself by a white dot, and the 4 attracting directions of Y_1 by 4 black petals. These petals are forward invariant under Y_1. Notice in particular that there number of objects marked is 5, which is the same as the multiplicity of the fixed point.*
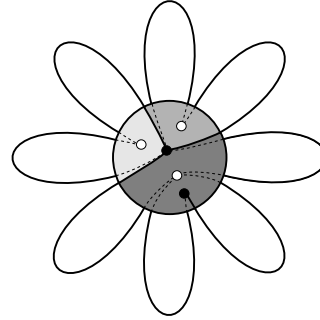
Figure 3.5: *The four forward trajectories $\{\gamma_{i,+,f}(t) \mid t > 0\}$ between them share two "black fixed points" at their ends. These forward trajectories will then chop $D_{r_0/2} \setminus \{\sigma_0, \ldots, \sigma_4\}$ into $\nu - s + 1 = 4 - 2 + 1 = 3$ pieces. Each of the three pieces must contain exactly one "white object" which will have to serve as the limits of the backward trajectories.*

We try to calculate the number of components of D' by taking it step by step. For each of the s black objects we start by marking out just one of the forward trajectories which enter it. So s forward trajectories have been marked, and D will still be in one piece. However each time we mark out one of the remaining $\nu - s$ forward trajectories, we are effectively chopping one of the remaining pieces of D in half. As a result, the ν trajectories of the form $\gamma_{i,+,f}([0, +\infty))$ will chop D into exactly $1 + (\nu - s)$ pieces.

Into each of these $\nu - s + 1$ of D', we can see that there will be at least one backward trajectory from one of the $z_{i,+}$, which (by Poincaré-Bendixson) converges to a "white object." Thus each of the $n - s + 1$ components of D' contains at least one white object, which is converged to by at least on backward trajectory.

Therefore we have shown that s black objects are entered by the forward trajectory from some $z_{i,+}$, and that at least $\nu - s + 1$ white points are converged to by at least one backward trajectory from some $z_{j,-}$. So a total of at least $s + (\nu - s + 1) = \nu + 1$ objects have been used, and we know that there are only $\nu + 1$ objects in total. Thus there is exactly one white point in each of the $\nu - s + 1$ pieces of D'.
(2.) is proved.

Take $z_- = \gamma_{i,-,f}(t) \in D_{r_0/2}$ for some large $t > 0$. Then there will be some s_0 (which is roughly $2/\nu r_0^{\nu}$) so that $z_+ := X_{s_0}(z_-) \in \ell_{i,+,f}$. (Make the change of coordinate $w = -1/\nu z^{\nu}$ to see this.) Now since $Y_t(z_+)$ converges to a fixed point σ, we can easily show that $X_{s_0}(Y_t(z_+)) - Y_t(z_+) \to 0$ as $t \to +\infty$, and so $Y_t(z_-) = X_{s_0} \circ Y_t(z_+) \to \sigma$. This implies that $\gamma_{i,-,f}(t) \to \sigma$. In the same way we can show that $\gamma_{j,-,f}(t)$ and $\gamma_{j+1,+,f}(t)$ must converge to the same fixed point as $t \to -\infty$. Thus **(3.)** holds.

Now we prove part **(4.)**. Fix $i \in \mathbb{Z}/\nu\mathbb{Z}$ and consider $\gamma_{i,+,f}$. If $\gamma_{i,+,f}(+\infty) = \gamma_{i,-,f}(-\infty)$ then there is nothing to prove. So we assume that $\gamma_{i,+,f}$ has distinct fixed points at its ends, and aim to show the existence of some $j \in \mathbb{Z}/\nu\mathbb{Z}$ such that $\gamma_{j,-,f}$ has the same fixed points at its ends.

Now let $L_s(t) := X_s(\gamma_{i,+,f}(t))$ for all $s > 0$ for which this is well defined. Since X_s and Y_t commute, L_s will always be a trajectory for $\dot{z} = i[f(z) - z]$, and it has the same ends as $L_0 = \gamma_{i,+,f}$. Let P be the closed set of points bounded away from infinity by $L_0(\mathbb{R}) \cup \partial D_{r_0/2}$. (See Figure 3.6.)

We assume for contradiction that $L_s(\mathbb{R})$ is well defined and contained in P for all $s > 0$. Then we set

$$C := \bigcup_{s \geqslant 0} L_s(\mathbb{R})$$

which will must exist and be contained in P.
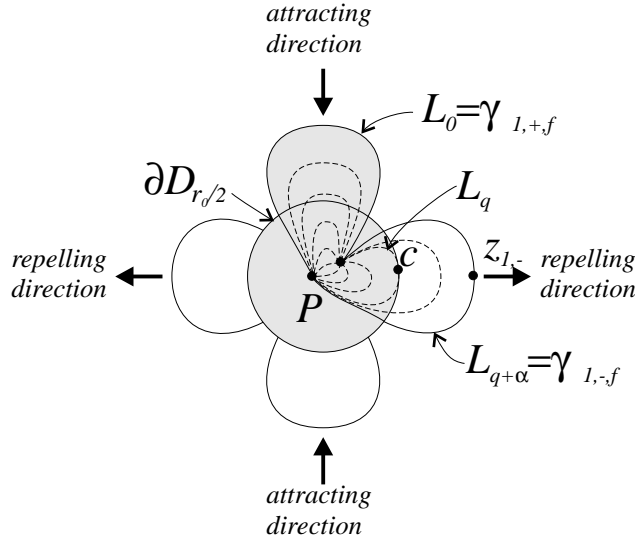
Figure 3.6: *The shaded region in the above figure is the set P. The point c in $L_q(\mathbb{R}) \cap \partial D_{r_0/2}$ is close to one of the repelling directions for f_0 at 0.*

$\partial C \setminus L_0(\mathbb{R})$ has some slightly strange properties—it is almost invariant under the flows for $\dot{z} = f(z) - z$ and $\dot{z} = i[f(z) - z]$. More specifically, for every $z \in \partial C \setminus L_0(\mathbb{R})$ and a small enough $\varepsilon > 0$ we have $X_s \circ Y_t(z) \in \partial C \setminus L_0(\mathbb{R})$ for all $s, t \in [-\varepsilon, \varepsilon]$. However, if ε is very small and z is not a singular point for either of the orthogonal vector fields $\dot{z} = f(z) - z$ and $\dot{z} = i[f(z) - z]$ then the set $\{X_s \circ Y_t(z) \mid s, t \in [-\varepsilon, +\varepsilon]\}$ should be a deformed square, and certainly cannot be contained in ∂C. This implies that z is a singular point for one (and in fact both) of the vector fields, so it is a fixed point for f. But this contradicts the fact that there are only finitely many fixed points in K_0, and infinitely many $z \in \partial C \setminus L_0(\mathbb{R})$.

Thus $L_s(\mathbb{R})$ cannot be contained in P for all $s > 0$, so we set

$$q := \inf \{s_0 > 0 \mid L_s(\mathbb{R}) \subset P \ \forall s < s_0\}.$$

$L_q(\mathbb{R})$ intersects the circle $\partial D_{r_0/2}$, and is easily shown to be tangent to that circle at some $c = L_q(t_0) \in \partial D_{r_0/2}$ where $t_0 \in \mathbb{R}$.

Let $I(z) := -\frac{1}{\nu z^\nu}$. Then $I \circ L_q$ is tangent to $\partial D_{|I(r_0/2)|}$ at $I(c)$. Lemma 3.3.4 implies that $(I \circ L_q)'(t_0) \approx i$, so $I(c)$ is close to either $+|I(r_0/2)|$ or $-|I(r_0/2)|$. Now notice that if $\varepsilon > 0$ is small then $L_{q-\varepsilon}(t_0) \in D_{r_0/2}$. Therefore $I \circ L_{q-\varepsilon}(t_0) \in \mathbb{C} \setminus \overline{D_{|I(r_0/2)|}}$, and by Lemma 3.3.4 it is just to the left of $I(c)$. Thus $I \circ L_q(t_0)$ is close to $-|I(r_0/2)|$. This implies that there is some $j \in \mathbb{Z}/\nu\mathbb{Z}$ such that c is close to $\frac{1}{2}z_{j,-}$. (See Figure 3.6.)

With a little more work we can show that there is an $\alpha > 0$ such that $L_{q+\alpha} = \gamma_{j,-,f}$ as required. (See Figure 3.6.)

Part **(5.)** is assured by Lemma 3.3.7 below. ∎

The final part of proof of Proposition 2.3.2 (part **(5.)**) is given by the following Lemma.

Lemma 3.3.7 (Existence of fundamental regions) *If $f \in \mathcal{WB}$ then for every $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$ we will find that if $\gamma = \gamma_{i,s,f}$ and $\ell = \gamma(\mathbb{R})$ then we get $\ell \cap f(\ell) = \varnothing$ and $f(\ell) \subset K_0$.*

Thus the fundamental regions $S'_{i,s,f}$ is well defined for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$. These fundamental regions are pairwise disjoint.

Proof. Let

$$S^* = S^*_{i,s,f} := \bigcup_{a \in [0,2]} X_a(\ell) = \bigcup_{(a,b) \in [0,2] \times \mathbb{R}} X_a \circ Y_b(z_{i,s}).$$

It is easily shown that $S^* \subset K_0 = \overline{D_{2r_0}}$, since $\ell \subset D_{3r_0/2}$ and $|X_a(z) - z| \ll r_0$ for any $a \in [0, 2]$ and $z \in D_{3r_0/2}$.

We want to show that for each $z \in S^*$ there is a **unique** $(a, b) \in [0, 2] \times \mathbb{R}$ such that $z = X_a \circ Y_b(z_{i,s})$.

Suppose for contradiction that there is some $a \in [0, 2]$ and some distinct $b, b' \in \mathbb{R}$ such that $X_a \circ Y_b(z_{i,s}) = X_a \circ Y_{b'}(z_{i,s})$. But this would imply that $t \mapsto X_a \circ Y_t(z_{i,s})$ is a periodic solution of $\dot{z} = i[f(z) - z]$. And since the flows X_a and Y_t commute, this would imply that γ is a periodic solution, which we know cannot be true.

Now assume for contradiction that there are some distinct $a, a' \in [0, 2]$ and some b, b' (not necessarily distinct) such that $X_a \circ Y_b(z_{i,s}) = X_{a'} \circ Y_{b'}(z_{i,s})$. Then it is easily shown that $z_0 := X_{a-a'}(z_{i,s}) = Y_{b'-b}(z_{i,s})$. But then it is clear that z_0 is "fairly close" to $z_{i,s}$ (since $|a - a'| \leqslant 2$), so $z_0 \in \ell \setminus D_{r_0/2}$. This contradicts the definition of f being well behaved.

Thus it is indeed true that for each $z \in S^*$ there is a unique $(a, b) \in [0, 2] \times \mathbb{R}$ such that $z = X_a \circ Y_b(z_{i,s})$. Thus $\Psi(z) := \int_{z_{i,s}}^{z} \frac{d\zeta}{f(\zeta) - \zeta}$ is well defined on S^* and $\Psi(X_a \circ Y_b(z_{i,s})) = a + ib$ for each $(a, b) \in [0, 2] \times \mathbb{R}$. Thus $\Psi : S^* \to [0, 2] + i\mathbb{R}$ is bijective.

We can show that $\Psi(\ell) = i\mathbb{R}$ and that $\Psi \circ f(z) \approx \Psi(z) + 1$ for any $z \in \ell$ (see the proof of Lemma 3.3.13 below). Thus $\Psi \circ f(\ell) \cap \Psi(\ell) = \varnothing$. And since Ψ is bijective this implies that $f(\ell) \cap \ell = \varnothing$ as required. Also $S'_{i,s,f} \subset S^*_{i,s,f} \subset K_0$.

Now we need to show that the sets $S'_{i,s,f}$ are pairwise disjoint. It is clearly sufficient to show that all the $S^*_{i,s,f}$ are pairwise disjoint.

We can use that same kind of argument as above to show that if $S^*_{i,s,f} \cap S^*_{i',s',f} \neq \varnothing$ where $(i, s) \neq (i', s')$ then there will be some $t \in \mathbb{R}$ such that $\gamma_{i,s,f}(t)$ is close to $z_{i',s'}$, which again implies that f is not well behaved, another contradiction. ∎

Proof of Proposition 2.3.5 on page 11 (All gate structures are admissible) This comes from Proposition 2.3.2 and some simple combinatorics ∎

Proof of Proposition 2.3.8 on page 12 (Numbers of fixed points and open gates) Consider a "simple tree structure" as shown in Figure 3.7. It is simple combinatorics to show that if there are r lines, then there are $r + 1$ nodes.

Proposition 2.3.2 implies that for any $f \in \mathcal{WB}(\mathsf{G})$ the topological picture of

$$\bigcup_{\mathsf{G}_i \neq \star} \ell_{i,+,f} \cup \mathrm{Fix}(f)$$

where points in $\mathrm{Fix}(f)$ are treated as "marked nodes" will be a simple tree structure. Thus there are indeed $r + 1$ fixed points.
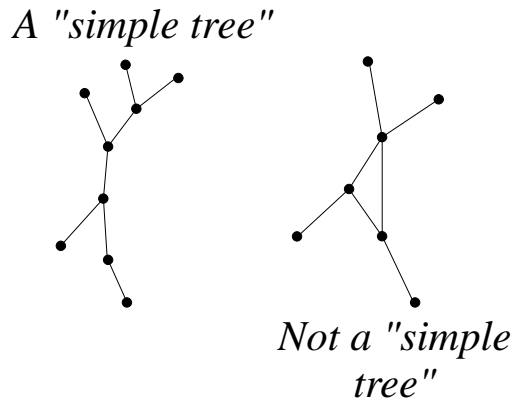

Figure 3.7: *On the left is a "simple tree." The picture on the right is not a simple tree because it contains a closed loop.*

Suppose that σ has multiplicity $m + 1 > 1$. Then it will have m attracting directions (and m repelling directions) and using Lemma 2.2.2 (and a suitable affine change of coordinates) we can find m incoming fundamental regions for $f|_U$, where U is a small neighbourhood of σ. There will be associated trajectories p_k $(k = 1, \ldots, m)$. Using the same arguments that we did in the proofs of Lemma 3.3.5 and Proposition 2.3.2 we can show that for each $k = 1, \ldots, m$ there is some $i \in \mathbb{Z}/\nu\mathbb{Z}$, and $r_x, r_y \in \mathbb{R}$ such that $\gamma_{i,+,f} = X_{r_x} \circ Y_{r_y} \circ p_k$. Thus $\gamma_{i,+,f}(+\infty) = \gamma_{i,+,f}(-\infty)$ (since $p_k(+\infty) = p_k(-\infty)$).

Also for $i \in \mathbb{Z}/\nu\mathbb{Z}$ there can be no distinct $k_1, k_2 \in \{1, \ldots, p\}$ such that p_{k_1} and p_{k_2} are both "associated" to $\gamma_{i,+,f}$. (It is not too hard to show this by looking at the topological picture.)

Thus for a multiple fixed point σ of f

$$\text{mult}(f, \sigma) = 1 + \#\{i \mid \gamma_{i,+,f}(+\infty) = \gamma_{i,+,f}(-\infty)\}.$$

So since the topological picture of all the $\ell_{i,s,f}$'s and fixed points is uniquely determined by $\mathsf{gate}(f)$ we see that $\text{mult}(f, \gamma_{i,s,f}(\pm\infty))$ can be calculated from $\mathsf{gate}(f)$. ∎

3.3.4   Stability of trajectories: Proofs of Props 2.3.9 and 2.3.10

We would like to know that the forward and backward trajectories for the vector field which tend to fixed points will be stable under perturbation of the vector field and the starting point (as long as the fixed points do not split apart). We first prove the following.

Lemma 3.3.8 (No splitting of fixed points on $\mathcal{WB}(\mathsf{G})$) *Let $\sigma(f_1)$ be a fixed point of $f_1 \in \mathcal{WB}(\mathsf{G})$, of multiplicity $m \geqslant 1$. Suppose that B is a small open neighbourhood of $\sigma(f_1)$ so that $\overline{B}$ contains no other fixed points of f_1. Then any $f \in \mathcal{N}_0$ sufficiently close to f_1 will have m fixed points in B counted with multiplicity.*

However, if when we perturb f_1 in $\mathcal{WB}$ so that the multiple fixed point $\sigma(f_1)$ splits into more than one distinct fixed point, then the gate structure will have changed.

As a result there is a continuous map $f \mapsto \sigma(f)$ defined in a neighbourhood of f_1 in $\mathcal{WB}(\mathsf{G})$ such that $\sigma(f)$ is a fixed point of f of multiplicity m.

Proof.   If f is close enough to f_1 on ∂B, then Rouché's Theorem tells us that $f(z) - z$ has the same number of solutions in B (counted with multiplicity) as $f_1(z) - z$ in B. Thus f has m fixed points in B.

Now suppose that $f_1 \in \mathcal{WB}(\mathsf{G})$ and that the fixed point $\sigma(f_1)$ is of multiplicity $m > 1$.

We assume for contradiction that $f \in \mathcal{WB}(\mathsf{G})$ is very close to f_1, and contains at least two distinct fixed points of f in B.

If $\sigma(f_1)$ splits when we perturb f_1 to get f, then there will be some fixed point $\beta(f) \in B$ of f of multiplicity strictly less than m, and such that $\gamma_{i_0, s_0, f}(+\infty) = \beta(f)$ for some $i_0 \in \mathbb{Z}/\nu\mathbb{Z}$ and $s_0 \in \{+, -\}$.

We can restrict f_1 to a small neighbourhood of $\sigma(f_1)$. There will be an affine map $A(z) = az + b$ such that $h_1 := A \circ f_1 \circ A^{-1}$ is of the form $h_1(z) = z + z^{m+1} + \mathcal{O}(z^{m+2})$. Trajectories for $\dot{z} = i[f_1(z) - z]$ will be mapped by A to trajectories for $\dot{z} = i[h_1(z) - z]$. So then we can apply Lemma 3.3.10 to h_1, which implies that $\gamma_{i_0, s_0, f_1}(+\infty) = \sigma(f_1)$. However then the fixed point $\sigma(f_1) = \gamma_{i_0, s_0, f_1}(+\infty)$ of f_1 has multiplicity m, and the fixed point $\beta(f) = \gamma_{i_0, s_0, f}(+\infty)$ of f has multiplicity strictly less than m. This contradicts Proposition 2.3.8 and our assertion that $\mathsf{gate}(f) = \mathsf{gate}(f_1)$.

As a result "fixed points cannot split," so Rouché's Theorem implies that $f \mapsto \sigma(f)$ is continuous in some neighbourhood of f_1 in $\mathcal{WB}(\mathsf{G})$. ∎

The following standard theorem (c.f. for example [BR]), is used in the proof of Lemma 3.3.10 below and elsewhere.

Theorem 3.3.9 (Continuous dependence of solutions) *Let $D \subset \mathbb{C}$ and $f, g : D \to \mathbb{C}$ be continuous. Also let $z(t), w(t)$ be differentiable solutions of*

$$\dot{z} = f(z) \quad and \quad \dot{w} = g(w)$$

on an open interval I containing t_0.

If f is k-Lipschitz in D and $|f(z) - g(z)| \leqslant \mu$ for all $z \in$ then

$$|z(t) - w(t)| \leqslant |z(t_0) - w(t_0)|e^{k|t-t_0|} + \tfrac{\mu}{k}(e^{k|t-t_0|} - 1)$$

for $t \in I$.

Lemma 3.3.10 *Suppose that $h \in \mathcal{N}_0$ where $\mathcal{N}_0$ is a very small neighbourhood of $f_0(z) = z + z^{\nu+1}u_{f_0}(z)$, and that $f \mapsto z_0(f) \in D_{5r_0/4} \setminus D_{3r_0/2}$ is continuous on $\mathcal{N}_0$. For $f \in \mathcal{N}_0$ let p_f be the maximal trajectory for $\dot{z} = i[f(z) - z]$ with $p_f(0) = z_0(f)$.*
If $p_h(t) \in D_{5r_0/4} \setminus D_{3r_0/2}$ for all $t \geqslant 0$ then $p_{f_0}(t) \to 0$ as $t \to +\infty$.

Proof. Notice that if $p_h([0, +\infty)) \in D_{5r_0/4}$ then $p_f([0, +\infty)) \in K_0$ for all f close to f_0. Also, (by Poincaré-Bendixson) p_h must converge to some fixed point in $D_{r_0/2}$ as $t \to +\infty$.

We let $\{g_t\}_{t \in [0,1]}$ be a path in a small neighbourhood of f_0, such that $g_0 = f_0$ and $g_1 = h$. We call the small arc contained in $\partial D_{r_0/2}$ which connects $\ell_{i,+,f}$ to $\ell_{i,-,f}$ the "ith entrance" for f. See Figure 3.8.

We know that p_h will cross the ith entrance for some i. (That is to say that p_h enters $D_{r_0/2}$ "between $\ell_{i,+,f}$ and $\ell_{i,-,f}$.") In fact p_h will be almost perpendicular to the ith entrance as it crosses it. Also for any f close to f_0, any trajectory for $\dot{z} = i[f(z) - z]$ which crosses the ith entrance will do so almost perpendicularly. See Figure 3.8.
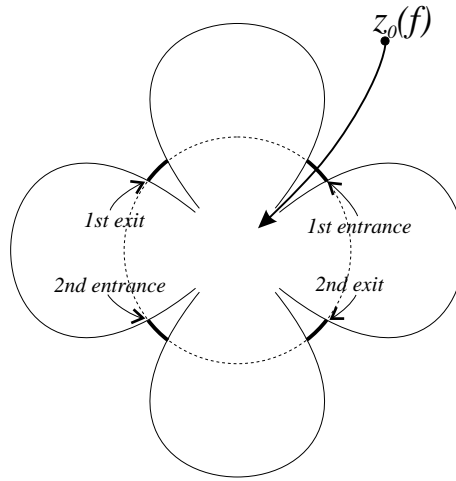
Figure 3.8:

Clearly when we perturb h to give $f \in \mathcal{WB}$, we will still have p_f crossing the ith entrance (for f) by the smoothness of the p_f and Theorem 3.3.9. It can be deduced that as we vary t from 1 down to 0, p_{g_t} will still always cross the ith entrance (for g_t).

Therefore p_{f_0} crosses the ith entrance for f_0. It is then simple to show that $p_{f_0}(t) \to 0$ (since we will be able to write $p_{f_0}(t) = X_s^{f_0} \circ \gamma_{i,+,f_0}(t + r)$ for some $r \in \mathbb{R}, s > 0$, where $X_s^{f_0}$ is the time-s flow for $\dot{z} = f_0(z) - z$). ∎

Lemma 3.3.11 *(Stability of trajectories) Suppose that we have an analytic function $f_1 : D \to \mathbb{C}$ with a fixed point $\sigma(f_1)$ of multiplicity m, and an analytic path $\gamma_{f_1} : [0, +\infty) \to D$ which solves*

$$\dot{z} = i[f_1(z) - z],$$

$\gamma_{f_1}(0) = z_0(f_1) \in D$, and $\gamma_{f_1}(t) \to \sigma(f_1)$ as $t \to +\infty$.
Suppose also that $\mathcal{M} \subset \mathcal{H}$ is a family of analytic maps (together with the compact-open topology) such that in a small enough neighbourhood of f_1 in $\mathcal{M}$ there is a continuous function $f \mapsto \sigma(f)$ such that $\sigma(f)$ is a fixed point of f with multiplicity m. Also let $f \mapsto z_0(f)$ be continuous in a neighbourhood of f_1 in $\mathcal{M}$.

For each $h \in \mathcal{M}$ let γ_h denote the maximal trajectory for

$$\dot{z} = i[h(z) - z],$$

with $\gamma_h(0) = z_0(h)$. Then we have $\gamma_f(t) \to \sigma(f)$ as $t \to +\infty$, for every $f \in \mathcal{M}$ which is close enough to f_1.

We will also have that $f \mapsto \gamma_f([0, +\infty)) \cup \{\sigma(f)\}$ is continuous with respect to the Hausdorff metric on a neighbourhood of f_1 in $\mathcal{M}$.

Proof.   We will first consider the case when $\sigma(f_1)$ is a simple fixed point. We already know that this fixed point is a sink for $\dot{z} = i[f_1(z) - z]$ (by Remark 3.3.6 and Corollary 3.3.3) and we must have $\operatorname{Im} f_1'(\sigma(f_1)) > 0$. So for f close enough to f_0 we must have $\operatorname{Im} f'(\sigma(f)) > 0$, and $\sigma(f)$ is also a sink for $\dot{z} = i[f(z) - z]$.

For $f \in \mathcal{M}$ let Y_t^f be the time t flow for the vector field $\dot{z} = i[f(z) - z]$. Because $\sigma(f_1)$ is a sink for $Y_t^{f_1}$, there will be some arbitrarily small closed neighbourhood G of σ, such that the closure of $Y_1^{f_1}(G)$ is contained in the interior of G.

We can easily show that $f \mapsto Y_1^f(G)$ is Hausdorff continuous (using Theorem 3.3.9), which implies that G will be mapped inside its interior by Y_1^f for any f sufficiently close to f_1.

So, we take a large $T > 0$, so that $\gamma_{f_1}(T) \in \overset{\circ}{G}$ (which is clearly possible, since $\gamma_{f_1}(t) \to \sigma \in \overset{\circ}{G}$). Then for f close enough to f_1 we know that $\gamma_f([0, T])$ is arbitrarily close to $\gamma_{f_1}([0, T])$, and that $\gamma_f([T, +\infty)) \subset G$ where G is arbitrarily small. Thus $f \mapsto \gamma_f([0, +\infty)) \cup \sigma(f)$ is continuous in a neighbourhood of f_1. (Compare with the argument using *collapsing traps* in [Do, §6].)

If $\sigma(f_1)$ has multiplicity $m > 1$ then we can use a similar argument. We can easily show that $\sigma(f)$ will then be a multiplicity m fixed point for the analytic map Y_1^f. There are arbitrarily small "attracting petals" (see [Mi, §7]) for Y_1^f at the multiple fixed point $\sigma(f)$, so that $\gamma_{f_1}(t)$ must enter one of these. If we let $G(f_1)$ be the closure of one of these then it will satisfy $Y_t^{f_1}(G(f_1)) \subset \overset{\circ}{G}(f_1) \cup \{\sigma(f_1)\}$.

It can shown that if f is close enough to f_1 then $Y_t^f(G(f)) \subset \overset{\circ}{G}(f) \cup \{\sigma(f)\}$ where $G(f) = G(f_1) + [\sigma(f) - \sigma(f_1)]$. Again, if we find T so that $\gamma_{f_1}(T)$ is in the interior of $G(f_1)$ then $\gamma_f(T) \subset \overset{\circ}{G}(f)$ if f is close enough to f_1 (since $f \mapsto [\sigma(f) - \sigma(f_1)]$ is continuous by Rouché's Theorem).

Thus we again find that $f \mapsto \gamma_f([0, +\infty)) \cup \sigma(f)$ is continuous in a neighbourhood of f_1. ■

Proof of Proposition 2.3.9 on page 14 (Continuity of the maps $f \mapsto \sigma_k(f)$) This is implied by Rouché's Theorem and Lemma 3.3.8 (which ensures that no multiple fixed points can split when we perturb $f \in \mathcal{WB}(\mathsf{G})$ in $\mathcal{WB}(\mathsf{G})$). ■

Proof of Proposition 2.3.10 on page 14 (The space of well behaved parameters is open) Take $(\mathbf{s}_0, u_0) \in P(\mathsf{G})$. Then for $(\mathbf{s}, u)$ close to $(\mathbf{s}_0, u_0)$ we have $f_{\mathbf{s}, u} \in \mathcal{N}_0$, since $f_{\mathbf{s}_0, u_0} \in \mathcal{N}_0$ and $\mathcal{N}_0$ is open.

Suppose that G has r open gates. Observe that $(\mathbf{s}, u) \in P(\mathsf{G})$ where $\mathbf{s} = (s_0, \ldots, s_r)$ implies that $s_0, \ldots, s_r$ are all distinct (otherwise there would be fewer than $r + 1$ fixed points in K_0 contradicting Proposition 2.3.8). Thus no fixed points will split when we perturb $(\mathbf{s}, u_0)$. But then Lemma 3.3.11 (and the definition for a map to be well behaved) implies that $f \in \mathcal{WB}(\mathsf{G})$ for all f close to $f_{\mathbf{s}_0, u_0} \in \mathcal{WB}(\mathsf{G})$. ■

3.3.5   Fatou coordinates: Proofs of Thm 2.3.12 and Prop 2.3.14

Lemma 3.3.12 (Existence of Φ_F) *Suppose that $Q_F \subset \mathbb{C}$ is a region bounded by either one or two (non-intersecting) differentiable paths $\gamma_i : \mathbb{R} \to \mathbb{C}$ where $\arg \gamma_i'(t) \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ for $t \in \mathbb{R}$ and each i. (See Figure 3.9.)*

If $F : Q_F \to \mathbb{C}$ is analytic and univalent and satisfies

$$
\begin{aligned}
|F(w) - (w+1)| &\leqslant \tfrac{1}{4}, \\
|F'(w) - 1| &\leqslant \tfrac{1}{4}
\end{aligned}
$$

on Q_F, and Q_F contains both $\ell = i\mathbb{R}$ and $F(\ell)$ then there is an analytic, univalent $\Phi_F : Q_F \to \mathbb{C}$ satisfying

$$
\Phi_F(F(w)) = \Phi_F(w) + 1 \qquad \text{if } w, F(w) \in Q_F.
$$

Φ_F will be unique up to addition by a constant. If $F \mapsto w_0(F)$ is continuous in a neighbourhood of F_0 in $\mathcal{H}$, and $w_0(F_0) \in Q_{F_0}$, then for F close to F_0 we can always normalise Φ_F by insisting that $\Phi_F(w_0(F)) = 0$. Then $F \mapsto \Phi_F$ will be continuous with respect to the compact-open topology in a neighbourhood of F_0.
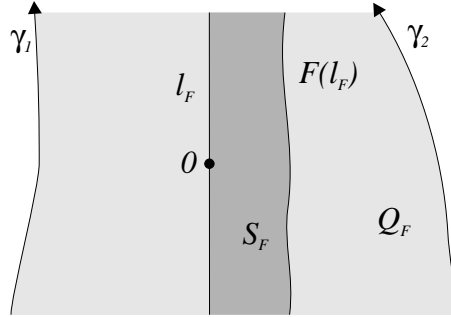
Figure 3.9:

Proof. See [Sh2, Prop. 2.5.2]—it depends on the Ahlfors-Bers Measurable Mapping Theorem. ∎

Lemma 3.3.13 (Main Lemma) *Let K be a closed Jordan domain and let $\mathcal{M} \subset \mathcal{H}$ be such that every $f \in \mathcal{M}$ is defined in a neighbourhood of K. Suppose that $f \mapsto z_0(f) \in \overset{\circ}{K}$ is a continuous map on $\mathcal{M}$.*

Now let $\mathcal{M}'$ be the set of $f \in \mathcal{M}$ such that the following are satisfied.

1. *$|f(z) - z| < \tfrac{1}{10}$ and $|f'(z) - 1| < \tfrac{1}{10}$ for every $z \in K$;*
2. *$\gamma_f : \mathbb{R} \to \overset{\circ}{K}$ solves $\dot{z} = i[f(z) - z]$ with $\gamma_f(0) = z_0(f)$ and $\gamma_f(t) \to \sigma_{\pm}(f) \in \overset{\circ}{K}$ as $t \to \pm\infty$ (for some $\sigma_-(f), \sigma_+(f)$);*
3. *$f(\ell_f) \subset \overset{\circ}{K}$ and $\ell_f \cap f(\ell_f) = \varnothing$, where $\ell_f := \gamma_f(\mathbb{R})$.*

Then for all $f \in \mathcal{M}'$ we can let S_f be the closed set bounded by the loop $\ell_f \cup f(\ell_f) \cup \{\sigma_+(f), \sigma_-(f)\}$ and let $S'_f := S_f \setminus \{\sigma_+, \sigma_-\}$ (a fundamental region).

