The Determinant Line Bundle for Fredholm Operators: Construction, Properties, and Classification

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Abstract

We provide a thorough construction of a system of compatible determinant line bundles over spaces of Fredholm operators, fully verify that this system satisfies a number of important properties, and include explicit formulas for all relevant isomorphisms between these line bundles. We also completely describe all possible systems of compatible determinant line bundles and compare the conventions and approaches used elsewhere in the literature.

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1 Introduction

A Fredholm operator between Banach vector spaces X and Y is a bounded homomorphism $D\colon X \longrightarrow Y$ such that

$$\operatorname{Im} D \equiv \big\{ Dx\colon x \in X \big\}$$

is closed in Y and the dimensions of its kernel and cokernel,

$$\kappa(D) \equiv \big\{ x \in X\colon Dx = 0 \big\} \qquad \text{and} \qquad \mathfrak{c}(D) \equiv Y/(\operatorname{Im} D),$$

are finite.[1] The space $\mathcal{F}(X,Y)$ of Fredholm operators is an open subspace of the space $\mathcal{B}(X,Y)$ of bounded linear operators $D\colon X \longrightarrow Y$ in the normed topology; see [13, Theorem A.1.5(ii)]. Quillen's construction, outlined in [15, Section 2], associates to each Fredholm operator D a $\mathbb{Z}_2$-graded one-dimensional vector space $\lambda(D) = \det D$, called the determinant line of D, and topologizes, in a systematic way, the set

$$\det{}_{X,Y} \equiv \bigsqcup_{D \in \mathcal{F}(X,Y)} \lambda(D)$$

as a line bundle over $\mathcal{F}(X,Y)$ for each pair (X,Y) of Banach vector spaces. There are in fact infinitely many compatible systems of such topologies, all of which we describe in Section 3.4; they are isomorphic pairwise. This is contrary to suggestions in many papers that there is a unique way of topologizing determinant line bundles in a systematic way and can be viewed as capturing the essence of the *unique up to a canonical isomorphism* statement in [11, Theorem 1]. We describe some intrinsic and not so-intrinsic ways of narrowing down the choices and of choosing a specific system at the end of Subsection 2.2 and at the end of Remark 3.1.

The determinant line bundle plays a prominent role in a number of geometric situations, but unfortunately there appears to be no thorough description of its construction and properties in the literature. The key issue in its construction is the existence of a collection of (set-theoretic) trivializations for $\det_{X,Y}$, such as $\widetilde{\mathcal{I}}_{D,T}$ in (2.26) and $\hat{\mathcal{I}}_{\Theta;D}$ in (3.2), that overlap continuously. The justification for the existence of such a collection in [15] consists of an allusion to some unspecified collection of compatible isomorphisms relating the determinant line bundles in the short exact triples

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{R}^{k+m} & \longrightarrow & \mathbb{R}^{k+m} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R}^{c+m} & \longrightarrow & \mathbb{R}^{c+m} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\tag{1.1}
$$

of homomorphisms, where the middle arrow is the projection onto the last m coordinates. Explicit formulas for such a collection of isomorphisms appear in [1, Section (f)], [9, Section 3.2.1], [12, Section 20.2], [13, Appendix A.2], [16, Section 2], and [17, Section (11a)], while [10, Appendix D.2] and [11, Chapter I] describe it more abstractly. The proof of [13, Theorem A.2.2] uses them to describe trivializations for determinant line bundles for Fredholm operators without checking that they overlap continuously, which in fact is not the case, as discovered by [14]; see Section 3.3 for more details. Key properties of such collections of isomorphisms necessary for the construction of the determinant line bundle are specified in [1], [12], [17], and the construction itself is then briefly

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[1] The first condition is implied by the other two, but is traditionally stated explicitly.

outlined. The discussion of the relevant linear algebra considerations is more extensive in [10], but it contains an important deficiency, which is described in Remark 4.9, and does not complete the construction. However, the general approach of [10, Appendix B] is well-suited for an explicit construction of the determinant line bundle and the analysis of its properties. Explicit formulas for the above collection are used directly to topologize determinant line bundles over spaces of Fredholm operators and for Kuranishi structures in [16] and [14], respectively. The latter are closely related to the two-term case of the bounded complexes of vector bundles for which a determinant line bundle is constructed in [11]. As explained in detail in Section 3.2, using [11, Theorem I], which predates [15], is perhaps the most efficient way for constructing the determinant line bundle and verifying its properties and would eliminate the need for most of our Section 4, but at the cost of explicit formulas for important isomorphisms (which may well be useful in specific applications) and of being self-contained. None of the above works explicitly considers most of the non-trivial properties of the determinant line bundle for Fredholm operators listed in Subsection 2.2.

This paper provides a comprehensive construction of a system of determinant line bundles and a complete verification of many important properties it satisfies. Section 2 sets up the necessary notation and precisely describes the properties we later show this system satisfies. Section 3.1 outlines the determinant line bundle construction carried out in this paper and three alternative approaches, while Section 3.2 provides more details for the approach based on the results obtained in [11]. Section 3.3 compares several conventions for the determinant line bundle that have appeared in the literature. Section 3.4 establishes Theorem 2, which describes all determinant line bundle systems satisfying the properties in Subsection 2.2 and shows that such systems correspond to collections of isomorphisms

$$A_{i,c} \colon \Lambda^c(\mathbb{R}^c) \longrightarrow \mathbb{R}, \qquad i \in \mathbb{Z}, \ c \in \mathbb{Z}^+, \ c \geq -i \,, \tag{1.2}$$

that are orientation-preserving if $i, c \in 2\mathbb{Z}$. In contrast to the viewpoint of the previous paragraph, there are *no* compatibility conditions on the isomorphisms in these collections. By Theorem 2, the compatible systems of topologies on determinant line bundles correspond to the compatible systems of isomorphisms for the exact triples (1.1) and to the compatible collections of isomorphisms for exact triples of Fredholm operators. Section 4, which is motivated by [11, Section 1] and [10, Appendix D.2], deals with the relevant linear algebra. In particular, Subsection 4.2 provides explicit formulas for a collection of exact triple isomorphisms $\Psi_{\mathfrak{t}}$ as in (2.27) and dualization isomorphisms $\widetilde{\mathcal{D}}_D$ as in (2.36) satisfying all properties of Subsection 2.2; see (4.10) and (4.13), respectively. Section 5 concludes this paper with topological arguments; this section is motivated by the approach in [13, Appendix A.2]. Many of the individual steps that we describe in this paper are not new. However, even the full statement of Theorem 1 on page 14 does not seem to appear elsewhere.

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1.1   Post-publication updates

The present version of this paper contains one modification and two additions to the mathematical content of the published version, which are described below. The statements of the theorems, 1 on page 14 and 2 on page 35, and the discussions of the connections between various properties

have been updated in accordance with these changes. Subsections 4.3, 5.1, and 5.2 have been changed from the collections of specific isomorphisms $\Psi_{\mathfrak{t}}$ in (4.10) and $\widetilde{\mathcal{D}}_D$ in (4.13) to collections of such isomorphisms satisfying certain properties. Furthermore, Section 2 has been split into two subsections, and more details have been added to some arguments as well. The enumeration of theorems, propositions, etc. has not changed from the published version, but many equations numbers in Section 2 and some in Section 5 have changed.

Naturality II property broadened to quasi-isomorphisms. The Naturality II property on page 13 has been broadened from isomorphisms of exact triples of Fredholm operators in the published version to quasi-isomorphisms. In the present version of the paper, we refer to the previous version of this property as the *isomorphism* Naturality II property. Relatedly, the Naturality III and Normalization III, III* properties in the published version of the paper are now called Normalization III, IV, IV*, respectively.

While no issues have been discovered with any formal statements in the published version, the informal summary of connections between the various properties was off. In particular, the isomorphism Naturality II, Normalization II,III, and Compositions I,II properties do not imply the Exact Squares property. For example, the collection of exact triple isomorphisms $\Psi_{\mathfrak{t}}$ in (2.27) given by (4.10) satisfies the (full) Naturality II, Normalization II,III, and Compositions I,II properties. For each Fredholm operator D, let

$$
A_D = \begin{cases} -1, & \text{if } \mathfrak{c}(D) \neq 0, \ \dim\operatorname{dom}(D) = \infty; \\ 1, & \text{otherwise.} \end{cases}
$$

For each exact triple $\mathfrak{t}$ of Fredholm operators as in (2.6), let

$$
\Psi_{\mathfrak{t}}' = \begin{cases} \frac{A_{D'} A_{D''}}{A_D} \Psi_{\mathfrak{t}}, & \text{if } \dim\operatorname{dom}(D'), \dim\operatorname{dom}(D'') = \infty; \\ \Psi_{\mathfrak{t}}, & \text{otherwise.} \end{cases}
$$

The new collection of exact triple isomorphisms $\Psi_{\mathfrak{t}}'$ satisfies the isomorphism Naturality II, Normalization II,III, and Compositions I,II properties. However, this collection does not satisfy the full Naturality II property, for quasi-isomorphisms between exact triples with infinite- and finite-dimensional Fredholm operators. Since the isomorphism Naturality II, Normalization II,III, and Exact Squares properties imply the Naturality II property by Subsection 3.4, the above collection $\{\Psi_{\mathfrak{t}}'\}_{\mathfrak{t}}$ does not satisfy the Exact Squares property either.

Complex orientations. For a $\mathbb{C}$-linear Fredholm operator D between Banach vector spaces X and Y with complex structures, the determinant line $\lambda(D)$ has a canonical orientation. Such orientations are sometimes used to orient the determinant lines for other operators by transferring them along paths of Fredholm operators. The Complex Orientations, Complex Exact Triples, and Dual Complex Orientations properties on pages 11, 14, and 19 have been added to require the topologies on the determinant line bundles, isomorphisms for exact triples of Fredholm operators, and dualization isomorphisms to be compatible with these orientations. As indicated by the proof of Theorem 2 in Subsection 3.4, these properties do not cut down on the admissible systems of determinant line bundles significantly.

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Wall-crossing for orientations. If D is an isomorphism between Banach vector spaces X and Y, the determinant line $\lambda(D)$ again has a canonical orientation. By the Normalization I property on page 12, these orientations vary continuously over the space of isomorphisms. Subsection 5.3 has been added to give a criterion determining whether the extension of the canonical orientation for an isomorphism D over a generic path in $\mathcal{F}(X, Y)$ ending an another isomorphism D' restricts to the canonical orientation of $\lambda(D')$. The answer turns out to be independent of the choice of an admissible system of determinant line bundles. Along with the added complex orientation properties, this implies that the signs defined in certain geometric settings, such as in Gromov-Witten theory, by transferring orientations from $\mathbb{C}$-linear operators along paths do not depend on the choice of either $\mathbb{C}$-linear operators or an admissible system of determinant line bundles.

2 The determinant line bundle

2.1 Notation and terminology

All vector spaces we consider are over $\mathbb{R}$. We denote by $\mathfrak{d}(V)$ the dimension of a vector space V and by

$$\lambda(V) \equiv \Lambda^{\mathrm{top}} V \equiv \Lambda^{\mathfrak{d}(V)} V \qquad \text{and} \qquad \lambda^*(V) \equiv \big(\lambda(V)\big)^*$$

the top exterior power of V and its dual, whenever $\mathfrak{d}(V) < \infty$. We view $\lambda(V)$ and $\lambda^*(V)$ as graded lines of degrees

$$\deg \lambda(V), \deg \lambda^*(V) = \mathfrak{d}(V) + 2\mathbb{Z} \in \mathbb{Z}_2.$$

For any two $\mathbb{Z}_2$-graded lines L_1 and L_2, we define

$$\deg L_1 \otimes L_2 = \deg L_1 + \deg L_2,$$

$$R \colon L_1 \otimes L_2 \longrightarrow L_2 \otimes L_1, \qquad R(v_1 \otimes v_2) = (-1)^{(\deg L_1)(\deg L_2)} v_2 \otimes v_1. \tag{2.1}$$

If $\mathcal{L}_1, \mathcal{L}_2 \longrightarrow \mathcal{F}$ are $\mathbb{Z}_2$-graded line bundles (each fiber has a grading varying continuously over $\mathcal{F}$), the fiberwise isomorphisms R give rise to an isomorphism

$$R \colon \mathcal{L}_1 \otimes \mathcal{L}_2 \longrightarrow \mathcal{L}_2 \otimes \mathcal{L}_1$$

of $\mathbb{Z}_2$-graded line bundles over $\mathcal{F}$. If L is a line and $v \in L - 0$, we define $v^* \in L^*$ by $v^*(v) = 1$.

For a finite-dimensional vector space V, we define

$$\mathcal{P} \colon \lambda(V^*) \longrightarrow \lambda^*(V), \quad \big\{\mathcal{P}(\alpha_1 \wedge \ldots \wedge \alpha_n)\big\}(v_1 \wedge \ldots \wedge v_n) = (-1)^{\binom{n}{2}} \det\big(\alpha_i(v_j)\big)_{i,j=1,\ldots,n} \tag{2.2}$$

and denote the inverse of $\mathcal{P}$ also by $\mathcal{P}$. The advantages of the isomorphism (2.2) over the isomorphism induced by the first pairing in (3.10) are that the former respects complex orientations and fits better with short exact sequences; see (2.24) and the last statement of Lemma 4.2.

For a Fredholm operator $D \colon X \longrightarrow Y$, we define

$$\lambda(D) = \lambda(\kappa(D)) \otimes \lambda^*(\mathfrak{c}(D)) \tag{2.3}$$

with the grading

$$\deg \lambda(D) \equiv \mathrm{ind}\, D + 2\mathbb{Z} \equiv \mathfrak{d}(\kappa(D)) - \mathfrak{d}(\mathfrak{c}(D)) + 2\mathbb{Z} \in \mathbb{Z}_2.$$

This is the same definition as in [12, Section 20.2] and [14, Section 7.4]; we discuss alternative versions of (2.3) in Subsections 3.2 and 3.3.

Morphisms. A homomorphism between Fredholm operators $D\colon X \longrightarrow Y$ and $D'\colon X' \longrightarrow Y'$ is a pair of homomorphisms $\phi\colon X \longrightarrow X'$ and $\psi\colon Y \longrightarrow Y'$ so that $D'{\circ}\phi = \psi{\circ}D$. A quasi-isomorphism between Fredholm operators D and D' is a homomorphism $(\phi, \psi)\colon D \longrightarrow D'$ that induces isomorphisms

$$\phi_\kappa\colon \kappa(D) \longrightarrow \kappa(D') \qquad \text{and} \qquad \psi_{\mathfrak{c}}\colon \mathfrak{c}(D) \longrightarrow \mathfrak{c}(D').$$

In such a case, we denote by

$$\lambda(\phi_\kappa)\colon \lambda(\kappa(D)) \longrightarrow \lambda(\kappa(D')), \qquad \lambda(\psi_{\mathfrak{c}}^{-1})\colon \lambda(\mathfrak{c}(D')) \longrightarrow \lambda(\mathfrak{c}(D)),$$
$$\widetilde{\mathcal{I}}_{\phi,\psi;D}\colon \lambda(D) \longrightarrow \lambda(D'), \quad x \otimes \alpha \longrightarrow \big(\lambda(\phi_\kappa)x\big) \otimes \big(\alpha \circ \lambda(\psi_{\mathfrak{c}}^{-1})\big) \tag{2.4}$$

the induced isomorphisms of the associated lines. An isomorphism between Fredholm operators D and D' is a homomorphism $(\phi, \psi)\colon D \longrightarrow D'$ so that ϕ and ψ are isomorphisms.

Isomorphisms $\phi\colon X \longrightarrow X'$ and $\psi\colon Y \longrightarrow Y'$ between Banach vector spaces induce a homeomorphism

$$\mathcal{I}_{\phi,\psi}\colon \mathcal{F}(X,Y) \longrightarrow \mathcal{F}(X',Y'), \qquad \mathcal{I}_{\phi,\psi}(D) = \psi \circ D \circ \phi^{-1}.$$

In particular, $(\phi, \psi)\colon D \longrightarrow \mathcal{I}_{\phi,\psi}(D)$ is an isomorphism of Fredholm operators for each $D \in \mathcal{F}(X,Y)$. Putting the isomorphisms $\widetilde{\mathcal{I}}_{\phi,\psi;D}$ in (2.4) together, we obtain a bundle map

$$\widetilde{\mathcal{I}}_{\phi,\psi}\colon \det{}_{X,Y} \longrightarrow \mathcal{I}_{\phi,\psi}^* \det{}_{X',Y'} \tag{2.5}$$

covering the identity on $\mathcal{F}(X,Y)$.

Exact Triples. An exact triple of Fredholm operators,

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0,$$

is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \xrightarrow{\,i_X\,} & X & \xrightarrow{\,j_X\,} & X'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle D'} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D''} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\,i_Y\,} & Y & \xrightarrow{\,j_Y\,} & Y'' & \longrightarrow & 0
\end{array}
\tag{2.6}
$$

of homomorphisms between Banach vector spaces with exact rows and with Fredholm operators as columns. A quasi-isomorphism

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & D'_{\mathrm{T}} & \longrightarrow & D_{\mathrm{T}} & \longrightarrow & D''_{\mathrm{T}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\phi',\psi')} & & \downarrow{\scriptstyle (\phi,\psi)} & & \downarrow{\scriptstyle (\phi'',\psi'')} & & \\
0 & \longrightarrow & D'_{\mathrm{B}} & \longrightarrow & D_{\mathrm{B}} & \longrightarrow & D''_{\mathrm{B}} & \longrightarrow & 0
\end{array}
\tag{2.7}
$$

between exact triples

$$0 \longrightarrow D'_{\mathrm{T}} \longrightarrow D_{\mathrm{T}} \longrightarrow D''_{\mathrm{T}} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow D'_{\mathrm{B}} \longrightarrow D_{\mathrm{B}} \longrightarrow D''_{\mathrm{B}} \longrightarrow 0$$

of Fredholm operators are homomorphisms

$$
\begin{array}{ccccccccc}
0 \longrightarrow & X'_{\mathrm{T}} & \longrightarrow & X_{\mathrm{T}} & \longrightarrow & X''_{\mathrm{T}} & \longrightarrow 0 \\
& \downarrow{\phi'} & & \downarrow{\phi} & & \downarrow{\phi''} \\
0 \longrightarrow & X'_{\mathrm{B}} & \longrightarrow & X_{\mathrm{B}} & \longrightarrow & X''_{\mathrm{B}} & \longrightarrow 0
\end{array}
\qquad
\begin{array}{ccccccccc}
0 \longrightarrow & Y'_{\mathrm{T}} & \longrightarrow & Y_{\mathrm{T}} & \longrightarrow & Y''_{\mathrm{T}} & \longrightarrow 0 \\
& \downarrow{\psi'} & & \downarrow{\psi} & & \downarrow{\psi''} \\
0 \longrightarrow & Y'_{\mathrm{B}} & \longrightarrow & Y_{\mathrm{B}} & \longrightarrow & Y''_{\mathrm{B}} & \longrightarrow 0
\end{array}
$$

of short exact sequences of Banach vector spaces so that (ϕ',ψ'), (ϕ,ψ), and (ϕ'',ψ'') are quasi-isomorphisms between the Fredholm operators D'_{T} and D'_{B}, D_{T} and D_{B}, and D''_{T} and D''_{B}, respectively. An isomorphism between exact triples of Fredholm operators as above is a quasi-isomorphism between these exact triples as in (2.7) so that the homomorphisms $\phi',\psi',\phi,\psi,\phi'',\psi''$ are isomorphisms.