There is an analytic, injective map $\Phi_f : S'_f \to \mathbb{C}$ such that

$$
\Phi_f(f(z)) = \Phi_f(z) + 1 \qquad \text{for every } z \in \ell_f,
$$

and Φ_f is unique up to addition by a constant. We call Φ_f a Fatou coordinate. Also, the Écalle Cylinder S'_f/f is isomorphic to $\mathbb{C}/\mathbb{Z}$. We can normalise Φ_f such that $\Phi_f(z_0(f)) = 0$.

The map $f \mapsto S_f$ is Hausdorff lower semi-continuous on $\mathcal{M}'$. Also, the map $f \mapsto \Phi_f$ is continuous on $\mathcal{M}'$.

Proof. First let $\Psi_f : S'_f \to \mathbb{C}$ be defined as

$$
\Psi_f(z) := \int_{z_0(f)}^z \frac{d\zeta}{f(\zeta) - \zeta}
$$

where $z \in S'_f$. We let $F_f := \Psi_f \circ f \circ \Psi_f^{-1} : \Psi_f(S'_f) \to \mathbb{C}$ and will aim to prove that

1. $\Psi_f(\ell_f)$ is the vertical line $\{it \mid t \in \mathbb{R}\}$;

2. $|F_f(w) - (w+1)| < \frac{1}{4}$ if $w, F(w) \in Q_F$;

3. $|F_f'(w) - 1| < \frac{1}{4}$ if $w \in F(w)$.

If (1.), (2.) and (3.) do indeed hold then we can apply Lemma 3.3.12 to get $\Phi_{F_f} : \Psi_f(S_f') \to \mathbb{C}$ and then we can let $\Phi_f = \Phi_{F_f} \circ \Psi_f : S_f' \to \mathbb{C}$.

(1.) holds on $\Psi_f(S_f')$ because

$$\Psi_f(\gamma_f(t)) = \int_{\gamma_f(0)}^{\gamma_f(t)} \frac{d\zeta}{f(\zeta) - \zeta}$$
$$= \int_0^t \frac{\gamma_f'(s)}{f(\gamma_f(s)) - \gamma_f(s)} \, ds = \int_0^t i \, dt = it.$$

If $z \in \ell$ is fixed and $p(t) = (1-t)z + tf(z)$ where $t \in [0,1]$ then we can show that $p([0,1]) \subset S_f$. (This is fairly easy to show using condition (1.) in the statement of the Lemma.) And also

$$
\begin{aligned}
\left| [f(p(t)) - p(t)] - [f(z) - z] \right| &= \left| [f - \mathrm{id}](p(t)) - [f - \mathrm{id}](p(0)) \right| \\
&= \left| \int_0^t ([f - \mathrm{id}] \circ p)'(t) \, dt \right| \\
&= \left| \int_0^t [f - \mathrm{id}]'(p(t)) \cdot p'(t) \, dt \right| \\
&\leqslant \int_0^t |f'(p(t)) - 1| \cdot |f(z) - z| \, dt \leqslant \tfrac{1}{10}|f(z) - z|,
\end{aligned}
$$

since $\|f'(z) - 1\|_{K_0} \leqslant \frac{1}{10}$. This implies that

$$\left| \frac{f(p(t)) - p(t)}{f(z) - z} - 1 \right| \leqslant \tfrac{1}{10}. \tag{3.1}$$

So now if $w, F_f(w) \in Q_F$ and $w = \Psi_f(z)$ we calculate

$$F_f(w) - w = \Psi_f(f(z)) - \Psi_f(z) = \int_z^{f(z)} \frac{d\zeta}{f(\zeta) - \zeta} = \int_0^1 \frac{f(z) - z}{f(p(t)) - p(t)} \, dt,$$

and by (3.1) it is easy to see that $\left| [F_f(w) - w] - 1 \right| \leqslant \frac{1}{4}$. So **(2.)** is proven.

We can also see that

$$F_f'(w) = \Psi_f'(f(z)) \cdot f'(z) \cdot (\Psi_f^{-1})'(w) = \frac{\Psi_f'(f(z))}{\Psi_f'(z)} f'(z) = \frac{f(z) - z}{f^2(z) - f(z)} f'(z).$$

And applying (3.1) with $t = 1$ we must have $|F_f'(w) - 1| \leqslant \frac{1}{4}$. Thus **(3.)** is proven.

As stated above, **(1.)**, **(2.)** and **(3.)** imply the $\Phi_f : S_f' \to \mathbb{C}$ exists.

It is fairly easy to show that S_f'/f is isomorphic to $\mathbb{C}/\mathbb{Z}$. (See [Sh2, Lemma 2.5.4].)

Theorem 3.3.9 implies that $f \mapsto \gamma_f(t)$ is continuous for every t. Since $\ell_f = \{\gamma_f(t) \mid t \in \mathbb{R}\}$ it is therefore clear that $f \mapsto \overline{\ell_f}$ must be lower semi-continuous on $\mathcal{M}'$. It quickly follows that $f \mapsto S_f$ is lower semi-continuous on $\mathcal{M}'$.

It does not take too much effort to strengthen this to show that if $f_1 \in \mathcal{M}'$ then given any compact $G \subset \overset{\circ}{S}_{f_1}$ we will have $G \subset \overset{\circ}{S}_f$ if f is sufficiently close to f_1.

We can show that $f \mapsto (\Psi_f : S_f' \to \Psi_f(S_f'))$ and $f \mapsto (F_f : \Psi(S_f') \to \mathbb{C})$ are continuous. So if we set $w_0(f) = \Psi_f(z_0(f))$ then Lemma 3.3.12 tells us that $f \mapsto (\Phi_{F_f} : \Psi_f(S_f') \to \mathbb{C})$ is continuous. Thus $f \mapsto \Phi_F \circ \Psi_f =: \Phi_f$ is continuous also.   ■

Remark 3.3.14 *Suppose that* $f \in \mathcal{N}$ *has* r *distinct fixed points in* K_0 *which are* $\sigma_0, \ldots, \sigma_r$ *in* K_0, *and that* G *is the additive group* $\Gamma = 2\pi i \iota(f, \sigma_0)\mathbb{Z} + \cdots + 2\pi i \iota(f, \sigma_r)\mathbb{Z}$. *If we try to extend* Ψ_f *to* $K_0 \setminus \{\sigma_0, \ldots, \sigma_r\}$ *we will get a multi-valued function (because of the choice of paths over which we can integrate). In fact we will obtain* $\Psi_f^* : K_0 \setminus \{\sigma_0, \ldots, \sigma_r\} \to \mathbb{C}/\Gamma$.

Proof of Theorem 2.3.12 on page 14 (Existence and continuity of Fatou coordinates) Proposition 2.3.2 allows us to apply Lemma 3.3.13 with $z_0(f) = \gamma_{i,s,f}(0)$, which tells us that parts **(1.)**, **(2.)** and **(4.)** hold.

It also tells us that $f \mapsto S_{i,s,f}$ is lower semi-continuous on $\mathcal{WB}$. Lemma 3.3.11 implies that $f \mapsto \overline{\ell_{i,s,f}}$ is continuous on $\mathcal{WB}(\mathsf{G})$, from which it follows that $f \mapsto S_{i,s,f}$ is also continuous on $\mathcal{WB}(\mathsf{G})$. Thus **(3.)** holds. ∎

Proof of Proposition 2.3.14 on page 15 (Extending $\Phi_{i,s,f}$ to $U_{i,s,f}$) Lemma 3.3.11 will imply that $f \mapsto \overline{U_{i,s,f}}$ is continuous. And in particular we will be able to show that for any $f_1 \in \mathcal{WB}(\mathsf{G})$ and a compact $G \subset U_{i,s,f_1}$ we will find that $G \subset U_{i,s,f}$ for all f close enough to f_1.

The change of coordinate $\Psi_f(z) := \int_{z_{i,s}}^{z} \frac{d\zeta}{f(\zeta) - \zeta}$ (as used in the proof of Lemma 3.3.13) is well defined on $U_{i,s,f}$. Also, $f \mapsto (\Psi_f : U_{i,s,f} \to \mathbb{C})$ is continuous. F_f extends to $Q_{F_f} = \Psi_f(U_{i,s,f})$ and satisfies the conditions of Lemma 3.3.12. $f \mapsto (F_f : Q_{F_f} \to \mathbb{C})$ is continuous and Lemma 3.3.12 tells us that $f \mapsto (\Phi_{F_f} : Q_{F_f} \to \mathbb{C})$ is continuous. When we set $\Phi_{i,s,f} := \Phi_{F_f} \circ \Psi_f$, continuity of $f \mapsto (\Phi_f : U_{i,s,f} \to \mathbb{C})$ is assured, and the Lemma is proven. ∎

3.4 Lifted phases

The rest of this section is aimed at finding a convenient normalisation for the Fatou coordinates, and formulae for the lifted phases.

Much of the method and notation (from Lemma 3.4.1 onwards) parallels that used in [Sh1–3], but things are somewhat more complicated, and a lot of extra work needs to be done.

Proof of Lemma 2.4.8 on page 16 (Definition and properties of $\jmath(f, \sigma)$) Continuity of $f \mapsto \sum_{\sigma \in U} \iota(f, \sigma)$ is fairly trivial given the definition of the holomorphic index, the theory of residues (and Rouché's Theorem).

Note that

$$\begin{aligned}
\frac{-2\pi i}{\log(1 + z)} - \frac{-2\pi i}{z} &= 2\pi i \frac{\log(1 + z) - z}{z \log(1 + z)} \\
&= 2\pi i \frac{(z - \frac{z^2}{2} + \mathcal{O}(z^3)) - z}{z(z + \mathcal{O}(z^2))} \\
&= -\pi i + \mathcal{O}(z)
\end{aligned} \tag{3.2}$$

as $z \to 0$. Suppose that $\{h_k\}_{k \geqslant 0}$ is a sequence of maps, and $\{\sigma_k\}_{k \geqslant 0}$ is a sequence of points such that σ_k is a simple fixed point of h_k and for all $k \geqslant 0$, and that $h_k'(\sigma_k) \to 1$ as $k \to +\infty$. By (3.2) $\jmath(h_k, \sigma_k) = 2\pi i \iota(h_k, \sigma_k) - m\pi i + o(1)$ as $k \to +\infty$ (where $m = 1$ is the multiplicity of σ_k). And if σ is a multiplicity $m > 1$ fixed point of f then of course $\jmath(f, \sigma) = 2\pi i \iota(f, \sigma) - m\pi i$ by definition.

Thus if f_1 has M fixed points in U counted with multiplicity, and $H_k \to f_1$ uniformly on compact sets then

$$\begin{aligned}
\sum_{H_k(\sigma) = \sigma \in U} \jmath(H_k, \sigma) &= \sum_{H_k(\sigma) = \sigma \in U} 2\pi i \iota(H_k, \sigma) - M\pi i + o(1) \\
&= \left(\sum_{f_1(\sigma) = \sigma \in U} 2\pi i \iota(f_1, \sigma) + o(1) \right) - M\pi i + o(1) \\
&= \sum_{f_1(\sigma) = \sigma \in U} \jmath(f_1, \sigma) + o(1) \qquad \text{as } k \to +\infty.
\end{aligned}$$

As a result the map $f \mapsto \sum_{\sigma \in U} \jmath(f, \sigma)$ is continuous, and $\jmath(\cdot, \cdot)$ shares many of the same properties as the holomorphic index $\iota(\cdot, \cdot)$. ∎

Proof of Proposition 2.4.3 on page 16 (Continuity of the lifted phase) Suppose that $f \in \mathcal{WB}$, and that $f_k \to f$ in $\mathcal{WB}$. We need to show that for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ we have $\tilde{\tau}_i(f_k) \to \tilde{\tau}_i(f)$.

First we consider the case where $\mathsf{gate}_i(f) \neq \star$. We can find a $z \in \overset{\circ}{S}_{i,+,f}$ and $p \in \mathbb{N}$ such that $f^n(z) \in U_{i,+,f}$ for $n = 0, \ldots, p$ and $f^p(z) \in \overset{\circ}{S}_{j,-,f}$.

Note that the maps $f \mapsto \overline{\ell_{i,+,f}}$, $f \mapsto f(\overline{\ell_{i,+,f}})$, $f \mapsto \overline{\ell_{i,+,f}}$ and $f \mapsto f(\overline{\ell_{i,+,f}})$ are continuous with respect to the Hausdorff metric. (See the proof of Theorem 2.3.12.) Thus for k large we have $z \in \overset{\circ}{S}_{i,-,f_k}$ and $f_k^p(z) \in \overset{\circ}{S}_{j,-,f_k}$ and

$$\tilde{\tau}_i(f_k) = \Phi_{j,-,f_k}(f_k^p(z)) - \Phi_{i,+,f_k}(z) - m \to$$
$$\Phi_{j,-,f}(f^p(z)) - \Phi_{i,+,f}(z) - m = \tilde{\tau}_i(f)$$

as $k \to +\infty$ as required.

Now consider the case where $\mathsf{gate}_i(f) = \star$. Then $\tilde{\tau}_i(f) = \infty$, and we need to show that $|\tilde{\tau}_i(f_k)| \to +\infty$ as $k \to +\infty$.

So assume for contradiction that $f_k \to f$ in $\mathcal{WB}$, but $|\tilde{\tau}_i(f_k)| \not\to +\infty$. There must be an $M > 0$ and a subsequence $\{g_k\}$ of $\{f_k\}$ so that $|\tilde{\tau}_i(g_k)| < M$ for all k. But then by the compactness of $\overline{D_M}$ we can take another subsequence $\{h_k\}$ of $\{g_k\}$ so that $\tilde{\tau}_i(h_k) \to \tilde{\theta} \in \overline{D_M}$, and so that $\mathsf{gate}_i(f_k) = j \neq \star$ for all k (and some j).

Now let $G_k = \Phi_{j,-,h_k}^{-1} \circ T_{\tilde{\tau}_i(h_k)} \circ \Phi_{i,+,h_k}$ where $T_c(w) = w + c$ for $c \in \mathbb{C}$. G_k will be well defined upon $U_{i,+,h_k}$ and $G_k = \mathrm{id}_{U_{i,+,h_k}}$ (by the definition of $\tilde{\tau}_i(h_k)$).

It can also be shown that $G = \Phi_{j,-,f}^{-1} \circ T_{\tilde{\theta}} \circ \Phi_{i,+,f}$ is well defined on some subset Q of $U_{i,+,f}$. And because $f \mapsto \Phi_{a,s,f}$ is continuous for any a, s we can show that $G_k \to G$ on Q. However if $z \in Q$ then $G_k(z) = z \in U_{i,+,f}$, but $G(z) \in U_{j,-,f}$. So since $U_{i,+,f}$ and $U_{j,-,f}$ are disjoint this would imply that $G_k(z) \not\to G(z)$ which is a contradiction. ∎

Proof of Proposition 2.4.5 on page 16 (The size of the ith gate) See Lemma 3.7.8 below. ∎

Lemma 3.4.1 *Suppose that Φ and F are analytic, univalent functions defined on $\mathcal{U} = \{w \in \mathbb{C}^* \mid \theta_1 < \arg w < \theta_2\}$, where $\theta_2 < \theta_1 + 2\pi$, and satisfying*

$$\begin{aligned}
\Phi(F(w)) &= \Phi(w) + 1 && \text{if } w, F(w) \in \mathcal{U}, \\
|F(w) - (w+1)| &< \tfrac{1}{4} && \text{if } w \in \mathcal{U}, \\
|F'(w) - 1| &= \mathcal{O}(w^{-1-\beta}) && \text{as } w \in \infty \text{ in } \mathcal{U},
\end{aligned}$$

for some $\beta \in (0, 1]$.

Then, for any $w_0 \in \mathcal{U}$, and θ_1', θ_2' with $\theta_1 < \theta_1' < \theta_2' < \theta_2$

$$\Phi(w) = \int_{w_0}^{w} \frac{d\zeta}{F(\zeta) - \zeta} + const + \mathcal{O}(w^{-\beta})$$

as $w \to \infty$ with $\theta_1' < \arg w < \theta_2'$.

Proof. See [Sh2, Prop. 2.6.2] or [Zi, Lemma 2.2.4]. ∎

Definition 3.4.2 (The horns $S_{i,\pm,f}^u$ and $S_{i,\pm,f}^\ell$) *For an $f \in \mathcal{WB}$, we will define the upper and lower horns of $S_{i,\pm,f}'$ to be*

$$\begin{aligned}
S_{i,\pm,f}^u &= \{z \in S_{i,\pm,f}' \mid \mathrm{Im}\, \Phi_{i,\pm,f} > \eta\} && \text{and} \\
S_{i,\pm,f}^\ell &= \{z \in S_{i,\pm,f}' \mid \mathrm{Im}\, \Phi_{i,\pm,f} < -\eta\},
\end{aligned}$$

where $\eta > 0$ is a large constant independent of f. (See Figure 3.10.)

Remark 3.4.3 *If $\gamma_{i,s,f}(+\infty)$ is a simple fixed point then for any $z \in S^u_{i,s,f}$ there will be a minimum integer $p > 1$ such that $f^p(z) \in S_{i,s,f}$. This is because f in a neighbourhood of $\gamma_{i,s,f}(+\infty)$ is conjugate to the rotation $z \mapsto e^{2\pi i \alpha} z$ (where $\mathrm{Re}\,\alpha > 0$) and $S_{j,-,f}$ is a fundamental region. (See the proof of Lemma 3.4.4 below.)*

Similarly, if $\gamma_{i,s,f}(-\infty)$ is a simple fixed point then for any $z \in S^\ell_{i,s,f}$ there will be a minimum integer $q > 1$ such that $f^q(z) \in S_{i,s,f}$.
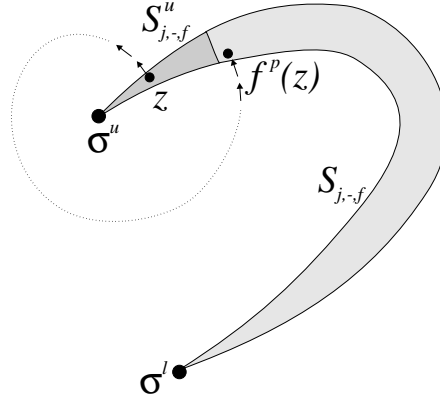
Figure 3.10: If $\eta > 0$ is sufficiently large then for every $z \in S^u_{j,-,f}$ the orbit of that point will circle σ^u and eventually fall in $S_{j,-,f}$ again.

Lemma 3.4.4 (Definition and properties of $\tilde{\mathcal{R}}^{(j,\cdot)}_f$) *Suppose that $f \in \mathcal{WB}$ with no multiple fixed points in K_0, and σ^u, σ^ℓ are the fixed points at the ends of the horns $S^u_{j,-,f}$ and $S^\ell_{j,-,f}$ respectively.*

If $z \in S^u_{j,-,f}$ is sufficiently close to σ^u then there will be a unique smallest integer $p > 1$ such that $f^p(z) \in S_{j,-,f}$ (and $f^k(z)$ stays in a small neighbourhood of σ^u, for $k = 0, \ldots, p$). We can then define $\tilde{\mathcal{R}}^{(j,u)}_f(w) := \Phi_{j,-,f}(f^p(z)) - p$ where $w = \Phi_{j,-,f}(z)$ which will satisfy

$$\tilde{\mathcal{R}}^{(j,u)}_f(w+1) = \tilde{\mathcal{R}}^{(j,u)}_f(w) + 1.$$

(See Figure 3.10.) Using this relation we can extend $\tilde{\mathcal{R}}^{(j,u)}_f$ to $\{w \mid \mathrm{Im}\,w > \eta_f\}$ for some large $\eta_f > 0$.

Also, if $z \in S^\ell_{j,-,f}$ is sufficiently close to σ^ℓ there will be some least integer $q > 1$ such that $f^q(z) \in S_{j,-,f}$. We can then define $\tilde{\mathcal{R}}^{(j,\ell)}_f(w) := \Phi_{j,-,f}(f^q(z)) - q$ where $w = \Phi_{j,-,f}(z)$, which extends to $\{w \mid \mathrm{Im}\,w < -\eta_f\}$ (provided that $\eta_f > 0$ was chosen large enough).

Then

$$\lim_{\mathrm{Im}\,w \to +\infty} \tilde{\mathcal{R}}^{(j,u)}_f(w) - w = \jmath(f, \sigma^u) \qquad \text{and}$$

$$\lim_{\mathrm{Im}\,w \to -\infty} \tilde{\mathcal{R}}^{(j,\ell)}_f(w) - w = -\jmath(f, \sigma^\ell).$$

Proof. Suppose that $S^u_{j,+,f}$ ends at a simple fixed point σ^u. Let $e^{2\pi i \alpha^u} = f'(\sigma)$ so that $\jmath(f, \sigma^u) = -\frac{1}{\alpha^u}$. If $M \gg 1$ and we let $Q^u = \{w \in \mathbb{C} \mid \mathrm{Im}\,\alpha^u w > M\}$ and $D^u = \{z \in \mathbb{C} \mid |z - \sigma^u| < e^{-2\pi M}\}$ then we can lift $z \in D^u$ to a $w \in Q^u$ via

$$z = L(w) := \sigma^u + e^{2\pi i \alpha^u w}.$$

There will be an $F^u : Q^u \to \mathbb{C}$ with $L \circ F^u = f \circ L$. We can show that

$$F^u(w) = w + 1 + \mathcal{O}(w^{-2}) \tag{3.3}$$

as $\mathrm{Im}\,\alpha^u w \to +\infty$. The change of coordinate L was chosen especially so that (3.3) would be satisfied. (Compare [Sh2, §3.3.3(iii)].) Notice that since $\sigma^u = \gamma_{j,-,f}(+\infty)$ is a sink for $\dot{z} = i[f(z) - z]$, we have $\mathrm{Im}\,f'(\sigma^u) > 1$. (Compare §3.3.1.) Therefore $\mathrm{Re}\,\jmath(f, \sigma^u) < 0$.

Let S^u be one of the connected components of $L^{-1}(S^u_{j,-,f} \cap D^u)$. Then there is an (inverse) $\hat{L} : S^u_{j,-,f} \cap D^u \to S^u$ such that $L \circ \hat{L} = \mathrm{id}$. By Lemma 3.3.1, $\gamma_{j,-,f}(t) \to \sigma^u$ exponentially as $t \to +\infty$. From this we can show that there is some $\eta \approx i$ such that $(\hat{L} \circ \gamma_{j,-,f})'(t) \to \eta$ exponentially as $t \to +\infty$. Thus $\hat{L} \circ \gamma_{j,-,f}$ is asymptotic to an almost vertical straight line as $t \to +\infty$.
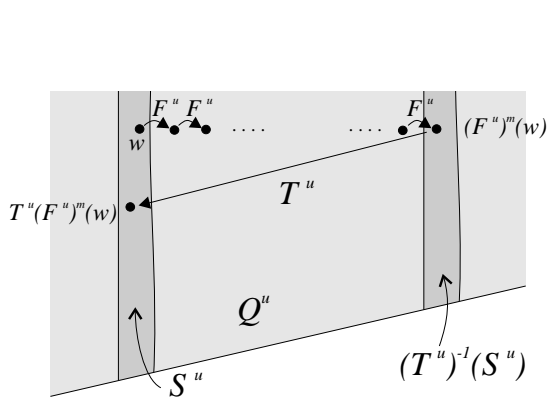
Figure 3.11:
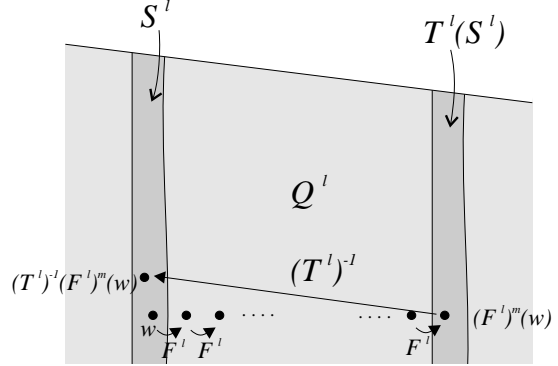
Figure 3.12:

We can let $\Phi^u = \Phi_{j,-,f} \circ \hat{L} : S^u \to \mathbb{C}$ which will satisfy $\Phi^u(F^u(w)) = \Phi^u(w) + 1$ if $w, F^u(w) \in S^u$. Using this relation we can extend it to some open sector $\mathcal{U}$ contained in Q^u. This $\mathcal{U}$ can be chosen so that if $w \in S^u$ and $\mathrm{Im}\, w$ is sufficiently large then $w \in \mathcal{U}$. Then we can apply Lemma 3.4.1 to get

$$\Phi^u(w) = w + c^u + o(1) \qquad \text{as } w \to \infty \text{ in } \mathcal{U},$$

for some constant $c^u \in \mathbb{C}$.

Let $T^u(w) := w + \jmath(f, \sigma^u)$. We can see that for any $m \in \mathbb{Z}$ we have $L\big((T^u)^m(S^u)\big) = L(S^u)$, and that $T^u(S^u)$ will lie to the **left** of S^u (because $\mathrm{Re}\,\jmath(f, \sigma^u) < 0$). If $w \in S^u$ and $\mathrm{Im}\, w$ is large enough then there will be some least integer $p > 1$ such that $(F^u)^p(w) \in (T^u)^{-1}(S^u) \cap \mathcal{U}$, and $(F^u)^k(w) \in Q^u$ for all $k = 0, \ldots, p$. See Figure 3.11. (Notice that p will be roughly $-\mathrm{Re}\,\jmath(f, \sigma^u)$, and certainly it will be bounded as $t \to +\infty$.) This implies that if $z \in S^u_{j,-,f}$ is close enough to σ^u, then there will be some least $p > 1$ such that $f^p \in S^u_{j,-,f}$, and $f^k(z) \in D^u$ for all $k = 0, \ldots, p$.

But now if $w \in S^u$ has $\mathrm{Im}\, w \gg 1$ and $z := L(w)$ then

$$\begin{aligned}
\Phi_{j,-,f}\big(f^p(z)\big) - \Phi_{j,-,f}(z) &= \Phi^u \circ T^u\big((F^u)^p(w)\big) - \Phi^u(w) \\
&= \Phi^u \circ T^u(w + p + o(1)) - \Phi^u(w) \\
&= \Phi^u\big(w + p + \jmath(f, \sigma^u) + o(1)\big) - \Phi^u(w) \\
&= [w + p + \jmath(f, \sigma^u) + c^u + o(1)] - [w + c^u + o(1)] \\
&= \jmath(f, \sigma^u) + p + o(1)
\end{aligned}$$

as $\mathrm{Im}\, w \to +\infty$. It follows that $\tilde{\mathcal{R}}^{(j,u)}_f(W) - W \to \jmath(f, \sigma^u)$ as $\mathrm{Im}\, W \to +\infty$.

Going through the same process as $\mathrm{Im}\, w \to -\infty$, we set $T^\ell(w) = w + \jmath(f, \sigma^\ell)$. We then see that $\mathrm{Re}\,\jmath(f, \sigma^\ell) > 0$. Therefore $T^\ell(S^\ell)$ is to the **right** (not left) of S^ℓ. See Figure 3.12. So, we find that for $w \in S^\ell$ (with $\mathrm{Im}\, w \ll 0$) there will be some least $q > 1$ such that $(F^\ell)^q(w) \in T^\ell(S^\ell)$. The calculation now becomes

$$\begin{aligned}
\Phi_{j,-,f}\big(f^q(z)\big) - \Phi_{j,-,f}(z) &= \Phi^\ell \circ (T^\ell)^{-1}\big((F^\ell)^q(w)\big) - \Phi^\ell(w) \\
&= [w + q - \jmath(f, \sigma^\ell) + c^\ell + o(1)] - [w + c^\ell + o(1)] \\
&= -\jmath(f, \sigma^\ell) + q + o(1)
\end{aligned}$$

as $\mathrm{Im}\, w \to -\infty$, implying that $\tilde{\mathcal{R}}^{(j,\ell)}_f(W) - W \to -\jmath(f, \sigma^\ell)$ as $\mathrm{Im}\, W \to -\infty$. ■

Lemma 3.4.5 (Definition and properties of $\tilde{\mathcal{E}}_f^{(j,\cdot)}$) If the $\eta > 0$ in Definition 3.4.2 is large enough then given any $z \in S_{j,-,f}^u$ there will be a unique smallest integer $p > 1$ such that $f^p(z) \in S_{j,+,f}$ and $f^k(z) \in K_0$ for $k = 0, \ldots, p$.

We can then define $\tilde{\mathcal{E}}_f^{(j,u)}(w) := \Phi_{j,+,f}(f^p(z)) - p$ where $w = \Phi_{j,-,f}(z)$. This will be well defined on $\Phi_{j,-,f}(S_{j,-,f}^u)$, and will satisfy

$$\tilde{\mathcal{E}}_f^{(j,u)}(w+1) = \tilde{\mathcal{E}}_f^{(j,u)}(w) + 1$$

where both sides are well defined. Using this relation we can extend $\tilde{\mathcal{E}}_f^{(j,u)}$ to $\{w \mid \operatorname{Im} w > \eta\}$.

In the same way, if $\eta > 0$ is sufficiently large then given any $z \in S_{j,-,f}^\ell$ there will be a unique smallest $q > 1$ such that $f^q(z) \in S_{j-1,+,f}$ and $f^k(z) \in K_0$ for $k = 0, \ldots, p$. We can define $\tilde{\mathcal{E}}_f^{(j,\ell)}(w) := \Phi_{j-1,+,f}(f^q(z)) - q$ where $w = \Phi_{j,-,f}(z)$. This extends to $\{w \mid \operatorname{Im} w < -\eta\}$.

There will be some $L_f^{(j,u)}$ and $L_f^{(j,\ell)}$ so that

$$\lim_{\operatorname{Im} w \to +\infty} \tilde{\mathcal{E}}_f^{(j,u)}(w) - w = L_f^{(j,u)} \qquad \text{and}$$

$$\lim_{\operatorname{Im} w \to -\infty} \tilde{\mathcal{E}}_f^{(j,\ell)}(w) - w = L_f^{(j,\ell)}.$$

There is a *preferred normalisation* of the Fatou coordinates under which all the limits $L_f^{(j,\cdot)}$ are equal to zero, except for $L^{(1,\ell)}$ (which will be equal to $-\sum_{\sigma = f(\sigma) \in K_0} J(f, \sigma)$). The continuity of $f \mapsto (\Phi_{i,\pm,f} : S'_{i,\pm,f} \to \mathbb{C})$ still holds on $\mathcal{WB}$ with this normalisation.

The $\eta > 0$ can be chosen independently of $f \in \mathcal{WB}$.

Proof. We only give a sketchy proof here, since much of the construction is the same as that used in the proof of Lemma 3.4.4.

Our first job is to make some kind of change of coordinate in a neighbourhood of σ^u.

If σ^u is a simple fixed point, then we can use the change of coordinate used in Lemma 3.4.4: $z = L(w) := \sigma^u + e^{2\pi i \alpha}$ (where $e^{2\pi i \alpha} = f'(\sigma^u)$ and α is close to 0).

If on the other hand σ^u is a fixed point of f of multiplicity $r + 1$ then f is of the form $f(z) = z + b(z - \sigma^u)^{r+1} + \mathcal{O}((z - \sigma^u)^{r+2})$ as $z \to \sigma^u$. We can then use the change of coordinate $w = -1/br(z - \sigma^u)^r$.