For Banach vector spaces X,Y,X',Y',X'',Y'', let

$$
\begin{aligned}
\mathcal{T}(X,Y&;X',Y';X'',Y'') \\
&\subset \mathcal{F}(X,Y)\times\mathcal{F}(X',Y')\times\mathcal{F}(X'',Y'')\times\mathcal{B}(X',X)\times\mathcal{B}(X,X'')\times\mathcal{B}(Y',Y)\times\mathcal{B}(Y,Y'')
\end{aligned}
\tag{2.8}
$$

be the subspace of tuples $(D,D',D'',\mathsf{i}_X,\mathsf{j}_X,\mathsf{i}_Y,\mathsf{j}_Y)$ so that (2.6) is an exact triple of Fredholm operators. Denote by

$$
\pi_{\mathrm{C}},\pi_{\mathrm{L}},\pi_{\mathrm{R}}\colon \mathcal{T}(X,Y;X',Y';X'',Y'') \longrightarrow \mathcal{F}(X,Y),\mathcal{F}(X',Y'),\mathcal{F}(X'',Y'')
$$

the restrictions of the projection maps. For $\star=',''$, denote by

$$
\mathcal{T}^{\star}(X,Y;X',Y';X'',Y'') \subset \mathcal{T}(X,Y;X',Y';X'',Y'')
$$

the subspace of diagrams (2.6) so that $D^{\star}$ is an isomorphism.

If $\mathsf{t}\in\mathcal{T}'(X,Y;X',Y';X'',Y'')$ is as in (2.6), $(\mathsf{j}_X,\mathsf{j}_Y)$ is a quasi-isomorphism between the Fredholm operators D and D''. With the notation as in (2.4), define

$$
\mathcal{I}'_{\mathsf{t}}\colon \lambda(D')\otimes\lambda(D'') \longrightarrow \lambda(D), \qquad \mathcal{I}'_{\mathsf{t}}\big((1\otimes 1^*)\otimes\sigma''\big) = \widetilde{\mathcal{I}}^{-1}_{\mathsf{j}_X,\mathsf{j}_Y;D}(\sigma'').
\tag{2.9}
$$

If $\mathsf{t}\in\mathcal{T}''(X,Y;X',Y';X'',Y'')$ is as in (2.6), $(\mathsf{i}_X,\mathsf{i}_Y)$ is a quasi-isomorphism between the Fredholm operators D' and D. Define

$$
\mathcal{I}''_{\mathsf{t}}\colon \lambda(D')\otimes\lambda(D'') \longrightarrow \lambda(D), \qquad \mathcal{I}''_{\mathsf{t}}\big(\sigma'\otimes(1\otimes 1^*)\big) = \widetilde{\mathcal{I}}_{\mathsf{i}_X,\mathsf{i}_Y;D'}(\sigma').
\tag{2.10}
$$

Direct Sums. For Banach vector spaces X',Y',X'',Y'', the direct sum operation

$$
\oplus\colon \mathcal{F}(X',Y')\times\mathcal{F}(X'',Y'') \longrightarrow \mathcal{F}(X'\oplus X'',Y'\oplus Y''), \qquad (D',D'') \longrightarrow D'\oplus D'',
$$

is a continuous map. For any $D\in\mathcal{F}(X,Y)$ and a Banach vector space Z, the projections

$$
(\phi,\psi)\colon (Z\oplus X,Z\oplus Y) \longrightarrow (X,Y) \qquad \text{and} \qquad (\phi,\psi)\colon (X\oplus Z,Y\oplus Z) \longrightarrow (X,Y)
$$

are quasi-isomorphism between the Fredholm operators $\mathrm{id}_Z\oplus D$, D, and $D\oplus\mathrm{id}_Z$. Via (2.4), they thus determine identifications

$$\lambda(\mathrm{id}_Z\oplus D)=\lambda(D)=\lambda(D\oplus\mathrm{id}_Z),\tag{2.11}$$

$$(0,x_1)\wedge\ldots\wedge(0,x_k)\otimes\big((0,y_1)\wedge\ldots\wedge(0,y_\ell)\big)^* \longleftrightarrow x_1\wedge\ldots\wedge x_k\otimes\big(y_1\wedge\ldots\wedge y_\ell\big)^*$$
$$\longleftrightarrow (x_1,0)\wedge\ldots\wedge(x_k,0)\otimes\big((y_1,0)\wedge\ldots\wedge(y_\ell,0)\big)^*.$$

We denote by

$$R_{X',X''}\colon X'\oplus X''\longrightarrow X''\oplus X'\qquad \text{and}$$
$$\mathcal{R}_{\mathcal{F}}\colon \mathcal{F}(X',Y')\times\mathcal{F}(X'',Y'')\longrightarrow \mathcal{F}(X'',Y'')\times\mathcal{F}(X',Y')$$

the maps interchanging the two factors. Let

$$\oplus'\colon \mathcal{F}(X',Y')\times\mathcal{F}(X'',Y'')\longrightarrow \mathcal{F}(X''\oplus X',Y''\oplus Y')\qquad\text{and}$$
$$\bigoplus\colon \mathcal{F}(X',Y')\times\mathcal{F}(X'',Y'')\times\mathcal{F}(X''',Y''')\longrightarrow \mathcal{F}(X'\oplus X''\oplus X''',Y'\oplus Y''\oplus Y''')$$

be the compositions

$$\oplus\circ\mathcal{R}_{\mathcal{F}}=\mathcal{I}_{R_{X',X''},R_{Y',Y''}}\circ\oplus\qquad\text{and}\tag{2.12}$$
$$\oplus\circ\oplus\times\mathrm{id}_{\mathcal{F}(X''',Y''')}=\oplus\circ\mathrm{id}_{\mathcal{F}(X',Y')}\times\oplus,\tag{2.13}$$

respectively.

We associate the direct sum $D'\oplus D''$ of Fredholm operators $D'\colon X'\longrightarrow Y'$ and $D''\colon X''\longrightarrow Y''$ with the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \xrightarrow{\ \mathrm{i}_X\ } & X'\oplus X'' & \xrightarrow{\ \mathrm{j}_X\ } & X'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle D'} & & \downarrow{\scriptstyle D'\oplus D''} & & \downarrow{\scriptstyle D''} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\ \mathrm{i}_Y\ } & Y'\oplus Y'' & \xrightarrow{\ \mathrm{j}_Y\ } & Y'' & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{l}
\mathrm{i}_X(x')=(x',0)\\
\mathrm{j}_X(x',x'')=x''\\[1em]
\mathrm{i}_Y(y')=(y',0)\\
\mathrm{j}_Y(y',y'')=y''.
\end{array}
\tag{2.14}
$$

This yields an embedding

$$\iota_\oplus\colon \mathcal{F}(X',Y')\times\mathcal{F}(X'',Y'')\longrightarrow \mathcal{T}(X'\oplus X'',Y'\oplus Y'';X',Y';X'',Y'')\qquad\text{s.t.}$$
$$\pi_{\mathrm{C}}\circ\iota_\oplus=\oplus,\quad \pi_{\mathrm{L}}\circ\iota_\oplus=\pi_1,\quad \pi_{\mathrm{R}}\circ\iota_\oplus=\pi_2,$$

where

$$\pi_1,\pi_2\colon \mathcal{F}(X',Y')\times\mathcal{F}(X'',Y'')\longrightarrow \mathcal{F}(X',Y'),\mathcal{F}(X'',Y'')$$

are the projection maps.

Compositions. For Banach vector spaces X_1,X_2,X_3, the composition map

$$\mathcal{C}_{X_2}\colon \mathcal{F}(X_1,X_2)\times\mathcal{F}(X_2,X_3)\longrightarrow \mathcal{F}(X_1,X_3),\qquad (D_1,D_2)\longrightarrow D_2\circ D_1,$$

is continuous as well. If X_4 is another Banach vector space, let

$$\mathcal{C}_{X_2,X_3}\colon \mathcal{F}(X_1,X_2)\times\mathcal{F}(X_2,X_3)\times\mathcal{F}(X_3,X_4)\longrightarrow\mathcal{F}(X_1,X_4)$$

denote the compositions

$$\mathcal{C}_{X_3}\circ\big\{\mathcal{C}_{X_2}\times\mathrm{id}_{\mathcal{F}(X_3,X_4)}\big\}=\mathcal{C}_{X_2}\circ\big\{\mathrm{id}_{\mathcal{F}(X_1,X_2)}\times\mathcal{C}_{X_3}\big\}. \tag{2.15}$$

With the notation as in (2.8), we denote by

$$\mathcal{C}_{\mathcal{T}}\colon \mathcal{T}(X_1,X_2;X_1',X_2';X_1'',X_2'')\times\mathcal{T}(X_2,X_3;X_2',X_3';X_2'',X_3'')\longrightarrow\mathcal{T}(X_1,X_3;X_1',X_3';X_1'',X_3'')$$

the continuous map sending commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1' & \xrightarrow{i_1} & X_1 & \xrightarrow{j_1} & X_1'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle D_1'} & & \downarrow{\scriptstyle D_1} & & \downarrow{\scriptstyle D_1''} & & \\
0 & \longrightarrow & X_2' & \xrightarrow{i_2} & X_2 & \xrightarrow{j_2} & X_2'' & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & X_2' & \xrightarrow{i_2} & X_2 & \xrightarrow{j_2} & X_2'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle D_2'} & & \downarrow{\scriptstyle D_2} & & \downarrow{\scriptstyle D_2''} & & \\
0 & \longrightarrow & X_3' & \xrightarrow{i_3} & X_3 & \xrightarrow{j_3} & X_3'' & \longrightarrow & 0
\end{array} \tag{2.16}$$

to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1' & \xrightarrow{i_1} & X_1 & \xrightarrow{j_1} & X_1'' & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle D_2'\circ D_1'} & & \Big\downarrow{\scriptstyle D_2\circ D_1} & & \Big\downarrow{\scriptstyle D_2''\circ D_1''} & & \\
0 & \longrightarrow & X_3' & \xrightarrow{i_3} & X_3 & \xrightarrow{j_3} & X_3'' & \longrightarrow & 0\,.
\end{array} \tag{2.17}$$

We note that

$$(\pi_{\mathrm{C}},\pi_{\mathrm{L}},\pi_{\mathrm{R}})\circ\mathcal{C}_{\mathcal{T}}=\big(\mathcal{C}_{X_2}\circ(\pi_{\mathrm{C}}\circ\pi_1,\pi_{\mathrm{C}}\circ\pi_2),\mathcal{C}_{X_2'}\circ(\pi_{\mathrm{L}}\circ\pi_1,\pi_{\mathrm{L}}\circ\pi_2),\mathcal{C}_{X_2''}\circ(\pi_{\mathrm{R}}\circ\pi_1,\pi_{\mathrm{R}}\circ\pi_2)\big), \tag{2.18}$$

where

$$\pi_1,\pi_2\colon \mathcal{T}(X_1,X_2;X_1',X_2';X_1'',X_2'')\times\mathcal{T}(X_2,X_3;X_2',X_3';X_2'',X_3'')$$
$$\longrightarrow\mathcal{T}(X_1,X_2;X_1',X_2';X_1'',X_2''),\mathcal{T}(X_2,X_3;X_2',X_3';X_2'',X_3'')$$

are the projection maps.

We associate the composition $D_2\circ D_1$ of Fredholm operators $D_1\colon X_1\longrightarrow X_2$ and $D_2\colon X_2\longrightarrow X_3$ with the exact triple

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{i_X} & X_1\oplus X_2 & \xrightarrow{j_X} & X_2 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle D_1} & & {\scriptstyle D_2\circ D_1\oplus\mathrm{id}_{X_2}} & & \downarrow{\scriptstyle D_2} & & \\
0 & \longrightarrow & X_2 & \xrightarrow{i_Y} & X_3\oplus X_2 & \xrightarrow{j_Y} & X_3 & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{l}
i_X(x_1)=(x_1,D_1x_1) \\
j_X(x_1,x_2)=D_1x_1-x_2 \\[1.2em]
i_Y(x_2)=(D_2x_2,x_2) \\
j_Y(x_3,x_2)=x_3-D_2x_2\,.
\end{array} \tag{2.19}$$

This yields an embedding

$$\iota_{\mathcal{C}}\colon \mathcal{F}(X_1,X_2)\times\mathcal{F}(X_2,X_3)\longrightarrow\mathcal{T}(X_1\oplus X_2,X_3\oplus X_2;X_1,X_2;X_2,X_3)\qquad\text{s.t.}$$
$$\pi_{\mathrm{C}}\circ\iota_{\mathcal{C}}(D_1,D_2)=\mathcal{C}_{X_2}(D_1,D_2)\oplus\mathrm{id}_{X_2},\quad \pi_{\mathrm{L}}\circ\iota_{\mathcal{C}}=\pi_1,\quad \pi_{\mathrm{R}}\circ\iota_{\mathcal{C}}=\pi_2.$$

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Combining the first identity above with (2.11), we obtain

$$\iota_{\mathcal{C}}^* \pi_{\mathrm{C}}^* \det\nolimits_{X_1 \oplus X_2, X_3 \oplus X_2} = \mathcal{C}_{X_2}^* \det\nolimits_{X_1, X_3}.$$

If $D \in \mathcal{F}(X, Y)$, the compositions $D \circ \mathrm{id}_X$ and $\mathrm{id}_Y \circ D$ correspond to elements of

$$\mathcal{T}'(X \oplus X, Y \oplus X; X, X; X, Y) \qquad \text{and} \qquad \mathcal{T}''(X \oplus Y, Y \oplus Y; X, Y; Y, Y),$$

respectively, with the isomorphisms $\mathcal{I}_{\mathfrak{t}}'$ and $\mathcal{I}_{\mathfrak{t}}''$ of (2.9) and (2.10) given by

$$\mathcal{I}_{\mathfrak{t}}'\big(1 \otimes 1^* \otimes x \otimes \beta\big) = x \otimes \beta \qquad \text{and} \qquad \mathcal{I}_{\mathfrak{t}}''\big(x \otimes \beta \otimes 1 \otimes 1^*\big) = x \otimes \beta$$

under the identifications (2.11).

Dualizations. For each Banach vector space X, let X^* denote the dual Banach vector space, i.e. the space $\mathrm{Hom}_{\mathbb{R}}(X, \mathbb{R})$ of bounded linear functionals $X \longrightarrow \mathbb{R}$. For each $D \in \mathcal{F}(X, Y)$, let $D^* \in \mathcal{F}(Y^*, X^*)$ denote the dual operator, i.e.

$$\{D^*\beta\}(x) = \beta(Dx) \qquad \forall\, \beta \in Y^*,\ x \in X.$$

The map

$$\mathcal{D} \colon \mathcal{F}(X, Y) \longrightarrow \mathcal{F}(Y^*, X^*), \qquad \mathcal{D}(D) = D^*,$$

is then continuous. For each $D \in \mathcal{F}(X, Y)$, the homomorphisms

$$
\begin{aligned}
\mathcal{D}_D &\colon \kappa(D) \longrightarrow \mathfrak{c}(D^*)^*, & \{\mathcal{D}_D(x)\}(\alpha + \mathrm{Im}\, D^*) &= \alpha(x) & \forall\, x \in \kappa(D),\ \alpha \in X^*, \\
\mathcal{D}_D &\colon \mathfrak{c}(D)^* \longrightarrow \kappa(D^*), & \{\mathcal{D}_D(\beta)\}(y) &= \beta(y + \mathrm{Im}\, D) & \forall\, \beta \in \mathfrak{c}(D)^*,\ y \in Y,
\end{aligned}
\tag{2.20}
$$

are isomorphisms.

For each exact triple $\mathfrak{t}$ of Fredholm operators as in (2.6), we define the dual triple $\mathfrak{t}^*$ to be given by the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y''^* & \xrightarrow{\ j_Y^*\ } & Y^* & \xrightarrow{\ i_Y^*\ } & Y'^* & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle D''^*} & & \downarrow{\scriptstyle D^*} & & \downarrow{\scriptstyle D'^*} & & \\
0 & \longrightarrow & X''^* & \xrightarrow{\ j_X^*\ } & X^* & \xrightarrow{\ i_X^*\ } & X'^* & \longrightarrow & 0\,.
\end{array}
\tag{2.21}
$$

This defines an embedding

$$
\begin{aligned}
\mathcal{D}_{\mathcal{T}} \colon \mathcal{T}(X, Y; X', Y'; X'', Y'') &\longrightarrow \mathcal{T}(Y^*, X^*; Y''^*, X''^*; Y'^*, X'^*) \qquad \text{s.t.} \\
\pi_{\mathrm{C}} \circ \mathcal{D}_{\mathcal{T}} = \mathcal{D} \circ \pi_{\mathrm{C}}, \quad &\pi_{\mathrm{L}} \circ \mathcal{D}_{\mathcal{T}} = \mathcal{D} \circ \pi_{\mathrm{R}}, \quad \pi_{\mathrm{R}} \circ \mathcal{D}_{\mathcal{T}} = \mathcal{D} \circ \pi_{\mathrm{L}}\,.
\end{aligned}
\tag{2.22}
$$

Complex Orientations. Let $\mathfrak{i}$ be a complex structure on a real vector space V. The homomorphism

$$\mathrm{Re} \colon \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \longrightarrow V^* \equiv \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}), \quad \{\mathrm{Re}(\alpha)\}(v) = \mathrm{Re}\big(\alpha(v)\big) \ \ \forall\, \alpha \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}),\ v \in V,$$

is an isomorphism. Via this isomorphism, $\mathfrak{i}$ induces a complex structure on V^*, which we still denote by $\mathfrak{i}$, so that

$$\{\mathfrak{i}\alpha\}(v) = \alpha(\mathfrak{i}v) \quad \forall\, \alpha \in V^*,\ v \in V.
\tag{2.23}$$

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If V is finite-dimensional with $\mathbb{C}$-basis $e_1, \ldots, e_n$, then $e_1, \mathrm{i}e_1, \ldots, e_n, \mathrm{i}e_n$ is an $\mathbb{R}$-basis for V determining the **complex orientation** of V. If $e_1^*, \ldots, e_n^*$ is the dual $\mathbb{C}$-basis for V^*, then

$$e_1^*, \mathrm{i}e_1^* = -(\mathrm{i}e_1)^*, \ldots, e_n^*, \mathrm{i}e_n^* = -(\mathrm{i}e_n)^*$$

is an $\mathbb{R}$-basis for V^* determining the complex orientation of V^*. Thus,

$$\big(e_1 \wedge \mathrm{i}e_1 \wedge \ldots \wedge e_n \wedge \mathrm{i}e_n\big)^* = \mathcal{P}\big(e_1^* \wedge \mathrm{i}e_1^* \wedge \ldots \wedge e_n^* \wedge \mathrm{i}e_n^*\big) \in \lambda^*(V), \tag{2.24}$$

i.e. the isomorphism (2.2) with n replaced by $2n$ intertwines the complex orientations of $\lambda(V^*)$ and $\lambda^*(V)$.