In either case we will be able to lift f in the z-coordinate, to give some $F : \mathcal{U} \to \mathbb{C}$ in the w-coordinate where $\mathcal{U} := \{w \in \mathbb{C} \mid \arg(w - w_0) \in (\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)\}$ and $\operatorname{Im} w_0 \gg 0$ and $\delta > 0$ small. In both cases we have $F(w) = w + 1 + \mathcal{O}(w^{-\beta})$ as $w \to \infty$ in $\mathcal{U}$, for some $\beta \in (0, 1)$.

We can show that if B is a sufficiently small neighbourhood of σ^u, we can lift $S_{j,\pm,f}^u \cap B$ to some "half strips" $S^\pm \subset \mathcal{U}$ where S^+ is to the right of S^-. If σ^u is a simple fixed point then we need to make sure that there are no other possible lifts of $S_{j,+,f}^u$ or $S_{j,-,f}^u$ in between S^+ and S^-.

On these we will have some $\Phi^\pm : S^\pm \to \mathbb{C}$ which satisfy $\Phi^\pm(F(w)) = \Phi(w) + 1$ if $w, F(w) \in S^\pm$. Using this relation, we can extend $\Phi^\pm$ to $Q := \{w \in \mathcal{U} \mid \operatorname{Im} w > \xi\}$ for some large $\xi > 0$.

With this construction we see that $\tilde{\mathcal{E}}_f^{(j,u)} = \Phi^+ \circ (\Phi^-)^{-1}$. Lemma 3.4.1 tells us that there are some constants $c_\pm(f)$ so that

$$\Phi^\pm(w) = \int_{w_0}^w \frac{d\zeta}{F(\zeta) - \zeta} + c_\pm(f) + o(1)$$

as $w \to \infty$ in Q. So clearly $\Phi^+(w) - \Phi^-(w) = c_+(f) - c_-(f) + o(1)$ as $w \to \infty$ in Q. And if $L_f^{(j,u)} := c_+(f) - c_-(f)$, then

$$\tilde{\mathcal{E}}_f^{(j,u)}(w) - w \to L_f^{(j,u)} \qquad \text{as } \operatorname{Im} w \to +\infty$$

because $\tilde{\mathcal{E}}_f^{(j,u)} = \Phi^+ \circ (\Phi^-)^{-1}$ when both sides are defined.

Just as in the proof of [Sh2, Lemma 3.4.2], suppose that $\eta_2 > \eta_1 > \xi$. Integrating over the rectangular contour with corners $i\eta_1, 1 + i\eta_1, 1 + i\eta_2, i\eta_2$, gives result 0 by Cauchy's Theorem. The periodicity of $\tilde{\mathcal{E}}_f^{(j,u)}(w) - w$ means that the integral over the vertical sides will cancel, implying that

$$\int_{i\eta_1}^{i\eta_1+1} (\tilde{\mathcal{E}}_f^{(j,u)}(w) - w)\, dw - \int_{i\eta_2}^{i\eta_2+1} (\tilde{\mathcal{E}}_f^{(j,u)}(w) - w)\, dw = 0.$$

And as $\eta_2 \to +\infty$, we have

$$\int_{i\eta_1}^{i\eta_1+1} (\tilde{\mathcal{E}}_f^{(j,u)}(w) - w)\, dw = L_f^{(j,u)}$$

(which will be true for any $\eta_1 > \xi$).

Initially we normalise the Fatou coordinates by insisting that $\Phi_{i,\pm,f}(z_{i,\pm}) = 0$ (as was originally done when proving Thm 2.3.12). By the continuity of $f \mapsto (\Phi_{j,\pm,f} : S'_{j,\pm,f} \to \mathbb{C})$ under this normalisation we can show that

$$f \mapsto \tilde{\mathcal{E}}_f^{(j,u)}(w) - w$$

is continuous for each $w \in [0,1] + i\eta_1$. It then becomes clear that

$$f \mapsto \int_{i\eta_1}^{i\eta_1+1} (\tilde{\mathcal{E}}_f^{(j,u)}(w) - w)\, dw = L_f^{(j,u)}$$

is continuous.

The same argument will show the existence of the limit $L_f^{(j,\ell)}$, which will again be continuous with respect to f.

Notice that if we replace $\Phi_{1,+,f}$ by $\Phi_{1,+,f} + L_f^{(1,u)}$ then we will have

$$\lim_{\mathrm{Im}\, w \to +\infty} \tilde{\mathcal{E}}_f^{(1,u)}(w) - w = 0.$$

We can go all the way round, changing the normalisations of $\Phi_{2,-,f}$, $\Phi_{2,+,f}$, ..., $\Phi_{\nu,-,f}$, $\Phi_{\nu,+,f}$ in turn so that the "new $L_f^{(j,\cdot)}$" are all zero except for $L_f^{(1,\ell)}$. Because the constants we have to add to each of the $\Phi_{i,\pm,f}$ are continuous with respect to $f \in \mathcal{WB}$, Theorem 2.3.12 part (4.) will still be true.

The value of $L_f^{(1,\ell)}$ is shown in the proof of Theorem 2.4.11 below.

It is not hard to show that $\eta > 0$ can be chosen independently of $f \in \mathcal{WB}$.   ■

Lemma 3.4.6 (Formula for $\jmath(f, \sigma)$ in terms of lifted phases) *Suppose that $f \in \mathcal{WB}$ has a simple fixed point $\sigma^u \in K_0$ which lies at the end of the upper horn $S_{j,-,f}^u$ (for some j).*

Then there will be some $r \leqslant \nu$ and $a_1, \ldots, a_{r+1} \in \mathbb{Z}/\nu\mathbb{Z}$ such that $a_1 = a_{r+1} = j$, $a_{k+1} = \mathsf{gate}_{a_k}(f)$ for $k = 1, \ldots, r$, and $j \notin \{a_2, \ldots, a_r\}$. (Notice that $\{S_{a_k,\pm,f}^u \mid k = 1, \ldots, r\}$ will consist of all those upper horns which end at σ^u.)

If we use the preferred normalisation of the Fatou coordinates given in Lemma 3.4.5 then

$$\jmath(f, \sigma^u) = \sum_{k=1}^{r} \tilde{\tau}_{a_k}(f). \tag{3.4}$$

Also if σ^ℓ is the fixed point that lies at the end of the lower horn $S_{j,-,f}^\ell$ then there will be some $s \leqslant \nu$ and $b_1, \ldots, b_{s+1} \in \mathbb{Z}/\nu\mathbb{Z}$ such that $b_1 = b_{s+1} = j$, $b_{k+1} = \mathsf{gate}_{b_k-1}(f)$ for $k = 1, \ldots, s$ and $j \notin \{b_2, \ldots, b_s\}$. Then

$$\jmath(f, \sigma^\ell) = \begin{cases} -\sum_{k=1}^{s} \tilde{\tau}_{b_k}(f) & \text{if } \sigma^\ell \neq \sigma_0(f); \\ -\sum_{k=1}^{s} \tilde{\tau}_{b_k}(f) - L^{(1,\ell)} & \text{if } \sigma^\ell = \sigma_0(f). \end{cases} \tag{3.5}$$

Proof.   Suppose σ^u and σ^ℓ are at the ends of the horns $S_{j,-,f}^u$ and $S_{j,-,f}^\ell$. Consider $z \in S_{i,+,f}^u$ very close to the fixed point σ^u.

Let $\tilde{\theta}_k = \tilde{\tau}_{a_k}(f)$ for $k = 1, \ldots, r$. Denote by T_c the translation given by $T_c(w) = w + c$, where $c \in \mathbb{C}$. We observe (from the definitions of $\tilde{\mathcal{R}}_f^{(j,u)}$, $\tilde{\mathcal{E}}_f^{(a_k,u)}$ and $\tilde{\theta}_k$) that

$$\tilde{\mathcal{R}}_f^{(j,u)} = T_{\tilde{\theta}_r} \circ \tilde{\mathcal{E}}_f^{(a_r,u)} \circ T_{\tilde{\theta}_{r-1}} \circ \tilde{\mathcal{E}}_f^{(a_{r-1},u)} \circ \cdots \circ T_{\tilde{\theta}_1} \circ \tilde{\mathcal{E}}_f^{(a_1,u)}$$

where both sides are defined.

So by Lemma 3.4.4 and Lemma 3.4.5 we now have

$$
\begin{aligned}
\jmath(f,\sigma^u) &= \lim_{\operatorname{Im} w \to +\infty} (\tilde{\mathcal{R}}_f^{(j,u)}(w) - w) \\
&= \sum_{k=1}^r \lim_{\operatorname{Im} w \to +\infty} (\tilde{\mathcal{E}}_f^{(a_k,u)}(w) - w) + \sum_{k=1}^r \tilde{\theta}_k \\
&= \sum_{k=1}^r L_f^{(a_k,u)} + \sum_{k=1}^r \tilde{\theta}_k .
\end{aligned}
\tag{3.6}
$$

Using the "preferred normalisation" $L_f^{(a_k,u)} = 0$ for all k, so we get (3.4).

In a similar way we can obtain

$$
\begin{aligned}
-\jmath(f,\sigma^\ell) &= \lim_{\operatorname{Im} w \to -\infty} (\tilde{\mathcal{R}}_f^{(j,\ell)}(w) - w) \\
&= \sum_{k=1}^s \lim_{\operatorname{Im} w \to -\infty} (\tilde{\mathcal{E}}_f^{(b_k,\ell)}(w) - w) + \sum_{k=1}^s \tilde{\varphi}_k \\
&= \sum_{k=1}^s L_f^{(b_k,\ell)} + \sum_{k=1}^s \tilde{\varphi}_k
\end{aligned}
\tag{3.7}
$$

where $\tilde{\varphi}_k = \tilde{\tau}_{b_k-1}(f)$ for $k = 1,\ldots,s$. Using the preferred normalisation we have $L_f^{(b_k,\ell)} = 0$ for $k = 2,\ldots,s$, and we will get (3.5). (Recall that $\sigma_0(f) := \gamma_{1,-,f}(-\infty)$, so if $\sigma^\ell = \sigma_0(f)$ then $b_k = 1$ for a unique $k \in \{1,\ldots,s\}$. On the other hand, if $\sigma^\ell \neq \sigma_0(f)$ then $b_k \neq 1$ for all $k = 1,\ldots,s$.) ∎

Proof of Theorem 2.4.11 on page 17   (Formula for the lifted phases) First suppose that $f \in \mathcal{WB}(\mathsf{G})$ has no multiple fixed points in K_0, which also means there are no closed gates.

Suppose that $\mathsf{gate}_i(f) = j \neq \star$ and let σ^u be the fixed point at the upper ends of $S_{i,+,f}$ and $S_{j,-,f}$, and let σ^ℓ be the one at the lower ends.

Notice that we can construct a "simple tree structure" for f as we did in the proof of Proposition 2.3.8. Each fixed point in $\operatorname{Fix}(f)$ corresponds to a node in the tree and each line between the nodes corresponds to one of the ν gates. See Figure 3.7 and Figure 3.13.

We can say that σ^u is at depth 0, and all nodes on the tree neighbouring it are at depth 1. Remaining nodes neighbouring the depth 1 nodes are said to be depth 2, etc. Notice that if σ is an even node then $\sigma = \gamma_{i,+,f}(+\infty)$ for some $i \in \mathbb{Z}/\nu\mathbb{Z}$. Similarly, if σ is of odd depth number then there is some $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that $\sigma = \gamma_{i,+,f}(-\infty)$.

Now we remove the line in the tree corresponding to the ith gate. This leaves us with two simple tree structures corresponding to $\operatorname{Fix}^u(i,f)$ and $\operatorname{Fix}^\ell(i,f)$.

Recall that $\sigma_0(f) := \gamma_{1,-,f}(-\infty)$. We want to calculate $\tilde{\tau}_i(f)$ using the definitions (3.4) and (3.5) in Lemma 3.4.6. These can be rewritten as the recursive definitions

$$
\tilde{\tau}_i(f) = \jmath(f,\sigma^u) - \sum_{k=1}^{r-1} \tilde{\tau}_{a_k}(f)
\tag{3.8}
$$

and

$$
\tilde{\tau}_i(f) = \begin{cases} -\jmath(f,\sigma^\ell) - \sum_{k=1}^{s-1} \tilde{\tau}_{b_k}(f) & \text{if } \sigma^\ell \neq \sigma_0(f), \\ -\bigl(\jmath(f,\sigma^\ell) + L_f^{(1,\ell)}\bigr) - \sum_{k=1}^{s-1} \tilde{\tau}_{b_k}(f) & \text{if } \sigma^\ell = \sigma_0(f), \end{cases}
\tag{3.9}
$$

where the a_k's and b_k's have the same definition as they were in Lemma 3.4.6, and $i = a_r$, $i = b_s$.

In effect we have a recursive algorithm which makes us traverse the tree associated with $\operatorname{Fix}^u(i,f)$, and at each node σ of depth d we add

$$
\begin{aligned}
C(\sigma) &= \begin{cases} (-1)^d \jmath(f,\sigma) & \text{if } d \text{ is even, by (3.8),} \\ (-1)^d[-\jmath(f,\sigma)] & \text{if } d \text{ is odd and } \sigma \neq \sigma_0(f), \text{ by (3.9),} \\ (-1)^d\bigl[-\bigl(\jmath(f,\sigma) + L_f^{(1,\ell)}\bigr)\bigr] & \text{if } d \text{ is odd and } \sigma = \sigma_0(f), \text{ by (3.9).} \end{cases} \\
&= \begin{cases} \jmath(f,\sigma) & \text{if } \sigma \neq \sigma_0(f), \\ \jmath(f,\sigma) + L_f^{(1,\ell)} & \text{if } \sigma = \sigma_0(f). \end{cases}
\end{aligned}
$$
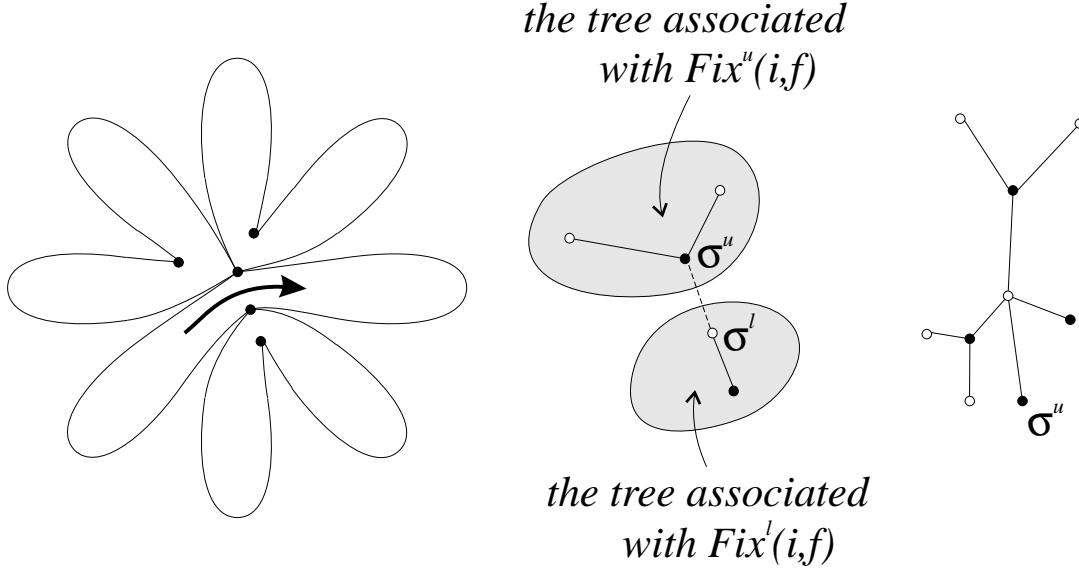
Figure 3.13: *On the right we show the arrangement of $\ell_{i,s,f}$'s for some f with gate structure $(2,3,1,4)$. In the middle we show the trees associated with the $\mathrm{Fix}^u(i,f)$ and $\mathrm{Fix}^\ell(i,f)$. The black nodes in the tree are "even," and the white nodes are "odd." On the right we show a more complicated tree.*

(Note that $\sigma_0(f) := \gamma_{1,-,f}(-\infty)$ must be odd.) Therefore

$$
\tilde{\tau}_i(f) = \begin{cases} \sum_{\sigma \in \mathrm{Fix}^u(i,f)} \jmath(f,\sigma) & \text{if } \sigma_0(f) \notin \mathrm{Fix}^u(i,f), \\ L^{(1,\ell)} + \sum_{\sigma \in \mathrm{Fix}^u(i,f)} \jmath(f,\sigma) & \text{if } \sigma_0(f) \in \mathrm{Fix}^u(i,f). \end{cases} \tag{3.10}
$$

Also, by considering the tree associated with $\mathrm{Fix}^\ell(i,f)$ we will traverse the tree and add $-C(\sigma)$ at each node σ. This gives

$$
\tilde{\tau}_i(f) = \begin{cases} -L^{(1,\ell)} - \sum_{\sigma \in \mathrm{Fix}^\ell(i,f)} \jmath(f,\sigma) & \text{if } \sigma_0(f) \in \mathrm{Fix}^\ell(i,f), \\ -\sum_{\sigma \in \mathrm{Fix}^\ell(i,f)} \jmath(f,\sigma) & \text{if } \sigma_0(f) \notin \mathrm{Fix}^\ell(i,f). \end{cases} \tag{3.11}
$$

As these two formulae for $\tilde{\tau}_i(f)$ must be equal we see that $-L^{(1,\ell)} = \sum_{\sigma \in \mathrm{Fix}(f)} \jmath(f,\sigma)$ since $\mathrm{Fix}(f) = \mathrm{Fix}^u(i,f) \sqcup \mathrm{Fix}^\ell(i,f)$. Thus the formulae in the Lemma have been proved for an f with no multiple fixed points.

If f has any multiple fixed points, we can always take a $\mathsf{G} \in \mathsf{Admissible}$ with no closed gates, and with $\mathsf{G}_i = \mathsf{gate}_i(f)$ for each i with $\mathsf{gate}_i(f) \neq \star$. Then we can construct a sequence $\{h_k\}$ in $\mathcal{WB}(\mathsf{G})$ and with $h_k \to f$. (See the proof of Lemma 3.7.12 for a way to do this.) Proposition 2.4.3 ensures that $\tilde{\tau}_i(f) = \lim_{k \to +\infty} \tilde{\tau}_i(h_k)$.

Now suppose that $\sigma_0(f) \notin \mathrm{Fix}^u(i,f)$. Lemma 2.4.8 implies that $g \mapsto \sum_{\sigma \in \mathrm{Fix}^u(i,g)} \jmath(g,\sigma)$ is continuous on $\mathcal{WB}(\mathsf{G}) \cup \mathcal{WB}(\mathsf{gate}(f))$. This along with Theorem 2.4.11 tells us that

$$
\tilde{\tau}_i(f) = \lim_{k \to +\infty} \tilde{\tau}_i(h_k) = \lim_{k \to +\infty} \sum_{\sigma \in \mathrm{Fix}^u(i,h_k)} \jmath(h_k,\sigma) = \sum_{\sigma \in \mathrm{Fix}^u(i,f)} \jmath(f,\sigma).
$$

Similarly, if $\sigma_0(f) \notin \mathrm{Fix}^\ell(i,f)$ then $\tilde{\tau}_i(f) = -\sum_{\sigma \in \mathrm{Fix}^\ell(i,f)} \jmath(f,\sigma)$.

This implies that the formulae (3.10) and (3.11) will both hold when f has a multiple fixed point. ∎

Proof of Proposition 2.4.12 on page 17 (Bijections between lifted phases, $\jmath$-indices and holomorphic indices) The existence of B comes from the formulae for the lifted phases given in Theorem 2.4.11. The fact that it is invertible is given by Lemma 3.4.6.

If σ is a fixed point of f of multiplicity m (with $f'(\sigma) \approx 1$), then one can check that

$$
J(f, \sigma) = \left\{
\begin{array}{ll}
\frac{-2\pi i}{\log(1 - 1/\iota(f,\sigma))} & \text{if } m = 1; \\
2\pi i[\iota(f, \sigma) - \frac{m}{2}] & \text{if } m > 1.
\end{array}
\right.
$$

Thus M exists and it is holomorphic and invertible.   ■

Proof of Proposition 2.4.13 on page 18 (Equivalent convergence criteria) Suppose that **(1.)** holds. Then (3.19) in Lemma 3.7.4 below implies that there will be a sequence $\rho_k \to 0$ such that f_k is ρ_k-well behaved. (See Definition 3.7.3 below.) This together with the lower semi-continuity of $f \mapsto \overline{\ell_{i,s,f}}$ will imply that $\overline{\ell_{i,s,f_k}} \to \overline{\ell_{i,s,f_0}}$. It is not too hard to show that this is equivalent to **(2.)**, to **(3.)** and to **(4.)**.

Also if $\overline{\ell_{i,s,f_k}} \to \overline{\ell_{i,s,f_0}}$ then (3.18) in Lemma 3.7.4 implies that **(1.)** holds. Thus **(1.)**–**(4.)** are all equivalent.   ■

3.5   The Lavaurs map g, and $\langle f_0, g\rangle$, $J(f_0, g)$, $K(f_0, g)$

Proof of Proposition 2.5.1 on page 18 (Partial Lavaurs map) Let $I(z) := -1/\nu z^\nu$ and $q := |I(r_0)| + 4$. Then let

$$
\Omega^+ := \{w \in \mathbb{C} \mid \arg(w - b) < 3\pi/4\}
$$

and $\Omega^- := -\Omega^+$. Now for $k \in \mathbb{Z}/\nu\mathbb{Z}$ let $\Omega_{k,\pm}$ be the component of $I^{-1}(\Omega^\pm)$ which contains $z_{k,\pm}$. (Compare [Sh2].) Notice that $U_{i,\pm,f_0} \subset \Omega_{i,\pm} \subset K_0$.

Now Let $I_{k,\pm} := I|_{\Omega_{k,\pm}}$ and $F^{k,\pm} := I_{k,\pm} \circ f_0 \circ I_{k,\pm}^{-1}$. The map $\Phi^{k,\pm} := \Phi_{k,\pm,f_0} \circ I_{k,\pm}^{-1}$ can be extended to $\Omega^\pm$ by using the relation

$$
\Phi^{k,\pm}(F^{k,\pm}(w)) = \Phi^{k,\pm}(w) + 1.
$$

Now since (by the proof of Lemma 2.2.2) $F^{k,\pm}(w) = w + 1 + \mathcal{O}(w^{-1/\nu})$ as $w \to \infty$ in $\Omega^\pm$ and $F^{k,\pm}(w) \approx w + 1$, we see that Lemma 3.4.1 implies that

$$
\Phi^{k,\pm}(w) = w + c^{k,\pm} + \mathcal{O}(w^{1/\nu}) \tag{3.12}
$$

as $w \to \infty$, for some $c^{k,\pm}$. Also $\left(\Phi^{k,\pm}\right)'(w) \approx 1$ for all $w \in \Omega^\pm$ (assuming that K_0 is small, which implies that w is big).

Let $H := \left(\Phi^{j,-}\right)^{-1} \circ T_{\tilde{\theta}} \circ \Phi^{i,+}$ wherever this is well defined, and $\Gamma := I_{i,+} \circ \gamma_{i,+,f_0}$. We have $\Gamma'(t) \approx i$ when $\Gamma(t) \in \Omega^+$. Thus $\Gamma(t) \in \Omega^+$ for all t, so $\Phi^{i,\pm} \circ \Gamma(t)$ is well defined for all $t \in \mathbb{R}$.

Define $Q^+ := \{w \mid |\arg(w - b')| < 2\pi/3\} = Q^+$ for some large $b' > 0$ and $Q^- := -Q^+$. Then $Q^\pm \subset \Phi^{k,\pm}(\Omega^\pm)$ for all $k \in \mathbb{Z}/\nu\mathbb{Z}$. (Compare [Sh2, §2.2.4].)

Now if $\eta > 0$ is large then by (3.12)

$$
\arg(\operatorname{Re} T_{\tilde{\theta}} \circ \Phi^{i,+} \circ \Gamma(0) + 2b') \in [\tfrac{3\pi}{4}, \tfrac{5\pi}{4}]
$$

and $\left(T_{\tilde{\theta}} \circ \Phi^{i,+} \circ \Gamma\right)'(t) \approx i$ for all $t \in \mathbb{R}$. Thus $T_{\tilde{\theta}} \circ \Phi^{i,+} \circ \Gamma(t) \in Q^-$ for all t, and $H(\Gamma(t))$ is well defined for all $t \in \mathbb{R}$.

So since $h_{i,j,\tilde{\theta}} = I_{j,-}^{-1} \circ H \circ I_{i,+}$ we see that $h_{i,j,\tilde{\theta}}$ is well defined on $\ell_{i,+,f_0}$. And moreover we can show that $h_{i,j,\tilde{\theta}}$ is well defined as a map $S'_{i,+,f_0} \to \Omega_{j,-}$ as required.

By the definition of $\tilde{\tau}_i(f)$, have we

$$
\Phi_{j,-,f}(f^N(z)) = \Phi_{j,-,f}(z) + N = (\Phi_{i,+,f}(z) + \tilde{\tau}_i(f)) + N,
$$

if $f^k(z) \in U_{i,+,f} = U_{j,-,f}$ for $k = 0, \ldots, N$.

As a result, we can we can write

$$
f^N = \Phi_{j,-,f}^{-1} \circ T_{N + \tilde{\tau}_i(f)} \circ \Phi_{i,+,f}.
$$

So clearly if $N_k + \tilde{\tau}_i(f_k) \to \tilde{\theta}$, for some sequence $f_k \to f_0$ in $\mathcal{WB}(\mathsf{G})$ with $\mathsf{G}_i = j$, and some sequence $N_k \to +\infty$, then we get

$$
\begin{aligned}
f_k^{N_k} &= \Phi_{j,-,f_k}^{-1} \circ T_{N_k + \tilde{\tau}_i(f_k)} \circ \Phi_{i,+,f_k} \to \\
&\quad \Phi_{j,-,f_0}^{-1} \circ T_{\tilde{\theta}} \circ \Phi_{i,+,f_0} = h_{i,j,\tilde{\theta}}
\end{aligned}
$$

uniformly on compact sets. Thus (2.3) holds.   ■

Proof of Corollary 2.5.4 on page 19 (Consequence of f_k approaching $\langle f_0, g \rangle$) There is almost immediate from Proposition 2.5.1 and the definition of "f_k approaches $\langle f_0, g \rangle$."   ■

3.6   Return maps and renormalised multipliers

Proof of Proposition 2.6.1 on page 21 (Renormalised multipliers) Suppose that σ^u is a simple fixed point of f at the upper end of $\ell_{j,-,f}$. Then if $\tilde{\mathcal{R}}_f^{(j,u)}$ is as in Lemma 3.4.4 and $\pi(w) = e^{2\pi i w}$ we have $\mathcal{R}_f^{(j,u)} \circ \pi = \pi \circ \tilde{\mathcal{R}}_f^{(j,u)}$. It is clear that "$\tilde{\mathcal{R}}_f^{(j,u)}(+i\infty) = +i\infty$" so we will have $\mathcal{R}_f^{(j,u)}(0) = 0$.

Also if $z = \pi(w)$ we can differentiate both sides of $\mathcal{R}_f^{(j,u)} \circ \pi = \pi \circ \tilde{\mathcal{R}}_f^{(j,u)}$ to get

$$
\begin{aligned}
\big(\mathcal{R}_f^{(j,u)}\big)'(z) \cdot \pi'(w) &= \pi'\big(\tilde{\mathcal{R}}_f^{(j,u)}(w)\big) \cdot \big(\tilde{\mathcal{R}}_f^{(j,u)}\big)'(w) \\
&= \pi'\big(w + \jmath(f, \sigma^u) + o(1)\big) \cdot \big(1 + o(1)\big) \\
&= e^{2\pi i \jmath(f, \sigma^u)} \pi'(w) + o(1)
\end{aligned}
$$

as $z \to 0$ and $\operatorname{Im} w \to +\infty$. (Recall that $\tilde{\mathcal{R}}_f^{(j,u)}(w) - w \to \jmath(f, \sigma^u)$ as $\operatorname{Im} w \to +\infty$.) Thus $\big(\mathcal{R}_f^{(j,u)}\big)'(0) = e^{2\pi i \jmath(f, \sigma^u)}$.

We can prove the statements for $\mathcal{R}_f^{(j,\ell)}$ in the same way.   ■

Proof of Proposition 2.6.2 on page 21 (Limits of return maps and renormalised multipliers) By Lemma 3.4.6, for any $f \in \mathcal{WB}(\mathsf{G})$ we can write $\jmath(f, \sigma_m(f)) = \pm \sum_{k=1}^r \tilde{\tau}_{a_k}(f)$ (for the same $a_1, \ldots, a_r \in \mathbb{Z}/\nu\mathbb{Z}$ used in the Lemma 3.4.6).

Now if f_m approaches $\langle f_0, g \rangle$ (where $\mathsf{gate}(g) = \mathsf{G}$) then $[\tilde{\tau}_{a_k}(f)]_{\mathbb{Z}} \to [\tilde{\theta}_{a_k}]_{\mathbb{Z}}$ as $m \to +\infty$ (where $0 \leqslant k < r$). This means that $[\jmath(f_m, \sigma_i(f))]_{\mathbb{Z}} \to \pm \sum_{k=0}^{r-1} [\tilde{\theta}_{a_k}]_{\mathbb{Z}} = \varphi_i$. Therefore the "$\Rightarrow$" direction of **(1.)** is proved. The "$\Longleftarrow$" direction comes immediately from Theorem 2.4.11. So **(1.)** is proved.

Recall that from Lemma 3.4.6 we have

$$
\tilde{\mathcal{R}}_f^{(j,u)} = T_{\tilde{\tau}_{a_r}(f)} \circ \tilde{\mathcal{E}}_f^{(a_r,u)} \circ T_{\tilde{\tau}_{a_{r-1}}(f)} \circ \tilde{\mathcal{E}}_f^{(a_{r-1},u)} \circ \cdots \circ T_{\tilde{\tau}_{a_1}(f)} \circ \tilde{\mathcal{E}}_f^{(a_1,u)},
$$

so we define

$$
\tilde{\mathcal{R}}_{\langle f_0, g \rangle}^{(j,u)} = T_{\tilde{\theta}_{a_r}} \circ \tilde{\mathcal{E}}_{f_0}^{(a_r,u)} \circ T_{\tilde{\theta}_{a_{r-1}}} \circ \tilde{\mathcal{E}}_{f_0}^{(a_{r-1},u)} \circ \cdots \circ T_{\tilde{\theta}_{a_1}} \circ \tilde{\mathcal{E}}_{f_0}^{(a_1,u)}.
$$

We know that $f \mapsto \tilde{\mathcal{E}}_f^{(j,u)}$ is continuous on $\mathcal{WB}$, for all $j \in \mathbb{Z}/\nu\mathbb{Z}$. Also "everything" commutes with $w \mapsto w + 1$, so it is clear that there exists some $\hat{\mathcal{R}}_f^{(j,u)}$ such that $\hat{\mathcal{R}}_f^{(j,u)}([w]_{\mathbb{Z}}) = [\tilde{\mathcal{R}}_f^{(j,u)}(w)]_{\mathbb{Z}}$, and some $\hat{\mathcal{R}}_{\langle f_0, g \rangle}^{(j,u)}$ such that $\hat{\mathcal{R}}_{\langle f_0, g \rangle}^{(j,u)}([w]_{\mathbb{Z}}) = [\tilde{\mathcal{R}}_{\langle f_0, g \rangle}^{(j,u)}(w)]_{\mathbb{Z}}$ are well defined. And since $\tilde{\mathcal{R}}_{f_k}^{(j,u)} \to \tilde{\mathcal{R}}_{\langle f_0, g \rangle}^{(j,u)}$ as $k \to +\infty$ we must also have that $\hat{\mathcal{R}}_{f_k}^{(j,u)} \to \hat{\mathcal{R}}_{\langle f_0, g \rangle}^{(j,u)}$.

Recall that $[w]_{\mathbb{Z}} \mapsto \pi(w) = e^{2\pi i w}$ is a conformal isomorphism between $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Thus $\hat{\mathcal{R}}_{\langle f_0, g \rangle}^{(j,u)}$ induces a map $\mathcal{R}_{\langle f_0, g \rangle}^{(j,u)}$. It can be checked that this is the limit we require. $\mathcal{R}_{\langle f_0, g \rangle}^{(j,\ell)}$ is defined by analogy, and **(2.)** must hold.

(3.) and **(4.)** follow easily.   ■

3.7 Realising maps with particular gate structures and lifted phases

The main aim of this section is to prove Theorem 2.7.1. However, this is still quite a long way away—a lot more ground work is needed before we can start on that proof.

3.7.1 Sufficient conditions for f to be well behaved

Definition 3.7.1 (Flower(f) and weakly well behaved) *Suppose that $f \in \mathcal{N}_0$. Given $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$, we can let $I_{i,s,f}$ be the largest interval containing 0 such that $\gamma_{i,s,f}(I_{i,s,f}) \cap D_{r_0/2} = \varnothing$. We can then let **Flower(f)** be the closed set bounded away from infinity by the union of $\partial D_{r_0/2}$ and all the $\gamma_{i,s,f}(I_{i,s,f})$. See Figure 3.14.*

*We then say that f is weakly well behaved if all forward and backward trajectories from the points $z_{i,s}$ stay inside the interior of **Flower(f)** once they enter that set. Clearly, if f is well behaved then it is also weakly well behaved. (Note that Figure 2.6 above shows the picture for a weakly well behaved which is not well behaved.)*

If f is weakly well behaved then statements (1.)–(4.) of Proposition 2.3.2 will still hold true. In particular this means that gate(f) will still be defined. (However the fundamental regions $S_{i,s,f}$ may not be well defined.)
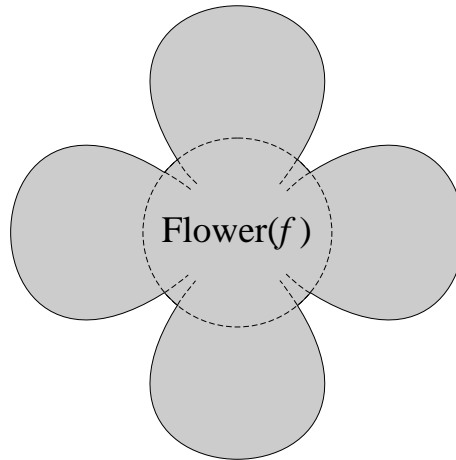
Figure 3.14:

Lemma 3.7.2 (Sufficient conditions for f to be weakly well behaved) *Suppose that $f \in \mathcal{N}_0$. There will be a constant P such that if for every X such that $\varnothing \subsetneq X \subsetneq \mathrm{Fix}(f)$ we have*

$$\left| \mathrm{Im} \sum_{\sigma \in X} \iota(f, \sigma) \right| > P,$$

then f will be weakly well behaved.

Proof. Let $P := 8/r_0^\nu$ and assume f satisfies that condition above. We will assume for contradiction that the forward trajectory $\{\gamma_{i,+,f}(t)\}_{t>0}$ escapes from **Flower(f)**. (The other possibilities we want to rule out can be disproved similarly.) This must mean that it chops **Flower(f)** into halves, and escapes through one of the "exits." (See Figure 3.15.)

As shown in Figure 3.15 we can construct a closed Jordan contour C from trajectories of the form $\{\gamma_{i,\pm,f}(t)\}_{t \in \mathbb{R}}$ and small arcs contained in $\partial D_{r_0/2}$, and so that C winds around one of "half" of $D_{r_0/2}$ (but around no part of the other). Let $p \in \mathbb{N}_0$ be the number such that C will wind around exactly p of the fixed points. It will wind around each of these only once.
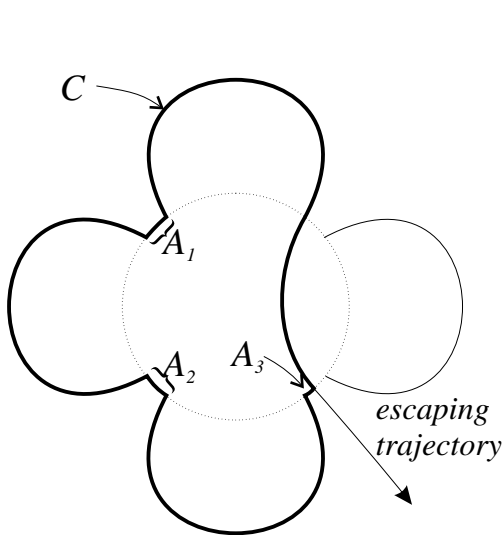
Figure 3.15: *We want to calculate the imaginary part of the integral of $\frac{1}{2\pi i} \cdot \frac{1}{z - f(z)}$ over the path C. This will be equal to the imaginary part of the sum of holomorphic indices of the fixed points around which C winds.*
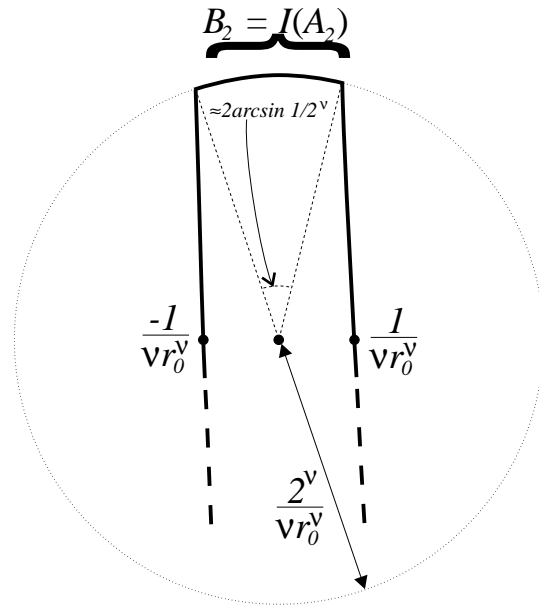
Figure 3.16: *We show the image of $C \cap \{z \mid \arg z \in [-\pi, -\frac{\pi}{2}]\}$ under the change of coordinate $w = I(z) := -\frac{1}{\nu z^\nu}$.*

Case $1 \leqslant p \leqslant \nu$: We are interested in the value of

$$\text{Int} := \frac{1}{2\pi i} \int_C \frac{dz}{z - f(z)}.$$

The condition in the Lemma gives us the estimate

$$|\operatorname{Im} \text{Int}| > P. \tag{3.13}$$

Now we try to calculate an upper bound for $|\operatorname{Im} \text{Int}|$ by integration over C.

As we have seen before

$$\operatorname{Im} \frac{1}{2\pi i} \int_p \frac{dz}{z - f(z)} = \operatorname{Im} \frac{1}{2\pi i} \int_a^b -i \, dt = 0, \tag{3.14}$$

if $p : [a, b] \to K_0$ is a solution of $\dot z = i[f(z) - z]$.

Each of the arcs of $C \cap \partial D_{r_0/2}$ is denoted by $A_r := \{\frac{1}{2} r_0 e^{i\theta} \mid \theta \in [\theta_r^-, \theta_r^+]\}$ where r is between 1 and m for some $m \leqslant 2\nu - 1$. (See Figure 3.15.)

Then we must have

$$\operatorname{Im} \text{Int} = \operatorname{Im} \sum_{r=1}^m \frac{1}{2\pi i} \int_{A_r} \frac{dz}{z - f(z)}. \tag{3.15}$$

We will show that the arcs A_r are all fairly small, and that integrating over them gives a contradiction of (3.13).

We make the change of coordinate $w = I(z) := -1/\nu z^\nu$ on smallish neighbourhoods of A_r and let $B_r = I(A_r)$ to get

$$\text{Int}_r = \frac{1}{2\pi i} \int_{A_r} \frac{dz}{z - f(z)} = \frac{1}{2\pi i} \int_{B_r} K(w) \, dw$$

where $K(w) = z^{\nu+1}/[z - f(z)] \approx 1$. Now since the "length" of B_r can be "no more than roughly" $2\pi \cdot 2^\nu / \nu r_0^\nu \cdot 2 \arcsin \frac{1}{2^\nu}$ (see Figure 3.16) we can be sure that the modulus of Int_r cannot be much more than $2^{\nu+1}/\nu r_0^\nu \cdot \arcsin \frac{1}{2^\nu}$, and certainly less than $4/\nu r_0^\nu$.

Also since there are fewer than 2ν of the arcs A_r, we can see by (3.15) that

$$|\operatorname{Im} \operatorname{Int}| < 2\nu \cdot \frac{4}{\nu r_0^\nu} = \frac{8}{r_0^\nu}. \tag{3.16}$$

This is a contradiction, as required.

Case $p = 0$ or $p = \nu + 1$: If C winds around no fixed points of f, then $\operatorname{Int} = 0$, and if it winds around all $\nu + 1$, then $\operatorname{Int} = \sum_{\sigma \in \operatorname{Fix}(f)} \iota(f, \sigma) \approx \iota(f_0, 0)$.

We can use similar ideas to those above to show that $\operatorname{Re} I$ is large in modulus, and certainly much larger than 0 or $|\iota(f_0, 0)|$. Thus we have another contradiction, and the Lemma is proved. ■

Now let $\alpha \gg 1$. It can be shown quite easily that if $f \in \mathcal{N}_0$ and

$$\rho(f) := \alpha \max_{\sigma = f(\sigma) \in K_0} |\sigma| \tag{3.17}$$

then Lemma 3.3.4 will still be true if (for any $i \in \mathbb{Z}/\nu\mathbb{Z}$) we instead let $R_{i,\pm} = \{z \in K_0 \setminus D_{\rho(f)} \mid |\arg(z/z_{i,\pm})| < 3\pi/4\}$ and $Q_\pm = I(R_{i,\pm})$. Thus we can make the following definition.

Definition 3.7.3 (ρ-well behaved) If $f \in \mathcal{N}_0$ and $\rho \in [\rho(f), \frac{1}{2}r_0]$, then we say that f is *ρ-well behaved* if every backward and forward trajectory through the points $z_{i,\pm}$ stay in the disc D_ρ once they have entered it.

Lemma 3.7.4 (Sufficient conditions for a weakly well behaved f to be ρ-well behaved) Suppose that f is weakly well behaved with gate structure G and $\rho \in [\rho(f), \frac{1}{2}r_0]$. If f is ρ-well behaved, then for every i such that $\mathsf{G}_i \neq \star$, we must have

$$\operatorname{Re} \tilde{\tau}_i(f) < -\frac{1}{3\nu \rho^\nu}. \tag{3.18}$$

Conversely, if for every i with $\mathsf{G}_i \neq \star$ we have

$$\operatorname{Re} \tilde{\tau}_i(f) < -\frac{2}{\nu \rho^\nu}, \tag{3.19}$$

then f will be ρ-well behaved.

Proof. Suppose that $f \in \mathcal{WB}(\mathsf{G})$ and $\mathsf{G}_i = j$. Recall that $\Psi(z) = \int_{z_{i,+}}^z \frac{dz}{f(z)-z}$ is well defined on $U_{i,+,f}$ and that $\Psi(\ell_{i,+,f})$ and $\Psi(\ell_{j,-,f})$ are vertical lines. Also trajectories for $\dot{z} = f(z) - z$ are mapped to horizontal lines by Ψ, so it is clear that there is some $T > 0$ so that $X_T(z_{i,+}) \in \ell_{j,-,f}$ where X_t is the flow for $\dot{z} = f(z) - z$. We first must prove that $T/\operatorname{Re} \tilde{\tau}_i(f) \approx -1$.

Notice that $\{X_t(z_{i,+})\}_{t \in [0,T]}$ chops $D_{r_0/2}$ into two disjoint pieces D^u and D^ℓ, containing $\operatorname{Fix}^u(i, f)$ and $\operatorname{Fix}^\ell(i, f)$ respectively. One can construct a closed Jordan contour C which winds once around the piece of D^u by using $\{X_t(z_{i,+})\}_{t \in [0,T]}$ and some of the lines $\ell_{k,s,f} \setminus D_{r_0/2}$ and some small arcs on $\partial D_{r_0/2}$. Notice that for any fixed point σ of f with $f'(\sigma) \approx 1$ we have $\operatorname{Im} \iota(f, \sigma) \approx -\operatorname{Re} \jmath(f, \sigma)/2\pi$.

We will assume that $\sigma_0(f) \notin \operatorname{Fix}^u(i, f)$. Integrating anti-clockwise over the contour C, Theorem 2.4.11 implies that we must have

$$\begin{aligned}
\operatorname{Imag} := \operatorname{Im} \frac{1}{2\pi i} \int_C \frac{dz}{z - f(z)} &= \operatorname{Im} \sum_{\sigma \in \operatorname{Fix}^u(i,f)} \iota(f, \sigma) \\
&\approx -\frac{1}{2\pi} \operatorname{Re} \sum_{\sigma \in \operatorname{Fix}^u(i,f)} \jmath(f, \sigma) = -\frac{1}{2\pi} \operatorname{Re} \tilde{\tau}_i(f).
\end{aligned} \tag{3.20}$$

We know that integrating over $\ell_{i,s,f}$ will contribute nothing to Imag. (Compare the proof of Lemma 3.7.2.) Also, the arcs which make up $\partial D_{r_0/2} \cap C$ are all very small. Using the same arguments that we did in the proof of Lemma 3.7.2 we can show that if $C_X(t) = X_t(z_{i,+})$ for $t \in [0, T]$ then

$$\operatorname{Imag}_X = \operatorname{Im} \frac{1}{2\pi i} \int_{C_X} \frac{dz}{z - f(z)} = -\frac{1}{2\pi} \operatorname{Re} \int_0^T \frac{f(C_X(t)) - C_X(t)}{C_X(t) - f(C_X(t))} dt = \frac{T}{2\pi}$$
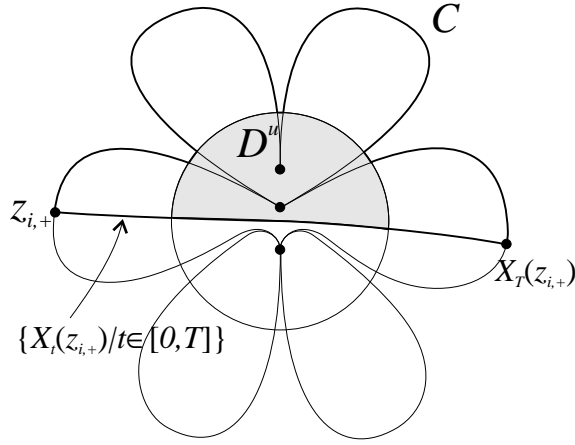
Figure 3.17: *We show the contour C around which we will integrate.*

and that $\mathrm{Imag}_X/\mathrm{Imag} \approx 1$. This and (3.20) imply that $T/\mathrm{Re}\,\tilde{\tau}_i(f) \approx -1$. This is also true when $\sigma_0(f) \in \mathrm{Fix}^u(i,f)$ since we can then integrate around D^ℓ instead D^u, so $T/\mathrm{Re}\,\tilde{\tau}_i(f) \approx -1$ as asserted.

Let $Q^+ := \{w \in I(K_0 \setminus D_{\rho(f)}) \mid |\arg w| < 3\pi/4\}$ and $Q^- := -Q^+$. Now let $R_{k,\pm} \subset K_0$ be the component of $I^{-1}(\Omega^\pm)$ containing $z_{k,\pm}$. Now define $I_{k,s} := I|_{R_{k,s}}$ for $k \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+,-\}$.

If $0 < r_0 \ll 1$ and $\alpha \gg 1$ (see (3.17) on p. 53) then for $z \in K_0 \setminus D_{\rho(f)}$ we have

$$\frac{\partial}{\partial t} I(X_t^f(z)) \approx 1, \tag{3.21}$$

where X_t^f is the time-t map for $\dot{z} = f(z) - z$.

Now if $f \in \mathcal{N}_0$ and $\rho \in (\rho(f), \frac{r_0}{2})$ then $X_t(z_{i,+})$ must enter D_ρ for some least $t \in (0,T)$. Therefore, $I(X_t(z_{i,+}))$ is an almost horizontal line which must leave $\overline{D_{1/\nu\rho^\nu}}$ for some t. Since $I(z_{i,+}) = 1/\nu r_0^\nu \in Q^+$ we see that (3.21) implies that $T > \frac{3}{4} \cdot (\frac{1}{\nu\rho^\nu} - \frac{1}{\nu r_0^\nu})$. So, since $\rho \leqslant \frac{1}{2}r_0$, this means that $\mathrm{Re}\,\tilde{\tau}_i(f) < -\frac{1}{3\nu\rho^\nu}$.

Conversely, suppose that f is weakly well behaved and $\mathrm{Re}\,\tilde{\tau}_i(f) < -\frac{2}{\nu\rho^\nu}$ for every $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that $\mathsf{gate}_i(f) \neq \star$. We want to show that if $\gamma_{k,s,f}(t)$ escapes from D_ρ (for some k,s) then there is a contradiction.

Assume for contradiction that $\gamma_{k,s,f}$ "escapes" from D_ρ. This means that there will be an interval J such that $\gamma_{k,\pm,f}(J)$ is a component of $\ell_{k,s,f} \setminus D_\rho$, which does not contain $z_{k,s}$. There will be some $z_{i,s'}$ such that $\gamma_{k,s,f}(J)$ will be contained in the component of $U_{i,s',f} \setminus D_\rho$ containing $z_{i,s'}$.

To keep notation simple, we will only consider the case where $s' = +$. If $\mathsf{gate}_i(f) = \star$, then it is pretty easy to see that there must be a contradiction, so take the case where $\mathsf{gate}_i(f) = j \neq \star$.

Now since the orbit of any $z \in \ell_{i,+,f}$ must fall in $S_{j,-,f}$ before it falls in $S_{m,+,f}$ or $S_{m,-,f}$ (for any $m \neq j$), there must be another interval J' such that $\gamma_{j,-,f}(J')$ is a component $\ell_{j,-,f} \setminus D_\rho$ which is contained in the same component of $U_{i,+,f} \setminus D_\rho$ as $z_{i,+}$.

Again we take the least $T > 0$ such that $X_T(z_{i,+}) \in \ell_{j,-,f}$. We still have $\mathrm{Re}\,\tilde{\tau}_i(f)/T \approx -1$.

Note that $I(\gamma_{j,-,f}(J'))$ is an "almost vertical line" contained in Q^+ and "to the right of" the component of $I(\ell_{i,+} \cap D_\rho)$ containing $I(z_{i,+})$, which is also an almost vertical line. Take some $z_0 \in \gamma_{j,-,f}(J')$ (close to real line), and let $w_0 = I(z_0)$. Then by (3.21) $I \circ X_t^f(z_0) \in Q^+$ for $t \in [-T,0]$. And since $I(\ell_{i,+,f}) \cap D_{\rho(f)}$ is almost vertical we see that $\mathrm{Re}\,X_{-T}^f(z_0)/|I(r_0)| \approx 1$ and $\mathrm{Re}\,w_0 \leqslant |I(\rho)|$, implying that $T < \frac{3}{2}[I(\rho) - I(r_0)]$. This implies that $\mathrm{Re}\,\tilde{\tau}_i(f) > -\frac{5}{4} \cdot \frac{1}{\nu\rho^\nu}$, which contradicts (3.19). $\blacksquare$

Corollary 3.7.5 *(Sufficient conditions for f to be well behaved)* *Suppose that $f \in \mathcal{N}_0$ where $\mathcal{N}_0$ is a small neighbourhood of f_0, and that $Fix(f) = \{\sigma \in K_0 \mid f(\sigma) = \sigma\}$. There is a constant $M > 0$ such that if for every set X with $\varnothing \subsetneq X \subsetneq Fix(f)$ we have*

$$\left| \text{Im} \sum_{\sigma \in X} \iota(f, \sigma) \right| > M \qquad \left(or \; \left| \text{Re} \sum_{\sigma \in X} \jmath(f, \sigma) \right| > 2\pi \cdot M \right),$$

then $f \in \mathcal{WB}$.

Proof. Lemma 3.7.2 above was proved with the bound $8/r_0^\nu$. We can then apply Lemma 3.7.4 to prove the Proposition. ∎

Remark 3.7.6 *This condition is in no way necessary for f to be well behaved. For instance, if $f_a(z) = z + (z - ai)(z - 2ai)(z + ai)(z + 2ai)$ (where $a > 0$ is small) then by the symmetry we can quite easily show that f_a is well behaved with $\mathsf{gate}(f_a) = (2, 1, 3)$. However, $\text{Im}[\iota(f_a, ai) + \iota(f_a, -ai)] = 0$.*

If $\nu \geqslant 3$ and an explicitly given map $f \in \mathcal{N}$ does not satisfy the conditions in Corollary 3.7.5, then it is not too easy to determine whether or not f is well behaved well, unless we are lucky enough to have some kind of symmetry (like for f_a above).

3.7.2 The ith gate closes up as $\text{Re}\, \tilde{\tau}_i(f) \to -\infty$

The following technical Lemma is only used when proving the main result in this section, Lemma 3.7.8 below.

Lemma 3.7.7 *Suppose that $P_0 > 0$ is large, $f \in \mathcal{N}_0$ is of the form $f(z) = z + (z - \sigma_0) \ldots (z - \sigma_\nu)u(z)$ and that $\sigma_0, \ldots, \sigma_r \in D(a, R)$ (where $1 \leqslant r < \nu$) and $\sigma_{r+1}, \ldots, \sigma_\nu \in K_0 \setminus D(a, RP_0^2)$.*

Now suppose that there are between zero and $2r$ trajectories for $\dot{z} = i[f(z) - z]$ which pass into and then out of $D(a, 2RP_0)$. Each will chop $D(a, RP_0/2)$, and the pieces of the resulting partition of $D(a, RP_0/2)$ will be denoted by $M_1, \ldots, M_s$. Let $S_k = M_k \cap \{\sigma_0, \ldots, \sigma_r\}$.

Then if $\|u - 1\|_{K_0} \leqslant 1/P_0$ we will have (for $k = 1, \ldots, s$)

$$\left| \text{Im} \sum_{\sigma \in S_k} \iota(f, \sigma) \right| < \frac{1}{R^\nu}.$$

Also, the constant $P_0 = P_0(\nu)$ can be chosen so that it only depends on ν.

Proof. We start by making the change of coordinates $w = I(z) := -1/\beta r(z - a)^r$ on $D(a, 2RP_0) \setminus D(a, RP_0)$, where $\beta := (a - \sigma_{r+1}) \ldots (a - \sigma_\nu)u(a)$.

Assume that γ_R and γ_I are trajectories for $\dot{z} = f(z) - z$ and $\dot{z} = i[f(z) - z]$ respectively. It is easily shown that if P_0 is fairly large then $\gamma_R'(t) \approx 1$ if $\gamma_R(t) \in D(a, 2RP_0) \setminus D(a, RP_0)$, and $\gamma_I'(t) \approx i$ if $\gamma_R(t) \in D(a, 2RP_0) \setminus D(a, RP_0)$.

We can in some way consider $f|_{D(a, 2RP_0)}$ in the same way that we have considered $f \in \mathcal{N}_0$ up to now. We can replace K_0 by $\overline{D(a, 2RP_0)}$, and to take the place of $z_{i,\pm}$ we can define $c_{k,-} = a + RP_0 e^{2\pi i(k - \eta)/r}$ and $c_{k,+} = a + RP_0 e^{2\pi i(k + \frac{1}{2} - \eta)/r}$ for $k \in \mathbb{Z}/r\mathbb{Z}$, where $\eta = \arg \beta$. We can then consider the "maximal" trajectories for $\dot{z} = i[f(z) - z]$ which pass through the points $c_{k,\pm}$. (See Figure 3.18.)

To prove the Lemma we need to observe that

$$\text{Int} := \frac{1}{2\pi i} \int_C \frac{dz}{z - f(z)} = \sum_{\sigma \in S_k} \iota(f, \sigma)$$

where C is an anti-clockwise parameterisation of ∂M_k. Also it is integrating over the arcs contained in $\partial M_k \cap \partial D_{r_0/2}$ which can contribute anything to the imaginary part of Int.

In the same way as in Lemma 3.7.2 we construct a new closed Jordan contour C' from sections of $\ell_{i,s,f} \setminus D(a, RP_0)$ and small arcs contained in $\partial D(a, RP_0)$. These arcs will be denoted $A_1, \ldots A_s$ where $s \leqslant 2r$. See Figures 3.18 and 3.19.
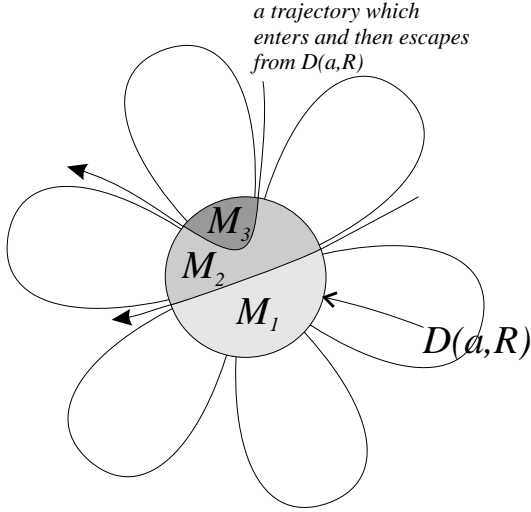
Figure 3.18: *The sets M_1, M_2 and M_3 are shown, and we want a bound for the associated values $\operatorname{Im} Int_1$, $\operatorname{Im} Int_2$ and $\operatorname{Im} Int_3$.*
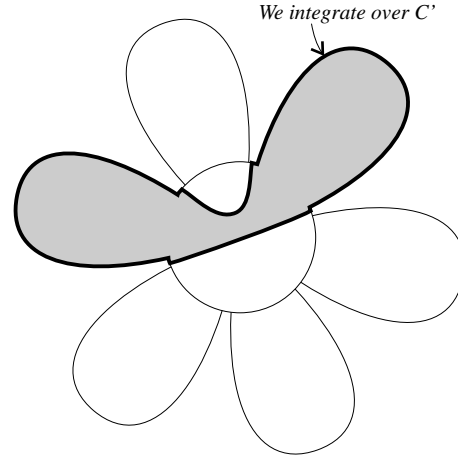
Figure 3.19: *The contour C' winds around the points contained in M_2, so we can integrate over this to give us Int_2.*

By analogy with (3.16) in the proof of Lemma 3.7.2 it can be shown that

$$|\operatorname{Im} Int| < \frac{8}{|\beta|(RP_0)^r}.$$

So because $|\beta| < (RP_0/2)^{\nu-r}$ and $P_0 > 8 \cdot 2^\nu$ we get the required inequality. ∎

The following Lemma is the main one of §3.7.2, and assures us that if the real parts of the lifted phases are all "very negative" then the fixed points must all be very close to one another.

Lemma 3.7.8 (*Closing of the ith gate as* $\operatorname{Re} \tilde{\tau}_i(f) \to -\infty$) *Fix* $\mathsf{G} \in$ Admissible *and an* $i \in \mathbb{Z}/\nu\mathbb{Z}$ *such that* $\mathsf{G}_i \neq \star$. *Then for* $f \in \mathcal{WB}(\mathsf{G})$ *we can then let* $\sigma^u(f) = \gamma_{i,+,f}(+\infty)$ *and* $\sigma^\ell(f) = \gamma_{i,+,f}(-\infty)$. *There is a constant* $C = C(\nu)$ *such that for all* $f \in \mathcal{WB}(\mathsf{G})$

$$|\sigma^u(f) - \sigma^\ell(f)|^\nu \leqslant \frac{C}{|\operatorname{Re} \tilde{\tau}_i(f)|}.$$

Proof. Let $f(z) = z + (z - \sigma_0)\dots(z - \sigma_\nu)u(z)$ where $\sigma_1, \dots, \sigma_\nu \in K_0$. Provided that r_0 was initially chosen small enough and $\mathcal{N}_0$, we can be sure that $|u_f(z) - 1| < 1/P_0$ for all $z \in K_0$ and $f \in \mathcal{N}_0$. We then set $\delta > 0$ so that $\delta^\nu = P_0^{3\nu^2}/|\operatorname{Re} \tilde{\tau}_i(f)|$.

We aim to decompose the set of fixed points into small "clusters" and then show that $\sigma^u(f)$ and $\sigma^\ell(f)$ belong to the same cluster.

For $k = 0, \dots, \nu$ we use the following algorithm to calculate the value $\rho_k > 0$.

1. Let $r := \delta/P_0^{2\nu}$.

2. If $[D(\sigma_k, rP_0^2) \setminus D(\sigma_k, r)] \cap \{\sigma_0, \dots, \sigma_\nu\} = \varnothing$ then go to step (4.).

3. Let $r := r \cdot P_0^2$ and then go back to step (2.).

4. Let $\rho_k := r$ and stop.

Note that for each k, we have $\rho_k \leqslant \delta = (\delta/P_0^{2\nu}) \cdot (P_0^2)^\nu$ since we can go through step (3.) at most ν times.

For $k = 0, \dots, \nu$ we let $S_k = \{\sigma_0, \dots, \sigma_\nu\} \cap D(\sigma_k, \rho_k)$, and note that $D(\sigma_k, \rho_k P_0^2) \setminus D(\sigma_k, \rho_k)$ does not intersect $\{\sigma_0, \dots \sigma_\nu\}$.

Now we need to show that we can define an equivalence relation on $\{\sigma_0, \ldots, \sigma_\nu\}$ by saying $\sigma \sim \sigma'$ if $\sigma, \sigma' \in S_a$ for some $a \in \{0, \ldots, \nu\}$.

Suppose that $s \in S_a \cap S_b$ and $s' \in S_b \setminus S_a$ for some $a, b \in \{0, \ldots, \nu\}$. We need to show that $S_a \subseteq S_b$.

We must have $s \in D(\sigma_a, \rho_a)$ and $s' \notin D(\sigma_a, \rho_a P_0^2)$. Thus $|s - s'| > \rho_a P_0^2 - \rho_a$. Now since $s, s' \in D(\sigma_b, \rho_b)$ we have $|s - s'| < 2\rho_b$. Therefore $\rho_a \ll \rho_a P_0^2 - \rho_a < 2\rho_b$ since $P_0 \gg 1$. However then for any $z \in D(\sigma_a, \rho_a)$

$$|z - \sigma_b| \leqslant |z - s| + |s - s'| + |s' - \sigma_b| \leqslant \rho_a + 2\rho_b + \rho_b \ll \rho_b P_0^2$$

implying that $D(\sigma_a, \rho_a) \subset D(\sigma_b, \rho_b P_0)$. This together with

$$S_a = \{\sigma_0, \ldots, \sigma_\nu\} \cap D(\sigma_a, \rho_a) \qquad\qquad \text{and}$$
$$S_b = \{\sigma_0, \ldots, \sigma_\nu\} \cap D(\sigma_b, \rho_b) = \{\sigma_0, \ldots, \sigma_\nu\} \cap D(\sigma_b, \rho_b P_0^2)$$

tells us that $S_a \subset S_b$ as required. Thus $\sim$ is indeed an equivalence relation.