Suppose X, Y are Banach vector spaces with complex structures. We then denote by

$$\mathcal{F}_{\mathbb{C}}(X, Y) \subset \mathcal{F}(X, Y)$$

the closed subspace of $\mathbb{C}$-linear Fredholm operators. For each $D \in \mathcal{F}_{\mathbb{C}}(X, Y)$, $\kappa(D)$ and $\mathfrak{c}(D)$ are finite-dimensional complex vector spaces. Thus, the real lines $\lambda(\kappa(D))$, $\lambda(\mathfrak{c}(D))$, and $\lambda(D)$ have canonical orientations, which we will call the **complex orientations**. In this case, the Banach vector spaces X^*, Y^* inherit complex structures from X, Y, $D^* \in \mathcal{F}_{\mathbb{C}}(Y^*, X^*)$, and the isomorphisms (2.20) are $\mathbb{C}$-linear.

If X, Y, X', Y', X'', Y'' are Banach vector spaces with complex structures, we denote by

$$\mathcal{T}_{\mathbb{C}}(X, Y; X', Y'; X'', Y'') \subset \mathcal{T}(X, Y; X', Y'; X'', Y'')$$

the subspace of exact triples as in (2.6) so that the Fredholm operators D', D, D'' and the homomorphisms $\mathrm{i}_X, \mathrm{j}_X, \mathrm{i}_Y, \mathrm{j}_Y$ are $\mathbb{C}$-linear.

2.2 Properties

The topologies on the line bundles $\det_{X,Y}$ should satisfy a number of important compatibility properties, which we now describe.

Naturality I. The bundle map $\widetilde{\mathcal{I}}_{\phi,\psi}$ in (2.5) is continuous for all isomorphisms $\phi \colon X \longrightarrow X'$ and $\psi \colon Y \longrightarrow Y'$ between Banach vector spaces.

Complex Orientations. If X, Y are Banach vector spaces with complex structures, the complex orientations of the lines $\lambda(D)$ with $D \in \mathcal{F}_{\mathbb{C}}(X, Y)$ determine an orientation on the restriction of $\det_{X,Y}$ to $\mathcal{F}_{\mathbb{C}}(X, Y)$.

The substance of the Complex Orientations property is that the complex orientations of the lines $\lambda(D)$, wherever defined, are continuous with respect to the topology of the relevant restriction of $\det_{X,Y}$.

If X, Y are Banach vector spaces and $T \in \mathcal{B}(Y, X)$, let

$$U_T = \big\{P \in \mathcal{B}(X, Y) \colon \|TP\| < 1\big\};$$

this is an open subset of $\mathcal{B}(X,Y)$. If in addition $P \in \mathcal{B}(X,Y)$, define

$$\Phi_{T;P} \in \mathcal{B}(X,X) \qquad \text{by} \qquad \Phi_{T;P}(x) = x + TPx.$$

The operator $\Phi_{T;P}$ is invertible for every $P \in U_T$ and the map

$$U_T \longrightarrow \mathcal{F}(X,X), \qquad P \longrightarrow \Phi_{T;P}^{-1} = \sum_{r=0}^{\infty} (-1)^r (TP)^r,$$

is continuous.

We define

$$\mathcal{F}^*(X,Y) = \{ D \in \mathcal{F}(X,Y) \colon \mathfrak{c}(D) = \{0\} \},$$
$$\pi \colon \kappa(X,Y) = \{ (D,x) \in \mathcal{F}^*(X,Y) \times X : Dx = 0 \} \longrightarrow \mathcal{F}^*(X,Y), \quad \pi(D,x) = D.$$

The first set above is an open subset of $\mathcal{F}(X,Y)$. For each $D \in \mathcal{F}^*(X,Y)$ and each right inverse $T \colon Y \longrightarrow X$ of D, the map

$$(D+U_T) \times X \longrightarrow (D+U_T) \times X, \qquad (D+P, x) \longrightarrow \big(D+P, \Phi_{T;P}(x) \big),$$

is continuous, linear on each fiber of the projection to the first component, and restricts to a bijection

$$\kappa(X,Y)\big|_{\pi^{-1}(D+U_T)} \longrightarrow (D+U_T) \times \kappa(D).$$

Thus, $\kappa(X,Y)$ is a subbundle of the trivial Banach bundle

$$\mathcal{F}^*(X,Y) \times X \longrightarrow \mathcal{F}^*(X,Y).$$

Normalization I. The topology of $\det_{X,Y} \big|_{\mathcal{F}^*(X,Y)}$ is the topology of the top exterior power of the vector bundle $\kappa(X,Y)$.

This condition can alternatively be described as follows. For $D \in \mathcal{F}^*(X,Y)$, each right inverse $T \colon Y \longrightarrow X$ of D, and $P \in \mathcal{B}(X,Y)$, let

$$\Phi_{D,T;P} \colon \kappa(D+P) \longrightarrow \kappa(D), \qquad \Phi_{D,T;P}(x) = \Phi_{T;P}(x). \tag{2.25}$$

This map is a bijection for every $P \in U_T$ and thus induces an isomorphism

$$\widetilde{\mathcal{I}}_{D,T;D+P} \colon \lambda(D) = \lambda(\kappa(D)) \otimes \mathbb{R} \longrightarrow \lambda \big(\kappa(D+P) \big) \otimes \mathbb{R} = \lambda(\kappa(D+P)).$$

Putting these isomorphisms together, we obtain a bundle map

$$\widetilde{\mathcal{I}}_{D,T} \colon (D+U_T) \times \lambda(D) \longrightarrow \det_{X,Y}\big|_{D+U_T} \tag{2.26}$$

covering the identity on the open neighborhood $D+U_T$ of D in $\mathcal{F}^*(X,Y)$. Normalization I is equivalent to the condition that the map $\widetilde{\mathcal{I}}_{D,T}$ is continuous for every $D \in \mathcal{F}^*(X,Y)$ and right inverse $T \colon Y \longrightarrow X$ of D.

If X, Y are Banach vector spaces with complex structures, every surjective $\mathbb{C}$-linear Fredholm operator D admits a $\mathbb{C}$-linear right inverse T. The isomorphism (2.25) is then $\mathbb{C}$-linear whenever $P \in U_T$ is $\mathbb{C}$-linear. Thus, the complex orientations of the lines $\lambda(D+T)$ with $P \in U_T$ $\mathbb{C}$-linear determine an orientation on the restriction of $\det_{X,Y}$ over the intersection of $D+U_T$ with

$$\mathcal{F}^*_{\mathbb{C}}(X,Y) \equiv \mathcal{F}_{\mathbb{C}}(X,Y) \cap \mathcal{F}^*(X,Y).$$

Thus, the Complex Orientations property over $\mathcal{F}^*_{\mathbb{C}}(X,Y)$ follows from the Normalization I property.

Exact Triples. There exists a collection of (continuous) line bundle isomorphisms

$$\Psi \colon \pi^*_{\mathrm{L}} \det_{X',Y'} \otimes \pi^*_{\mathrm{R}} \det_{X'',Y''} \longrightarrow \pi^*_{\mathrm{C}} \det_{X,Y} \tag{2.27}$$

over $\mathcal{T}(X,Y;X',Y';X'',Y'')$ parametrized by the tuples $(X,Y;X',Y';X'',Y'')$ of Banach vector spaces with the following properties.

Naturality II. The isomorphisms Ψ commute with the isomorphisms (2.4) induced by quasi-isomorphisms of exact triples of Fredholm operators, i.e. the diagram

$$
\begin{array}{ccc}
\lambda(D'_{\mathrm{T}}) \otimes \lambda(D''_{\mathrm{T}}) & \xrightarrow{\;\;\Psi_{\mathrm{T}}\;\;} & \lambda(D_{\mathrm{T}}) \\
{\scriptstyle \widetilde{\mathcal{I}}_{\phi',\psi';D'_{\mathrm{T}}} \otimes \widetilde{\mathcal{I}}_{\phi'',\psi'';D''_{\mathrm{T}}}} \Big\downarrow & & \Big\downarrow {\scriptstyle \widetilde{\mathcal{I}}_{\phi,\psi;D_{\mathrm{T}}}} \\
\lambda(D'_{\mathrm{B}}) \otimes \lambda(D''_{\mathrm{B}}) & \xrightarrow{\;\;\Psi_{\mathrm{B}}\;\;} & \lambda(D_{\mathrm{B}}),
\end{array}
$$

where Ψ_{T} and Ψ_{B} are the isomorphisms (2.27) for the top and bottom exact triples of Fredholm operators in (2.7), commutes for every quasi-isomorphism of exact triples of Fredholm operators as in (2.7).

Normalization II. For each $\mathfrak{t} \in \mathcal{T}(X,Y;X',Y';X'',Y'')$ as in (2.6) with $D' \in \mathcal{F}^*(X',Y')$ and $D'' \in \mathcal{F}^*(X'',Y'')$, the restriction $\Psi_{\mathfrak{t}}$ of Ψ to the fiber over $\mathfrak{t}$ is the canonical isomorphism $\wedge_{\kappa(D)}$ of Lemma 4.1 for the short exact sequence

$$0 \longrightarrow \kappa(D') \longrightarrow \kappa(D) \longrightarrow \kappa(D'') \longrightarrow 0$$

of finite-dimensional vector spaces.

Normalization III. For each $\star =',''$ and $\mathfrak{t} \in \mathcal{T}^{\star}(X,Y;X',Y';X'',Y'')$, the restriction $\Psi_{\mathfrak{t}}$ of Ψ to the fiber over $\mathfrak{t}$ is the corresponding isomorphism $\mathcal{I}^{\star}_{\mathfrak{t}}$ of (2.9) or (2.10).

With $\iota_{\oplus}$ as below (2.14), $D' \in \mathcal{F}(X',Y')$, and $D'' \in \mathcal{F}(X'',Y'')$, let

$$\widetilde{\oplus}_{D',D''} \equiv \Psi_{\iota_{\oplus}(D',D'')} \colon \lambda(D') \otimes \lambda(D'') \longrightarrow \lambda(D' \oplus D'')$$

be the corresponding exact triples isomorphism (2.27). With ι_C as below (2.19), $D_1 \in \mathcal{F}(X_1, X_2)$, and $D_2 \in \mathcal{F}(X_2, X_3)$, let

$$\widetilde{\mathcal{C}}_{D_1,D_2} \equiv \Psi_{\iota_{\mathcal{C}}(D_1,D_2)} \colon \lambda(D_1) \otimes \lambda(D_2) \longrightarrow \lambda(D_2 \circ D_1)$$

be the corresponding isomorphism (2.27). By the next four properties, these isomorphisms provide liftings of (2.12), (2.13), (2.15), and (2.18) to determinant line bundles.

Direct Sums I. For all $D' \in \mathcal{F}(X', Y')$ and $D'' \in \mathcal{F}(X'', Y'')$, the diagram

$$
\begin{array}{ccc}
\lambda(D') \otimes \lambda(D'') & \xrightarrow{\widetilde{\oplus}_{D',D''}} & \lambda(D' \oplus D'') \\
{\scriptstyle R} \downarrow & & \left| \; \widetilde{\mathcal{I}}_{R_{X',X''}, R_{Y',Y''}; D' \oplus D''} \right. \\
\lambda(D'') \otimes \lambda(D') & \xrightarrow{\widetilde{\oplus}_{D'',D'}} & \lambda(D'' \oplus D')
\end{array}
\tag{2.28}
$$

commutes.

Direct Sums II. For all $D' \in \mathcal{F}(X', Y')$, $D'' \in \mathcal{F}(X'', Y'')$, and $D''' \in \mathcal{F}(X''', Y''')$, the diagram

$$
\begin{array}{ccc}
\lambda(D') \otimes \lambda(D'') \otimes \lambda(D''') & \xrightarrow{\mathrm{id} \otimes \widetilde{\oplus}_{D'',D'''}} & \lambda(D') \otimes \lambda(D'' \oplus D''') \\
{\scriptstyle \widetilde{\oplus}_{D',D''} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \widetilde{\oplus}_{D',D'' \oplus D'''}} \\
\lambda(D' \oplus D'') \otimes \lambda(D''') & \xrightarrow{\widetilde{\oplus}_{D' \oplus D'',D'''}} & \lambda(D' \oplus D'' \oplus D''')
\end{array}
\tag{2.29}
$$

commutes.

Compositions I. For all $D_1 \in \mathcal{F}(X_1, X_2)$, $D_2 \in \mathcal{F}(X_2, X_3)$, and $D_3 \in \mathcal{F}(X_3, X_4)$, the diagram

$$
\begin{array}{ccc}
\lambda(D_1) \otimes \lambda(D_2) \otimes \lambda(D_3) & \xrightarrow{\mathrm{id} \otimes \widetilde{\mathcal{C}}_{D_2,D_3}} & \lambda(D_1) \otimes \lambda(D_3 \circ D_2) \\
{\scriptstyle \widetilde{\mathcal{C}}_{D_1,D_2} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \widetilde{\mathcal{C}}_{D_1,D_3 \circ D_2}} \\
\lambda(D_2 \circ D_1) \otimes \lambda(D_3) & \xrightarrow{\widetilde{\mathcal{C}}_{D_2 \circ D_1, D_3}} & \lambda(D_3 \circ D_2 \circ D_1)
\end{array}
\tag{2.30}
$$

commutes.

Compositions II. For all $\mathfrak{t}_1 \in \mathcal{T}(X_1, X_2; X_1', X_2'; X_1'', X_2'')$ and $\mathfrak{t}_2 \in \mathcal{T}(X_2, X_3; X_2', X_3'; X_2'', X_3'')$ as in (2.16), the diagram

$$
\begin{array}{ccc}
\lambda(D_1') \otimes \lambda(D_1'') \otimes \lambda(D_2') \otimes \lambda(D_2'') & \xrightarrow{\Psi_{\mathfrak{t}_1} \otimes \Psi_{\mathfrak{t}_2}} & \lambda(D_1) \otimes \lambda(D_2) \\
{\scriptstyle \widetilde{\mathcal{C}}_{D_1',D_2'} \otimes \widetilde{\mathcal{C}}_{D_1'',D_2''} \circ \mathrm{id} \otimes R \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \widetilde{\mathcal{C}}_{D_1,D_2}} \\
\lambda(D_2' \circ D_1') \otimes \lambda(D_2'' \circ D_1'') & \xrightarrow{\Psi_{\mathcal{C}_{\mathcal{T}}(\mathfrak{t}_1,\mathfrak{t}_2)}} & \lambda(D_2 \circ D_1)
\end{array}
\tag{2.31}
$$

commutes.

Complex Exact Triples. If X, Y, X', Y', X'', Y'' are Banach vector spaces with complex structures and $\mathfrak{t} \in \mathcal{T}_{\mathbb{C}}(X, Y; X', Y'; X'', Y'')$, the restriction $\Psi_{\mathfrak{t}}$ of Ψ to the fiber over $\mathfrak{t}$ intertwines the complex orientations of $\lambda(D'), \lambda(D''), \lambda(D)$.

Theorem 1. *There exist a collection of topologies on the line bundles $\det_{X,Y} \longrightarrow \mathcal{F}(X,Y)$ corresponding to pairs (X,Y) of Banach spaces and a collection of continuous line-bundle isomorphisms (2.27) which satisfy the Naturality I,II, Complex Orientations, Normalization I,II,III, Direct Sums I,II, Compositions I,II, and Complex Exact Triples properties.*

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We will refer to the Naturality II property restricted to the isomorphisms of exact triples of Fredholm operators as the **isomorphism Naturality II** property.

Any family of exact triple isomorphisms $\Psi_{\mathfrak{t}}$ as in (2.27) satisfying Normalization II also satisfies Normalization III for triples of surjective Fredholm operators and Complex Exact Triples for triples of surjective $\mathbb{C}$-linear Fredholm operators. By Lemma 5.1, such a family induces a continuous bundle map over the subspace

$$\mathcal{T}^*(X, Y; X', Y'; X'', Y'') \subset \mathcal{T}(X, Y; X', Y'; X'', Y'')$$

of exact triples as in (2.6) with surjective Fredholm operators D, D', D'' with respect to the topologies determined by the Normalization I property.

For an isomorphism $(\phi, \psi)\colon D \longrightarrow D'$ between Fredholm operators $D\colon X \longrightarrow Y$ and $D'\colon X' \longrightarrow Y'$ as above (2.4), define

$$\widetilde{\mathcal{I}}_{\phi,\psi;D}\colon \lambda(\phi^{-1}) \otimes \lambda(D) \otimes \lambda(\psi) \longrightarrow \lambda(D'), \quad \widetilde{\mathcal{I}}_{\phi,\psi;D}\big((1 \otimes 1^*) \otimes \sigma \otimes (1 \otimes 1^*)\big) = \widetilde{\mathcal{I}}_{\phi,\psi;D}(\sigma).$$

By Normalization III, the diagram

$$
\begin{array}{ccc}
\lambda(\phi^{-1}) \otimes \lambda(D) \otimes \lambda(\psi) & \xrightarrow{\ \mathrm{id}\otimes\widetilde{\mathcal{C}}_{D,\psi}\ } & \lambda(\phi^{-1}) \otimes \lambda(\psi\circ D) \\
\Big\downarrow{\scriptstyle \widetilde{\mathcal{C}}_{\phi^{-1},D}\otimes\mathrm{id}} \quad {\scriptstyle \widetilde{\mathcal{I}}_{\phi,\psi;D}} \searrow & & \Big\downarrow{\scriptstyle \widetilde{\mathcal{C}}_{\phi^{-1},\psi\circ D}} \\
\lambda(D\circ\phi^{-1}) \otimes \lambda(\psi) & \xrightarrow[{\scriptstyle \widetilde{\mathcal{C}}_{D\circ\phi^{-1},\psi}}]{} & \lambda(\psi\circ D \circ \phi^{-1}) = \lambda(D')
\end{array}
$$

commutes.

Some of the properties listed in Theorem 1 are similarly implied by other properties:

- Naturality I follows from the continuity of Ψ in (2.27) and Normalization III applied to the diagrams

$$
\begin{array}{ccccccc}
0 & \longrightarrow & X & \xrightarrow{\ \phi\ } & X' & \longrightarrow & \{0\} & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle D'} & & \Big\downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{\ \psi\ } & Y' & \longrightarrow & \{0\} & \longrightarrow & 0
\end{array}
$$

- Complex Orientations follows from the continuity of Ψ, Normalization I, and Complex Exact Triples;

- the isomorphism Naturality II property follows from Compositions II and Normalization II,III applied to the diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X'_B & \longrightarrow & X_B & \longrightarrow & X''_B & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \phi'^{-1}} & & \Big\downarrow{\scriptstyle \phi^{-1}} & & \Big\downarrow{\scriptstyle \phi''^{-1}} & & \\
0 & \longrightarrow & X'_T & \longrightarrow & X_T & \longrightarrow & X''_T & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle D'_T} & & \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle D''_T} & & \\
0 & \longrightarrow & Y'_T & \longrightarrow & Y_T & \longrightarrow & Y''_T & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & X'_B & \longrightarrow & X_B & \longrightarrow & X''_B & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle D'_T\circ\phi'^{-1}} & & \Big\downarrow{\scriptstyle D_T\circ\phi^{-1}} & & \Big\downarrow{\scriptstyle D''_T\circ\phi''^{-1}} & & \\
0 & \longrightarrow & Y'_T & \longrightarrow & Y_T & \longrightarrow & Y''_T & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \psi'} & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \psi''} & & \\
0 & \longrightarrow & Y'_B & \longrightarrow & Y_B & \longrightarrow & Y''_B & \longrightarrow & 0
\end{array}
$$

- the Exact Squares property below is implied by the Naturality II, Normalization II, and Compositions II properties (see Corollary 4.13);

- the two Direct Sums properties follow from the Exact Squares property applied to the two diagrams in Figure 1 and the isomorphism Naturality II property applied to the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & D'' & \longrightarrow & D' \oplus D'' & \longrightarrow & D' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & D'' \oplus D' & \longrightarrow & D' & \longrightarrow & 0
\end{array}
$$

- Compositions II is implied by the Normalization III and Exact Squares properties (see Subsection 3.2);

- Compositions I is implied by the isomorphism Naturality II, Normalization III, and Exact Squares properties (see Subsection 3.2);

- the full Naturality II property is implied by the isomorphism Naturality II, Normalization II,III, and Exact Squares properties (see Subsection 3.4).

By Proposition 5.3, a collection of exact triple isomorphisms Ψ_t as in (2.27) determines topologies on the line bundles $\det_{X,Y}$ which satisfy the Naturality I and Normalization I properties if this collection satisfies the Normalization II,III and Compositions I,II properties. By Corollary 5.4, all these isomorphisms Ψ_t are continuous in the resulting topologies if in addition they satisfy the Naturality II property. By the above, the full Naturality and Compositions I,II property can be replaced by the isomorphism Naturality II and Exact Squares properties.

In summary, a collection of exact triple isomorphisms $(\Psi_t)_t$ as in (2.27) determines a system of topologies on the determinant line bundles $\det_{X,Y}$ which satisfy the Naturality I and Normalization I properties and in which the isomorphisms Ψ_t are continuous if this collection satisfies the Normalization II,III properties along with either

- the Exact Squares property or

- the Naturality II and Compositions I,II properties.

In either case, such a collection of isomorphisms necessarily satisfies the Exact Squares, Naturality II, Compositions I,II, and Direct Sums I,II properties. This is consistent with [11, Theorem 1]; see Subsection 3.2 for more details. If a collection of isomorphisms as above also satisfies the Complex Exact Triples properties, then the resulting topologies satisfy the Complex Orientations property as well. A collection of isomorphisms Ψ_t satisfying all of the above properties is specified by (4.10). Theorem 2 on page 35 describes all other collections of isomorphisms Ψ_t with these properties.

Exact Squares. For every commutative diagram

$$
\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & D_{\mathrm{TL}} & \xrightarrow{i_{\mathrm{T}}} & D_{\mathrm{TM}} & \xrightarrow{j_{\mathrm{T}}} & D_{\mathrm{TR}} & \longrightarrow 0 \\
& \downarrow{\scriptstyle i_{\mathrm{L}}} & & \downarrow{\scriptstyle i_{\mathrm{M}}} & & \downarrow{\scriptstyle i_{\mathrm{R}}} & \\
0 \longrightarrow & D_{\mathrm{CL}} & \xrightarrow{i_{\mathrm{C}}} & D_{\mathrm{CM}} & \xrightarrow{j_{\mathrm{C}}} & D_{\mathrm{CR}} & \longrightarrow 0 \\
& \downarrow{\scriptstyle j_{\mathrm{L}}} & & \downarrow{\scriptstyle i_{\mathrm{M}}} & & \downarrow{\scriptstyle j_{\mathrm{R}}} & \\
0 \longrightarrow & D_{\mathrm{BL}} & \xrightarrow{i_{\mathrm{B}}} & D_{\mathrm{BM}} & \xrightarrow{j_{\mathrm{B}}} & D_{\mathrm{BR}} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}
\tag{2.32}
$$

of exact rows and columns of Fredholm operators, the diagram

$$
\begin{array}{ccc}
\lambda(D_{\mathrm{TL}}) \otimes \lambda(D_{\mathrm{BL}}) \otimes \lambda(D_{\mathrm{TR}}) \otimes \lambda(D_{\mathrm{BR}}) & \xrightarrow{\Psi_{\mathrm{T}} \otimes \Psi_{\mathrm{B}} \,\circ\, \mathrm{id} \otimes R \otimes \mathrm{id}} & \lambda(D_{\mathrm{TM}}) \otimes \lambda(D_{\mathrm{BM}}) \\
\downarrow{\scriptstyle \Psi_{\mathrm{L}} \otimes \Psi_{\mathrm{R}}} & & \downarrow{\scriptstyle \Psi_{\mathrm{M}}} \\
\lambda(D_{\mathrm{CL}}) \otimes \lambda(D_{\mathrm{CR}}) & \xrightarrow{\Psi_{\mathrm{C}}} & \lambda(D_{\mathrm{CM}})
\end{array}
\tag{2.33}
$$

of graded lines, where $\Psi_\star$ are the isomorphisms (2.27) corresponding to the top, center, and bottom rows and left, middle, and right columns of the diagram (2.32), commutes.

By the proof of Lemma 5.6, the Normalization I property can be replaced by a dual version. Let

$$
\mathcal{F}'(X,Y) \equiv \big\{ D \in \mathcal{F}(X,Y) \colon \kappa(D) = 0 \big\}
$$

be the space of injective Fredholm operators. For each $D_0 \in \mathcal{F}'(X,Y)$, right inverse $S \colon \mathfrak{c}(D_0) \longrightarrow Y$ for

$$
q_{D_0} \colon Y \longrightarrow \mathfrak{c}(D_0), \qquad q_{D_0}(y) = y + \operatorname{Im} D_0,
$$

and $D \in \mathcal{F}(X,Y)$ sufficiently close to D_0, the homomorphism

$$
q_D \circ S \colon \mathfrak{c}(D_0) \longrightarrow \mathfrak{c}(D)
$$

is an isomorphism and thus induces an isomorphism

$$
\widetilde{\mathcal{I}}_{D_0,S;D} \colon \lambda(D_0) \longrightarrow \lambda(D), \quad \widetilde{\mathcal{I}}_{D_0,S;D}(1 \otimes \alpha) = 1 \otimes \big(\alpha \circ \lambda (q_D \circ S)^{-1} \big).
\tag{2.34}
$$

Putting these isomorphisms together, we obtain a bundle map

$$
\widetilde{\mathcal{I}}_{D_0,S} \colon U_{D_0,S} \times \lambda(D_0) \longrightarrow \det\nolimits_{X,Y}|_{U_{D_0,S}}
\tag{2.35}
$$

covering the identity on an open neighborhood $U_{D_0,S}$ of D_0 in $\mathcal{F}'(X,Y)$.

Normalization I'. The map $\widetilde{\mathcal{I}}_{D_0,S}$ is continuous for every $D_0 \in \mathcal{F}'(X,Y)$, right inverse $S \colon \mathfrak{c}(D_0) \longrightarrow Y$ of q_{D_0}, and sufficiently small open neighborhood $U_{D_0,S}$ of D_0 in $\mathcal{F}'(X,Y)$.

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & D' & \longrightarrow & D' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & D'\oplus D'' & \longrightarrow & D' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & D'' & \longrightarrow & 0 & \longrightarrow & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D' & \longrightarrow & D' & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'\oplus D'' & \longrightarrow & D'\oplus D''\oplus D''' & \longrightarrow & D''' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & D''\oplus D''' & \longrightarrow & D''' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 1: Exact squares of Fredholm operators corresponding to the two Direct Sums properties

The determinant line bundle is also compatible with dualizations of Fredholm operators. Let $\mathcal{D}_D$ and $\mathcal{P}$ be as in (2.20) and (2.2), respectively.

Dualizations. There exists a collection of (continuous) line bundle isomorphisms

$$\widetilde{\mathcal{D}}\colon \det{}_{X,Y} \longrightarrow \mathcal{D}^*\det{}_{Y^*,X^*} \tag{2.36}$$

over $\mathcal{F}(X,Y)$ parametrized by pairs (X,Y) of Banach vector spaces with the following properties.

Normalization IV. For every homomorphism $\delta\colon L \longrightarrow \{0\}$ from a line,

$$\widetilde{\mathcal{D}}_\delta\big(x \otimes 1^*\big) = 1 \otimes \mathcal{P}\big(\mathcal{D}_\delta(x)\big) \qquad \forall\, x \in \lambda(L) = L.$$

Dual Exact Triples. The isomorphisms (2.27) and (2.36) provide a lifting of (2.22) to determinant line bundles, i.e. the diagram

$$
\begin{array}{ccc}
\lambda(D') \otimes \lambda(D'') & \xrightarrow{\ \Psi_{\mathfrak{t}}\ } & \lambda(D) \\
{\scriptstyle \widetilde{\mathcal{D}}_{D''} \otimes \widetilde{\mathcal{D}}_{D'} \circ R} \downarrow & & \downarrow {\scriptstyle \widetilde{\mathcal{D}}_D} \\
\lambda(D''^*) \otimes \lambda(D'^*) & \xrightarrow{\ \Psi_{\mathfrak{t}^*}\ } & \lambda(D^*)
\end{array}
\tag{2.37}
$$

commutes for every $\mathfrak{t} \in \mathcal{T}(X, Y; X', Y'; X'', Y'')$ as in (2.8).

Dual Complex Orientations. If X, Y are Banach vector spaces with complex structures and $D \in \mathcal{F}_{\mathbb{C}}(X, Y)$, the isomorphism (2.36) intertwines the complex orientations of $\lambda(D)$ and $\lambda(D^*)$.

By Corollary 5.8 and Section 3.4, each determinant line bundle system as in Theorem 1 on page 14 determines a unique system of isomorphisms $\widetilde{\mathcal{D}}$ satisfying the above three properties. Furthermore, there is a somewhat smaller family of determinant line bundle systems that satisfy a stronger version of the Normalization IV property:

Normalization IV*. For each $D \in \mathcal{F}^*(X, Y)$, $\widetilde{\mathcal{D}}_D$ is the canonical isomorphism induced by the first equation in (2.20) and the pairing (2.2):

$$
\lambda(D) \longrightarrow \lambda(D^*), \qquad x \otimes 1^* \longrightarrow 1 \otimes \mathcal{P}\big(\lambda(\mathcal{D}_D)x\big).
\tag{2.38}
$$

By Lemma 5.7, the isomorphisms (2.38) give rise to a continuous bundle map over $\mathcal{F}^*(X, Y)$ for any system of topologies on determinant line bundles as in Theorem 1. In the proof of Corollary 5.8, we use this to show that the continuity of (2.36) is implied by the Dual Exact Triples property. However, the isomorphisms (2.36) are compatible with the Normalization IV* property only for some determinant line bundle systems, including the one specified by the isomorphisms $\Psi_{\mathfrak{t}}$ of (4.10).

The dualization isomorphisms $\widetilde{\mathcal{D}}_D$ given by (4.13) and the identity isomorphisms $A_{i,1}$ in (1.2) seem rather natural. However, by Theorem 2 on page 35, the number of systems of topologies on determinant line bundles compatible with these choices is still infinite.

Combining the Dual Exact Triples property with the isomorphism Naturality II property applied to the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & D''^* & \xrightarrow{\ j^*\ } & (D' \oplus D'')^* & \xrightarrow{\ i^*\ } & D'^* & \longrightarrow & 0 \\
& & {\scriptstyle \text{id}}\downarrow & & {\scriptstyle (T_X, T_Y)}\downarrow & & {\scriptstyle \text{id}}\downarrow & & \\
0 & \longrightarrow & D''^* & \xrightarrow{\ i\ } & D''^* \oplus D'^* & \xrightarrow{\ j\ } & D'^* & \longrightarrow & 0,
\end{array}
\qquad
\begin{array}{l}
T_X(\beta) = (\beta|_{Y''}, \beta|_{Y'}) \\[2em]
T_Y(\alpha) = (\alpha|_{X''}, \alpha|_{X'})
\end{array}
\tag{2.39}
$$

where $\mathfrak{i} = (\mathfrak{i}_X, \mathfrak{i}_Y)$ and $\mathfrak{j} = (\mathfrak{j}_X, \mathfrak{j}_Y)$ are as in (2.14), we find that the diagram

$$
\begin{array}{ccc}
\lambda(D') \otimes \lambda(D'') & \xrightarrow{\ \widetilde{\oplus}_{D', D''}\ } & \lambda(D' \oplus D'') \\
{\scriptstyle \widetilde{\mathcal{D}}_{D''} \otimes \widetilde{\mathcal{D}}_{D'} \circ R} \downarrow & & \downarrow {\scriptstyle \widetilde{\mathcal{I}}_{T_X, T_Y; (D' \oplus D'')^*} \circ \widetilde{\mathcal{D}}_{D' \oplus D''}} \\
\lambda(D''^*) \otimes \lambda(D'^*) & \xrightarrow{\ \widetilde{\oplus}_{D''^*, D'^*}\ } & \lambda(D''^* \oplus D'^*)
\end{array}
$$

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commutes, i.e. the dualization and direct sum isomorphisms, $\widetilde{\mathcal{D}}$ and $\widetilde{\oplus}$, on the determinant lines are compatible. Combining the Dual Exact Triples property with the isomorphism Naturality II property applied to (2.39) with $(D', D'') = (D_2 \circ D_1, \mathrm{id}_{X_2})$, we find that the diagram

$$
\begin{array}{ccc}
\lambda(D_1) \otimes \lambda(D_2) & \xrightarrow{\;\widetilde{\mathcal{C}}_{D_1,D_2}\;} & \lambda(D_2 \circ D_1) \\
\Big\downarrow{\scriptstyle \widetilde{\mathcal{D}}_{D_2} \otimes \widetilde{\mathcal{D}}_{D_1} \circ R} & & \Big\downarrow{\scriptstyle \widetilde{\mathcal{D}}_{D_2 \circ D_1}} \\
\lambda(D_2^*) \otimes \lambda(D_1^*) & \xrightarrow{\;\widetilde{\mathcal{C}}_{D_2^*,D_1^*}\;} & \lambda(D_1^* \circ D_2^*)
\end{array}
$$

commutes, i.e. the dualization and composition isomorphisms, $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{C}}$, on the determinant lines are compatible.

Section 4.2 provides explicit formulas for the above isomorphisms Ψ_t, $\widetilde{\oplus}_{D',D''}$, $\widetilde{\mathcal{C}}_{D_1,D_2}$, and $\widetilde{\mathcal{D}}_D$; see (4.10), (4.12), (4.22), and (4.13), respectively. Such formulas may be useful in some applications.

3 Conceptual considerations and comparison of conventions

3.1 Topologizing determinant line bundles

For any Banach vector spaces X and Y, the overlap maps between the trivializations $\widetilde{\mathcal{I}}_{D,T}$ of $\det_{X,Y}$ in (2.26) are continuous. Thus, the trivializations $\widetilde{\mathcal{I}}_{D,T}$ topologize $\det_{X,Y}\big|_{\mathcal{F}^*(X,Y)}$ as a line bundle over $\mathcal{F}^*(X,Y)$, as required by the Normalization I property on page 12. By Lemma 5.1, the resulting topology is compatible with the Normalization II property on page 13.

For any Banach vector space X and $N \in \mathbb{Z}^{\geq 0}$, let $\iota_{X;N} \colon X \longrightarrow X \oplus \mathbb{R}^N$ be the natural inclusion. If Y is another Banach vector space, $D \in \mathcal{F}(X,Y)$, and $\Theta \colon \mathbb{R}^N \longrightarrow Y$ is any homomorphism, define

$$
\iota_\Theta \colon \mathcal{F}(X,Y) \longrightarrow \mathcal{F}(X \oplus \mathbb{R}^N, Y) \qquad \text{by} \quad \iota_\Theta(D) = D_\Theta, \quad D_\Theta(x, u) = Dx + \Theta(u);
$$

the map ι_Θ is an embedding. The exact triple

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\;\iota_{X;N}\;} & X \oplus \mathbb{R}^N & \xrightarrow{\;\pi_2\;} & \mathbb{R}^N & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle D_\Theta} & & \Big\downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{\;\mathrm{id}_Y\;} & Y & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\tag{3.1}
$$

and (2.27) give rise to the isomorphism

$$
\hat{\mathcal{I}}_{\Theta;D} \colon \lambda(D) \longrightarrow \lambda(D_\Theta), \qquad \hat{\mathcal{I}}_{\Theta;D}(\sigma) = \Psi_t\big(\sigma \otimes \Omega_N \otimes 1^*\big),
\tag{3.2}
$$

where Ω_N is the standard volume tensor on $\mathbb{R}^N$, i.e.

$$
\Omega_N = e_1 \wedge \ldots \wedge e_N
$$

if $e_1, \ldots, e_N$ is the standard basis for $\mathbb{R}^N$.

By the continuity requirement on the family of isomorphisms $\Psi_{\mathfrak{t}}$ in (2.27) and the Normalization I property, the isomorphisms $\hat{\mathcal{I}}_{\Theta;D}$ topologize $\det_{X,Y}$ over the open subset

$$U_{X;\Theta} \equiv \big\{D \in \mathcal{F}(X,Y)\colon \mathfrak{c}(D_{\Theta}) = 0\big\}.$$

Since these open subsets cover $\mathcal{F}(X,Y)$ as Θ ranges over all homomorphisms $\mathbb{R}^N \longrightarrow Y$ and N ranges over all nonnegative integers, the isomorphisms $\hat{\mathcal{I}}_{\Theta;D}$ completely specify the topology on $\det_{X,Y}$. However, the overlap map

$$\hat{\mathcal{I}}_{\Theta_2;D} \circ \hat{\mathcal{I}}_{\Theta_1;D}^{-1}\colon \iota_{\Theta_1}^* \det_{X \oplus \mathbb{R}^{N_1},Y} \longrightarrow \iota_{\Theta_2}^* \det_{X \oplus \mathbb{R}^{N_2},Y}$$

must be continuous over $U_{X;\Theta_1} \cap U_{X;\Theta_2}$ for any pair of homomorphisms $\Theta_1 : \mathbb{R}^{N_1} \longrightarrow Y$ and $\Theta_2 : \mathbb{R}^{N_2} \longrightarrow Y$. By Proposition 5.3, this is indeed the case if the collection of isomorphisms $\Psi_{\mathfrak{t}}$ satisfies the Normalization II,III and Compositions I,II properties. By Corollary 5.4, every isomorphism in such a collection is continuous in the resulting topologies if this collection also satisfies the Naturality II property. A collection of isomorphisms $\Psi_{\mathfrak{t}}$ satisfying all these properties is provided by (4.10). All other such collections are described by Theorem 2 on page 35.