We can let $A_0, \ldots, A_r$ (where $r \leqslant \nu$) denote the equivalence classes. These we will call the "clusters." Associated to each A_k there will be some $a_k \in K_0$ and $R_k \in (\delta/P_0^{2\nu}, \delta)$ such that $A_k = D(a_k, R_k) \cap \{\sigma_0, \ldots, \sigma_\nu\}$ and $D(a_k, R_k P_0^2) \setminus D(a_k, R_k)$ contains none of the fixed points $\sigma_0, \ldots, \sigma_\nu$.

So each cluster A_k has diameter at most 2δ. We now have to show that there is some k such that $\sigma^u(f), \sigma^\ell(f) \in A_k$.
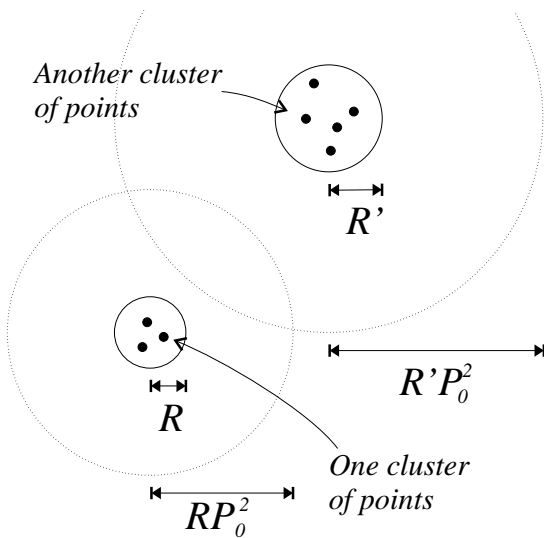
Figure 3.20: *We show two clusters of points. The first cluster is contained in a disc of radius R, and these are the only marked points in a larger disc of radius $R P_0^2$ with the same centre. The second cluster is contained in a disc of radius R', and these are the only marked points in a larger disc of radius $R' P_0^2$.*
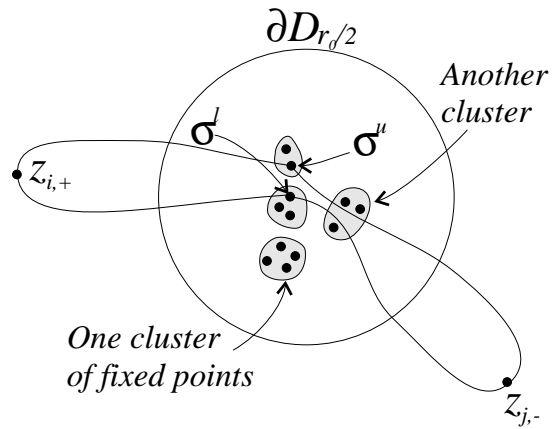
Figure 3.21: *The ith gate of f is shown, with the fixed points of f separated into clusters, each of very small diameter. Actually if we suppose that σ^u and σ^ℓ belong to different clusters then this will lead to a contradiction.*

Assume for contradiction that $\sigma^u(f)$ and $\sigma^\ell(f)$ do not belong to the same cluster. We will try to calculate an upper bound for $|\operatorname{Re} \tilde{\tau}_i(f)|$.

For each k we consider the sets $A_k^u = \operatorname{Fix}^u(i, f) \cap A_k$ and $A_k^\ell = \operatorname{Fix}^\ell(i, f) \cap A_k$.

Notice that it is basically the trajectories $\gamma_{i,+,f}$ and $\gamma_{j,-,f}$ which separate $\operatorname{Fix}^u(i, f)$ from $\operatorname{Fix}^\ell(i, f)$. We can then partition $D(a_k, R_k)$ into pieces by chopping it using trajectories for $\dot{z} = i[f(z) - z]$, and so that each piece contains only elements of A_k^u, or only elements of A_k^ℓ. We can do this so that $D(a_k, R_k)$ is chopped into at most ν pieces. We now apply Lemma 3.7.7 to show that $|\operatorname{Im} \sum_{\sigma \in A_k^u} \iota(f, \sigma)| < \nu/R_k^\nu$.

Since there are no more than ν of these sets A_k^u, when we sum up we must have

$$\left| \text{Im} \sum_{\sigma \in \text{Fix}^u(i,f)} \iota(f,\sigma) \right| < \nu^2/R_k^\nu,$$

which implies that $|\text{Re}\,\tilde{\tau}_i(f)| < 8\nu^2/R_k^\nu$ by Theorem 2.4.11 and the fact that "$\text{Re}\,\jmath(f,\sigma) \approx -2\pi\,\text{Im}\,\iota(f,\sigma)$." Now since $1/R_k^\nu < P_0^{2\nu^2}/\delta^\nu$ this implies that $|\text{Re}\,\tilde{\tau}_i(f)| < 8\nu^2 \cdot P_0^{2\nu^2}/\delta^\nu$.

However since we can assume that $P_0 > 8\nu^2$, substituting $\delta^\nu = P_0^{3\nu^2}/|\text{Re}\,\tilde{\tau}_i(f)|$ implies that $|\text{Re}\,\tilde{\tau}_i(f)| < |\text{Re}\,\tilde{\tau}_i(f)|$ which is a clear contradiction.

Therefore our assumption was wrong, and $\sigma^u(f)$ and $\sigma^\ell(f)$ must both lie in the same cluster. And since clusters have size at most 2δ the Lemma has been proved where $C(\nu) = 2^\nu[P_0(\nu)]^{3\nu^2}$. ∎

3.7.3   Realising a map with a particular gate structure

In this section we will want to show that we can realise an $f \in \mathcal{WB}(\mathsf{G})$ for an arbitrary $\mathsf{G} \in \mathsf{Admissible}$. But first we will prove this in the simplest case where G has only one open gate.

The following Lemma is needed.

Lemma 3.7.9 (The value of the holomorphic index) If σ is a multiplicity-r fixed point of f, then

$$\iota(f,\sigma) = \lim_{z \to \sigma} \left\{ \frac{1}{(r-1)!} \frac{d^{r-1}}{dz^{r-1}} \frac{(z-\sigma)^r}{z-f(z)} \right\}. \tag{3.22}$$

Proof. Notice that the holomorphic index is related to the Cauchy residue by $\iota(f,\sigma) = \text{res}(\frac{1}{z-f(z)}, \sigma)$. The formula for calculating the Cauchy residue (see [ST, Lemma 12.3]) gives us (3.22). ∎

Lemma 3.7.10 Suppose that G has a single open gate, with $\mathsf{G}_i = j$, where $i, j \in \mathbb{Z}/\nu\mathbb{Z}$.

Then we can find an $f \in \mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ of the form $f(z) = z + z^r(z-\sigma)^{\nu-r+1}u(z)$ (for some u close to u_{f_0}) arbitrarily close to f_0, and such that $|\text{Re}\,\jmath(f,0)| \approx |\text{Re}\,\jmath(f,\sigma)|$ is arbitrarily large.

Proof. Recall the significance of the holomorphic family $\{v_\mathbf{s}\}_\mathbf{s}$ from §2.7. Let $u_\sigma := v_{(\sigma,0,\ldots,0)}$ and $h_\sigma(z) = z + z^r(z-\sigma)^{\nu-r+1}u_\sigma(z)$, where $1 \leqslant r \leqslant \nu$ and $r + \nu\mathbb{Z} = i - j + \nu\mathbb{Z}$. Then clearly $h_\sigma \to f_0$ as $\sigma \to 0$ and by Lemma 3.7.9 we have that

$$\iota(h_\sigma, 0) = \frac{B}{\sigma^\nu}(1 + o(1)) \qquad \text{as } \sigma \to 0, \tag{3.23}$$

for some $B \in \mathbb{Q} \setminus \{0\}$.

Take $N \gg 0$ and let

$$s = \left(-\frac{2\pi i B}{N} \right)^{\frac{1}{\nu}}$$

where $\arg s \in (0, 2\pi/\nu)$. (3.23) implies that $\iota(h_s, 0) = -N/2\pi i \cdot (1 + o(1))$ as $N \to +\infty$, which implies that $\jmath(h_s, 0) = -N(1 + o(1))$. Similarly, $\jmath(h_s, s) = N(1 + o(1))$.

Corollary 3.7.5 then ensures that h_s is indeed well behaved.

However, we cannot be sure that $\mathsf{gate}(h_s) = \mathsf{G}$ because we had a choice of ν different roots when we defined s.

It is clear that 0 is a multiplicity-r fixed point and has r attracting and r repelling directions.

Because there is only one open gate the arrangement of fundamental regions is very simple. Also $\text{mult}(h_s, 0) = r$ and it can be shown that there will be an $m \in \mathbb{Z}/\nu\mathbb{Z}$ so that the r repelling directions at 0 will be "contained" in the petals

$$U_{m,-,h_s}, U_{m-1,-,h_s}, \ldots, U_{m-r+1,-,h_s}$$

and so that the r attracting directions at σ will be contained in

$$U_{m-1,+,h_s}, U_{m-2,+,p}, \ldots, U_{m-r,+,h_s}.$$

We must then have $\mathsf{gate}_m(h_s) = m - r$ (modulo ν).

However, we can "rotate" the picture so that we do get the single open gate in the right place. To do this we let $\sigma = e^{2\pi i(i-m)/\nu}s$ and then let $f = h_\sigma$. It can be checked that $\mathsf{gate}(f) = \mathsf{G}$, and we still have that $\jmath(h_\sigma, 0) = -N(1 + o(1))$ and $\jmath(h_\sigma, \sigma) = N(1 + o(1))$ as required. ∎

The following technical Lemma is needed in the proof of Lemma 3.7.12 when we want to apply Corollary 3.7.5.

> **Lemma 3.7.11** *Suppose that $M \gg 1$ and $a_0, \ldots, a_r \in \mathbb{R} \setminus \{0\}$ have $|a_1| > M|a_2|, \ldots, |a_{r-1}| > M|a_r|$, and $M|a_0 + \cdots + a_r| < |a_r|$. Then for any X such that $\varnothing \subsetneq X \subsetneq \{0, \ldots, r\}$ we must have*
> $$\left| \sum_{i \in X} a_i \right| \geq |a_r|$$

Proof. This fairly easy to show (using $|x + y| > \big||x| - |y|\big|$ for $x, y \in \mathbb{R}$). ∎

Now we can prove the main Lemma of §3.7.3.

> **Lemma 3.7.12 (An $f \in \mathcal{WB}(\mathsf{G})$ can be realised)** *Suppose that $\mathsf{G} \in$ Admissible. Then we can find an $f \in \mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ arbitrarily close to f_0 such that $\operatorname{Re} \tilde{\tau}_i(f) \ll 0$ is "arbitrarily negative" for each i with $\mathsf{G}_i \neq \star$.*

Proof. We take one of the $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that $\mathsf{G}_i = j \neq \star$. Then we proceed as in the proof of Proposition 3.7.10, to give us a well behaved $g_1(z) = z + z^r(z - \lambda)^{\nu-r+1}u_{g_1}(z)$ with $\operatorname{Re} \jmath(g_1, 0) \ll 0$ and $\operatorname{Re} \jmath(g_1, \lambda) \gg 0$. Thus $\tilde{\tau}_i(g_1)$ is arbitrarily large.

We then take an $i' \neq i$ with $\mathsf{G}_{i'} = j' \neq \star$ (if there is one left). $\ell_{i',+,g_1}$ will have the same multiplicity-m fixed point, σ, at both its ends, where σ is either 0 or λ. We restrict $g_1 : K_0 \to \mathbb{C}$ to a small neighbourhood K_{h_1} of σ, to give us $h_1 : K_{h_1} \to \mathbb{C}$ of the form $h_1(z) = z + a(z - \sigma)^m u_{h_1}(z)$ for $m = \mathrm{mult}(g_1, \sigma)$, and some $a \in \mathbb{C} \setminus \{0\}$, where $u_{h_1}(z) \approx 1$ for all $z \in K_{h_1}$. (See Figure 3.22.)

h_1 can be treated in exactly the same way as f_0, and there will be m attracting and m repelling directions for the multiple point σ. There will be corresponding fundamental regions for h_1 contained within K_{h_1}.

$\ell_{i',+,g_1}$ has both its ends at σ and there will be a corresponding $\ell_{i,s,h_1} \subset K_{h_1}$ "inside" it, such that $\ell_{i,s,h_1} \cup f(\ell_{i,\pm,h_1}) \cup \{\sigma\}$ is the boundary of one of the fundamental regions for h_1. Also $\ell_{j',-,g_1}$ will have a corresponding $\ell_{j',-,h_1}$ inside it.

Again using Lemma 3.7.10 we split apart the fixed point σ of h_1 to get an $h_2 : K_{h_1} \to \mathbb{C}$ with two new fixed points in K_{h_1}, and so that the closure of $\ell_{i',+,h_2} \cup \ell_{j',-,h_2}$ is homeomorphic to a circle. See Figure 3.22.

There is a corresponding g_2 which is an extension of h_2 such that g_2 and g_1 share a fixed point outside K_{h_1}, and such that g_2 has two fixed points in K_{h_1}. We can denote by s_0, s_1 the fixed points in K_{h_1}, and by s_2 the fixed point in $K_0 \setminus K_{h_1}$.

If g_2 is close enough to g_1, then $\jmath(g_1, s_2) \approx \jmath(g_2, s_2)$ (by Lemma 2.4.8). Also we can certainly make sure that $|\operatorname{Re} \jmath(g_2, s_1)| \gg |\operatorname{Re} \jmath(g_2, s_2)|$, and it is clear that $|\operatorname{Re} \jmath(g_2, s_0) + \operatorname{Re} \jmath(g_2, s_1) + \operatorname{Re} \jmath(g_2, s_2)| \approx |\operatorname{Re} \jmath(f_0, 0)| \ll |\operatorname{Re} \jmath(g_2, s_2)|$. Therefore Lemma 3.7.11 can be applied, allowing us to apply Corollary 3.7.5. Thus g_2 is well behaved. Using Lemma 3.3.11 we can deduce that $\mathsf{gate}_i(g_2) = \mathsf{gate}_i(g_1) = j$, and that $\mathsf{gate}_{i'}(g_2) = j'$ as required, and these are the only open gates of $\mathsf{gate}(g_2)$. Lemma 3.7.11, Lemma 3.7.4 and Theorem 2.4.11 imply that $\operatorname{Re} \tilde{\tau}_{i'}(g_1) \ll 0$ and $\operatorname{Re} \tilde{\tau}_i(g_1) \ll 0$.

We can continue splitting the remaining multiple fixed points until we obtain a g_r (where G has r open gates) with the gate structure desired. As long as each successive perturbation was much smaller than the previous ones, g_r will indeed be well behaved (again using Lemma 3.7.11 and Corollary 3.7.5), and the real parts of all the lifted phases will be "very negative."

Also by being a bit more careful we can make sure that $f \in \mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$. ∎
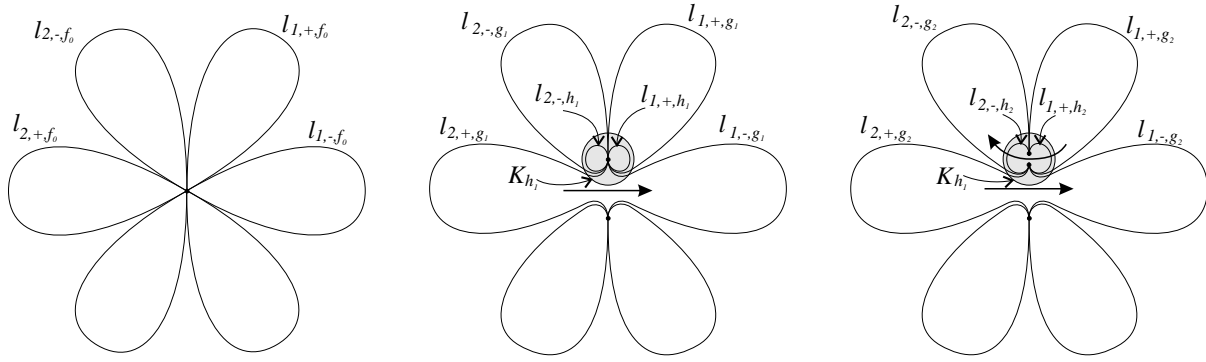
Figure 3.22: *On the left we have the lines $\ell_{i,\pm,f_0}$. In the middle we have split the 0 to give two double fixed points, and we show all the $\ell_{i,\pm,g_1}$ and $\ell_{1,+,h_1}, \ell_{1,-,h_1} \subset K_{h_1}$. On the right we have split the upper double fixed point apart to give us a second gate, and we show the lines $\ell_{i,\pm,g_2}$ and $\ell_{1,+,h_2}, \ell_{1,-,h_2} \subset K_{h_1}$. We have $\mathsf{gate}(f_0) = (\star, \star, \star)$, $\mathsf{gate}(g_1) = (\star, 1, \star)$ and $\mathsf{gate}(g_2) = (2, 1, \star)$.*

3.7.4   The Jacobian is non-zero when G has no closed gates

The following Lemma corresponds to a result in [DES] which states that "if two $(\nu+1)$-degree polynomials V_1, V_2 are normalised suitably, have the same combinatorics and the same integral invariants, then $V_1 = V_2$."

> **Lemma 3.7.13 (Uniqueness of holomorphic indices for $(\nu+1)$-degree polynomials)**
> *Suppose that f_1, f_2 are weakly well behaved, $\mathsf{G} = \mathsf{gate}(f_1) = \mathsf{gate}(f_2)$, $\sigma_0(f_1) = \sigma_1(f_2) = 0$ and $u_{f_1} = u_{f_2} = 1$. If $\iota(f_1, \sigma_k(f_1)) = \iota(f_2, \sigma_k(f_2))$ for every $k = 1, \ldots, r$ (where G has r open gates) then $f_1 = f_2$.*

| (a) | (b) | (c) |
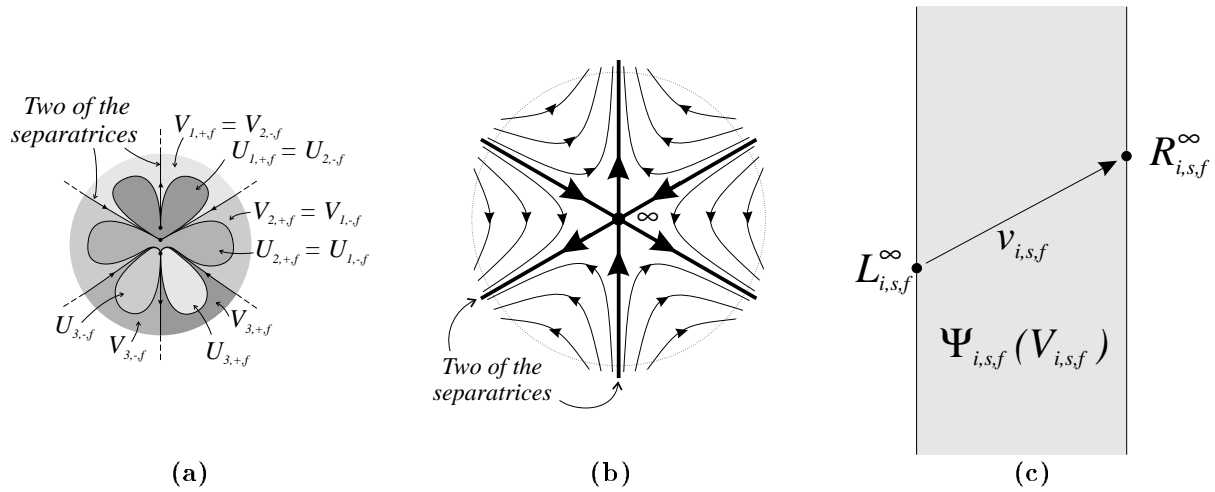
Figure 3.23: *In (a) we show the sets $V_{i,s,f}$ (where $i \in \mathbb{Z}/3\mathbb{Z}$ and $s \in \{+, -\}$) in a neighbourhood of 0, for some $f \in \mathcal{WB}(\mathsf{G})$ where $\mathsf{G} = (2, 1, \star)$. $\{V_{i,s,f}\}_{i,s}$ is a partition of $\mathbb{C}$, and for each $i \in \mathbb{Z}/3\mathbb{Z}$ and $s \in \{+, -\}$ the set $V_{i,s,f}$ contains $U_{i,s,f}$ defined earlier. The boundary of each $V_{i,s,f}$ is a union of separatrices. In (b) we show trajectories for the vector field $\dot{z} = i[f(z) - z]$ in a neighbourhood of ∞, with the separatrices emphasised. In (c) we show the image of $V_{i,s,f}$ under the straightening coordinate $\Psi_{i,s,f}$ (in the case where $s = +$ and $\mathsf{gate}_i(f) \neq \star$, or where $s = -$ and $\mathsf{gate}_j(f) = i$ for some $j \in \mathbb{Z}/\nu\mathbb{Z}$). $v_{i,s,f}$ is the vector between the points $L^\infty_{i,s,f}$ and $R^\infty_{i,s,f}$.*

Proof.   Since $u_{f_1} = u_{f_2} = 1$, the functions f_1, f_2 are weakly well behaved $(\nu + 1)$-degree polynomials defined on the whole of $\mathbb{C}$. Thus we can tackle the problem from the point of view of [DES]. (We will not go through all the details.)

For a $(\nu + 1)$-degree polynomial f and a $z \in \mathbb{C}$ we can let $\gamma_z : I_z \to \mathbb{C}$ be the maximal solution of $\dot{z} = i[f(z) - z]$ on $\mathbb{C}$. (See Definition 2.2.1.) Then $\sup I = +\infty$ if and only if $|\gamma_z(t)| \to +\infty$ as $t \to +\infty$ and $\inf I = -\infty$ if and only if $|\gamma_z(t)| \to +\infty$ as $t \to -\infty$. We can then let $\mathrm{Traj}(z) = \overline{\gamma_z(I_z)} \subset \overline{\mathbb{C}}$.

Because of the topological picture of the $\ell_{i,s,f}$'s (which is ensured by Proposition 2.3.2) and the fact that "trajectories cannot cross," it can be shown that there are no limit cycles in $\mathbb{C}$ and no "homoclinic links" (i.e. trajectories $\gamma : I \to \mathbb{C}$ such that $|\gamma(t)| \to +\infty$ as $t \to +\infty$ and $t \to -\infty$.) Also, for any $z \in \mathbb{C}$ there are fixed points $\sigma_-, \sigma_+ \in \overline{\mathbb{C}}$ such that $\gamma_z(t) \to \sigma_\pm$ as $t \to \pm\infty$.

If $\infty \notin \mathrm{Traj}(z_0)$ then $z \mapsto \mathrm{Traj}(z)$ is continuous in a neighbourhood of z_0 with respect the Hausdorff metric. And in fact if z_1 is close to z_0 then there are some small $\alpha, \beta \in \mathbb{R}$ such that $\gamma_{z_1} = X_\alpha \circ Y_\beta \circ \gamma_{z_0}$.

Where $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$ we can let

$$a_{i,s} := \inf\{r_0 < 0 \mid \infty \notin \mathrm{Traj}(X_s(z_{i,s})) \ \forall \ r \in [r_0, 0]\}$$

$$b_{i,s} := \sup\{r_0 > 0 \mid \infty \notin \mathrm{Traj}(X_s(z_{i,s})) \ \forall \ r \in [0, r_0]\}$$

(where we allow $a_{i,s} = -\infty$ or $b_{i,s} = +\infty$). Then we let

$$V_{i,s,f} := \bigcup_{r \in (a_{i,s}, b_{i,s})} X_r(\ell_{i,s,f})$$

If $\mathsf{G}_i \neq \star$ and $s \in \{+, -\}$ then there will be points $a_1, \ldots, a_4 \in \mathbb{C}$ so that $\partial V_{i,s,f} = \mathrm{Traj}(a_1) \cup \cdots \cup \mathrm{Traj}(a_4)$. We call these $\mathrm{Traj}(a_k)$ *separatrices* and each will contain ∞ at one end, and either $\gamma_{i,s,f}(+\infty)$ or $\gamma_{i,s,f}(-\infty)$ at the other. If $s = +$ and $\mathsf{G}_i = \star$ (or if $s = -$ and $\mathsf{G}_j \neq i$ for all $i \in \mathbb{Z}/\nu\mathbb{Z}$) then $V_{i,s,f}$ is bounded by two separatrices, both with ∞ at one end, and with $\gamma_{i,s,f}(+\infty) = \gamma_{i,s,f}(-\infty)$ at the other. (There are exactly 2ν separatrices, and these are "almost asymptotic at ∞" to one of the lines $e^{\pi i(2k+1)/2\nu}\mathbb{R}_+$ for some $k = 1, \ldots, 2\nu$. See [DES].)

It can be shown that $U_{i,s,f} \subset V_{i,s,f}$ and $V_{i,+,f} = V_{j,-,f}$ iff $U_{i,+,f} = U_{j,-,f}$ iff $\mathsf{G}_i = j \neq \star$. We find there is a separatrix contained in the boundaries of both $V_{i,+,f}$ and $V_{i,-,f}$ which links $\gamma_{i,+,f}(+\infty) = \gamma_{i,-,f}(+\infty)$ to ∞. Also $V_{i,+,f}$ and $V_{i-1,-,f}$ will share a separatrix which links $\gamma_{i,+,f}(-\infty) = \gamma_{i-1,-,f}(-\infty)$ to ∞. We can show that the union of the closures of the $V_{i,s,f}$ is the whole complex sphere, and that $V_{i,s,f} \cap V_{i',s',f} \neq \varnothing$ only if $V_{i,s,f} = V_{i',s',f}$. Therefore we have a partition of $\overline{\mathbb{C}}$.

We can then define $\Psi_{i,s,f} : V_{i,s,f} \to \mathbb{C}$ to be a straightening coordinate (or approximate Fatou coordinate) of the form

$$w = \Psi_{i,s,f}(z) := \int_{z_{i,s}}^{z} \frac{d\zeta}{f(\zeta) - \zeta}$$

where we only integrate over paths in $V_{i,s,f}$. If $\mathsf{G}_i \neq \star$ then $\Psi_{i,s,f}(V_{i,s,f})$ will be an "open" vertical strip. On the left and right boundary lines of the strip $\Psi_{i,s,f}(V_{i,s,f})$ there will be points $L^\infty_{i,s,f}$ and $R^\infty_{i,s,f}$ which correspond to ∞ back in the z-coordinate. (That is to say, if $\{w_k\}_k \subset \Psi_{i,s,f}(V_{i,s,f})$ and $w_k \to L^\infty_{i,s,f}$ or $w_k \to R^\infty_{i,s,f}$, then $\Psi^{-1}_{i,s,f}(w_k) \to \infty$.)

We will now show that the vector from the left "infinity" point $L^\infty_{i,s,f}$ to the right "infinity" point $R^\infty_{i,s,f}$ must be given by

$$v_{i,s,f} := R^\infty_{i,s,f} - L^\infty_{i,s,f} = -2\pi i \sum_{\sigma \in \mathrm{Fix}^u(i,f)} \iota(f, \sigma).$$

Suppose that $\mathrm{Im}[R^\infty_{i,s,f} - L^\infty_{i,s,f}] \geqslant 0$. Then we let $P : (0, S) \to \mathbb{C}$ be defined as $P(s) = L^\infty_{i,s,f} + s$ where $S = \mathrm{Re}[R^\infty_{i,s,f} - L^\infty_{i,s,f}]$, and $Q : (0, T) \to \mathbb{C}$ where $T = \mathrm{Im}[R^\infty_{i,s,f} - L^\infty_{i,s,f}]$ be defined as $Q(t) = P(S) + it$. If $p = \Psi^{-1}_{i,s,f} \circ P : (0, S) \to \mathbb{C}$ and $q = \Psi^{-1}_{i,s,f} \circ Q : (0, T) \to \mathbb{C}$ then these paths will be trajectories for the vector fields $\dot{z} = f(z) - z$ and $\dot{z} = i[f(z) - z]$ respectively.

Also $\lim_{s \to 0} p(s) = \infty$, $\lim_{s \to S} p(s) = \lim_{t \to 0} q(t)$ and $\lim_{t \to T} q(t) = \infty$. Thus we can basically stick the paths p and q together to give a closed Jordan contour $C : [0, S + T] \to \overline{\mathbb{C}}$ such that $C(r) = p(r)$ if $r \in (0, S)$ and $C(r) = q(r - S)$ if $r \in (S, S + T)$. (Also $C(0) = C(S + T) = \infty$.) Then it is easily shown that

$$\mathrm{Int} = \frac{1}{2\pi i} \int_C \frac{d\zeta}{z - f(\zeta)} = -\frac{S + iT}{2\pi i} = -\frac{R^\infty_{i,s,f} - L^\infty_{i,s,f}}{2\pi i}$$

This contour C winds once anti-clockwise around each element of $\mathrm{Fix}^u(i, f)$. It is easily shown (by the theory of residues) that

$$\mathrm{Int} = \sum_{\sigma \in \mathrm{Fix}^u(i,f)} \iota(f, \sigma)$$

We can show this in much the same way when $\mathrm{Im}[R_{i,s,f}^\infty - L_{i,s,f}^\infty] \leqslant 0$.

So now we see that for f_1, f_2 with the same gate structure and same holomorphic indices, we must have $v_{i,s,f_1} = v_{i,s,f_2}$ for all $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $s \in \{+, -\}$. (Also, if $\mathsf{G}_i = j \neq \star$ then $v_{i,+,f_1} = v_{j,-,f_1}$ and $v_{i,+,f_2} = v_{j,-,f_2}$.)

Thus for each element of the partition $\{V_{i,s,f_1}\}$ such that $\Psi_{i,s,f_1}(V_{i,s,f_1})$ is a strip (as opposed to a half-plane) the translation $T_{i,s}(w) = w + \left(L_{i,s,f_2}^\infty - L_{i,s,f_1}^\infty\right)$ maps the strip $\Psi_{i,s,f_1}(V_{i,s,f_1})$ onto the strip $\Psi_{i,s,f_2}(V_{i,s,f_2})$, and maps L_{i,s,f_1}^∞ to L_{i,s,f_2}^∞, and R_{i,s,f_1}^∞ to R_{i,s,f_2}^∞. Similarly if $\Psi_{i,s,f_1}(V_{i,s,f_1})$ is a half plane, then there is a $T_{i,s}$ which either maps L_{i,s,f_1}^∞ to L_{i,s,f_2}^∞ or maps R_{i,s,f_1}^∞ to R_{i,s,f_2}^∞.

As a result for any i, s there is a "natural" way to construct conformal maps $h_{i,s} : V_{i,s,f_1} \to V_{i,s,f_2}$ by setting $h_{i,s} = \Psi_{i,s,f_2}^{-1} \circ T_{i,s} \circ \Psi_{i,s,f_1}$. By their construction these satisfy $h_{i,s}(\gamma_{i,s,f_1}(+\infty)) = \gamma_{i,s,f_2}(+\infty)$ and $h_{i,s}(\gamma_{i,s,f_1}(-\infty)) = \gamma_{i,s,f_2}(-\infty)$.

These $h_{i,s}$ can be patched together and extended analytically to the whole of $\mathbb{C}$. This gives us a conformal mapping $I : \mathbb{C} \to \mathbb{C}$ fixing 0, such that (for any i, s) $I(V_{i,s,f_1}) = V_{i,s,f_2}$ and by the construction we will have $\sigma_k(f_2) = I(\sigma_k(f_1))$ for $k = 1, \ldots, r$ (where G has r open gates).

However any conformal map $\mathbb{C} \to \mathbb{C}$ is an affine map. Thus $I(z) = \alpha z$ for some $\alpha \in \mathbb{C} \setminus \{0\}$ since $I(0) = 0$. We want to show that $\alpha = 1$.