For $D \in U_{X;\Theta}$, the exact triple (3.1) induces an exact sequence

$$0 \longrightarrow \kappa(D) \longrightarrow \kappa(D_{\Theta}) \longrightarrow \mathbb{R}^N \longrightarrow \mathfrak{c}(D) \longrightarrow 0$$

of vector spaces. A homomorphism $\delta : V \longrightarrow W$ between finite-dimensional vector spaces also induces an exact sequence

$$0 \longrightarrow \kappa(\delta) \longrightarrow V \xrightarrow{\ \delta\ } W \longrightarrow \mathfrak{c}(\delta) \longrightarrow 0$$

of vector spaces. There is an isomorphism

$$\mathcal{I}_{\delta}\colon \lambda(\delta) \longrightarrow \lambda(\mathbf{0}) \equiv \lambda(V) \otimes \lambda^*(W). \tag{3.3}$$

As suggested in [15], a suitable collection of these isomorphisms is fundamental to constructing a system of determinant line bundles for Fredholm operators. Unfortunately, [15] makes no mention of what properties of a system of isomorphisms (3.3) are needed for such a construction and gives no explicit formula for these isomorphisms. The discussion in [15] is also limited to Cauchy-Riemann operators on Riemann surfaces.

In the convention (2.3), which is also used in [12, Section 20.2] and [14, Section 7.4], the exact triple isomorphisms (4.10) correspond to the isomorphisms (3.3) given by

$$\mathcal{I}_{\delta}\colon \lambda(\delta) \longrightarrow \lambda(\mathbf{0}), \qquad x \otimes y^* \longrightarrow (-1)^{(\mathfrak{d}(W) - \mathfrak{d}(\mathfrak{c}(\delta)))\mathfrak{d}(\mathfrak{c}(\delta))} (x \wedge_V v) \otimes \big(\lambda(\delta)v \wedge_W y\big)^*, \tag{3.4}$$

$$\forall\ x \in \lambda\big(\kappa(\delta)\big) - 0,\ y \in \lambda\big(\mathfrak{c}(\delta)\big) - 0,\ v \in \lambda\!\left(\frac{V}{\kappa(\delta)}\right) - 0,$$

where $\wedge_V$ and $\wedge_W$ are the isomorphisms of Lemma 4.1; see Remark 4.6. This is precisely the isomorphism of [14, Lemma 7.4.7] and is used directly to topologize determinant line bundles in the proof of [14, Proposition 7.4.8].[2] While the properties of (3.4) necessary for this construction are verified in [14], few of the important properties of the resulting determinant line bundles

---

[2]As shown in the proof of Proposition 5.3 in this paper, the restriction to injective homomorphisms Θ in the proof of [14, Proposition 7.4.8] is unnecessary.

are checked in [14]. The isomorphism (3.4) appears only indirectly in the construction of this paper.

There are alternative ways of constructing a system of determinant line bundles satisfying the properties in Subsection 2.2.

(1) A system of determinant line bundles for bounded complexes of vector bundles and isomorphisms for exact triples of such complexes is constructed in [11, Chapter I]. A system of determinant line bundles for Fredholm operators can then be obtained by associating each Fredholm operator with a two-term complex, deducing the Exact Squares property for Fredholm operators from that for bounded complexes and the two algebraic Compositions properties from the Exact Squares property, and deriving explicit formulas for all isomorphisms. This approach is described in detail in Section 3.2.

(2) One could explicitly specify a collection of isomorphisms $\hat{\mathcal{I}}_{\Theta;D}$ as in (3.2) that are compatible with compositions. This is essentially the approach taken in [13], [14], [16], and [17] to topologize determinant line bundles, without verifying the properties in Subsection 2.2. The isomorphisms (3.2) can be used to define Exact Triples isomorphisms (2.27) from the Normalization II property, imposing the commutativity property of Lemma 4.12 by definition, and to derive an explicit formula for these isomorphisms. The Exact Squares property for Fredholm operators can then be obtained from the basic Exact Squares property of Lemma 4.3 as in the proof of Corollary 4.13 and used to confirm the two algebraic Compositions properties.

(3) The commutativity property of Lemma 4.12 could be verified for the isomorphism (4.10) directly, without using Proposition 4.10, and used to obtain the Exact Squares property as in the proof of Corollary 4.13. The two algebraic Compositions properties could then be deduced either from the Exact Squares property or from the corresponding properties for vector spaces by an argument similar to the proof of Corollary 4.13. Unfortunately, the proof of the special case of Proposition 4.10 corresponding to Lemma 4.12 is as elaborate as the proof of Proposition 4.10 itself; the former involves a bit less notation, but exactly the same steps.

In all three approaches, the Dual Exact Triples property can be either checked directly or deduced from more general considerations. The above listed alternatives can be used to replace parts of Section 4 in this paper, but most of Section 5 would still be needed. It appears the overall approach of this paper is more efficient than the three alternatives described above.

The equivalence of the topologies arising from the algebraic approach of [11] and the analytic approach of [15] in many complex-geometric settings is shown in the trilogy [2, 3, 4]; see [2, Theorem 0.1] in particular. Combined with earlier work [6, 7], this trilogy leads to an arithmetic version of the Grothendick-Riemann-Roch Theorem; see [8]. A thorough discussion of the determinant line bundle in Arakelov geometry, which is outside of the scope of this paper, is contained in the books [5, 18].

3.2 Relation with Knudsen-Mumford

The existence of a determinant line bundle system satisfying the properties in Subsection 2.2 follows most readily (but still with some work) from the proof of [11, Theorem 1], which constructs determinant line bundles for bounded complexes of vector bundles. Unfortunately, a complete

construction of a determinant line bundle based on [11] with a verification of all of the properties listed in Subsection 2.2 and with explicit formulas for the relevant isomorphisms does not seem to appear elsewhere; we describe it below.

For each homomorphism $\Theta\colon \mathbb{R}^N \longrightarrow Y$,

$$\mathcal{K}_\Theta \equiv \left\{ (D, x, u) \in U_{X;\Theta} \times X \oplus \mathbb{R}^N \colon (x, u) \in \kappa(D_\Theta) \right\} \longrightarrow U_{X;\Theta}$$

is a vector bundle. For each $D \in U_{X;\Theta}$, the commutative diagram (3.1) gives rise to an exact sequence

$$0 \longrightarrow \kappa(D) \longrightarrow \kappa(D_\Theta) \xrightarrow{\delta_\Theta} \mathbb{R}^N \xrightarrow{\Theta} \mathfrak{c}(D) \longrightarrow 0. \tag{3.5}$$

Thus, each homomorphism $\Theta\colon \mathbb{R}^N \longrightarrow Y$ determines a two-term graded complex

$$\ldots \longrightarrow 0 \longrightarrow \mathcal{K}_\Theta \xrightarrow{\delta_\Theta} U_{X;\Theta} \times \mathbb{R}^N \longrightarrow 0 \longrightarrow \ldots \tag{3.6}$$

of vector bundles over $U_{X;\Theta}$, with $\mathcal{K}_\Theta$ placed at the 0-th position, and a $\mathbb{Z}_2$-graded line bundle

$$\mathcal{L}_\Theta \equiv \lambda(\mathcal{K}_\Theta) \otimes \lambda^*(U_{X;\Theta} \times \mathbb{R}^N),$$

the determinant line bundle of the two-term complex (3.6).

For each $D \in U_{X;\Theta}$, let $\Xi\colon \mathfrak{c}(D) \longrightarrow \mathbb{R}^N$ be a right inverse for the surjective map

$$\mathbb{R}^N \longrightarrow \mathfrak{c}(D), \qquad u \longrightarrow \Theta(u) + \operatorname{Im} D. \tag{3.7}$$

The diagram

is then a quasi-isomorphism of graded complexes over $\{D\}$, i.e. a homomorphism of graded complexes of vector bundles that induces an isomorphism in homology. By [11, Theorem 1], there is then a canonical isomorphism

$$\hat{\mathcal{I}}'_{\Theta;D}\colon \lambda(D) \longrightarrow \mathcal{L}_\Theta|_D \approx \lambda(D_\Theta).$$

Since any other right inverse for the homomorphism (3.7) is of the form $\Xi + \delta_\Theta\tilde{\Xi}$ for some homomorphism $\tilde{\Xi}\colon \mathfrak{c}(D) \longrightarrow \kappa(D_\Theta)$, $\hat{\mathcal{I}}'_{\Theta;D}$ is independent of the choice of Ξ by [11, Proposition 2]. If $\Theta'\colon \mathbb{R}^{N'} \longrightarrow Y$ is another homomorphism and $\iota\colon \mathbb{R}^N \longrightarrow \mathbb{R}^{N'}$ is a homomorphism such that $\Theta = \Theta' \circ \iota$,

is also a quasi-isomorphism of graded complexes. By the proof of [11, Theorem 1], it also induces a canonical isomorphism

$$\mathcal{I}_{\Theta',\Theta}\colon \mathcal{L}_\Theta \longrightarrow \mathcal{L}_{\Theta'}$$

of line bundles over $U_{X;\Theta}$. By the functoriality of the determinant construction of [11, Theorem 1],

$$\hat{\mathcal{I}}'_{\Theta';D} = \mathcal{I}_{\Theta',\Theta} \circ \hat{\mathcal{I}}'_{\Theta;D} \colon \lambda(D) \longrightarrow \mathcal{L}_{\Theta'}|_D \approx \lambda(D_{\Theta'}).$$

Since the line bundle maps $\mathcal{I}_{\Theta',\Theta}$ are continuous, the isomorphisms $\hat{\mathcal{I}}'_{\Theta;D}$ topologize $\det_{X,Y}$ over $U_{X;\Theta}$ and endow $\det_{X,Y}$ with a well-defined topology of a line bundle over $\mathcal{F}(X,Y)$, which satisfies the Normalization I and Naturality I properties.

The proof of [11, Theorem 1] produces analogues of the isomorphisms (2.27) for exact triples of graded complexes (3.6) of vector bundles. These isomorphisms satisfy analogues of the Normalization II,III, Naturality II, and Exact Squares properties. By the proof of Corollary 5.4, an exact triple of Fredholm operators gives rise to an exact triple of two-term complexes (over a point). By the analogue of the Naturality II property for two-term complexes, the isomorphisms of [11, Theorem 1] then induce via the isomorphisms $\hat{\mathcal{I}}'_{\Theta;D}$ isomorphisms $\Psi_{\mathfrak{t}}$ for exact triples of Fredholm operators which satisfy the Normalization II,III and Naturality II properties. These isomorphisms depend continuously on $\mathfrak{t}$ by the proofs of Lemma 5.1 and Corollary 5.4. By the proof of Corollary 4.13, an exact square of Fredholm operators as in (2.32) gives rise to an exact square of two-term complexes. By the analogue of the Exact Squares property for two-term complexes and the proof of Corollary 4.13, the induced isomorphisms for exact triples of Fredholm operators satisfy the Exact Squares property for Fredholm operators. The proof of [11, Theorem 1] implies the existence of the bundle maps $\widetilde{\mathcal{D}}_D$ as in (2.36) satisfying the analogue of the Dual Exact Triples property on page 19 for two-term complexes. These bundle maps $\widetilde{\mathcal{D}}_D$ satisfy the analogue of the Normalization IV* property on page 19 in the case of the system explicitly constructed in the proof of [11, Theorem 1]; this can be seen from the last paragraph of this section and Section 3.4.

We show below that the Compositions I,II properties on page 14 follow from the isomorphism Naturality II, Normalization III, and Exact Squares properties. By Subsection 3.4, the full Naturality II property follows from the isomorphism Naturality II, Normalization II,III, and Exact Squares properties. Thus, [11, Theorem 1] gives rise to a determinant line bundle system satisfying all properties in Subsection 2.2, with the possible exceptions of the Complex Orientations, Complex Exact Triples, and Dual Complex Orientations properties.

Exact Squares and Normalization III imply Compositions II. Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be exact triples as in (2.31). For $\star = ',''$ or blank, let

$$\Psi^\star \colon \lambda(D_1^\star) \otimes \lambda(D_2^\star) \longrightarrow \lambda\big(D_2^\star \circ D_1^\star \oplus \mathrm{id}_{X_2^\star}\big)$$

be the isomorphism (2.27) for the exact triple (2.19) corresponding to the composition $D_2^\star \circ D_1^\star$. For $i = 1, 3$, define

$$\iota_{X_i^\star; X_2^\star} \colon X_i^\star \longrightarrow X_i^\star \oplus X_2^\star, \qquad \iota_{X_i^\star; X_2^\star}(x) = (x, 0).$$

Thus, $(\iota_{X_1^\star; X_2^\star}, \iota_{X_3^\star; X_2^\star})$ is a quasi-isomorphism from $D_2^\star \circ D_1^\star$ to $D_2^\star \circ D_1^\star \oplus \mathrm{id}_{X_2^\star}$. Let

$$\mathcal{I}^\star = \widetilde{\mathcal{I}}_{\iota_{X_1^\star; X_2^\star}, \iota_{X_3^\star; X_2^\star}; D_2^\star \circ D_1^\star} \colon \lambda(D_2^\star \circ D_1^\star) \longrightarrow \lambda\big(D_2^\star \circ D_1^\star \oplus \mathrm{id}_{X_2^\star}\big)$$

be the corresponding isomorphism (2.4). In the top diagram of Figure 2, the rows are the exact triple $\mathfrak{t}_1$, the direct sum of $\mathcal{C}_\mathcal{T}(\mathfrak{t}_1, \mathfrak{t}_2)$ with

$$0 \longrightarrow \mathrm{id}_{X_2'} \longrightarrow \mathrm{id}_{X_2} \longrightarrow \mathrm{id}_{X_2''} \longrightarrow 0,$$

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and the exact triple $\mathfrak{t}_2$. The columns in this commutative square of Fredholm operators are the exact triples (2.19) corresponding to the compositions $D_2'{\circ}D_1'$, $D_2{\circ}D_1$, and $D_2''{\circ}D_1''$.

Applying the Exact Squares property to this diagram, we obtain the top commutative square in the last diagram in Figure 2. Applying the Exact Squares and Normalization III properties to the center diagram in Figure 2 and using the identification

$$\lambda(D_2^\star{\circ}D_1^\star) \longrightarrow \lambda(D_2^\star{\circ}D_1^\star){\otimes}\lambda(\mathrm{id}_{X_2^\star}), \qquad \sigma \longrightarrow \sigma \otimes 1{\otimes}1^*, \tag{3.8}$$

with $\star{=}',''$ or blank, we obtain the bottom commutative square in this diagram. The two round arrows are the vertical arrows in (2.31); the two half-disk diagrams commute by the definition of $\widetilde{\mathcal{C}}_{D_1^\star,D_2^\star}$. Thus, the diagram (2.31), which consists of the outermost arrows in the last diagram in Figure 2, commutes. $\qquad\square$

Exact Squares, Normalization III, and isomorphism Naturality II imply Compositions I. Let D_1, D_2, D_3 be Fredholm operators as in (2.30). Let

$$\Psi_{1,2}\colon \lambda(D_1){\otimes}\lambda(D_2) \longrightarrow \lambda(D_2{\circ}D_1{\oplus}\mathrm{id}_{X_2}), \quad \Psi_{1,23}\colon \lambda(D_1){\otimes}\lambda(D_3{\circ}D_2) \longrightarrow \lambda(D_3{\circ}D_2{\circ}D_1 {\oplus}\mathrm{id}_{X_2}),$$
$$\Psi_{2,3}\colon \lambda(D_2){\otimes}\lambda(D_3) \longrightarrow \lambda(D_2{\circ}D_3{\oplus}\mathrm{id}_{X_3}), \quad \Psi_{12,3}\colon \lambda(D_2{\circ}D_1){\otimes}\lambda(D_3) \longrightarrow \lambda(D_3{\circ}D_2{\circ}D_1 {\oplus}\mathrm{id}_{X_3})$$

be the isomorphisms (2.27) for the exact triple (2.19) corresponding to the compositions $D_2{\circ}D_1$, $(D_3{\circ}D_2){\circ}D_1$, $D_3{\circ}D_2$, and $D_3{\circ}(D_2{\circ}D_1)$. We define isomorphisms

$$\mathcal{I}_{1,2}\colon \lambda(D_2{\circ}D_1) \longrightarrow \lambda\big(D_2{\circ}D_1{\oplus}\mathrm{id}_{X_2}\big), \quad \mathcal{I}_{1,23}\colon \lambda(D_3{\circ}D_2{\circ}D_1) \longrightarrow \lambda\big(D_3{\circ}D_2{\circ}D_1{\oplus}\mathrm{id}_{X_2}\big),$$
$$\mathcal{I}_{2,3}\colon \lambda(D_3{\circ}D_2) \longrightarrow \lambda\big(D_3{\circ}D_2{\oplus}\mathrm{id}_{X_3}\big), \quad \mathcal{I}_{12,3}\colon \lambda(D_3{\circ}D_2{\circ}D_1) \longrightarrow \lambda\big(D_3{\circ}D_2{\circ}D_1{\oplus}\mathrm{id}_{X_3}\big)$$

analogously to $\mathcal{I}^\star$ above.

The left column in the top diagram of Figure 3, the bottom row in this diagram, and the top row in the middle diagram of Figure 3 are the exact triples (2.19) corresponding to the compositions $D_2{\circ}D_1$, $D_3{\circ}D_2$, and $D_3{\circ}(D_2{\circ}D_1)$, respectively. The middle rows in these two figures are the direct sum of the exact triple (2.19) corresponding to the composition $D_3{\circ}(D_2{\circ}D_1)$ with the exact triple

$$0 \longrightarrow \mathrm{id}_{X_2} \longrightarrow \mathrm{id}_{X_2} \longrightarrow 0 \longrightarrow 0.$$

The homomorphisms $\mathfrak{i}{\equiv}(\mathfrak{i}_X,\mathfrak{i}_Y)$ and $\mathfrak{i}{\equiv}(\mathfrak{j}_X,\mathfrak{j}_Y)$ in the center column of the top diagram of Figure 3 are given by

$$\mathfrak{i}_X(x_1) = (x_1, D_1 x_1, D_2 D_1 x_1), \qquad \mathfrak{j}_X(x_1,x_2,x_3) = (D_1 x_1 {-} x_2, x_3 {-} D_2 x_2),$$
$$\mathfrak{i}_Y(x_2) = (D_3 D_2 x_2, x_2, D_2 x_2), \qquad \mathfrak{j}_Y(x_4,x_2,x_3) = (x_4 {-} D_3 D_2 x_2, x_3 {-} D_2 x_2).$$

The left and center columns in the middle and bottom diagrams of this figure and the top and middle rows in the bottom diagram are the direct sums of the obvious exact triples.