We can use Lemma 3.7.9 to show that there are some heterogeneous polynomials (see Definition 3.7.15 below) $P(\cdot), Q(\cdot) \in \mathbb{C}[S_1, \ldots, S_r]$ so that

$$\iota(f_1, 0) = \frac{P\big(\sigma_1(f_1), \ldots, \sigma_r(f_1)\big)}{Q\big(\sigma_1(f_1), \ldots, \sigma_r(f_1)\big)} \qquad \text{and} \qquad \iota(f_2, 0) = \frac{P\big(\sigma_1(f_2), \ldots, \sigma_r(f_2)\big)}{Q\big(\sigma_1(f_2), \ldots, \sigma_r(f_2)\big)}.$$

Also $\deg Q - \deg P = \nu$. So since $\sigma_k(f_2) = \alpha\sigma_k(f_1)$ (for $k = 0, \ldots, r$) we see that $\iota(f_2, 0) = \frac{1}{\alpha^\nu}\iota(f_1, 0)$. And since we assumed that $\iota(f_1, \sigma_k(f_1)) = \iota(f_2, \sigma_k(f_2))$ for all k, we see that $\alpha^\nu = 1$.

But by its construction I must map each $z_{i,s} \in V_{i,s,f_1}$ to some point in V_{i,s,f_2}. Recall the definition of the points $z_{i,s}$. It is clear that (since $\alpha^\nu = 1$) $\alpha \neq 1$ would imply that $I(z_{i,s}) = z_{j,s} \notin V_{i,s,f_2}$ for some $j \neq i$, which is a contradiction.

Thus $\alpha = 1$, implying that $f_1 = f_2$ as required. ∎

Corollary 3.7.14 (*The Jacobian* $\mathrm{Jac}_\mathbf{h}(\mathbf{s})$ *is non-zero if* $u = 1$) *Let* $\mathsf{G} \in \mathrm{Admissible}$ *have no closed gates,* $\mathbf{s} = (s_1, \ldots, s_r) \in K_0^r$ *and* $f_\mathbf{s}(z) = z + z(z - s_1)\ldots(z - s_\nu)$. *Now define*

$$\mathbf{h}(\mathbf{s}) = \big(h_1(\mathbf{s}), \ldots, h_\nu(\mathbf{s})\big) := \big(f_\mathbf{s}'(s_1), \ldots, f_\mathbf{s}'(s_\nu)\big).$$

If $0, s_1, \ldots, s_r$ *are all distinct (and* $f_\mathbf{s} \in \mathcal{WB}(\mathsf{G})$*) then*

$$\mathrm{Jac}_\mathbf{h}(\mathbf{s}) = \det \left(\frac{\partial h_i}{\partial s_j}(\mathbf{s})\right)_{\substack{1 \leqslant j \leqslant \nu \\ 1 \leqslant j \leqslant \nu}} \neq 0.$$

Proof. Supposing for contradiction that $\mathrm{Jac}_\mathbf{h}(\mathbf{s}) = 0$, then $\mathbf{h}$ is not injective in any neighbourhood of $\mathbf{s}$, so there are some distinct $\mathbf{s}^a, \mathbf{s}^b \in \mathbb{C}^\nu$ arbitrarily close to $\mathbf{s}$ such that $\mathbf{h}(\mathbf{s}^a) = \mathbf{h}(\mathbf{s}^b)$. But since $\{\mathbf{s} \in \mathbb{C}^\nu \mid f_\mathbf{s} \in \mathcal{WB}(\mathsf{G})\}$ is open (by Proposition 2.3.10) we can also assume that $f_{\mathbf{s}^a}, f_{\mathbf{s}^b} \in \mathcal{WB}(\mathsf{G})$. But this would contradict Lemma 3.7.13 above. ∎

Definition 3.7.15 (The polynomial ring, $\mathbb{C}[Z_1, \ldots, Z_n]$) *A polynomial $P \in \mathbb{C}[Z_1, \ldots, Z_n] = \mathbb{C}[\mathbf{Z}]$ is of the form*

$$P(Z_1, \ldots, Z_n) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} a_{i_1 \ldots i_n} Z_1^{i_1} \cdots Z_n^{i_n}$$

where the coefficients $a_{i_1 \ldots i_n}$ are all in $\mathbb{C}$, and $Z_1, \ldots, Z_n$ are "indeterminates." $\mathbb{C}[\mathbf{Z}]$ forms a ring where addition and multiplication are defined in the natural way. Each $P \in \mathbb{C}[\mathbf{Z}]$ can be treated as a map $\mathbb{C}^n \to \mathbb{C}$, so for $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we let $P(\mathbf{z}) \in \mathbb{C}^n$ be the evaluation of P at $\mathbf{z}$. Also, the degree of P is defined to be

$$\deg P := \max\{i_1 + \cdots + i_n \mid a_{i_1 \ldots i_n} \neq 0\}.$$

If $i_1 + \cdots + i_n = \deg P$ for all $i_1, \ldots, i_n$ with $a_{i_1 \ldots i_n} \neq 0$ then P is said to be a heterogeneous polynomial.

Definition 3.7.16 (The ring of convergent power series, $\mathbb{C}\{Z_1, \ldots, Z_n\}$) *Let $\mathbb{C}\{Z_1, \ldots, Z_n\} = \mathbb{C}\{\mathbf{Z}\}$ be the set of "convergent" power series of the form*

$$f(\mathbf{Z}) = \sum_{j \geqslant 0} P_j(\mathbf{Z})$$

where each $P_j(\mathbf{Z})$ is a degree-j heterogeneous polynomial. (See [Na] or [Ka].)
  If $P_j(\mathbf{Z}) = 0$ for all $j = 0, \ldots, j_0 - 1$ then we say that

$$f(\mathbf{Z}) = P_{j_0}(\mathbf{Z}) + \text{"higher terms"}.$$

$\mathbb{C}\{\mathbf{Z}\}$ is a ring which contains $\mathbb{C}[\mathbf{Z}]$ as a sub-ring. $\mathbb{C}\{\mathbf{Z}\}$ is basically equivalent to the ring of holomorphic germs at $\mathbf{0}$. (That is to say that any $f \in \mathbb{C}\{\mathbf{Z}\}$ defines a holomorphic map in a neighbourhood of $\mathbf{0}$, and that every holomorphic map defined in a neighbourhood of $\mathbf{0}$ has an associated convergent power series.)
  For $\mathbf{z} \in \mathbb{C}^n$ we let $f(\mathbf{z})$ be the evaluation of f at $\mathbf{z}$.

Lemma 3.7.17 *If $f \in \mathbb{C}[Z_1, Z_2]$ (resp. $f \in \mathbb{C}\{Z_1, Z_2\}$) then $A - B$ is a factor of $f(A, B) - f(B, A)$ in $\mathbb{C}[A, B]$ (resp. $\mathbb{C}\{A, B\}$.)*

Proof. Note that $(A - B)(A^m B^0 + \cdots + A^0 B^m) = A^{m+1} - B^{m+1}$, so $A - B$ is a factor of $A^{m+1} - B^{m+1}$ in $\mathbb{C}[A, B]$. It follows quickly that if $f(Z_1, Z_2) \in \mathbb{C}[Z_1, Z_2]$ then $A - B$ is a factor of $f(A, B) - f(B, A)$ in $\mathbb{C}[A, B]$, as required.

If $f(Z_1, Z_2) \in \mathbb{C}\{Z_1, Z_2\}$ then $\frac{f(A,B) - f(B,A)}{A - B}$ has an obvious formal power series. The fact that this power series is convergent can also be shown with a little more effort. ∎

Lemma 3.7.18 (Prime elements) *$\mathbb{C}[\mathbf{Z}]$ and $\mathbb{C}\{\mathbf{Z}\}$ are both unique factorisation domains (UFDs). Any (non-zero) degree one heterogeneous polynomial is prime in both $\mathbb{C}[\mathbf{Z}]$ and $\mathbb{C}\{\mathbf{Z}\}$.*
  If P, Q are two non-zero degree one heterogeneous polynomials then either P and Q are coprime, or else they are constant multiples of each other. (That is there is a $\lambda \in \mathbb{C} \setminus \{0\}$ so that $P = \lambda Q$.)

Proof. $\mathbb{C}[\mathbf{Z}]$ and $\mathbb{C}\{\mathbf{Z}\}$ are both UFDs by [Ka, 23.5, 23.6]. It is then sufficient to show that a non-zero degree one heterogeneous polynomial $P_1(\mathbf{z})$ is irreducible in $\mathbb{C}\{\mathbf{Z}\}$ (since irreducible elements in a UFD are also prime, and $\mathbb{C}[\mathbf{Z}]$ is a sub-ring of $\mathbb{C}\{\mathbf{Z}\}$).

Notice that if $U = \{f \in \mathbb{C}\{\mathbf{Z}\} \mid f(0, \ldots, 0) \neq 0\}$, then $\frac{1}{f} \in \mathbb{C}\{\mathbf{Z}\}$ if and only if $f \in U$. Thus U is the set of units of $\mathbb{C}\{\mathbf{Z}\}$ (since the "inverse" of f in $\mathbb{C}\{\mathbf{Z}\}$ must be $\frac{1}{f}$ if it actually exists).

Suppose for contradiction that $A, B \in \mathbb{C}\{\mathbf{Z}\}$ are non-units and that $A(\mathbf{Z}) \cdot B(\mathbf{Z}) = P_1(\mathbf{Z})$. Then we can write A and B as power series

$$A(\mathbf{Z}) = \sum_{i \geqslant 0} A_i(\mathbf{Z}) \qquad \text{and} \qquad B(\mathbf{Z}) = \sum_{i \geqslant 0} B_i(\mathbf{Z})$$

where each A_i and B_i is a degree i heterogeneous polynomial. However $A, B \notin U$ implies that $A_0(\mathbf{Z}) = 0$ and $B_0(\mathbf{Z}) = 0$. But then multiplying the power series term by term, and collecting together degree one terms we see that $P_1(\mathbf{Z}) = A_0(\mathbf{Z}) \cdot B_1(\mathbf{Z}) + A_1(\mathbf{Z}) \cdot B_0(\mathbf{Z}) = 0$. This is a contradiction, as required.   ∎

It is then easy to see that P and Q are either coprime or constant multiples of each other.   ∎

For $\mathbf{s} = (s_1, \ldots, s_\nu)$ in a small neighbourhood $\mathcal{N}(\mathbf{0}) \subset \overline{\mathbb{C}}^\nu$ of $\mathbf{0} = (0, \ldots, 0)$ let $\{v_\mathbf{s}\}_{\mathbf{s} \in \mathcal{N}(\mathbf{0})}$ be the holomorphic family of maps defined in §2.7.

Again let

$$f_\mathbf{s}(z) = z + z(z - s_1) \ldots (z - s_\nu)v_\mathbf{s}(z),$$

and recall in particular that if $\mathbf{s}, \mathbf{s}' \in \mathcal{N}(\mathbf{0})$ are permutations of each other then $v_\mathbf{s} = v_{\mathbf{s}'}$.

Lemma 3.7.19 ($\mathrm{Jac}_\mathbf{h}(\mathbf{s})$ *is non-zero*) Define

$$\mathbf{h}(\mathbf{s}) = \big(h_1(\mathbf{s}), \ldots, h_\nu(\mathbf{s})\big) = \big(f_\mathbf{s}'(s_1), \ldots, f_\mathbf{s}'(s_\nu)\big).$$

The Jacobian of $\mathbf{h}$ *at* $\mathbf{s}$ *will be*

$$\mathrm{Jac}_\mathbf{h}(\mathbf{s}) = \det \left(\frac{\partial h_i}{\partial s_j}(\mathbf{s}) \right)_{\substack{1 \leqslant i \leqslant \nu \\ 1 \leqslant j \leqslant \nu}} = R(\mathbf{s}) \cdot \prod_{i < j}(s_i - s_j)^2.$$

for some holomorphic map $R(\cdot) : K_0^\nu \to \mathbb{C}$. *If* K_0 *is sufficiently small then for all* $\mathbf{s} \in K_0^\nu$ *we have* $R(\mathbf{s}) \approx m_0$ *for some* $m_0 \in \mathbb{Z} \setminus \{0\}$.

Proof.   We let $A(\mathbf{s}) = \big(a_{ij}(\mathbf{s})\big)$ be the $\nu \times \nu$ matrix associated with the above Jacobian. Then for example, $h_1(\mathbf{s}) = 1 + s_1(s_1 - s_2) \ldots (s_1 - s_\nu)v_\mathbf{s}(s_1)$.

We treat $\mathrm{Jac}_\mathbf{h}$ as a power series in $\mathbb{C}\{\mathbf{S}\}$. We first aim to prove that $(S_1 - S_2)^2$ is a factor of $\mathrm{Jac}_\mathbf{h}(\mathbf{S})$ in $\mathbb{C}\{\mathbf{S}\}$.

For $1 \leqslant i, j \leqslant \nu$ and $i \neq j$, let

$$C_{ij}(\mathbf{S}) = \begin{pmatrix} a_{1i}(\mathbf{S}) & a_{1j}(\mathbf{S}) \\ a_{2i}(\mathbf{S}) & a_{2j}(\mathbf{S}) \end{pmatrix}.$$

It is not too hard to show that we can (partially) expand $\mathrm{Jac}_\mathbf{h}(\mathbf{S}) = \det A(\mathbf{S})$ in such a way that every term in the resulting sum will have $\det C_{ij}(\mathbf{S})$ as a factor, for some i, j. Therefore it is sufficient to show that $(S_1 - S_2)^2$ is a factor of each of the $\det C_{ij}(\mathbf{S})$.

Notice that there is some $B \in \mathbb{C}\{Z_1, \ldots, Z_\nu\}$ such that

$$h_1(\mathbf{s}) = 1 + (S_1 - S_2)B(S_1, S_2, S_3, \ldots, S_\nu) \qquad \text{and}$$
$$h_2(\mathbf{s}) = 1 + (S_2 - S_1)B(S_2, S_1, S_3, \ldots, S_\nu).$$

(For this we need the fact that $v_\mathbf{s} = v_{\mathbf{s}'}$ if $\mathbf{s}'$ is a permutation of $\mathbf{s}$.)

We can check that if $C := \frac{\partial B}{\partial Z_1}$ and $D := \frac{\partial B}{\partial Z_2}$ then

$$C_{12}(\mathbf{S}) = \begin{pmatrix} B(\mathbf{S}) + (S_1 - S_2)C(\mathbf{S}) & -B(\mathbf{S}) + (S_1 - S_2)D(\mathbf{S}) \\ -B(\mathbf{S}') + (S_2 - S_1)D(\mathbf{S}') & B(\mathbf{S}') + (S_2 - S_1)C(\mathbf{S}') \end{pmatrix}$$

where $\mathbf{S} = (S_1, S_2, S_3, \ldots, S_\nu)$ and $\mathbf{S}' = (S_2, S_1, S_3, \ldots, S_\nu)$. Lemma 3.7.17 implies that $(S_1 - S_2)^2$ is a factor of $\det C_{12}(\mathbf{S})$.

Also for each $i = 3, \ldots, \nu$ there is some $V_i \in \mathbb{C}\{\mathbf{Z}\}$ such that

$$a_{1i}(\mathbf{S}) = (S_1 - S_2)V_i(\mathbf{S}) \qquad \text{and} \qquad a_{2i}(\mathbf{S}) = (S_2 - S_1)V_i(\mathbf{S}').$$

It is not hard to show that $(S_i - S_j)^2$ is a factor of $C_{ij}(\mathbf{S})$ for all i, j. Thus $(S_i - S_j)^2$ is indeed a factor of $\mathrm{Jac}_\mathbf{h}(\mathbf{S})$ in $\mathbb{C}\{\mathbf{S}\}$.

Lemma 3.7.18 tells us that $(S_i - S_j)^2$ and $(S_k - S_\ell)^2$ are coprime if $\{i, j\} \neq \{k, \ell\}$. Thus if we let $\zeta(\mathbf{S}) = \prod_{i < j}(S_i - S_j)^2 \in \mathbb{C}[\mathbf{S}]$ we see that $\zeta(\mathbf{S})$ is a factor of $\mathrm{Jac}_\mathbf{h}(\mathbf{S})$ in $\mathbb{C}\{\mathbf{S}\}$.

We still need to show that $R \approx m_0 \in \mathbb{Z} \setminus \{0\}$. Let $f^*(z) = z + z(z - S_1) \ldots (z - S_\nu)$ and

$$\mathbf{h}^*(\mathbf{S}) = \big(h_1^*(\mathbf{S}), \ldots, h_\nu^*(\mathbf{S})\big) = \big((f^*)'(S_1), \ldots, (f^*)'(S_\nu)\big).$$

We can then let $a_{ij}^*(\mathbf{S}) = \frac{\partial h_i}{\partial S_j}(\mathbf{S})$, so that $\mathrm{Jac}_{\mathbf{h}^*}(\mathbf{S}) = \det\big(a_{ij}^*(\mathbf{S})\big)$. Both $\mathrm{Jac}_{\mathbf{h}^*}(\mathbf{S})$ and $\zeta(\mathbf{S})$ are heterogeneous and of degree $\nu(\nu-1)$. Using the argument above and Lemma 3.7.14 we see that there is some $m_0 \in \mathbb{Z} \setminus \{0\}$ such that $\mathrm{Jac}_{\mathbf{h}^*}(\mathbf{S}) = m_0\zeta(\mathbf{S})$.

Note that for any i we have $v_{\mathbf{S}}(S_i) = v_{\mathbf{0}}(0) + $ "higher terms", so

$$a_{ij}(\mathbf{S}) = a_{ij}^*(\mathbf{S})v_{\mathbf{0}}(0) + \text{"higher terms"}.$$

This implies that

$$\mathrm{Jac}_{\mathbf{h}}(\mathbf{S}) = \mathrm{Jac}_{\mathbf{h}^*}(\mathbf{S})\big(v_{\mathbf{0}}(0)\big)^\nu + \text{"higher terms"}.$$

Thus

$$R(\mathbf{S}) := \frac{\mathrm{Jac}_{\mathbf{h}}(\mathbf{S})}{\zeta(\mathbf{S})} = \frac{\mathrm{Jac}_{\mathbf{h}^*}(\mathbf{S})}{\zeta(\mathbf{S})}\big(v_{\mathbf{0}}(0)\big)^\nu + \text{"higher terms"} = m_0\big(v_{\mathbf{0}}(0)\big)^\nu + \text{"higher terms"}.$$

as required. ■

Corollary 3.7.20 *(**The Jacobian** $\mathrm{Jac}_{\mathbf{p}}(\mathbf{s})$ **is non-zero if** G **has no closed gates**) Now define $\mathbf{p}(\mathbf{s}) = \big(\jmath(f_{\mathbf{s}}, s_1), \ldots, \jmath(f_{\mathbf{s}}, s_\nu)\big)$.*
   If all the $0, s_1, \ldots, s_\nu \in K_0$ are distinct then $\mathrm{Jac}_{\mathbf{p}}(\mathbf{s}) \neq 0$.

Proof. This comes directly from Lemma 3.7.19 and the observation that there is a biholomorphic map d defined in a neighbourhood of 1 such that $d(f_{\mathbf{s}}'(s_i)) = \jmath(f_{\mathbf{s}}, s_i)$ for all i. This is given by $d(\lambda) = \frac{-2\pi i}{\log \lambda}$. ■

3.7.5 Realising a map with the correct lifted phases

To show the existence of maps with particular lifted phases we will be using the following.

Theorem 3.7.21 *(**Inverse Mapping Theorem**) Suppose that $X \subset \mathbb{C}^m$ is open, $f : X \to \mathbb{C}^m$ is holomorphic and $\mathbf{a} \in X$. Then f is a biholomorphic mapping from an open neighbourhood of $\mathbf{a}$ onto an open neighbourhood of $f(\mathbf{a})$ if and only if $\mathrm{Jac}_f(\mathbf{a}) \neq 0$. ($\mathrm{Jac}_f(\mathbf{a})$ is the Jacobian of f at $\mathbf{a}$.)*

Throughout the rest of this section we will be using the following notation.

Notation 3.7.22 *Fix $\mathsf{G} \in \mathsf{Admissible}$, and let r be the number of open gates that G possesses.*
   Then there will be multiplicities $m_0, \ldots, m_r$ (dependent upon G) so that $m_k = \mathrm{mult}(g, \sigma_k(g))$ for all $g \in \mathcal{WB}(\mathsf{G})$. Now for $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r)$ in a small neighbourhood of $(0, \ldots, 0) \in \mathbb{C}^r$ define

$$\ell(\boldsymbol{\sigma}) := (\overbrace{0, \ldots, 0}^{m_0-1}, \overbrace{\sigma_1, \ldots, \sigma_1}^{m_1}, \ldots, \overbrace{\sigma_r, \ldots, \sigma_r}^{m_r}) \in \mathbb{C}^\nu$$

and $w_{\boldsymbol{\sigma}} := v_{\ell(\boldsymbol{\sigma})}$ where $\{v_{\mathbf{s}}\}$ is the family of maps defined in a neighbourhood of K_0 which comes from §2.7. Also define $f_{\boldsymbol{\sigma}} : K_0 \to \mathbb{C}$ as

$$f_{\boldsymbol{\sigma}}(z) := z + z^{m_0}(z - \sigma_1)^{m_1} \ldots (z - \sigma_r)^{m_r} w_{\boldsymbol{\sigma}}(z).$$

Now we define

$$S^{*,r} := \{(\sigma_1, \ldots, \sigma_r) \in \mathbb{C}^r \mid 0, \sigma_1, \ldots, \sigma_r \text{ are not all distinct}\}.$$

Then the map $\mathbf{p} = \mathbf{p}(\mathsf{G}) : K_0^r \setminus S^{*,r} \to \mathbb{C}^r$ defined as

$$\mathbf{p}(\sigma_1, \ldots, \sigma_r) := \big(\jmath(f_{\boldsymbol{\sigma}}, \sigma_1), \ldots, \jmath(f_{\boldsymbol{\sigma}}, \sigma_r)\big)$$

is well defined and holomorphic.

We can also find $a_1, \ldots, a_r \in \mathbb{Z}/\nu\mathbb{Z}$ so that $\{a_1, \ldots, a_r\} = \{i \mid \mathsf{G}_i \neq \star\}$ and $a_1 < a_2 < \cdots < a_r$. (See Proposition 2.4.12.) Let $T : \mathcal{WB}(\mathsf{G}) \to \mathbb{C}^r$ be defined as

$$T(f) = \left(\tilde{\tau}_{a_1}(f), \ldots, \tilde{\tau}_{a_r}(f)\right).$$

Now we let $B = B(\mathsf{G}) : \mathbb{C}^r \to \mathbb{C}^r$ be the invertible linear map given in Proposition 2.4.12. Then $\Theta = \Theta(\mathsf{G}) : K_0^r \setminus S^{*,r} \to \mathbb{C}$ is holomorphic if it is defined as

$$\Theta := B \circ \mathbf{p},$$

and we have $\Theta(\boldsymbol{\sigma}) = T(f_{\boldsymbol{\sigma}})$ for all $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r) \in K_0^r$ such that $f_{\boldsymbol{\sigma}} \in \mathcal{WB}(\mathsf{G})$ and $\sigma_k = \sigma_k(f_{\boldsymbol{\sigma}})$ for $k = 1, \ldots, r$.

We also need the following technical Lemma.

Lemma 3.7.23 *Fix* $\mathsf{G} \in$ Admissible *and use the notation in Notation 3.7.22. Let* $\eta \gg 0$ *and* $\varepsilon > 0$. *Then there is a* $\xi > 0$ *such that the following hold.*

1. *There is a closed neighbourhood* $N(S^{*,r})$ *of* $S^{*,r}$ *in* $\mathbb{C}^r$ *so that if* $\boldsymbol{\sigma} \in K_0^r \setminus S^{*,r}$ *and* $\|\Theta(\boldsymbol{\sigma})\| < \eta$ *then* $\boldsymbol{\sigma} \notin N(S^{*,r})$. $(\|\cdot\|$ *is the Euclidean norm.)*

2. *Suppose that* $\boldsymbol{\sigma} : [0, t_1] \to \mathbb{C}$ *is a continuous path in* $D_{2\varepsilon}^r$ *such that* $f_{\boldsymbol{\sigma}(0)} \in \mathcal{WB}(\mathsf{G})$, *and* $\Theta(\boldsymbol{\sigma}(t)) \in H_\xi^r$ *for all* $t \in [0, t_1]$. *Then* $f_{\boldsymbol{\sigma}(t)}$ *is well behaved and* $\boldsymbol{\sigma}(t) \in D_\varepsilon^r$ *for all* $t \in [0, t_1]$.

Proof. If we have a bound $\|\Theta(\boldsymbol{\sigma})\| < \eta$ then there is some $\eta' > 0$ so that $\|\mathbf{p}(\boldsymbol{\sigma})\| < \eta'$ (by the linearity of the invertible map $B = B(\mathsf{G})$ and the fact that $\Theta = B \circ \mathbf{p}$). This implies that the holomorphic indices of all the fixed points of $f_{\boldsymbol{\sigma}}$ (close to 0) are also bounded.

However, when a multiple fixed point is perturbed so that it splits apart, at least one of the holomorphic indices will be very large, and in fact arbitrarily large for a sufficiently small perturbation. (See Lemma 3.7.24 below.) Thus there is no sequence $\boldsymbol{\sigma}_k$ in $D_{2\varepsilon}^r$ accumulating on $S^{*,r}$ such that $\|\Theta(\boldsymbol{\sigma}_k)\| < \eta$ for all k. Therefore the neighbourhood $N(S^{*,r})$ of $S^{*,r}$ exists and **(1.)** is proven.

Lemma 3.7.8 implies that if $\xi \gg 1$, $f_{\boldsymbol{\sigma}(t)} \in \mathcal{WB}(\mathsf{G})$ and $\Theta(\boldsymbol{\sigma}(t)) \in H_\xi^r$ for all $t \in [0, 1]$ then $\boldsymbol{\sigma}(t) \in D_\varepsilon^r$ for all $t \in [0, 1]$.

Now let $I = \{t_1 \in [0, 1] \mid f_{\boldsymbol{\sigma}(t)} \in \mathcal{WB}(\mathsf{G}) \; \forall t \in [0, t_1]\}$. Then I is an interval. If we can also show that I is open and closed in $[0, 1]$, then $I = [0, 1]$ and the Lemma will be proved.

Proposition 2.3.10 implies that I is open. To show that I is closed, it is sufficient to show that $f_{\boldsymbol{\sigma}(t_+)}$ is in neither $\mathcal{WB} \setminus \mathcal{WB}(\mathsf{G})$ nor $\mathcal{N}_0 \setminus \mathcal{WB}$.

Suppose that G has r open gates. Now suppose for contradiction that $f_{\boldsymbol{\sigma}(t_+)} \in \mathcal{WB}(\mathsf{G}')$ where $\mathsf{G} \neq \mathsf{G}' \in$ Admissible. In the case where $\boldsymbol{\sigma}(t_+) \notin S^{*,r}$ Proposition 2.3.10 implies that $f_{\boldsymbol{\sigma}(t)} \in \mathcal{WB}(\mathsf{G}')$ for t sufficiently close to t_+, which contradicts the definition of t_+. The other possibility is that $\boldsymbol{\sigma}(t_+) \in S^{*,r}$. However, the fact that $\Theta(\boldsymbol{\sigma}(t))$ is bounded, together with part (1.) imply that this is not possible. Thus $f_{\boldsymbol{\sigma}(t_+)} \notin \mathcal{WB} \setminus \mathcal{WB}(\mathsf{G})$.

Now suppose for contradiction that $f = f_{\boldsymbol{\sigma}(t_+)} \in \mathcal{N}_0 \setminus \mathcal{WB}$. Then there is an $i \in \mathbb{Z}/\nu\mathbb{Z}$, $s \in \{+, -\}$ and $r, r' \in \mathbb{R}$ so that $\gamma_{i,s,f}(r) \notin D_{r_0/2}$, $\gamma_{i,s,f}(r') \in D_{r_0/8}$ and r' lies between 0 and r.

Then for $\bar{f}$ close enough to $f_{\boldsymbol{\sigma}(t_+)}$ we will have $\gamma_{i,s,\bar{f}}(r) \notin D_{r_0/4}$ and $\gamma_{i,s,\bar{f}}(r') \in D_{r_0/4}$. This implies that $\bar{f}$ is not $\frac{r_0}{4}$-well behaved. (See Definition 3.7.3.)

However, Lemma 3.7.4 and the fact that $T(f_{\boldsymbol{\sigma}(t)}) \in H_\xi^r$ for all $t \in [0, t_+)$ implies that $f_{\boldsymbol{\sigma}(t)}$ is $\frac{r_0}{4}$-well behaved for all $t \in [0, t_+)$ which is a contradiction. Thus I is closed, and part **(2.)** is proven. ■

Lemma 3.7.24 *Suppose that* $f : D \to \mathbb{C}$ *is holomorphic on a domain* $D \subset \mathbb{C}$, *and has a single fixed point* σ *in* D, *with multiplicity* $m > 1$. *Now suppose that* $f_k \to f$ *uniformly on compact sets and that there is a sequence* $\sigma_k \to \sigma$ *such that* σ_k *is a fixed point of* f_k *of multiplicity strictly less than* m.

Then there is a sequence $\sigma'_k \to \sigma$ *such that* σ'_k *is a fixed point of* f_k *for all large* k *and such that* $|\iota(f_k, \sigma'_k)| \to +\infty$ *and* $|\jmath(f_k, \sigma'_k)| \to +\infty$ *as* $k \to +\infty$.

Proof. We can assume without loss of generality that $\sigma = 0$ and $f(z) = f_0(z) = z + z^{\nu+1} + \mathcal{O}(z^{\nu+1})$. We can choose associated K_0, r_0, $\mathcal{N}_0$ and so on.

Consider the case where all $f_k \in \mathcal{WB}$ for all k. Proposition 2.4.3 can be applied to show that $|\tilde{\tau}_i(f_k)| \to +\infty$ as $k \to +\infty$ for all $i \in \mathbb{Z}/\nu\mathbb{Z}$. Then Proposition 2.4.12 implies that the $\jmath$-indices (and therefore the holomorphic indices) of the fixed points cannot all be bounded. Thus the sequence σ_k must exist.

Now consider the case where $f_k \notin \mathcal{WB}$ for all k. Then for each large k there is a trajectory $\gamma_k : [t_{k,-}, t_{k,+}] \to \overline{D_{r_0/2}}$ for $\dot{z} = i[f_k(z) - z]$ which has $\gamma_k(t_{k,-}), \gamma_k(t_{k,-}) \in \partial D_{r_0/2}$, and passes very close to 0 (if k is very large). Let

$$\mathrm{Int}_k(\theta_-, \theta_+) := \frac{1}{2\pi i} \int_{C(\theta_-, \theta_+)} \frac{dz}{z - f_k(z)}$$

where $C(\theta_-, \theta_+)(t) := \frac{r_0}{2} \exp(i[(1-t)\theta_- + t\theta_+])$ for $t \in [0, 1]$. Then $\mathrm{Int}_k(\theta_-, \theta_+)$ is bounded if k is large and $\theta_-, \theta_+ \in [0, 4\pi]$, since $\frac{1}{z - f_k(z)}$ is bounded on $K_0 \setminus D_{r_0/2}$.