Applying the Exact Squares and Normalization III properties to the top exact square in Figure 3 and using identifications similar to (3.8), we find that the top left quadrilateral in Figure 4, where $\widetilde{\Psi}_{12,3}$ and Ψ_{M} are the isomorphisms (2.27) corresponding to the middle row and the center column in this square, commutes. The commuting bottom left quadrilateral in Figure 4, where

Top diagram (3×3 grid):

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_1' & \longrightarrow & D_1 & \longrightarrow & D_1'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2'\circ D_1' \oplus \mathrm{id}_{X_2'} & \longrightarrow & D_2\circ D_1 \oplus \mathrm{id}_{X_2} & \longrightarrow & D_2''\circ D_1'' \oplus \mathrm{id}_{X_2''} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2' & \longrightarrow & D_2 & \longrightarrow & D_2'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Middle diagram (3×3 grid):

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2'\circ D_1' & \longrightarrow & D_2\circ D_1 & \longrightarrow & D_2''\circ D_1'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2'\circ D_1' \oplus \mathrm{id}_{X_2'} & \longrightarrow & D_2\circ D_1 \oplus \mathrm{id}_{X_2} & \longrightarrow & D_2''\circ D_1'' \oplus \mathrm{id}_{X_2''} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{id}_{X_2'} & \longrightarrow & \mathrm{id}_{X_2} & \longrightarrow & \mathrm{id}_{X_2''} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Bottom diagram:

$$
\begin{array}{ccc}
\lambda(D_1')\otimes\lambda(D_1'')\otimes\lambda(D_2')\otimes\lambda(D_2'') & \xrightarrow{\;\Psi_{t_1}\otimes\Psi_{t_2}\;} & \lambda(D_1)\otimes\lambda(D_2) \\[2pt]
\Big\downarrow{\scriptstyle \Psi'\otimes\Psi''\circ\mathrm{id}\otimes R\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle \Psi} \\[2pt]
\lambda(D_2'\circ D_1' \oplus \mathrm{id}_{X_2'}) \otimes \lambda(D_2''\circ D_1'' \oplus \mathrm{id}_{X_2''}) & \longrightarrow & \lambda(D_2\circ D_1 \oplus \mathrm{id}_{X_2}) \\[2pt]
\Big\uparrow{\scriptstyle \mathcal{I}'\otimes\mathcal{I}''} & & \Big\uparrow{\scriptstyle \mathcal{I}} \\[2pt]
\lambda(D_2'\circ D_1')\otimes\lambda(D_2''\circ D_1'') & \xrightarrow{\;\Psi_{\mathcal{C}_{\mathcal{T}}(t_1,t_2)}\;} & \lambda(D_2\circ D_1)
\end{array}
\qquad \widetilde{c}_{D_1,D_2}
$$

Figure 2: Derivation of the Compositions II property from the Exact Squares and Normalization III properties

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_1 & \xrightarrow{\text{id}} & D_1 & \longrightarrow & \mathbf{0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & D_2\circ D_1 \oplus \text{id}_{X_2} & \longrightarrow & D_3\circ D_2\circ D_1 \oplus \text{id}_{X_2}\oplus\text{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \text{id}} & & \\
0 & \longrightarrow & D_2 & \longrightarrow & D_3\circ D_2 \oplus \text{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2\circ D_1 & \longrightarrow & D_3\circ D_2\circ D_1 \oplus \text{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \text{id}} & & \\
0 & \longrightarrow & D_2\circ D_1 \oplus \text{id}_{X_2} & \longrightarrow & D_3\circ D_2\circ D_1 \oplus \text{id}_{X_2}\oplus\text{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{id}_{X_2} & \xrightarrow{\text{id}} & \text{id}_{X_2} & \longrightarrow & \mathbf{0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_3\circ D_2\circ D_1 & \longrightarrow & D_3\circ D_2\circ D_1 \oplus \text{id}_{X_3} & \longrightarrow & \text{id}_{X_3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \text{id}} & & \\
0 & \longrightarrow & D_3\circ D_2\circ D_1 \oplus \text{id}_{X_2} & \longrightarrow & D_3\circ D_2\circ D_1 \oplus \text{id}_{X_2}\oplus\text{id}_{X_3} & \longrightarrow & \text{id}_{X_3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{id}_{X_2} & \xrightarrow{\text{id}} & \text{id}_{X_2} & \longrightarrow & \mathbf{0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 3: Exact squares of Fredholm operators used in the derivation of the Compositions I property
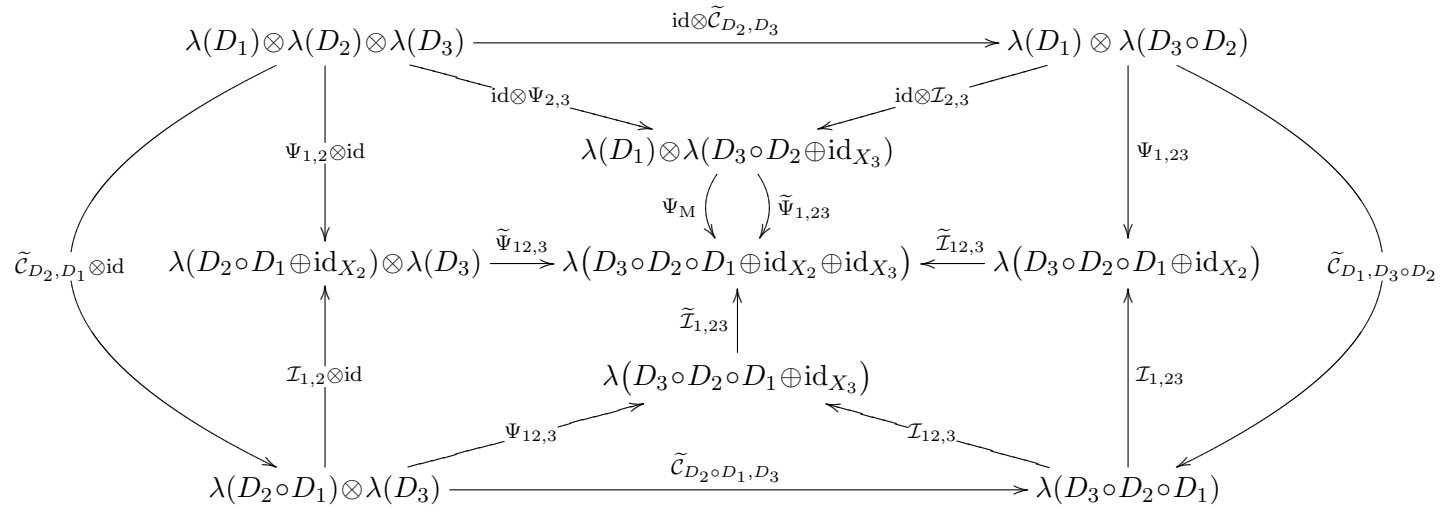
Figure 4: Commutative diagram used in the derivation of the Compositions I property from the Exact Squares, Normalization III, and isomorphism Naturality II properties

$\widetilde{\mathcal{I}}_{1,23}$ is the isomorphism (2.27) corresponding to the center column in the second diagram of Figure 3, is obtained from this diagram by applying the Exact Squares and Normalization III properties as well. A similar exact square gives the commuting top right quadrilateral in Figure 4, where $\widetilde{\Psi}_{1,23}$ and $\widetilde{\mathcal{I}}_{12,3}$ are the isomorphisms (2.27) corresponding to the direct sum of the exact triple (2.19) for the composition $(D_3 \circ D_2) \circ D_1$ with the exact triple

$$0 \longrightarrow 0 \longrightarrow \mathrm{id}_{X_3} \longrightarrow \mathrm{id}_{X_3} \longrightarrow 0$$

and to the middle row in the last diagram in Figure 3, respectively. The bottom right quadrilateral in Figure 4 arises from the last diagram in Figure 3.

The two arrows that run between the same objects in the middle of Figure 4 are related by the isomorphism of exact triples of Fredholm operators,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & D_1 & \longrightarrow & D_3 \circ D_2 \circ D_1 \oplus \mathrm{id}_{X_2} \oplus \mathrm{id}_{X_3} & \longrightarrow & D_3 \circ D_2 \oplus \mathrm{id}_{X_3} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle (\phi,\psi)} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & D_1 & \xrightarrow{\;i\;} & D_3 \circ D_2 \circ D_1 \oplus \mathrm{id}_{X_2} \oplus \mathrm{id}_{X_3} & \xrightarrow{\;j\;} & D_3 \circ D_2 \oplus \mathrm{id}_{X_3} & \longrightarrow & 0 \,,
\end{array}
$$

where the top row is the exact triple (2.19) corresponding to the composition $(D_3 \circ D_2) \circ D_1$ augmented by id_{X_3},

$$\phi(x_1, x_2, x_3) = (x_1, x_2, x_3 + D_2 x_2), \qquad \psi(x_4, x_2, x_3) = (x_4, x_2, x_3 + D_2 x_2).$$

Since $\widetilde{\mathcal{I}}_{\phi,\psi;D_3 \circ D_2 \circ D_1 \oplus \mathrm{id}_{X_2} \oplus \mathrm{id}_{X_3}} = \mathrm{id}$, these two arrows are in fact the same by the isomorphism Naturality II property. The two half-disk and two triangular diagrams in Figure 4 commute by the definition of $\widetilde{\mathcal{C}}$. Thus, the diagram (2.30), which consists of the outermost arrows of the diagram in Figure 4, commutes. $\qquad\square$

The determinant for a complex of vector bundles in [11, p31] corresponds to reversing the two factors in (2.3) and thus interchanges the roles of the kernels and cokernels of linear operators. The isomorphism (3.4) should then be replaced by

$$
\begin{aligned}
\lambda^*\big(\mathfrak{c}(\delta)\big) \otimes \lambda\big(\kappa(\delta)\big) &\longrightarrow \lambda^*(W) \otimes \lambda(V), \\
y^* \otimes x &\longrightarrow (-1)^{(\mathfrak{d}(V) - \mathfrak{d}(\kappa(\delta)))\mathfrak{d}(\kappa(\delta))} \big(\lambda(\delta) v \wedge_W y\big)^* \otimes x \wedge_V v,
\end{aligned}
\tag{3.9}
$$

with x, y, v as before. This isomorphism differs from the isomorphism (3.4) conjugated by the isomorphisms (2.1) by (-1) to the power of $\mathfrak{d}(\operatorname{Im} \delta)$, which equals $N - \mathfrak{d}(\mathfrak{c}(D))$ in the case of (3.5). The dependence on $\mathfrak{d}(\mathfrak{c}(D))$ drops out when taking the overlap maps, analogous to $\hat{\mathcal{I}}_{\Theta_2;D} \circ \hat{\mathcal{I}}_{\Theta_1;D}^{-1}$ on page 21, for the trivializations of the new version of the determinant line bundle, and so the isomorphisms (3.9) still give rise to a well-defined topology on this bundle. The two versions of the determinant line bundle are isomorphic by the maps (2.1) composed with the multiplication by $A_{\mathrm{ind}D,\mathfrak{d}(\mathfrak{c}(D))} \equiv (-1)^{\mathfrak{c}(D)}$ in the fiber over $D \in \mathcal{F}(X, Y)$. Neither of the last two maps is continuous, but the composite is continuous; see Section 3.4 for a systematic discussion of such isomorphisms. The isomorphism $\Psi_{\mathfrak{t}}$ for exact triples of Fredholm operators described by (4.10) for the topology on $\det_{X,Y}$ specified by (3.4) should then be conjugated by the above isomorphism between the two versions of the determinant line bundle. In particular, this changes the sign exponent in (4.12) to $(\mathrm{ind}\, D')\mathfrak{d}(\mathfrak{c}(D''))$, in addition to interchanging the kernel and cokernel factors.

3.3 Other conventions

In [9, Section 3.1], $\lambda(D)$ is defined as the tensor product of $\lambda(\kappa(D))$ and $\lambda(\mathfrak{c}(D)^*)$. In [13, Appendix A.2], $\lambda(D)$ is defined as the tensor product of $\lambda(\kappa(D))$ and $\lambda(\kappa(D^*))$. In light of the second isomorphism in (2.20), these two conventions are essentially identical. They implicitly identify $\lambda^*(\mathfrak{c}(D))$ with $\lambda(\mathfrak{c}(D)^*)$. Such an identification is determined by a pairing of $\lambda(V^*)$ with $\lambda(V)$ for a finite-dimensional vector space V. There are two such standard pairings:

$$\lambda(V^*) \otimes \lambda(V) \longrightarrow \mathbb{R}, \qquad \alpha_1 \wedge \ldots \wedge \alpha_n \otimes v_1 \wedge \ldots \wedge v_n \longrightarrow \det\left(\alpha_i(v_j)\right)_{i,j=1,\ldots,n} \quad \text{and}$$
$$\longrightarrow (-1)^{\binom{n}{2}} \det\left(\alpha_i(v_j)\right)_{i,j=1,\ldots,n}. \tag{3.10}$$

Along with (3.4), these two pairings topologize the new version of the determinant line bundle in two different ways. The resulting line bundles are isomorphic by the multiplication by (-1) to the power of $\binom{\mathfrak{d}(\mathfrak{c}(D))}{2}$ in the fiber over $D \in \mathcal{F}(X,Y)$. Under the second pairing in (3.10), the isomorphism (3.4) precisely corresponds to the isomorphism [9, (3.1)]. On the other hand, the analogue of (3.4) used in the proof of [13, Theorem A.2.2] corresponds under the first pairing in (3.10) to (3.4) without the sign; see [13, Exercise A.2.3]. In the case of (3.5), the exponent of this sign is $(N - \mathfrak{c}(\mathfrak{d}))\mathfrak{c}(D)$, which changes the overlap maps between the trivializations of the determinant line bundle by (-1) to the power of $(N' - N)\mathfrak{d}(\mathfrak{c}(D))$. The overlap maps in the proof of [13, Theorem A.2.2] thus need not be continuous if $N - N'$ is odd and so do not topologize the determinant line bundles.[3]

In [17, Section (11a)], $\lambda(D)$ is defined as the tensor product of $\lambda(\mathfrak{c}(D)^*)$ and $\lambda(\kappa(D))$. In [16, Section 1.2], $\lambda(D)$ is defined as the tensor product of $\lambda(\kappa(D^*))$ and $\lambda(\kappa(D))$. In light of the second isomorphism in (2.20), these conventions are essentially identical. Under the second pairing in (3.10), the isomorphism (3.9) becomes [17, (11.3)]. Under the same pairing, the isomorphism (3.9) corresponds to the isomorphism of [16, Theorem 2.1] multiplied by (-1) to the power of

$$\left(\mathfrak{d}(W) - \mathfrak{d}(\mathfrak{c}(\delta))\right)(\operatorname{ind}\delta) + \mathfrak{d}\left(\kappa(\delta)\right)\mathfrak{d}\left(\mathfrak{c}(\delta)\right) \cong \mathfrak{d}(W)(\operatorname{ind}\delta) + \mathfrak{c}(\delta) \mod 2.$$

In the case of (3.5), the sign exponent reduces to $N(\operatorname{ind}D) + \mathfrak{d}(\mathfrak{c}(D))$. The dependence on $\mathfrak{d}(\mathfrak{c}(D))$ drops out when taking the overlap maps for the trivializations of this version of the determinant line bundle, and so the isomorphism of [16, Theorem 2.1] gives rise to a well-defined topology on this bundle. It is isomorphic to the determinant line bundle of [17, Section (11a)] by the multiplication by $(-1)^{\mathfrak{c}(D)}$ in the fiber over $D \in \mathcal{F}(X,Y)$. The interchange of factors in $\lambda(D)$ accounts for the change of the sign exponent in the direct sum formulas, [16, (3)] and [17, (11.2)], from (4.12), as explained at the end of the last paragraph in Section 3.2.

In [15, Section 1] and [10, Appendix D.2], $\lambda(D)$ is defined to be either

$$\lambda(\kappa(D)^*) \otimes \lambda(\mathfrak{c}(D)) \qquad \text{or} \qquad \lambda^*(\kappa(D)) \otimes \lambda(\mathfrak{c}(D));$$

---

[3]The 2017 revision of [13] contains a modification to address this issue. There is inconsistency in the definition of $\lambda(D)$ at the beginning of the revised Appendix A.2 in [13] and the notation in Theorem A.2.1ab. Assuming the intended definition is as stated, Theorem A.2.1ab defines a collection of exact triple isomorphisms Ψ_t corresponding to the collection $A_{i,c} \equiv (-1)^{ic}$ of our Theorem 2 on page 35 under the second pairing in (3.10); this pairing is consistent with the notation in Theorem A.2.1ab. These isomorphisms correspond to the isomorphisms (3.4) with the sign as in (3.9). This sign is more consistent with reversing the factors in the definition of $\lambda(D)$ in [13], as in Theorem A.2.1ab and in [11].

the notation is somewhat ambiguous, but looks more like the former. The usage in [15] is more consistent with the latter convention; the usage in [10] is sometimes more consistent with the latter and sometimes more consistent with the former.[4] The latter definition of $\lambda(D)$ is used in [1, Section (f)]. While $\lambda(\kappa(D)^*)$ and $\lambda^*(\kappa(D))$ are canonically isomorphic, there are at least two choices of such canonical isomorphisms, the two provided by the pairings (3.10). The "construction" of the determinant line bundle in [15] consists of mentioning that each homomorphism $\delta : V \longrightarrow W$ between finite-dimensional vector spaces gives rise to a natural isomorphism

$$\lambda(\kappa(D)^*) \otimes \lambda(\mathfrak{c}(D)) \longrightarrow \lambda(V^*) \otimes \lambda(W) \qquad \text{or} \qquad \lambda^*(\kappa(D)) \otimes \lambda(\mathfrak{c}(D)) \longrightarrow \lambda^*(V) \otimes \lambda(W),$$

but no indication is given what it is. In the proof of [10, Proposition D.2.2], this isomorphism is described as a composition of other isomorphisms, but some of them are not specified.[5] The construction in [10, Appendix D.2] is fundamentally based on [10, Proposition D.2.6], though its proof appears to be incomplete; see Remark 4.9 for details. However, the statement of this proposition is the basis for the construction of the determinant line bundle in this paper and a close cousin of this proposition, Proposition 4.10, is used to verify the continuity of the bundle map (2.27) for families of exact triples of Fredholm operators. The construction in [1] is limited to Hilbert spaces and still omits some details. Neither [1, Section (f)], [10, Appendix D], nor [15] confirms most of the properties of the determinant line bundle stated in Subsection 2.2.

As noted in [15, Section 2], the section of $\det_{X,Y}$ in the definitions of [15, Section 1] and [10, Appendix D.2] given by

$$\sigma(D) = \begin{cases} 1^* \otimes 1, & \text{if } D \text{ is isomorphism;} \\ 0, & \text{otherwise;} \end{cases} \tag{3.11}$$

is continuous. There is no such section if $\det_{X,Y}$ is defined as in (2.3), [9], [14], [16], or [17]. The definition of $\det_{X,Y}$ in [15, Section 1] and [10, Appendix D.2] thus comes with a natural normalization for the topology, but it does not restrict the topology of $\det_{X,Y}$ any further than the properties in Subsection 2.2; see Section 3.4. The alternative definitions seem more natural from the geometric viewpoint, as typically the spaces $\kappa(D)$ describe tangent spaces of some, ideally smooth, moduli spaces, and so it seems desirable not to dualize them. The alternative definitions also lead to a somewhat nicer appearance of formulas describing key properties of the determinant line bundle system. For example, [10, Proposition D.2.2] reverses the order of the factors in the isomorphism of Lemma 4.1.

3.4 Classification of determinant line bundles

For each exact triple $\mathfrak{t}$ of Fredholm operators, we denote by $\Psi_{\mathfrak{t}}$ the isomorphism (4.10). By Subsections 4.2-5.2, this collection of exact triple isomorphisms satisfies all properties for systems of determinant line bundles listed in Subsection 2.2. Let $\{\Psi'_{\mathfrak{t}}\}_{\mathfrak{t}}$ be another collection of isomorphisms for exact triples of Fredholm operators satisfying the isomorphism Naturality II, Normalization II,III, and Exact Squares properties. We show that this collection is as in (3.17) for some $A_{i,c} \in \mathbb{R}^*$ and

---

[4]For example, the last equality in the last displayed expression in the proof of [10, Proposition D.2.2] uses the latter definition, while [10, (D.2.9)] uses the former.