There are some $\theta_{k,-}, \theta_{k,+} \in [0, 4\pi]$ so that the path "$p_k := \gamma_k + C(\theta_{k,-}, \theta_{k,+})$" is a closed Jordan contour (i.e. a loop). We want to show that the modulus of $\mathrm{Int}_k := \frac{1}{2\pi i} \int_{p_k} \frac{dz}{z - f_k(z)}$ tends to $+\infty$, since this will imply that the modulus of the sum the holomorphic indices of all the fixed points which p_k winds around must converge to $+\infty$. But since $\mathrm{Int}_k(\theta_{k,-}, \theta_{k,+})$ is bounded for all k, it is sufficient to show that the modulus of $\mathrm{Int}'_k := \frac{1}{2\pi i} \int_{\gamma_k} \frac{dz}{z - f_k(z)}$ tends to $+\infty$.

Let $w = I(z) := -\frac{1}{\nu z^\nu}$, $\alpha \gg 1$ and $\rho(f) := \alpha \max\{|\sigma| \mid f(\sigma) = \sigma \in K_0\}$ for $f \in \mathcal{N}$. Lemma 3.3.4 can be extended to show that $(I \circ \gamma_k)'(t) \approx i$ for all those t such that $\gamma_k(t) \in K_0 \setminus D_{\rho(f_k)}$. So in particular we see that γ_k crosses from the "outside" of the annulus $D_{r_0/2} \setminus D_{\rho(f_k)}$ to the "inside" (and then back out again).

Thus the path $I \circ \gamma_k$ crosses from the inside of the annulus $\overline{D_{|I(\rho(f_k))|}} \setminus \overline{D_{|I(r_0/2)|}}$ to the outside. But then $t_{k,+} - t_{k,-} > \frac{1}{2} |I(\rho(f_k)) - I(r_0/2)|$ (since $|(I \circ \gamma_k)'(t)| \approx 1$.) Now since $|I(\rho(f_k))| \to +\infty$ as $k \to +\infty$, we see that $|\mathrm{Int}'_k| \geqslant \frac{1}{2\pi} |t_{k,+} - t_{k,-}| \to +\infty$ as $k \to +\infty$. Thus for all k large, we can always find a fixed point σ_k which p_k winds around so that $|\iota(f_k, \sigma_k)| \to +\infty$. The definition of $\jmath(\cdot, \cdot)$ implies that $|\jmath(f_k, \sigma_k)| \to +\infty$ also.

Thus the Lemma holds in general. ∎

We are now ready to take our first step in proving Theorem 2.7.1.

Lemma 3.7.25 *Given* $G \in$ Admissible *with no closed gates,* $\xi \gg 1$ *and* $(\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu) \in H_\xi^\nu$ *there is an* $f \in \mathcal{WB}(G)$ *such that* $T(f) = (\tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)$.

Proof. Define $S^{*, \nu}$, $\Theta : K_0^\nu \setminus S^{*, \nu} \to \mathbb{C}$, $B : \mathbb{C}^\nu \to \mathbb{C}^\nu$, $\mathbf{p} : K_0^\nu \setminus S^{*, \nu} \to \mathbb{C}$ and $f_{\boldsymbol{\sigma}}$ as they were in Notation 3.7.22.

Given $\mathbf{w} \in H_\xi^r$ we will aim to find a well behaved $f \in \mathcal{WB}(G)$ so that $T(f) = \mathbf{w}$.

Let $\varepsilon > 0$ be very small. Then $\boldsymbol{\sigma} \in D_{2\varepsilon}^\nu$ implies that $f_{\boldsymbol{\sigma}} \in \mathcal{N}_0$. Now let $\xi := C/\varepsilon^\nu$ where $C \gg 1$.

We know by Lemma 3.7.12 that there is an $\boldsymbol{\sigma}^* \in D_\varepsilon^\nu$ such that $f_{\boldsymbol{\sigma}^*} \in \mathcal{WB}(G)$ and $\Theta(\boldsymbol{\sigma}^*) \in H_\xi^\nu$. Now define $\mathbf{w}(t) = (1 - t)\Theta(\boldsymbol{\sigma}^*) + t\mathbf{w} \in H_\xi^\nu$ for $t \in [0, 1]$ and

$$\mathcal{G}(t_0) := \left\{ \boldsymbol{\sigma} : [0, t_0] \to D_{2\varepsilon}^\nu \setminus S^{*, \nu} \,\middle|\, \begin{array}{c} \boldsymbol{\sigma}(\cdot) \text{ is continuous,} \\ \Theta \circ \boldsymbol{\sigma}(t) = \mathbf{w}(t) \; \forall t \in [0, t_0] \end{array} \right\}$$

where $t_0 \in [0, 1]$. Also let I be the interval

$$I := \{t_0 \in [0, 1] \mid \exists \, \boldsymbol{\sigma}(\cdot) \in \mathcal{G}(t_0)\}.$$

If $I = [0, 1]$ then (by Lemma 3.7.23 part **(2.)**) we can simply let $f = f_{\boldsymbol{\sigma}(1)} \in \mathcal{WB}(G)$, and we must have $(\tilde{\tau}_1(f), \ldots, \tilde{\tau}_\nu(f)) = \mathbf{w}$ as required.

So now we will show that $I = [0, 1]$ by showing that I is both open and closed in $[0, 1]$.

Open: Take $t_1 \in I$. We know there is an $\boldsymbol{\sigma}(\cdot) \in \mathcal{G}(t_1)$. We aim to show that if $t_1 < 1$ then $\boldsymbol{\sigma}(\cdot)$ can be extended to give $\mathbf{r}(\cdot) \in \mathcal{G}(t_1 + \varepsilon)$ for some small $\varepsilon > 0$. Since $\mathrm{Jac}_\Theta(\boldsymbol{\sigma}(t_1)) \neq 0$ by Corollary 3.7.20, there is a local inverse $\hat{\Theta}$ of $\Theta = B \circ \mathbf{p}$ from a small neighbourhood of $\mathbf{w}(t_1)$ to a neighbourhood of $\boldsymbol{\sigma}(t_1)$. If $\varepsilon > 0$ is small enough then we let $\mathbf{r}(t) = \boldsymbol{\sigma}(t)$ for $t \in [0, t_1]$, and $\mathbf{r}(t) = \hat{\Theta}(\mathbf{w}(t))$ for $t \in [t_1, \varepsilon)$. Then $\mathbf{r}(\cdot) \in \mathcal{G}(t_1 + \varepsilon)$, so I is open in $[0, 1]$.

Closed: Let $t_+ = \sup I$, and take a sequence $t_k \to t_+$. There will be associated paths $\boldsymbol{\sigma}_k(\cdot) \in \mathcal{G}(t_k)$.
Lemma 3.7.23 part (**2.**), we can let $\boldsymbol{\sigma}_+ \in \overline{D_\varepsilon}^\nu \setminus S^{*,\nu}$ be an accumulation point of $\{\boldsymbol{\sigma}_k(t_k)\} \subset \overline{D_\varepsilon}^\nu$.
Then $\Theta(\boldsymbol{\sigma}_+) = \mathbf{w}(t_+)$ by continuity. Since $\mathrm{Jac}_\Theta(\boldsymbol{\sigma}_+) \neq 0$ by Corollary 3.7.20 there is a local
inverse $\hat\Theta$ of Θ from a convex neighbourhood B of $\mathbf{w}(t_+)$ to a neighbourhood of $\boldsymbol{\sigma}_+$. We can find a
k so that $\mathbf{w}(t_k) \in B$. Now let $\mathbf{r}(t) = \boldsymbol{\sigma}_k(t)$ for $t \in [0, t_k]$, and $\mathbf{r}(t) = \hat\Theta(\mathbf{w}(t))$ for $t \in [t_k, t_+]$. Then
$\mathbf{r}(\cdot) \in \mathcal{G}(t_+)$, so I is closed in $[0,1]$.

Therefore the required $f \in \mathcal{WB}(\mathsf{G})$ exists. ∎

Lemma 3.7.26 *Given* $\mathsf{G} \in \mathsf{Admissible}$, $\xi \gg 1$ *and* $(\tilde\theta_1, \ldots, \tilde\theta_\nu) \in \mathbf{H}(\mathsf{G}, \xi)$ *there is an* $f \in$
$\mathcal{WB}(\mathsf{G})$ *such that for each* $i \in \mathbb{Z}/\nu\mathbb{Z}$ *we have* $\tilde\tau_i(f) = \tilde\theta_i$.

Proof. There will be some $\mathsf{G}' = (\mathsf{G}'_1, \ldots, \mathsf{G}'_\nu) \in \mathsf{Admissible}$ with no closed gates, and such that for each
i with $\mathsf{G}_i \neq \star$ we have $\mathsf{G}'_i = \mathsf{G}_i$.
Now for $i \in \mathbb{Z}/\nu\mathbb{Z}$ and $k > \xi$ let

$$\tilde\theta_{i,k} = \begin{cases} \tilde\theta_i & \text{if } \mathsf{G}_i \neq \star, \\ -k & \text{if } \mathsf{G}_i = \star. \end{cases}$$

Then by Lemma 3.7.25 for all $k > \xi$ there is a $\mathbf{c}_k = (c_{1,k}, \ldots, c_{\nu,k}) \in K_0^\nu$ so that if

$$h_k(z) = z + z(z - c_{1,k}) \ldots (z - c_{\nu,k}) v_{\mathbf{c}_k}(z)$$

then $h_k \in \mathcal{WB}(\mathsf{G}')$ and $\tilde\tau_i(h_k) = \tilde\theta_{i,k}$ for all $i \in \mathbb{Z}/\nu\mathbb{Z}$. There must then be some $\mathbf{c}^*$ very close to $\mathbf{0}$ and
some subsequence $\{\mathbf{c}_{k_n}\}_n$ of $\{\mathbf{c}_k\}_k$ such that $\mathbf{c}_{k_n} \to \mathbf{c}^*$. Now let $f := h_{\mathbf{c}^*}$ and $f_n := h_{k_n}$. Clearly $f_n \to f$
as $n \to +\infty$.

Assume for contradiction that $f \notin \mathcal{WB}$. Then there is some $i \in \mathbb{Z}/\nu\mathbb{Z}$, $s \in \{+, -\}$ and $t, t' \in \mathbb{R}$ so
that $\gamma_{i,s,f}(t) \notin D_{r_0/2}$ and $\gamma_{i,s,f}(t') \in D_{r_0/8}$ where t' is between 0 and t. But then Theorem 3.3.9 implies
that for all n large we have $\gamma_{i,s,f_n}(t) \notin D_{r_0/4}$ and $\gamma_{i,s,f_n}(t') \in D_{r_0/4}$. This implies that f_n is not $\frac{r_0}{4}$-well
behaved for n large.
However Lemma 3.7.4 implies that f_n is $\frac{r_0}{4}$-well behaved for all n (because $\mathrm{Re}\,\tilde\theta_i < -\xi$ for all $i \in \mathbb{Z}/\nu\mathbb{Z}$
and $\xi \gg 1$). This is a contradiction, so we have $f \in \mathcal{WB}$.
Proposition 2.4.3 implies that $f \in \mathcal{WB}(\mathsf{G})$ (since $\mathsf{gate}_i(f) = \star \iff \tilde\tau_i(f) = \infty$) and that all the
lifted phases are correct. ∎

3.7.6   The Jacobian is non-zero in general

Lemma 3.7.27 (**Uniqueness of** $f(\mathsf{G}; \tilde\theta_1, \ldots, \tilde\theta_\nu)$ **when** G **has no closed gates**) *Let* $\xi > 0$
be large, and fix $\mathsf{G} \in \mathsf{Admissible}$ *with no closed gates. Let* H_ξ *and* $\mathcal{T}$ *be defined as they are*
in §2.7.
If $f^a, f^b \in \mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ *and* $\mathcal{T}(f^a) = \mathcal{T}(f^b) \in H_\xi^\nu$ *then* $f^a = f^b$.

Proof. We let $v_\mathbf{s} := u_{f^a} = u_{f^b}$. For $\mathbf{s} = (s_1, \ldots, s_\nu) \in \mathbb{C}^\nu$ close to $\mathbf{0}$ set

$$f_\mathbf{s}(z) := z + z(z - s_1) \ldots (z - s_\nu) v_\mathbf{s}(z) \qquad \text{and}$$
$$g_\mathbf{s}(z) := z + z(z - s_1) \ldots (z - s_\nu).$$

Let $S^{*,\nu}$, $T : \mathcal{WB}(\mathsf{G}) \to \mathbb{C}^\nu$ and $\Theta : K_0^\nu \setminus S^{*,\nu} \to \mathbb{C}^\nu$ be defined as they were in Notation 3.7.22.
Now let $\mathbf{s}^a = (\sigma_1(f^a), \ldots, \sigma_\nu(f^a))$ and $\mathbf{s}^b = (\sigma_1(f^b), \ldots, \sigma_\nu(f^b))$. Finally let $\mathbf{w}(t) = (1 - t)T(g_{\mathbf{s}^a}) +$
$tT(g_{\mathbf{s}^b})$ for $t \in [0, 1]$.

We aim to construct a path $\{\mathbf{s}(t)\}_{t \in [0,1]}$ so that $f_{\mathbf{s}(0)} = f^a$ and $f_{\mathbf{s}(1)} = f^b$ and $f_{\mathbf{s}(t)} \in \mathcal{WB}(\mathsf{G})$ for all
$t \in [0, 1]$. First of all we need to show that $T(g_{\mathbf{s}^a}), T(g_{\mathbf{s}^b}) \in H_{\xi/2}^\nu$.
Let $j = \mathsf{G}_i$. Also let $p(t) = X_t^{f^a}(z_{i,+})$ where $t \in [0, T]$ and

$$T := \sup\{t_0 > 0 \mid X_t^{f^a}(z_{i,+}) \notin \gamma_{j,-,f^a}(\mathbb{R}) \; \forall t \in [0, t_0]\}.$$

Now let C be the Jordan contour in Figure 3.17 in the proof of Lemma 3.7.4 which winds anti-clockwise around $\text{Fix}^u(i, f^a)$ (where "f" has been replaced by "f^a").

It is not difficult to show using the same kinds of arguments that we have before (for instance in the proof of Lemma 3.7.4) that

$$\frac{\tilde{\tau}_i(f^a)}{\tilde{\tau}_i(g_{\mathbf{s}^a})} \approx \frac{\int_C \frac{dz}{z - f^a(z)}}{\int_C \frac{dz}{z - g_{\mathbf{s}^a}(z)}} \approx \frac{\int_p \frac{dz}{z - f^a(z)}}{\int_p \frac{dz}{z - g_{\mathbf{s}^a}(z)}}.$$

Now since p is a solution of $\frac{dz}{dt} = f^a(z) - z$ we have $p'(t) = f^a(p(t)) - p(t)$ and

$$\int_p \frac{dz}{z - f^a(z)} = \int_0^T \frac{p'(t)\, dt}{p(t) - f^a(p(t))} = \int_0^T -1\, dt = -T.$$

Notice that if $\eta(t) := \frac{p'(t)}{p(t) - g_{\mathbf{s}^a}(p(t))}$ then there is a small $\varepsilon > 0$ so that $|\eta(t) + 1| < \varepsilon$ for all $t \in [0, T]$, and we will have

$$\int_p \frac{dz}{z - g_{\mathbf{s}^a}(z)} = \int_0^T \frac{p'(t)\, dt}{p(t) - g_{\mathbf{s}^a}(p(t))} = \int_0^T \eta(t)\, dt \in -T(1 + D_\varepsilon).$$

Thus $\tilde{\tau}_i(f^a)/\tilde{\tau}_i(g_{\mathbf{s}^a}) \approx 1$. Since this can be done for all i with $\mathsf{G}_i \neq \star$, we must have $T(g_{\mathbf{s}^a}) \in H^\nu_{\xi/2}$ since $T(f^a) \in H^\nu_\xi$. And similarly $T(g_{\mathbf{s}^b}) \in H^\nu_{\xi/2}$.

We can use the proof of Lemma 3.7.26 to construct a path $\{\mathbf{s}(t)\}_{t \in [0,1]} \subset \mathbb{C}^\nu$ so that $\mathbf{s}(0) = \mathbf{s}^a$ and so that $T(g_{\mathbf{s}(t)}) = \mathbf{w}(t) \in H^\nu_{\xi/2}$ and $g_{\mathbf{s}(t)} \in \mathcal{WB}(\mathsf{G})$ for all $t \in [0, 1]$. Lemma 3.7.13 and Proposition 2.4.12 imply that $\mathbf{s}(1) = \mathbf{s}^b$. By Lemma 3.7.8 $\mathbf{s}(t)$ is close to $\mathbf{0}$ for all $t \in [0, 1]$.

Using the same kind of argument as above we can show that $T(f_{\mathbf{s}(t)}) \in H^\nu_{\xi/4}$ for all $t \in [0, 1]$ (since we already know that $T(g_{\mathbf{s}(t)}) \in H^\nu_{\xi/2}$ for all t).

Recall that Θ is defined and holomorphic from $K^\nu_0 \setminus S^{*,\nu} \to \mathbb{C}^\nu$. We let $X = H_{\xi/4}$ and $\tilde{Y} = \Theta^{-1}(X)$. Now since X is open, and $\text{Jac}_\Theta(\mathbf{s}) \neq 0$ for all $\mathbf{s} \in \tilde{Y}$ we see that the Inverse Mapping Theorem (Theorem 3.7.21) implies that $\tilde{Y}$ is open. We now let $\tilde{X}$ be the path connected component of $\tilde{Y}$ which contains $\mathbf{s}(0)$. Clearly $\tilde{X}$ must also be open.

The proof of Lemma 3.7.26 implies that the restriction $\Theta : \tilde{X} \to X$ is surjective. (Actually, the proof says that Θ is surjective onto H^ν_ξ instead of $H^\nu_{\xi/4}$, but since we can make ξ arbitrarily large this does not matter.)

Now we want to show that $(\tilde{X}, \Theta)$ is a covering space of X. That is we need to show that "the boundary of $\tilde{X}$ maps to the boundary of X."

Assume for contradiction that there is a sequence $\{\mathbf{s}_k\}_{k \geqslant 1}$ in $\tilde{X}$ such that $\mathbf{s}_k \to \mathbf{s}^\# \in \partial\tilde{X}$ and $\Theta(\mathbf{s}_k) \not\to \partial X$. Then there is a $\mathbf{w}^\# \in \overset{\circ}{X}$ and a subsequence $\{\mathbf{s}_{k_n}\}_{n \geqslant 1}$ converging to $\mathbf{s}^\#$ such that $\Theta(\mathbf{s}_{k_n}) \to \mathbf{w}^\#$ as $n \to +\infty$.

Then the Lemma 3.7.23 part (**1.**) implies that $\mathbf{s}^\# \in K^\nu_0 \setminus S^{*,\nu}$. Since $\tilde{X}$ is path connected, we find a continuous path $\mathbf{r} : [0, 1] \to \tilde{X} \cup \{\mathbf{s}^\#\}$ so that $\mathbf{r}(1 - \frac{1}{n}) = \mathbf{s}_{k_n}$ and $\mathbf{r}(1) = \mathbf{s}^\#$. We now have $\Theta(\mathbf{r}(t)) \in X$ for $t \in [0, 1]$ (since $\Theta(\tilde{X} \cup \{\mathbf{s}^\#\}) = X \cup \{\mathbf{w}^\#\} = X$). This implies that $\mathbf{r}([0, 1]) \subset \Theta^{-1}(X) = \tilde{Y}$. But then the definition of $\tilde{X}$ implies that $\mathbf{s}^\# = \mathbf{r}(1) \in \tilde{X}$, which is a contradiction since $\tilde{X}$ is open and $\mathbf{s}^\# \in \partial\tilde{X}$. Thus $(\tilde{X}, \Theta)$ is indeed a covering space for X.

Note that $\Theta \circ \mathbf{s}(0) = \Theta \circ \mathbf{s}(1)$ and X is path connected and simply connected, so we can use the following Lemma. (See [Ma, Lemma 3.3].)

> **Lemma** Let $(\tilde{X}, p)$ be a covering space of X and let $\gamma_1, \gamma_2 : [0, 1] \to \tilde{X}$ be paths in $\tilde{X}$ with the same initial point. If $p \circ \gamma_1$ is homotopic to $p \circ \gamma_2$ then γ_1 is homotopic to γ_2; in particular, γ_1 and γ_2 have the same terminal point.

By this and the fact that any "loop" (such as $\Theta \circ \mathbf{s}$) in a simply connected space is homotopic to a single point, we see that $\mathbf{s} : [0, 1] \to \tilde{X}$ must be homotopic to the trivial path $\bar{\mathbf{s}} : [0, 1] \to \tilde{X}$ given by

$\bar{\mathbf{s}}(t) := \mathbf{s}(0)$. Thus $\mathbf{s}(0) = \mathbf{s}(1)$, implying that $f^a = f^b$, as required.   ■

Definition 3.7.28 (Analytic sets) *Let* Ω *be open in* $\mathbb{C}^m$. *An analytic set* A *in* Ω *is a subset of* Ω *so that for each* $\mathbf{a} \in \Omega$ *there is an open* $U \subset \Omega$, *an* $n \in \mathbb{N}$ *and a holomorphic map* $\mathbf{f} : \Omega \to \mathbb{C}^n$ *such that*
$$A \cap U = \{\mathbf{z} \in U \mid \mathbf{f}(\mathbf{z}) = \mathbf{0}\}.$$

Lemma 3.7.29 (The Jacobian $\mathrm{Jac_p}(\boldsymbol{\sigma})$ **is non-zero)** *Fix* $\mathsf{G} \in$ Admissible, *and let* r *be the number of open gates that* G *has. Let* $f_{\boldsymbol{\sigma}} : K_0 \to \mathbb{C}$ *and* $\mathbf{p} : K_0^r \setminus S^{*,r} \to \mathbb{C}$ *be defined as they were in Notation 3.7.22.*
    Then $\mathrm{Jac_p}(\boldsymbol{\sigma}) \neq 0$ *for all* $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_r) \in K_0^r \setminus S^{*,r}$ *such that* $f_{\boldsymbol{\sigma}} \in \mathcal{WB}(\mathsf{G})$.

Proof.   Notice that if $f_{\boldsymbol{\sigma}} \in \mathcal{WB}(\mathsf{G})$ then $\boldsymbol{\sigma} \notin S^{*,r}$ by Proposition 2.3.8.

Recall that if $\mathrm{Jac_p}(\boldsymbol{\sigma}) = 0$ then $\mathbf{p}$ is not injective in any neighbourhood of $\boldsymbol{\sigma}$ by the Inverse Mapping Theorem (Theorem 3.7.21). Thus (using Proposition 2.3.10) it will be sufficient to show that if $\mathbf{a}, \mathbf{b} \in K_0^r \setminus S^{*,r}$, $\mathbf{p}(\mathbf{a}) = \mathbf{p}(\mathbf{b})$ and $f_{\mathbf{a}}, f_{\mathbf{b}} \in \mathcal{WB}(\mathsf{G})$ then $f_{\mathbf{a}} = f_{\mathbf{b}}$.

We can assume without loss of generality (by reordering the entries of the vectors $\mathbf{a}$ and $\mathbf{b}$) that
$$\mathbf{a} = \big(\sigma_1(f_{\mathbf{a}}), \dots, \sigma_r(f_{\mathbf{a}})\big) \qquad \text{and} \qquad \mathbf{b} = \big(\sigma_1(f_{\mathbf{b}}), \dots, \sigma_r(f_{\mathbf{b}})\big).$$

Let $J := \mathbf{p}(\mathbf{a}) = \mathbf{p}(\mathbf{b})$. We first need to show that
$$X := \big\{\boldsymbol{\sigma} \in \mathbb{C}^r \setminus S^{*,r} \mid \mathbf{p}(\boldsymbol{\sigma}) = J, \ f_{\boldsymbol{\sigma}} \in \mathcal{WB}(\mathsf{G}), \ \sigma_k(f_{\boldsymbol{\sigma}}) = \sigma_k \text{ for } k = 1, \dots, r\big\}$$
is a compact analytic subset of $\mathbb{C}^r$, and that $\mathbf{a}$ and $\mathbf{b}$ belong to the same path connected component of X.

Fix $\mathbf{x} \in X$. Since $\mathbf{x} \notin S^{*,r}$, Proposition 2.3.10 and Proposition 2.3.9 imply that there is a small open neighbourhood $U \subset \mathbb{C}^r$ of $\mathbf{x}$ so that $X \cap U = \{\mathbf{z} \in U \mid \mathbf{p}(\mathbf{z}) = J\}$.

We now need to know that X is closed in $\mathbb{C}^r$. Suppose that $\mathbf{x} \in \partial X$. Then by Lemma 3.7.23, $X \subset \overline{D_\varepsilon^r} \setminus N(S^{*,r})$ for some very small $\varepsilon > 0$ and a neighbourhood $N(S^{*,r})$ of $S^{*,r}$ in $\mathbb{C}^r$. Thus we see that $\mathbf{x} \in \overline{D_\varepsilon^r} \setminus S^{*,r}$. Now take a sequence $\{\mathbf{x}_n\}_{n \geq 0} \subset X$ converging to $\mathbf{x}$.

Lemma 2.4.8 implies that $\mathbf{p}(\mathbf{x}) = \lim \mathbf{p}(\mathbf{x}_n) = J$, so if $f_{\mathbf{x}} \in \mathcal{WB}(\mathsf{G})$ then Proposition 2.3.9 implies that $\mathbf{x} \in X$ as required. So now we must show that $f_{\mathbf{x}} \in \mathcal{WB}(\mathsf{G})$. This is assured by the definition of a well behaved map, Theorem 3.3.9 and by Lemma 3.7.4. Thus X is indeed closed.

As a result, if $\mathbf{x} \in \mathbb{C}^r \setminus X$ then there is a small neighbourhood U of $\mathbf{x}$ such that $X \cap U = \varnothing = \{\mathbf{z} \in U \mid 1 = 0\}$. Thus X is an analytic subset of $\mathbb{C}^r$ as required. Lemma 3.7.8 implies that X is bounded, and this (together with the fact that X is closed) implies that X is compact.

Now we need to construct a continuous path in X between $\mathbf{a}$ and $\mathbf{b}$. Now let $\mathsf{G}' = (\mathsf{G}'_1, \dots, \mathsf{G}'_\nu) \in$ Admissible be such that it has no closed gates and so that $\mathsf{G}'_i = \mathsf{G}_i$ for all i such that $\mathsf{G}_i \neq \star$.

For $\mathbf{s} = (s_1, \dots, s_\nu)$ we define $h_{\mathbf{s}}(z) = z + z(z - s_1) \dots (z - s_\nu) v_{\mathbf{s}}(z)$. Now define (using Notation 3.7.22) $\Theta_r := \Theta(\mathsf{G}) : K_0^r \setminus S^{*,r} \to \mathbb{C}^r$ and $\Theta_\nu := \Theta(\mathsf{G}') : K_0^\nu \setminus S^{*,\nu} \to \mathbb{C}^\nu$.

Using the method in the proof of Lemma 3.7.12 we can find sequences $\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k \in \mathbb{C}^\nu$ so that $h_{\boldsymbol{\alpha}_k} \to f_{\mathbf{a}}$, $h_{\boldsymbol{\beta}_k} \to f_{\mathbf{b}}$ as $k \to +\infty$ and $h_{\boldsymbol{\alpha}_k}, h_{\boldsymbol{\beta}_k} \in \mathcal{WB}(\mathsf{G}')$ for all k. (Notice that $\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k \in \mathbb{C}^\nu$ but that $\mathbf{a}, \mathbf{b} \in \mathbb{C}^r$.) We can also make sure that for each i with $\mathsf{G}_i = \star$ we have $\mathrm{Re}\, \tilde{\tau}_i(h_{\boldsymbol{\alpha}_k}) \to -\infty$ and $\mathrm{Re}\, \tilde{\tau}_i(h_{\boldsymbol{\beta}_k}) \to -\infty$ as $k \to +\infty$. Also, the continuity of $f \mapsto \tilde{\tau}_i(f)$ (see Proposition 2.4.3) implies that for each i with $\mathsf{G}_i \neq \star$ we have $\tilde{\tau}_i(h_{\boldsymbol{\alpha}_k}) \to \tilde{\tau}_i(f_{\mathbf{a}})$ and $\tilde{\tau}_i(h_{\boldsymbol{\beta}_k}) \to \tilde{\tau}_i(f_{\mathbf{b}})$ as $k \to +\infty$.

We can assume without loss of generality that
$$\boldsymbol{\alpha}_k = \big(\sigma_1(h_{\boldsymbol{\alpha}_k}), \dots, \sigma_\nu(h_{\boldsymbol{\alpha}_k})\big) \qquad \text{and} \qquad \boldsymbol{\beta}_k = \big(\sigma_1(h_{\boldsymbol{\beta}_k}), \dots, \sigma_\nu(h_{\boldsymbol{\beta}_k})\big). \tag{3.24}$$

Using the method in the proof of Lemma 3.7.26, for each k we can construct a path $\{\mathbf{c}_k(t)\}_{t \in [0,1]} \subset K_0^\nu$ so that $\mathbf{c}_k(0) = \boldsymbol{\alpha}_k$, $f_{\mathbf{c}_k(t)} \in \mathcal{WB}(\mathsf{G})$ for all $t \in [0,1]$ and $\Theta_\nu(\mathbf{c}_k(t)) = (1 - t)\Theta_\nu(\boldsymbol{\alpha}_k) + t\Theta_\nu(\boldsymbol{\beta}_k) \in H_\xi^\nu$. By the uniqueness assured by Lemma 3.7.27, and (3.24) we see that $\mathbf{c}_k(1) = \boldsymbol{\beta}_k$.

Also, by Lemma 3.7.8 we see that for all $t \in [0,1]$ and k large we have $\mathbf{c}_k(t)$ close to $\mathbf{0}$.

Recall that if A, B are compact metric spaces then $C(A, B) = \{f : A \to B \mid f \text{ is continuous}\}$ is compact with respect to the uniform metric. Thus there is a continuous path $\mathbf{c} : [0,1] \to \mathbb{C}^\nu$ and a

subsequence $\{\mathbf{c}_{k_n}\}_n$ of $\{\mathbf{c}_k\}_k$ so that $\mathbf{c}_{k_n}(t) \to \mathbf{c}(t)$ uniformly on $[0,1]$. Note that $\mathbf{c}_{k_n}([0,1]) \subset D_\varepsilon^\nu$ for some small $\varepsilon > 0$ and all n (by Lemma 3.7.8), so $\mathbf{c}([0,1]) \subset \overline{D_\varepsilon}$. Also, $h_{\mathbf{c}(0)} = f_{\mathbf{a}}$, $h_{\mathbf{c}(1)} = f_{\mathbf{b}}$. One can show (using the arguments from the proof of Lemma 3.7.26) that $h_{\mathbf{c}(t)} \in \mathcal{WB}(\mathsf{G})$ for all $t \in [0,1]$. Thus there is a continuous path $\boldsymbol{\sigma} : [0,1] \to \mathbb{C}^r$ so that $f_{\boldsymbol{\sigma}(t)} = h_{\mathbf{c}(t)} \in \mathcal{WB}(\mathsf{G})$ for all $t \in [0,1]$, with $\boldsymbol{\sigma}(0) = \mathbf{a}$ and $\boldsymbol{\sigma}(1) = \mathbf{b}$.

Since $\Theta_\nu(\mathbf{c}_k(t)) = (1-t)\Theta_\nu(\boldsymbol{\alpha}_k) + t\Theta_\nu(\boldsymbol{\beta}_k) \in H_\xi^\nu$ for all $t \in [0,1]$, $k \in \mathbb{N}$ and the fact that for each i with $\mathsf{G}_i \neq \star$ we have $\mathrm{Re}\,\tilde{\tau}_i(h_{\boldsymbol{\alpha}_k})$, $\mathrm{Re}\,\tilde{\tau}_i(h_{\boldsymbol{\beta}_k}) \to \tilde{\tau}_i(f_{\mathbf{a}}) = \tilde{\tau}_i(f_b)$, we see that $\Theta_r(\boldsymbol{\sigma}(t)) = \Theta_r(\mathbf{a}) = \Theta_r(\mathbf{b})$ for all $t \in [0,1]$. Proposition 2.4.12 then implies that $\mathbf{p}(\boldsymbol{\sigma}(t)) = \mathbf{p}(\mathbf{a}) = \mathbf{p}(\mathbf{b}) = J$ for all $t \in [0,1]$.

Thus $\{\boldsymbol{\sigma}(t)\}_{t \in [0,1]}$ is indeed a path through X from $\mathbf{a}$ to $\mathbf{b}$.

Now [Na, Corollary III.1] states that any compact analytic subset of $\mathbb{C}^r$ is a finite set, so X is finite. And since we have shown that $\mathbf{a}$ and $\mathbf{b}$ belong to the same path connected component of X, it is clear that $\mathbf{a} = \mathbf{b}$ as required. $\blacksquare$

> **Corollary 3.7.30** (**Uniqueness of** $f(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)$) *Let* $\xi > 0$ *be large, and fix* $\mathsf{G} \in$ Admissible. *Let* $\mathbf{H}(\mathsf{G}, \xi)$ *and* $\mathcal{T}$ *be defined as they are in* §2.7.
> *If* $f_1, f_2 \in \mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ *and* $\mathcal{T}(f_1) = \mathcal{T}(f_2) \in \mathbf{H}(\mathsf{G}, \xi)$ *then* $f_1 = f_2$.

Proof. This is immediate from the proof of the above Lemma. $\blacksquare$

> **Corollary 3.7.31** *If* $f_1, f_2 \in \mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ *and* $\mathcal{T}^{\#}(f_1) = \mathcal{T}^{\#}(f_2)$ *then* $f_1 = f_2$.

Proof. This is just the same as Corollary 3.7.30, just without the condition that the lifted phases are in $\mathbf{H}(\mathsf{G}, \xi)$.

Recall that when we defined $\mathcal{WB}$ we had to choose some small $r_0 > 0$ and $\mathcal{N}_0$. Therefore we can write $\mathcal{WB}(\mathsf{G}, r_0, \mathcal{N}_0)$ instead of just $\mathcal{WB}(\mathsf{G})$. Also we can denote by $\mathcal{WWB}(\mathsf{G}, r_0, \mathcal{N}_0)$ the set of weakly well behaved maps with gate structure G.

We will show that r_0 and $\mathcal{N}_0$ can be replaced by some $r_0' \in (0, r_0)$ and $\mathcal{N}_0' \subset \mathcal{N}_0$ such that

$$\mathcal{WB}(\mathsf{G}, r_0', \mathcal{N}_0') \subset \mathcal{WB}(\mathsf{G}, r_0, \mathcal{N}_0)$$

and that for any $f \in \mathcal{WB}(\mathsf{G}, r_0', \mathcal{N}_0')$ we have

$$\left(\tilde{\tau}_1(f), \ldots, \tilde{\tau}_\nu(f)\right) \in \mathbf{H}(\mathsf{G}, \xi).$$

Thus if $f_1, f_2 \in \mathcal{WB}(\mathsf{G}, r_0', \mathcal{N}_0') \cap \mathcal{F}$ then Corollary 3.7.30 can be applied to show that $f_1 = f_2$. Therefore the Corollary is proved if we replace the original r_0 and $\mathcal{N}_0$ by r_0' and $\mathcal{N}_0'$.

We choose $r_0' > 0$ much smaller than r_0 such that $\frac{2}{\nu(r_0'/2)^\nu} > \xi$, and a very small neighbourhood $\mathcal{N}_0' \subset \mathcal{N}_0$ of f_0. (Notice that the values of the lifted phases are independent of the choice of $r_0, \mathcal{N}_0$.)

Then there is a family of maps $\mathcal{WWB}(\mathsf{G}, r_0', \mathcal{N}_0')$ associated to our choice of r_0' and $\mathcal{N}_0'$. It is fairly clear (from the fact that "trajectories for $\dot{z} = i[f(z) - z]$ cannot cross one another") that $\mathcal{WWB}(\mathsf{G}, r_0', \mathcal{N}_0') \subset \mathcal{WWB}(\mathsf{G}, r_0, \mathcal{N}_0)$.

Then Lemma 3.7.4 implies that $\mathcal{WB}(\mathsf{G}, r_0', \mathcal{N}_0') \subset \mathcal{WB}(\mathsf{G}, r_0, \mathcal{N}_0)$. Lemma 3.7.4 also implies that if $f \in \mathcal{WB}(\mathsf{G}, r_0', \mathcal{N}_0')$ then $\left(\tilde{\tau}_1(f), \ldots, \tilde{\tau}_\nu(f)\right) \in \mathbf{H}(\mathsf{G}, \xi)$ as required. $\blacksquare$

3.7.7 Proofs of Thms 2.7.1 and 2.7.4

Proof of Theorem 2.7.1 on page 24 (**Injectivity of** $\mathcal{T}$) Lemma 3.7.26, Corollary 3.7.31 and Lemma 3.7.4 imply the result. $\blacksquare$

Proof of Corollary 2.7.2 on page 24 (**Existence of** $f(\mathsf{G}, \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu)$) Part (1.) is immediate from Theorem 2.7.1.

Part (2.) follows from the fact that if $f_{\boldsymbol{\sigma}}$ is defined as it is in Notation 3.7.22 then $\boldsymbol{\sigma} \mapsto \mathcal{T}(f_{\boldsymbol{\sigma}})$ is holomorphic (on a suitable domain of definition). $\blacksquare$

Proof of Corollary 2.7.3 on page 24 (**Every $\langle f_0, g \rangle$ is approached by some sequence $\{f_k\}$**) By Theorem 2.7.1 and Proposition 2.4.5 there is a sequence $f_k \to f_0$ in $\mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ such that for all k we have $\tilde{\tau}_i(f_k) = \tilde{\theta}_i - k$ for each i with $\mathsf{G}_i \neq \star$.

Proposition 2.5.1 then tells us that $f_k^k \to g$ as $k \to +\infty$ uniformly on compact sets. ∎

We will use the following technical Lemma in the proof of Theorem 2.7.4.

> **Lemma 3.7.32** *Suppose that $D \in \mathbb{C}^m$ is a neighbourhood of $\mathbf{0}$ and $\boldsymbol{\alpha}_0 : D \to \mathbb{C}^m$ is holomorphic with $\boldsymbol{\alpha}_0(\mathbf{0}) = \mathbf{0}$ and the Jacobian $\mathrm{Jac}_{\boldsymbol{\alpha}_0}(\mathbf{0}) \neq 0$. Suppose also that we have a sequence $\{\boldsymbol{\alpha}_k : D \to \mathbb{C}^m\}$ such that $\boldsymbol{\alpha}_k(\mathbf{z}) \to \boldsymbol{\alpha}_0(\mathbf{z})$ uniformly on compact sets.*
>
> *Then there exists a sequence $\mathbf{z}_k \to \mathbf{0}$ such that $\boldsymbol{\alpha}_k(\mathbf{z}_k) = \mathbf{0}$ for all large k. For a large fixed k_0 the sequence $\{\mathbf{z}_k\}_{k \geqslant k_0}$ is unique.*

Proof. Weierstrass' Theorem implies that the Jacobians converge $\mathrm{Jac}_{\boldsymbol{\alpha}_k} \to \mathrm{Jac}_{\boldsymbol{\alpha}_0}$ as $k \to +\infty$ uniformly on compact sets. Also of course, $\mathbf{z} \mapsto \mathrm{Jac}_{\boldsymbol{\alpha}_k}(\mathbf{z})$ is continuous for all $k \geqslant 0$.

Thus we can find a small open neighbourhood B of $\mathbf{0} = (0, \ldots, 0)$ and a $C > 0$ so that $|\mathrm{Jac}_{\boldsymbol{\alpha}_0}(\mathbf{z})| > C > 0$ for every $\mathbf{z} \in B$. We can also find a very large k_0 so that $|\mathrm{Jac}_{\boldsymbol{\alpha}_k}(\mathbf{z})| > C/2$ for every $\mathbf{z} \in B$ and $k \geqslant k_0$.

Assuming that B is sufficiently small, $\boldsymbol{\alpha}_0$ must map $\overline{B}$ biholomorphically onto $\boldsymbol{\alpha}_0(\overline{B})$, by the Inverse Mapping Theorem (Theorem 3.7.21).

We want to show that for all $k \geqslant k_0$ large, we will have $\mathbf{0} \in \boldsymbol{\alpha}_k(B)$. Let $G \subset \boldsymbol{\alpha}_0(B)$ be a small compact path connected neighbourhood of $\mathbf{0}$. Then in particular we have $\partial \boldsymbol{\alpha}_0(B) \cap G = \varnothing$. And since $\boldsymbol{\alpha}_0$ is biholomorphic on $\overline{B}$ we see that $\boldsymbol{\alpha}_0(\partial B) = \partial \boldsymbol{\alpha}_0(B)$. So then $\boldsymbol{\alpha}_0(\partial B) \cap G = \varnothing$, and if $k \geqslant k_0$ (and k_0 is large enough) then $\boldsymbol{\alpha}_k(\partial B) \cap G = \varnothing$.

Notice that the Inverse Mapping Theorem implies that $\partial \boldsymbol{\alpha}_k(B) \subset \boldsymbol{\alpha}_k(\partial B)$ for all k. Therefore $\partial \boldsymbol{\alpha}_k(B) \cap G = \varnothing$ for all $k \geqslant k_0$ (if k_0 is large enough).

Therefore for $k \geqslant k_0$, either $G \subset \boldsymbol{\alpha}_k(B)$ or $\boldsymbol{\alpha}_k(B) \cap G = \varnothing$. But since $\boldsymbol{\alpha}_k(\mathbf{0}) \to \mathbf{0}$ and G is a neighbourhood of $\mathbf{0}$, we see that $\boldsymbol{\alpha}_k(\mathbf{0}) \cap G \neq \varnothing$ for all k large. Thus $G \subset \boldsymbol{\alpha}_k(B)$ for $k \geqslant k_0$ (if k_0 is large enough), implying that $\left(\boldsymbol{\alpha}_k|_B\right)^{-1}(\mathbf{0}) \neq \varnothing$.

Thus we can take a sequence $\{\mathbf{z}_k\}_{k \geqslant k_0} \subset B$ such that $\boldsymbol{\alpha}_k(\mathbf{z}_k) = \mathbf{0}$ for $k \geqslant k_0$. Supposing for contradiction that $\mathbf{z}_k \not\to \mathbf{0}$, there will be an accumulation point $\mathbf{z}^* \in \overline{B} \setminus \{\mathbf{0}\}$ of $\{\mathbf{z}_k\}_{k \geqslant k_0}$. But then by continuity we must have $\boldsymbol{\alpha}_0(\mathbf{z}^*) = \mathbf{0}$, which contradicts the assertion the $\boldsymbol{\alpha}_0$ is biholomorphic on $\overline{B}$. Thus $\mathbf{z}_k \to \mathbf{0}$ as required.

Now suppose for contradiction that for all k_0 the sequence $\{\mathbf{z}_k\}_{k \geqslant k_0}$ is not unique. Then there is some $\{m_n\}_{n \geqslant 1} \subset \mathbb{N}$ so that $m_n \to +\infty$, and sequences $\{\mathbf{a}_n\}_{n \geqslant 1}$ and $\{\mathbf{b}_n\}_{n \geqslant 1}$ in B such that $\boldsymbol{\alpha}_{m_n}(\mathbf{a}_n) = \boldsymbol{\alpha}_{m_n}(\mathbf{b}_n) = \mathbf{0}$ and $\mathbf{a}_n \neq \mathbf{b}_n$ for all $n \in \mathbb{N}$, and with $\mathbf{a}_n, \mathbf{b}_n \to \mathbf{0}$ as $n \to +\infty$.

So then let $F(\mathbf{w}) = \boldsymbol{\alpha}_0(\mathbf{w})$, $F_n(\mathbf{w}) = \boldsymbol{\alpha}_{m_n}(\mathbf{w} + \mathbf{b}_{m_n})$ and $\mathbf{w}_n = \mathbf{a}_{m_n} - \mathbf{b}_{m_n}$. Then we can use Lemma 3.7.33 below which contradicts the fact that $\mathrm{Jac}_{\boldsymbol{\alpha}_0}(\mathbf{0}) \neq 0$. Thus $\{\mathbf{z}_k\}_{k \geqslant k_0}$ is unique if k_0 is large enough. ∎

> **Lemma 3.7.33** *Suppose that $D \subset \mathbb{C}^m$ is a domain containing $\mathbf{0}$, and that $F : D \to \mathbb{C}^m$ is holomorphic with $F(\mathbf{0}) = \mathbf{0}$. Now suppose that F_k converges to F uniformly on compact sets and there is a sequence $\{\mathbf{w}_k\}_{k \geqslant 0} \subset D \setminus \{\mathbf{0}\}$ converging to $\mathbf{0}$ with $F_k(\mathbf{w}_k) = F_k(\mathbf{0}) = \mathbf{0}$ for all $k \geqslant 0$.*
>
> *Then $\mathrm{Jac}_F(\mathbf{0}) = 0$.*

Proof. Let $A : \mathbb{C}^m \to \mathbb{C}^m$ and $A_k : \mathbb{C}^m \to \mathbb{C}^m$ be the linear maps associated with the "Jacobian matrices at $\mathbf{0}$." These satisfy $\det A = \mathrm{Jac}_F(\mathbf{0})$, $\det A_k = \mathrm{Jac}_{F_k}(\mathbf{0})$, and

$$\frac{\|F(\mathbf{w}) - A\mathbf{w}\|}{\|\mathbf{w}\|} \to 0 \qquad \text{and} \qquad \frac{\|F_k(\mathbf{w}) - A_k\mathbf{w}\|}{\|\mathbf{w}\|} \to 0 \tag{3.25}$$

as $\mathbf{w} \to \mathbf{0}$ (where $\| \cdot \|$ is the Euclidean norm).

Suppose for contradiction that $\det A \neq 0$. Then there is a $\rho > 0$ such that $\|A\mathbf{w}\| > \rho\|\mathbf{w}\|$ for all $\mathbf{w} \in \mathbb{C}^m$. For k large we must also have $\|A_k\mathbf{w}\| > \frac{\rho}{2}\|\mathbf{w}\|$ for all $\mathbf{w} \in \mathbb{C}^m$.

Let $G(\mathbf{w}) = F(\mathbf{w}) - A\mathbf{w}$ and $G_k(\mathbf{w}) = F_k(\mathbf{w}) - A_k\mathbf{w}$ for $\mathbf{w} \in D$. It is clear that $G_k \to G$ uniformly on compact sets. Then by (3.25) there is a neighbourhood $D' \subset D$ of $\mathbf{0}$ so that $\|G(\mathbf{w})\| \leqslant \frac{\rho}{8}\|\mathbf{w}\|$ for all $\mathbf{w} \in D'$ and k large. Also we will have $\|G_k(\mathbf{w})\| \leqslant \frac{\rho}{4}\|\mathbf{w}\|$ for all $\mathbf{w} \in D'$ if k is large enough.

Therefore for $\mathbf{w} \in D' \setminus \{\mathbf{0}\}$ and k large we have

$$
\begin{aligned}
\|F_k(\mathbf{w})\| &= \|A_k(\mathbf{w}) + G_k(\mathbf{w})\| \\
&\geqslant \|A_k(\mathbf{w})\| - \|G_k(\mathbf{w})\| \\
&\geqslant \tfrac{\rho}{2}\|\mathbf{w}\| - \tfrac{\rho}{4}\|\mathbf{w}\| \\
&= \tfrac{\rho}{4}\|\mathbf{w}\| > 0.
\end{aligned}
$$

This contradicts the existence of $\mathbf{w}_k \to \mathbf{0}$ in $D \setminus \{\mathbf{0}\}$ with $F_k(\mathbf{w}_k) = 0$ for all k. Thus our assumption that $\mathrm{Jac}_F(\mathbf{0}) \neq 0$ was wrong. ∎

Proof of Theorem 2.7.4 on page 25 (Simultaneous orbit correspondence) To keep the notation simple we assume that $\mathsf{G} \in \mathsf{Admissible}$ has no closed gates, but the same argument works even if this is not so.

Then for each $i \in \mathbb{Z}/\nu\mathbb{Z}$ we let $\tilde{\theta}_i := \Phi_{j,-,f_0}(b_j(f_0)) - \Phi_{i,+,f_0}(a_i(f_0))$ if $\mathsf{G}_i = j$. If $\mathbf{z} = (z_1, \ldots, z_\nu) \in \mathbb{C}^\nu$, we set $h_{k,\mathbf{z}} := f(\mathsf{G}; \tilde{\theta}_1 - k - z_1, \ldots, \tilde{\theta}_\nu - k - z_\nu)$ (in the notation of Corollary 2.7.2).

Now for $i \in \mathbb{Z}/\nu\mathbb{Z}$ set $j = \mathsf{G}_i$ we let

$$
\begin{aligned}
\alpha_k^{(i)}(\mathbf{z}) :=& \Phi_{j,-,h_{k,\mathbf{z}}}(b_j(h_{k,\mathbf{z}})) - \Phi_{j,-,h_{k,\mathbf{z}}}(h_{k,\mathbf{z}}^k(a_i(h_{k,\mathbf{z}}))) \\
=& \Phi_{j,-,h_{k,\mathbf{z}}}(b_j(h_{k,\mathbf{z}})) - [\Phi_{i,+,h_{k,\mathbf{z}}}(a_i(h_{k,\mathbf{z}})) + k + \tilde{\tau}_i(h_{k,\mathbf{z}})] \\
=& [\Phi_{j,-,h_{k,\mathbf{z}}}(b_j(h_{k,\mathbf{z}})) - \Phi_{i,+,h_{k,\mathbf{z}}}(a_i(h_{k,\mathbf{z}}))] - k - \tilde{\tau}_i(h_{k,\mathbf{z}}) \\
=& [\tilde{\theta}_i + o(1)] - k - [\tilde{\theta}_i - k - z_i] \\
=& z_i + o(1) = \alpha_0^{(i)}(\mathbf{z}) + o(1)
\end{aligned}
$$

as $k \to +\infty$. Notice that if $\alpha_k^{(i)}(\mathbf{z}) = 0$ then $h_{k,\mathbf{z}}^k(a_i(h_{k,\mathbf{z}})) = b_j(h_{k,\mathbf{z}})$.

If $\boldsymbol{\alpha}_k(\mathbf{z}) = (\alpha_k^{(1)}(\mathbf{z}), \ldots, \alpha_k^{(\nu)}(\mathbf{z}))$ we find that for fixed $\mathbf{z}$ we will have $\boldsymbol{\alpha}_k(\mathbf{z}) \to \boldsymbol{\alpha}_0(\mathbf{z}) = \mathbf{z}$ as $k \to +\infty$ uniformly on compact sets. So then we can apply Lemma 3.7.32 to show that there will be a sequence $\mathbf{z}_k \to \mathbf{0}$ as $k \to +\infty$, such that $\boldsymbol{\alpha}_k(\mathbf{z}_k) = \mathbf{0}$. So if we set $f_k = h_{k,\mathbf{z}_k}$ then everything will work as it is supposed to.

This sequence $\{f_k\}_{k \geqslant k_0}$ then satisfies **(1.)**, **(2.)** (and converges to f_0).

We still need to prove that this is unique for a sufficiently large k_0. Suppose for contradiction that we have sequences $\{G_k\}_k$ and $\{H_k\}_k$ in $\mathcal{WB}(\mathsf{G}) \cap \mathcal{F}$ satisfying **(1.)** and **(2.)** and that there is a strictly increasing sequence of integers $\{k_n\}_n$ such that $G_{k_n} \neq H_{k_n}$ for all n. (Note that we do not assume that $G_{k_n} \to f_0$ or $H_{k_n} \to f_0$.)

We know that for all $i \in \mathbb{Z}_\nu$, $s \in \{+, -\}$ the map $f \mapsto \Phi_{i,s,f}$ on $\mathcal{WB}$ is continuous in the compact-open topology. It follows that there is some $M > 0$ such that for each $i \in \mathbb{Z}_\nu$ we have $\|\Phi_{i,+,f}\|_{X_i} < M$ and $\|\Phi_{i,-,f}\|_{Y_i} < M$ for all $f \in \mathcal{WB}$ sufficiently close to f_0. So if $i, j \in \mathbb{Z}/\nu\mathbb{Z}$ with $\mathsf{G}_i = j$ then

$$
\begin{aligned}
\tilde{\tau}_i(G_{k_n}) &= \Phi_{j,-,G_{k_n}}\big(b_j(G_{k_n})\big) - \Phi_{i,+,G_{k_n}}\big(b_j(G_{k_n})\big) \\
&= \Phi_{j,-,G_{k_n}}\big(b_j(G_{k_n})\big) - \Phi_{i,+,G_{k_n}}\big(G_{k_n}^{k_n}(a_i(G_{k_n}))\big) \\
&= \big[\Phi_{j,-,G_{k_n}}\big(b_j(G_{k_n})\big) - \Phi_{i,+,G_{k_n}}\big(a_i(G_{k_n})\big)\big] - k_n = \mathcal{O}(1) - k_n
\end{aligned}
$$

as $n \to +\infty$ and similarly, $\tilde{\tau}_i(H_{k_n}) = \mathcal{O}(1) - k_n$. Then Lemma 3.7.8 implies that $G_{k_n} \to f_0$ and $H_{k_n} \to f_0$ as $n \to +\infty$. The continuity of the Fatou coordinates then ensures that $\tilde{\tau}_i(G_{k_n}) = [\tilde{\theta}_i + o(1)] - k_n$ and $\tilde{\tau}_i(H_{k_n}) = [\tilde{\theta}_i + o(1)] - k_n$ as $n \to +\infty$. So if

$$
a_{k_n,i} := \tilde{\theta}_i - k_n - \tilde{\tau}_i(G_{k_n}), \qquad b_{k_n,i} := \tilde{\theta}_i - k_n - \tilde{\tau}_i(H_{k_n})
$$

and $\mathbf{a}_{k_n} := (a_{k_n,1} \ldots, a_{k_n,\nu})$, $\mathbf{b}'_{k_n} := (b'_{k_n,1} \ldots, b'_{k_n,\nu})$ then $\mathbf{a}_{k_n} \to \mathbf{0}$ and $\mathbf{b}_{k_n} \to \mathbf{0}$, as $n \to +\infty$. But Corollary 2.7.2 implies that $h_{k_n,\mathbf{a}_k} = G_{k_n}$ and $h_{k_n,\mathbf{b}_k} = H_{k_n}$ for n large. Thus $\boldsymbol{\alpha}_{k_n}(\mathbf{z}_{k_n}) = \boldsymbol{\alpha}_{k_n}(\mathbf{z}'_{k_n}) = \mathbf{0}$ for all n large. This contradicts the uniqueness in Lemma 3.7.32. ∎

3.8   Parameterisation of the well behaved maps

Proof of Theorem 2.8.1 on page 25   (Injectivity of $\mathcal{T}^{\#}$) For a fixed σ_0 close to 0 and u close to u_{f_0}, consider the holomorphic family $\mathcal{F}'$ of maps of the form

$$f_{\mathbf{s}}(z) = z + (z - \sigma_0)(z - s_1)\ldots(z - s_\nu)u(z)$$

where $\mathbf{s} = (s_1, \ldots, s_\nu)$ is close to $\mathbf{0}$. One can show that $\mathcal{T}$ is injective on $\mathcal{WB}(\mathsf{G}) \cap \mathcal{F}'$ with basically the same proof that was used for Theorem 2.7.1.

The rest of the proof follows easily.   ∎

Proof of Corollary 2.8.2 on page 25   (Existence of $f(\mathsf{G}; \tilde{\theta}_1, \ldots, \tilde{\theta}_\nu; \sigma_0; u)$) Simple extension of Corollary 2.7.2.   ∎

Appendix A

Fundamental Regions for Non-Well Behaved f's

Here we give a couple of examples of f's which are not well behaved, but still have "Fatou coordinates" of some kind. No details are given.

We can also consider $h_{s,0}(z) = z + z(z - s)^2$ where $s > 0$. This $h_{s,0}$ will not be well behaved. Significantly, $\gamma_{i,\pm,h_{s,0}}(t) \in K_0$ for all i. We can then try to define the fundamental regions $S_{i,\pm,h_{s,0}}$ as the closed region bounded by the closure of $\gamma_{i,\pm,h_{s,0}}(\mathbb{R}) \cup h_{s,0}(\gamma_{i,\pm,h_{s,0}}(\mathbb{R}))$.

Then we get $S_{1,-,h_{s,0}}$ and $S_{1,+,h_{s,0}} = S_{2,+,h_{s,0}}$ which are "fundamental regions," and $S_{2,-,h_{s,t}}$ which is an annulus. (This is true even if $s > 0$ is relatively large.) We can then perturb the double fixed point s to give $h_{s,t}(z) = z + z(z - s + it)(z - s - it)$ with $t > 0$ small, which will have the dynamics shown in Figure 1.1(d), and have the fundamental regions shown in Figure A.1.

Fatou coordinates can be defined upon the fundamental regions $S_{1,-,h_{s,t}}$ and $S_{1,+,h_{s,t}} = S_{2,+,h_{s,t}}$ in the usual way. However $S = S_{2,-,h_{s,t}}$ is an annulus, and if we try to define a coordinate on this, then the coordinate must be interpreted modulo $\eta := \jmath(f,\sigma)$, where σ is the fixed point inside S. That is to say there is an analytic $\Phi : S \to \mathbb{C}/\eta\mathbb{Z}$ such that $\Phi(f(z)) = \Phi(z) + 1 \pmod{\eta}$ for all $z \in S$. (This comes from the fact that $h_{s,0}$ in a neighbourhood of σ is conjugate to $z \mapsto e^{2\pi i \eta} z$ in some neighbourhood of 0.) This extends to the whole "punctured disc" U which is bounded by $h_{s,t}^2(\ell_{2,-,h_{s,t}})$ and punctured at σ. This Φ is unique up to addition by a constant.

The bifurcation of $h_{s,0}$ to give $h_{s,t}$ is used in [La] to prove the non-local connectivity of the cubic connectedness locus. (See also [EY, Appendix B].)

We find however that $J(h_{s,t}) \to J(h_{0,0})$ and $K(h_{s,t}) \to K(h_{0,0})$ as $(s,t) \to 0$, which is not really very interesting.
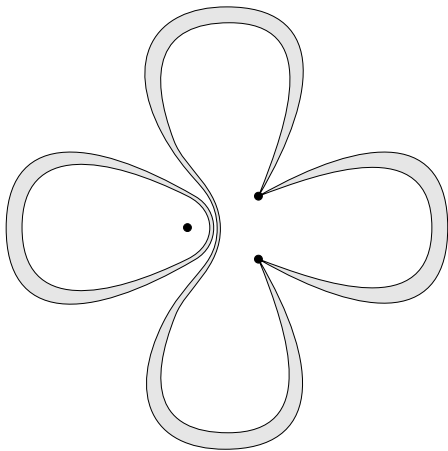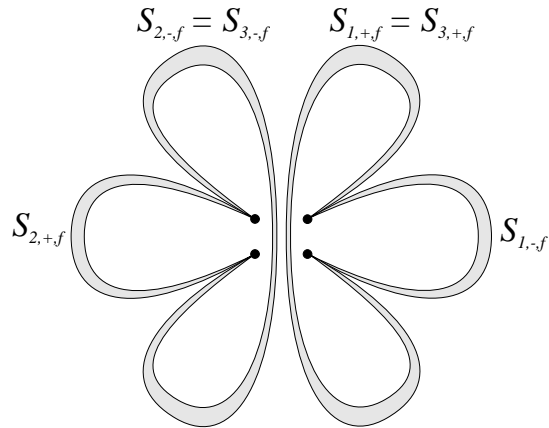

Figure A.1:

Figure A.2:

75

However we can give an example of a sequence of non-well behaved functions converging to f_0, which have a non-trivial limit behaviour.

Consider the map $f_a(z) = z + (z - a)(z + a)(z - \bar{a})(z + \bar{a})$ close to $f_0(z) = z + z^4$, where $\mathrm{Re}\, a > 0$ and $\mathrm{Im}\, a > 0$. Clearly the restriction $f_a : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function and the critical point on the real line must escape.

By the symmetry we can show that $S_{1,+,f_a} = S_{3,+,f_a}$ and $S_{2,-,f_a} = S_{3,-,f_a}$, as shown in Figure A.2. Note that for any a we will have $T(a) = 2\pi i[\iota(f_a, a) + \iota(f_a, -\bar{a})] \in \mathbb{R}$. There will be coordinates $\Phi_{i,\pm,f_a} : S'_{i,\pm,f_a} \to \mathbb{C}$, and we can normalise these so that $\Phi_{1,-,f_a}(z_{1,-}) = 0$ and $\Phi_{2,+,f_a}(z_{2,+}) = 0$ for all k. Then we have $\Phi_{1,-,f_a} \to \Phi_{1,-,f_0}$ and $\Phi_{2,+,f_a} \to \Phi_{2,+,f_0}$ as $a \to 0$.

Suppose that $a_k \to 0$ is a sequence and there is a small $\varepsilon > 0$ such that $\arg a_k \in [\varepsilon, \frac{\pi}{2} - \varepsilon]$ for all k. Suppose also that $N_k \to +\infty$ is a sequence of integers such that $N_k + T(a_k) \to \tilde{\theta} \in \mathbb{R}$ it can be shown that

$$f_{a_k}^{N_k} \to g$$

uniformly on compact subsets of $U_{2,+,f}$, where $g = g(\mathsf{G}; \infty, \tilde{\theta}, \infty)$ is the Lavaurs map with $\mathsf{G} = (\star, 1, \star)$. (Note that we are not using the preferred normalisation here.) This g maps $\{x \mid x < 0\}$ onto $\{x \mid x > 0\}$. It is then to be expected that $K(f_{a_k})$ converges in some way to $K(f_0, g)$. See Figures A.3 and A.4.
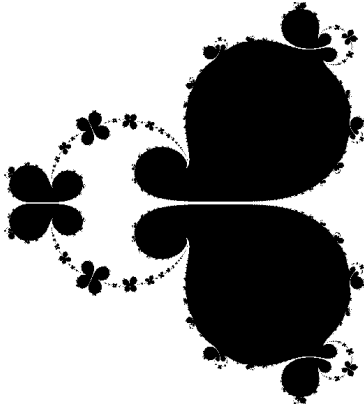

Figure A.3: $K(f_{a_k})$, where a_k is small. f_{a_k} has two repelling fixed points $-a_k, -\bar{a}_k$ to the left of the imaginary axis, and two attracting fixed points $a_k, \bar{a}_k$ to the right.
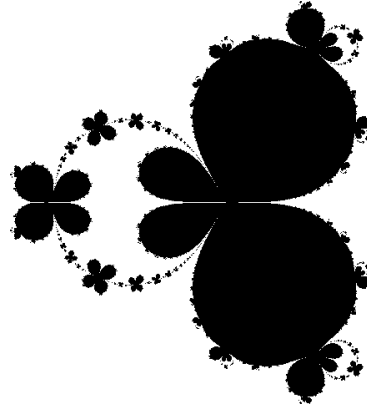
Figure A.4: *The associated $K(f_0, g)$.*

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Department of Mathematics
Hiroshima University
Kagamiyama 1-3-1
Higashi-Hiroshima 739-8526
JAPAN

e-mail: **roudkerk@mail.com** or **oudkerk@math.sci.hiroshima-u.ac.jp**