[5]In addition, $(\det F)^{-1}$ should be $\det F$ at the end of the statement of this proposition and $\det H_2$ should be $(\det H_2)^*$ in the second-to-last displayed equation in the proof; the first change is necessary for the section (3.11) to be continuous in the finite-dimensional case.

satisfies the remaining properties for systems of determinant line bundles listed in Subsection 2.2, if $A_{i,c}$ are chosen appropriately. Theorem 2 describes all collections of isomorphisms for exact triples of Fredholm operators satisfying all properties listed in Subsection 2.2.

For $i \in \mathbb{Z}$ and $c \in \mathbb{Z}^{\geq 0}$ with $c \geq -i$, let $\Psi_{i,c}$ be the isomorphism $\Psi_{\mathfrak{t}}$ in (4.10) for the exact triple

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}^{i+c} & \longrightarrow & \mathbb{R}^{i+c} \oplus \mathbb{R}^c & \longrightarrow & \mathbb{R}^c & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathbf{0}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R}^c & \longrightarrow & \mathbb{R}^c & \longrightarrow & 0 & \longrightarrow & 0 \,,
\end{array}
\tag{3.12}
$$

where the top right and middle arrows are the projections onto the last c coordinates. Thus,

$$
\Psi_{i,c}\big(\Omega_{i+c} \otimes \Omega_c^* \otimes \Omega_c \otimes 1^*\big) = (-1)^c\, \Omega_{i+2c} \otimes 1^*.
$$

Let $\Psi_{i,c}'$ be the isomorphism $\Psi_{\mathfrak{t}}'$ for the exact triple (3.12) and $A_{i,c} \in \mathbb{R}^*$ be such that

$$
\Psi_{i,c}' = A_{i,c} \Psi_{i,c} \,.
\tag{3.13}
$$

In particular,

$$
\Psi_{i,c}'\big(\Omega_{i+c} \otimes \Omega_c^* \otimes \Omega_c \otimes 1^*\big) = (-1)^c A_{i,c}\, \Omega_{i+2c} \otimes 1^*.
\tag{3.14}
$$

By the Normalization II property on page 13, $A_{i,0} = 1$ for all $i \in \mathbb{Z}^{\geq 0}$. By the Complex Exact Triples property, $A_{i,c} \in \mathbb{R}^+$ if $i, c \in 2\mathbb{Z}$.

Let $\mathfrak{t}$ be an exact triple as in (2.6) and

$$
\Theta' : \mathbb{R}^{N'} \longrightarrow Y' \qquad \text{and} \qquad \tilde{\Theta}'' : \mathbb{R}^{N''} \longrightarrow Y
$$

be homomorphisms such that $D' \in U_{X';\Theta'}$ and $D'' \in U_{X'';j_Y \circ \tilde{\Theta}''}$. Let $N = N' + N''$, $i : \mathbb{R}^{N'} \longrightarrow \mathbb{R}^N$ be the inclusion as $\mathbb{R}^{N'} \times 0^{N''}$, and $j : \mathbb{R}^N \longrightarrow \mathbb{R}^{N''}$ be the projection onto the last N'' coordinates. We define

$$
\begin{aligned}
\Theta : \mathbb{R}^N \longrightarrow X, \qquad & \Theta(x', x'') = i_Y\big(\Theta'(x')\big) + \tilde{\Theta}''(x'') && \forall\, (x', x'') \in \mathbb{R}^{N'} \oplus \mathbb{R}^{N''}, \\
\Theta'' : \mathbb{R}^{N''} \longrightarrow X'', \qquad & \Theta''(x'') = j_Y\big(\tilde{\Theta}''(x'')\big) && \forall\, x'' \in \mathbb{R}^{N''}.
\end{aligned}
$$

Thus, the first diagram in Figure 5, where the right column is the exact triple

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}^{N'} & \overset{i}{\longrightarrow} & \mathbb{R}^N & \overset{j}{\longrightarrow} & \mathbb{R}^{N''} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle j_{N'}} & & \downarrow{\scriptstyle j_N} & & \downarrow{\scriptstyle j_{N''}} & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \,,
\end{array}
$$

is an exact square of Fredholm operators. By the Exact Squares and Normalization II properties, the collection $\{\Psi_{\mathfrak{t}}'\}$ is thus determined by the isomorphisms

$$
\hat{\mathcal{I}}_{\Theta;D}' : \lambda(D) \longrightarrow \lambda(D_\Theta), \qquad \hat{\mathcal{I}}_{\Theta;D}'(\sigma) = \Psi_{\mathfrak{t}}'\big(\sigma \otimes \Omega_N \otimes 1^*\big),
$$

corresponding to the exact triples (3.1) with $D \in U_{X;\Theta}$.

First diagram (top):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D' & \longrightarrow & D'_{\Theta'} & \longrightarrow & j_{N'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & D_{\Theta} & \longrightarrow & j_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & D''_{\Theta''} & \longrightarrow & j_{N''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Second diagram (middle):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dot{D} & \longrightarrow & D & \longrightarrow & \mathbf{0}_{\kappa(D),\mathfrak{c}(D)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dot{D} & \longrightarrow & D_{\Theta_D} & \longrightarrow & \left(\mathbf{0}_{\kappa(D),\mathfrak{c}(D)}\right)_{\Theta_D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{0} & \longrightarrow & j_{N_D} & \longrightarrow & j_{N_D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Third diagram (bottom):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & D_{\Theta_D} & \longrightarrow & j_{N_D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & D_{\Theta} & \longrightarrow & j_N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{0} & \longrightarrow & j_{N-N_D} & \longrightarrow & j_{N-N_D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 5: Exact squares of Fredholm operators specifying a determinant line bundle system

Given $D \in \mathcal{F}(X,Y)$, let $\dot{X} \subset X$ be a closed linear subspace such that the operator

$$\dot{D} \colon \dot{X} \longrightarrow \operatorname{Im} D, \qquad \dot{D}(x) = Dx,$$

is an isomorphism and $\Theta_D \colon \mathbb{R}^{N_D} \longrightarrow Y$ be a homomorphism inducing an isomorphism to $\mathfrak{c}(D)$ when composed with the projection $Y \longrightarrow \mathfrak{c}(D)$. There is an exact square of Fredholm operators as in the second diagram in Figure 5, where the right column is the exact triple

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \kappa(D) & \longrightarrow & \kappa(D) \oplus \mathbb{R}^{N_D} & \longrightarrow & \mathbb{R}^{N_D} & \longrightarrow & 0 \\
& & \Big\downarrow{\mathbf{0}} & & \Big\downarrow{\mathbf{0}_{\Theta_D}} & & \Big\downarrow{j_{N_D}} & & \\
0 & \longrightarrow & \mathfrak{c}(D) & \longrightarrow & \mathfrak{c}(D) & \longrightarrow & 0 & \longrightarrow & 0 \,.
\end{array}
$$

By the Exact Squares, isomorphism Naturality II, and Normalization III properties and (3.13), the isomorphisms $\hat{\mathcal{I}}'_{\Theta_D;D}$ above are determined by the isomorphisms $\Psi'_{i,c}$ and satisfy

$$\hat{\mathcal{I}}'_{\Theta_D;D} = A_{\operatorname{ind} D, \mathfrak{d}(\mathfrak{c}(D))} \hat{\mathcal{I}}_{\Theta_D;D} \,.$$

For each homomorphism $\Theta \colon \mathbb{R}^N \longrightarrow Y$ with $D \in U_{X;\Theta}$, there is an exact square of Fredholm operators as in the last diagram in Figure 5. By the Exact Squares and Normalization II,III properties and the above equation,

$$\hat{\mathcal{I}}'_{\Theta;D} = A_{\operatorname{ind} D, \mathfrak{d}(\mathfrak{c}(D))} \hat{\mathcal{I}}_{\Theta;D} \,. \tag{3.15}$$

The overlap maps $\hat{\mathcal{I}}'_{\Theta_2;D} \circ \hat{\mathcal{I}}'^{-1}_{\Theta_1;D}$ between the above isomorphisms are still $\hat{\mathcal{I}}_{\Theta_2;D} \circ \hat{\mathcal{I}}^{-1}_{\Theta_1;D}$ and in particular are continuous. By Remark 4.6, the isomorphisms (3.15) are compatible with the isomorphisms (3.4) given by

$$\mathcal{I}'_\delta = \frac{A_{\mathfrak{d}(V) - \mathfrak{d}(W), \mathfrak{d}(\mathfrak{c}(\delta))}}{A_{\mathfrak{d}(V) - \mathfrak{d}(W), \mathfrak{d}(W)}} \, \mathcal{I}_\delta \colon \lambda(\delta) \longrightarrow \lambda(\mathbf{0}) \,, \tag{3.16}$$

whenever $\delta \colon V \longrightarrow W$ is a homomorphism between finite-dimensional vector spaces. The isomorphisms

$$\mathcal{I}_D \colon \lambda(D) \longrightarrow \lambda(D), \qquad \sigma \longrightarrow A^{-1}_{\operatorname{ind} D, \mathfrak{d}(\mathfrak{c}(D))} \sigma \,,$$

give rise to continuous isomorphisms between the determinant line bundles in the original and new topologies. The suitable exact triples and dualization isomorphisms are given by

$$
\begin{aligned}
\Psi'_{\mathfrak{t}} &= \mathcal{I}_D \circ \Psi_{\mathfrak{t}} \circ \mathcal{I}^{-1}_{D'} \otimes \mathcal{I}^{-1}_{D''} = \frac{A_{\operatorname{ind} D', \mathfrak{d}(\mathfrak{c}(D'))} A_{\operatorname{ind} D'', \mathfrak{d}(\mathfrak{c}(D''))}}{A_{\operatorname{ind} D, \mathfrak{d}(\mathfrak{c}(D))}} \, \Psi_{\mathfrak{t}} \,, \\
\widetilde{\mathcal{D}}'_D &= A^{\operatorname{ind} D}_{-1,1} \, \mathcal{I}_{D^*} \circ \widetilde{\mathcal{D}}_D \circ \mathcal{I}^{-1}_D = A^{\operatorname{ind} D}_{-1,1} \, \frac{A_{\operatorname{ind} D, \mathfrak{d}(\mathfrak{c}(D))}}{A_{-\operatorname{ind} D, \mathfrak{d}(\kappa(D))}} \, \widetilde{\mathcal{D}}_D,
\end{aligned}
\tag{3.17}
$$

if $\mathfrak{t}$ is as in (2.6). The extra factors of $A_{-1,1}$ in the second equation above are needed to achieve the Normalization IV property on page 18, while preserving the Dual Exact Triples property. In the case of the exact triple (3.1), $\Psi'_{\mathfrak{t}} = \hat{\mathcal{I}}_{\Theta;D}$, as the case should be. The new determinant line bundle system also satisfies the Normalization IV* property if and only if $A_{-k,k} = A^k_{-1,1}$ for every $k \in \mathbb{Z}^+$.

The above argument also implies that the Normalization IV and Dual Exact Triples properties on page 13 determine the dualization isomorphisms $\widetilde{\mathcal{D}}_D$ completely. Putting everything together, we obtain a complete description of systems of determinant line bundles.

Theorem 2. *The map specified by (3.14) sends each system of determinant line bundles satisfying the properties in Subsection 2.2, other than Normalization IV*, to the functions*

$$\big\{(i,c)\colon i\in\mathbb{Z},\ c\in\mathbb{Z}^+,\ c\geq -i\big\}\longrightarrow \mathbb{R}^*,\qquad (i,c)\longrightarrow A_{i,c},\quad A_{i,c}\in\mathbb{R}^+\ \text{if}\ i,c\in 2\mathbb{Z},$$

and is a bijection with the set of all such functions. The determinant line bundle systems that also satisfy the Normalization IV* property correspond to the subset of the above functions satisfying $A_{-k,k}=A^k_{-1,1}$ for all $k\in\mathbb{Z}^+$. In particular, the compatible systems of topologies on determinant line bundles are in one-to-one correspondence with the admissible systems of isomorphisms $\mathcal{I}_\delta$ as in (3.3), (3.4), and (3.16) and with admissible systems of isomorphisms $\Psi_{\mathfrak{t}}$ as in (2.27).

By Theorem 2 and the preceding discussion, the section S of $\det^*_{X,Y}$ given by

$$S_D(\sigma)=\begin{cases}c, & \text{if }\sigma=c\,1\otimes 1^*;\\ 0, & \text{if }D\text{ is not an isomorphism};\end{cases}$$

is continuous. This is the analogue of the section (3.11) for the convention (2.3).

Remark 3.1. According to [17, Remark 11.1], there are two possible sign conventions for the determinant line bundle and the sign convention in [17, Section (11a)] is the same as in [11]. As noted in Subsection 3.3, the setup in [17, Section (11a)] corresponds to the setup in [11, Chapter I] via the second pairing in (3.10). The alternatives for [17, (11.2)] and [17, (11.3)] specified in [17, Remark 11.1] for the "other" sign convention do not satisfy the key commutativity requirement on the preceding page in [17]. In order for this requirement to be satisfied, the sign in [17, (11.2)] must be kept precisely the same (contrary to what is explicitly stated in [17, Remark 11.1]); This "other" convention would then correspond to the setup in [11, Chapter I] via the first pairing in (3.10). Furthermore, by Theorem 2, there are infinitely many possible sign conventions, at least several of which seem quite natural. The isomorphisms (3.15) satisfy the two requirements above the diagram on page 150 in [17] provided $A_{0,1}>0$. These systems of isomorphisms can be narrowed down by replacing the Normalization IV property on page 18 with the Normalization IV* property ($A_{-k,k}=A^k_{-1,1}$ for all $k\in\mathbb{Z}^+$), by specifying the dualization or direct sum isomorphisms, i.e.

$$A_{-i,i+c}=A^i_{-1,1}A_{i,c}\quad\text{or}\quad A_{i,c}=A^c_{0,1}\qquad \forall\, i\in\mathbb{Z},\ c\in\mathbb{Z}^{\geq 0},\ c\geq -i,$$

and/or by requiring the isomorphisms $\mathcal{I}_\delta$ to be given by

$$\mathcal{I}_\delta\colon \lambda(\delta)\longrightarrow \lambda(\mathbf{0}),\qquad 1\otimes 1^*\longrightarrow (\det\delta)^{-1}\,v\otimes v^*,$$

whenever $\delta\colon V\longrightarrow V$ is an isomorphism and $v\in\lambda(V)-0$ ($A_{0,c}=1$ for all $c\in\mathbb{Z}^+$). The strongest of these additional conditions, specifying the isomorphisms for direct sums of Fredholm operators, seems to be the least natural requirement to make.

4 Linear algebra

4.1 Finite-dimensional vector spaces

In this subsection, we make a number of purely algebraic observations concerning finite-dimensional vector spaces that lie behind the determinant line construction.

Lemma 4.1 ([11, Proposition 1(i)]). *Every short exact sequence*

$$0 \longrightarrow V' \stackrel{\mathrm{i}}{\longrightarrow} V \stackrel{\mathrm{j}}{\longrightarrow} V'' \longrightarrow 0 \tag{4.1}$$

induces a natural isomorphism

$$\wedge_V \colon \lambda(V') \otimes \lambda(V'') \longrightarrow \lambda(V).$$

Proof. If $v'_1, \ldots, v'_k$ is a basis for V' and $v_1, \ldots, v_\ell \in V$ are such that $\mathrm{j}(v_1), \ldots, \mathrm{j}(v_\ell)$ is a basis for V'', $v'_1 \wedge \ldots \wedge v'_k$ and $\mathrm{j}(v_1) \wedge \ldots \wedge \mathrm{j}(v_\ell)$ span $\lambda(V')$ and $\lambda(V'')$, respectively. By the exactness of (4.1), $\mathrm{i}(v'_1), \ldots, \mathrm{i}(v'_k), v_1, \ldots, v_\ell$ is basis for V and so the map

$$\wedge_V \colon v'_1 \wedge \ldots \wedge v'_k \otimes \mathrm{j}(v_1) \wedge \ldots \wedge \mathrm{j}(v_\ell) \longrightarrow \mathrm{i}(v'_1) \wedge \ldots \wedge \mathrm{i}(v'_k) \wedge v_1 \wedge \ldots \wedge v_\ell \tag{4.2}$$

induces an isomorphism $\lambda(V') \otimes \lambda(V'') \longrightarrow \lambda(V)$. By the exactness of (4.1), each $v_i \in V$ is determined by $\mathrm{j}(v_i) \in V''$ up to a linear combination of $\mathrm{i}(v'_1), \ldots, \mathrm{i}(v'_k)$, and so the right-hand side of (4.2) is determined by $v'_1, \ldots, v'_k \in V'$ and $\mathrm{j}(v_1), \ldots, \mathrm{j}(v_\ell) \in V''$. Changing the collections $v'_1, \ldots, v'_k \in V'$ and $v_1, \ldots, v_\ell \in V$ by a $k \times k$-matrix A' and an $\ell \times \ell$-matrix A, respectively, changes the wedge products of the first k vectors and the last ℓ vectors by $\det A'$ and $\det A$, respectively, on both sides of (4.2). Thus, the isomorphism induced by (4.2) is independent of the choices of collections $v'_1, \ldots, v'_k \in V'$ and $v_1, \ldots, v_\ell \in V$ as above. It clearly commutes with isomorphisms of short exact sequences. $\square$

The next lemma follows immediately from the definitions of $\mathcal{P}$ in (2.2) and of $\wedge_V$ above.

Lemma 4.2. *For every finite-dimensional vector space V,*

$$\mathcal{P}(v^*) = (\mathcal{P}v)^* \qquad \forall\, v \in \lambda(V) - 0. \tag{4.3}$$

For every isomorphism $\delta \colon V \longrightarrow W$ between finite-dimensional vector spaces,

$$\lambda(\delta^*) \mathcal{P}\big((\lambda(\delta)v)^*\big) = \mathcal{P}(v^*) \qquad \forall\, v \in \lambda(V) - 0. \tag{4.4}$$

For every short exact sequence (4.1),

$$\mathcal{P}\big((\lambda(\mathrm{i})v' \wedge_V v'')^*\big) = \mathcal{P}(v''^*) \wedge_{V^*} \lambda(\mathrm{i}^*)^{-1} \mathcal{P}(v'^*) \quad \forall\, v' \in \lambda(V') - 0,\, v'' \in \lambda\big(V/\mathrm{i}(V')\big) - 0. \tag{4.5}$$

From (4.2), we immediately find that the isomorphisms $\wedge_V$ of Lemma 4.1 satisfy graded commutativity, as described by the next lemma. Corollary 4.4 below is a special case of this lemma (either $V_{\mathrm{TR}} = 0$ or $V_{\mathrm{BL}} = 0$).

Lemma 4.3 ([11, Proposition 1(ii)]). *For every commutative diagram*

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of exact rows and columns, the diagram

$$\lambda(V_{\mathrm{TL}}) \otimes \lambda(V_{\mathrm{BL}}) \otimes \lambda(V_{\mathrm{TR}}) \otimes \lambda(V_{\mathrm{BR}}) \xrightarrow{\wedge_{V_{\mathrm{TM}}} \otimes \wedge_{V_{\mathrm{BM}}} \,\circ\, \mathrm{id} \otimes R \otimes \mathrm{id}} \lambda(V_{\mathrm{TM}}) \otimes \lambda(V_{\mathrm{BM}})$$

$$\downarrow{\scriptstyle \wedge_{V_{\mathrm{CL}}} \otimes \wedge_{V_{\mathrm{CR}}}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \wedge_{V_{\mathrm{CM}}}}$$

$$\lambda(V_{\mathrm{CL}}) \otimes \lambda(V_{\mathrm{CR}}) \xrightarrow{\qquad\qquad\wedge_{V_{\mathrm{CM}}}\qquad\qquad} \lambda(V_{\mathrm{CM}})$$

commutes.

Corollary 4.4. *For every commutative diagram*

of 4 exact short sequences, the diagram

$$\lambda(V_{\mathrm{LL}}) \otimes \lambda(V_{\mathrm{LR}}) \otimes \lambda(V_{\mathrm{RR}}) \xrightarrow{\wedge_{V_{\mathrm{LC}}} \otimes \mathrm{id}} \lambda(V_{\mathrm{LC}}) \otimes \lambda(V_{\mathrm{RR}})$$

$$\downarrow{\scriptstyle \mathrm{id} \otimes \wedge_{V_{\mathrm{CR}}}} \qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \wedge_{V_{\mathrm{CC}}}}$$

$$\lambda(V_{\mathrm{LL}}) \otimes \lambda(V_{\mathrm{CR}}) \xrightarrow{\qquad\wedge_{V_{\mathrm{CC}}}\qquad} \lambda(V_{\mathrm{CC}})$$

commutes.

4.2 Exact triples of Fredholm operators

We begin this subsection by extending the isomorphism of Lemma 4.1 to exact triples of Fredholm operators. It is immediate from the explicit formula (4.10) for the new isomorphism that it satisfies the Naturality II, Normalization II,III, Direct Sums I,II, and Complex Exact Triples properties in Subsection 2.2. We verify that it also satisfies the Dual Exact Triples property with $\widetilde{\mathcal{D}}_D$ given by (4.13) and the Compositions I,II properties.

We will use the natural pairing of a one-dimensional vector space L with its dual given by

$$L^* \otimes L \longrightarrow \mathbb{R}, \qquad \alpha \otimes v \longrightarrow \alpha(v).$$

If V is a finite-dimensional vector space and $v \in \lambda(V)$, we denote by

$$\langle v \rangle \equiv \dim V + 2\mathbb{Z} \in \mathbb{Z}_2$$

the degree of v as an element of the $\mathbb{Z}_2$-line $\lambda(V)$.

Proposition 4.5 ([10, Proposition D.2.3])**.** *Every exact triple $\mathfrak{t}$ of Fredholm operators as in (2.6) induces a natural isomorphism*

$$\Psi_{\mathfrak{t}} \colon \lambda(D') \otimes \lambda(D'') \longrightarrow \lambda(D).$$

Proof. By the Snake Lemma, (2.6) induces an exact sequence

$$0 \longrightarrow \kappa(D') \xrightarrow{i_X} \kappa(D) \xrightarrow{j_X} \kappa(D'') \xrightarrow{\delta} \mathfrak{c}(D') \xrightarrow{i_Y} \mathfrak{c}(D) \xrightarrow{j_Y} \mathfrak{c}(D'') \longrightarrow 0. \qquad (4.6)$$

By Lemma 4.1, there are then natural isomorphisms

$$\lambda(\kappa(D)) \approx \lambda(\kappa(D')) \otimes \lambda(\operatorname{Im}j_X), \qquad \lambda(\kappa(D'')) \approx \lambda(\operatorname{Im}j_X) \otimes \lambda(\operatorname{Im}\delta),$$
$$\lambda(\mathfrak{c}(D')) \approx \lambda(\operatorname{Im}\delta) \otimes \lambda(\operatorname{Im}i_Y), \qquad \lambda(\mathfrak{c}(D)) \approx \lambda(\operatorname{Im}i_Y) \otimes \lambda(\mathfrak{c}(D'')). \qquad (4.7)$$

Putting these isomorphisms together and using the natural evaluation isomorphisms, we obtain

$$\lambda(D') \otimes \lambda(D'') \equiv \lambda(\kappa(D')) \otimes \lambda^*(\mathfrak{c}(D')) \otimes \lambda(\kappa(D'')) \otimes \lambda^*(\mathfrak{c}(D''))$$
$$\approx \lambda(\kappa(D)) \otimes \lambda^*(\operatorname{Im}j_X) \otimes \lambda^*(\operatorname{Im}i_Y) \otimes \lambda^*(\operatorname{Im}\delta) \qquad (4.8)$$
$$\otimes \lambda(\operatorname{Im}j_X) \otimes \lambda(\operatorname{Im}\delta) \otimes \lambda^*(\mathfrak{c}(D)) \otimes \lambda(\operatorname{Im}i_Y) \approx \lambda(\kappa(D)) \otimes \lambda^*(\mathfrak{c}(D)).$$

This establishes the claim. $\qquad\qquad\square$

For computational purposes, it is essential to specify the isomorphism of Proposition 4.5 explicitly. With the notation as in (2.6) and (4.6), let

$$\epsilon_{\mathfrak{t}} = (\operatorname{ind}D'')\mathfrak{d}(\mathfrak{c}(D')) + \mathfrak{d}(\mathfrak{c}(D))\mathfrak{d}(\operatorname{Im}\delta). \qquad (4.9)$$

For $\mathfrak{t}$ corresponding to (2.6), we define

$$\Psi_{\mathfrak{t}}\big(x \otimes (\lambda(\delta)v \wedge_{\mathfrak{c}(D')} w)^* \otimes (\lambda(j_X)u \wedge_{\kappa(D'')} v) \otimes (\lambda(j_Y)y)^*\big)$$
$$= (-1)^{\epsilon_{\mathfrak{t}}}\big(\lambda(i_X)x \wedge_{\kappa(D)} u\big) \otimes \big(\lambda(i_Y)w \wedge_{\mathfrak{c}(D)} y\big)^*, \qquad (4.10)$$

whenever

$$x \in \lambda(\kappa(D')), \quad u \in \lambda\left(\frac{\kappa(D)}{i_X(\kappa(D'))}\right), \quad v \in \lambda\left(\frac{\kappa(D'')}{j_X(\kappa(D))}\right), \quad w \in \lambda\left(\frac{\mathfrak{c}(D')}{\delta(\kappa(D''))}\right), \quad y \in \lambda\left(\frac{\mathfrak{c}(D)}{i_Y(\mathfrak{c}(D'))}\right),$$
$$x, u, v, w, y \neq 0.$$

In particular, $\Psi_{\mathfrak{t}}$ satisfies the Naturality II and Normalization II,III properties. By (2.24), it also satisfies the Complex Exact Triples property.

Remark 4.6. If $\delta : V \longrightarrow W$ is a homomorphism between finite-dimensional vector spaces, the isomorphism (4.10) applied to the exact sequence

$$0 \longrightarrow 0 \longrightarrow \kappa(\delta) \longrightarrow V \xrightarrow{\delta} W \xrightarrow{q} \mathfrak{c}(\delta) \longrightarrow 0 \longrightarrow 0 \qquad (4.11)$$

induces the isomorphism

$$\Psi_{\delta} : \lambda^*(W) \otimes \lambda(V) \longrightarrow \lambda(\delta), \qquad \Psi_{\delta}(\beta \otimes x) = \Psi_{\mathfrak{t}_{\delta}}(1 \otimes \beta \otimes x \otimes 1^*),$$

where $\Psi_{\mathfrak{t}_{\delta}}$ is the isomorphism (4.10) for the exact sequence (4.11). Explicitly,

$$\Psi_{\delta}\big((\lambda(\delta)v \wedge_W w)^* \otimes (u \wedge_V v)\big) = (-1)^{\mathfrak{d}(V)\mathfrak{d}(W) + (\mathfrak{d}(W) - \mathfrak{d}(\mathfrak{c}(\delta)))\mathfrak{d}(\mathfrak{c}(\delta))} u \otimes w^*,$$
$$\text{if} \quad u \in \lambda(\kappa(\delta)) - 0, \quad v \in \lambda(V/\kappa(\delta)) - 0, \quad w \in \lambda(\mathfrak{c}(\delta)) - 0.$$

Thus,

$$\Psi_{\mathbf{0}} \circ \Psi_{\delta}^{-1} : \lambda(\delta) \longrightarrow \lambda(\mathbf{0}) \equiv \lambda(V) \otimes \lambda^*(W),$$
$$u \otimes w^* \longrightarrow (-1)^{(\mathfrak{d}(W) - \mathfrak{d}(\mathfrak{c}(\delta)))\mathfrak{d}(\mathfrak{c}(\delta))}(u \wedge_V v) \otimes (\lambda(\delta)v \wedge_W w)^*,$$

is precisely the isomorphism (3.4).

For any $D' \in \mathcal{F}(X', Y')$ and $D'' \in \mathcal{F}(X'', Y'')$, let

$$\widetilde{\oplus}_{D',D''} \colon \lambda(D') \otimes \lambda(D'') \longrightarrow \lambda(D' \oplus D'')$$

be the isomorphism $\Psi_{\mathfrak{t}}$ in (4.10) corresponding to the diagram (2.14). Thus,

$$
\begin{aligned}
\widetilde{\oplus}_{D',D''} &\big((x_1' \wedge \ldots \wedge x_{k'}') \otimes (y_1' \wedge \ldots \wedge y_{\ell'}')^* \otimes (x_1'' \wedge \ldots \wedge x_{k''}'') \otimes (y_1'' \wedge \ldots \wedge y_{\ell''}'')^* \big) \\
&= (-1)^{(\mathrm{ind}\, D'')\mathfrak{d}(\mathfrak{c}(D'))} \big((x_1', 0) \wedge \ldots \wedge (x_{k'}', 0) \wedge (0, x_1'') \wedge \ldots \wedge (0, x_{k''}'') \big) \\
&\qquad\qquad \otimes \big((y_1', 0) \wedge \ldots \wedge (y_{\ell'}', 0) \wedge (0, y_1'') \wedge \ldots \wedge (0, y_{\ell''}'') \big)^*,
\end{aligned}
\tag{4.12}
$$

whenever

$$
\begin{aligned}
x_1' \wedge \ldots \wedge x_{k'}' &\in \lambda(\kappa(D')) - 0, & y_1' \wedge \ldots \wedge x_{\ell'}' &\in \lambda(\mathfrak{c}(D')) - 0, \\
x_1'' \wedge \ldots \wedge x_{k''}'' &\in \lambda(\kappa(D'')) - 0, & y_1'' \wedge \ldots \wedge y_{\ell''}'' &\in \lambda(\mathfrak{c}(D'')) - 0.
\end{aligned}
$$

The two Direct Sums properties on page 14 follow immediately from (4.12).

The isomorphism

$$\widetilde{\mathcal{D}}_D \colon \lambda(D) \longrightarrow \lambda(D^*), \qquad x \otimes \alpha \longrightarrow (-1)^{(\mathrm{ind}\, D)\mathfrak{d}(\mathfrak{c}(D))} \lambda(\mathcal{D}_D)(\mathcal{P}\alpha) \otimes \mathcal{P}\big(\lambda(\mathcal{D}_D)x\big), \tag{4.13}$$

satisfies the Normalization IV* property on page 19. By (2.24), it satisfies the Dual Complex Orientations properties as well. The next proposition shows that it also satisfies the Dual Exact Triples property. The extra factor of $(-1)^{\mathfrak{d}(\mathfrak{c}(D))}$ in (4.13) arises for the same reason as in the paragraph containing (3.9). Due to this extra factor, the compositions of $\widetilde{\mathcal{D}}_D$ with $\widetilde{\mathcal{D}}_{D^*}$ are the multiplication by $(-1)^{\mathrm{ind}\, D}$, not necessarily the identity, whenever the Banach spaces X and Y are reflexive.

Proposition 4.7 (Dual Exact Triples)**.** *For every exact triple (2.6) of Fredholm operators, the diagram (2.37) determined by the isomorphisms (4.10) and (4.13) commutes.*

Proof. With notation as in (2.6) and (2.37), we define

$$\epsilon_{\mathrm{L}} = (\mathrm{ind}\, D')(\mathrm{ind}\, D'') + (\mathrm{ind}\, D')\mathfrak{d}(\mathfrak{c}(D')) + (\mathrm{ind}\, D'')\mathfrak{d}(\mathfrak{c}(D'')) + \epsilon_{\mathfrak{t}^*}, \quad \epsilon_{\mathrm{R}} = \epsilon_{\mathfrak{t}} + (\mathrm{ind}\, D)\mathfrak{d}(\mathfrak{c}(D)).$$

The isomorphisms (2.20) intertwine the analogue of the exact sequence (4.6) for $\mathfrak{t}^*$ and the dual of (4.6):

$$
\begin{CD}
0 @>>> \kappa(D''^*) @>j_Y^*>> \kappa(D^*) @>i_Y^*>> \kappa(D'^*) @>\delta^*>> \mathfrak{c}(D''^*) @>j_X^*>> \mathfrak{c}(D^*) @>i_X^*>> \mathfrak{c}(D'^*) @>>> 0 \\
@. @AA\mathcal{D}_{D''}A @AA\mathcal{D}_D A @AA\mathcal{D}_{D'}A @VV\mathcal{D}_{D''}^*V @VV\mathcal{D}_D^*V @VV\mathcal{D}_{D'}^*V @. \\
0 @>>> \mathfrak{c}(D'')^* @>j_Y^*>> \mathfrak{c}(D)^* @>i_Y^*>> \mathfrak{c}(D')^* @>\delta^*>> \kappa(D'')^* @>j_X^*>> \kappa(D)^* @>i_X^*>> \kappa(D')^* @>>> 0.
\end{CD}
\tag{4.14}
$$

In particular,

$$\mathfrak{d}(\mathrm{Im}\, \delta^*) = \mathfrak{d}(\mathrm{Im}\, \delta) = \mathfrak{d}(\kappa(D')) + \mathfrak{d}(\kappa(D'')) + \mathfrak{d}(\kappa(D)) = \mathfrak{d}(\mathfrak{c}(D')) + \mathfrak{d}(\mathfrak{c}(D'')) + \mathfrak{d}(\mathfrak{c}(D))$$

and so $2 \mid (\epsilon_{\mathrm{L}} - \epsilon_{\mathrm{R}})$.

Let x, u, v, w, y be as in (4.10). By (4.14), we can compute $\Psi_{\mathfrak{t}^*}$ using

$$\check{x} = \lambda(\mathcal{D}_{D''})\mathcal{P}\big((\lambda(\mathsf{j}_Y)y)^*\big) \in \lambda\big(\kappa(D''^*)\big), \qquad \check{u} = \lambda(\mathcal{D}_D)\mathcal{P}\big((\lambda(\mathsf{i}_Y)w)^*\big) \in \lambda\bigg(\frac{\kappa(D^*)}{\mathsf{j}_Y^*(\kappa(D''^*))}\bigg),$$

$$\check{v} = \lambda(\mathcal{D}_{D'})\mathcal{P}\big((\lambda(\delta)v)^*\big) \in \lambda\bigg(\frac{\kappa(D'^*)}{\iota_Y^*(\kappa(D^*))}\bigg), \qquad \check{w} = \lambda(\mathcal{D}_{D''}^*)^{-1}\mathcal{P}\big((\lambda(\mathsf{j}_X)u)^*\big) \in \lambda\bigg(\frac{\mathfrak{c}(D''^*)}{\delta^*(\kappa(D'^*))}\bigg),$$

$$\check{y} = \lambda(\mathcal{D}_D^*)^{-1}\mathcal{P}\big((\lambda(\mathsf{i}_X)x)^*\big) \in \lambda\bigg(\frac{\mathfrak{c}(D^*)}{\mathsf{j}_X^*(\mathfrak{c}(D''^*))}\bigg).$$

By (4.3), (4.4), and the commutativity of the diagram (4.14),

$$\mathcal{P}\big(\lambda(\mathcal{D}_{D'})x\big) = \big(\lambda(\mathsf{i}_X^*)\check{y}\big)^*, \tag{4.15}$$

$$\lambda(\mathcal{D}_{D'})\mathcal{P}(w^*) = \lambda(\mathsf{i}_Y^*)\check{u}, \quad \lambda(\mathcal{D}_{D'})\lambda(\delta^*)^{-1}\mathcal{P}(v^*) = \check{v},$$

$$\lambda(\mathcal{D}_D)\mathcal{P}(y^*) = \lambda(\mathsf{j}_Y^*)\check{x}, \quad \lambda(\mathcal{D}_D)\lambda(\mathsf{i}_Y^*)^{-1}\mathcal{P}(w^*) = \check{u},$$

$$\lambda(\mathcal{D}_{D''})\lambda(\mathsf{j}_X)u = \mathcal{P}(\check{w}^*), \quad \lambda(\mathcal{D}_{D''})v = \lambda(\delta)^{-1}\mathcal{P}(\check{v}^*)$$

$$\lambda(\mathcal{D}_D)\lambda(\mathsf{i}_X)x = \mathcal{P}(\check{y}^*), \quad \lambda(\mathcal{D}_D)u = \lambda(\mathsf{j}_X)^{-1}\mathcal{P}(\check{w}^*).$$

Combining each pair of identities on the last four lines above with (4.5), we obtain

$$\lambda(\mathcal{D}_{D'})\mathcal{P}\big((\lambda(\delta)v \wedge_{\mathfrak{c}(D')} w)^*\big) = \lambda(\mathsf{i}_Y^*)\check{u} \wedge_{\kappa(D'^*)} \check{v}, \tag{4.16}$$

$$\lambda(\mathcal{D}_D)\mathcal{P}\big((\lambda(\mathsf{i}_Y)w \wedge_{\mathfrak{c}(D)} y)^*\big) = \lambda(\mathsf{j}_Y^*)\check{x} \wedge_{\kappa(D^*)} \check{u}, \tag{4.17}$$

$$\mathcal{P}\big(\lambda(\mathcal{D}_{D''})(\lambda(\mathsf{j}_X)u \wedge_{\kappa(D'')} v)\big) = \big(\lambda(\delta^*)\check{v} \wedge_{\mathfrak{c}(D''^*)} \check{w}\big)^*, \tag{4.18}$$

$$\mathcal{P}\big(\lambda(\mathcal{D}_D)(\lambda(\mathsf{i}_X)x \wedge_{\kappa(D)} u)\big) = \big(\lambda(\mathsf{j}_X^*)\check{w} \wedge_{\mathfrak{c}(D^*)} \check{y}\big)^*, \tag{4.19}$$

respectively. By (4.13), (4.15), (4.16), (4.18), and (4.10), the image of

$$x \otimes (\lambda(\delta)v \wedge_{\mathfrak{c}(D')} w)^* \otimes (\lambda(\mathsf{j}_X)u \wedge_{\kappa(D'')} v) \otimes (\lambda(\mathsf{j}_Y)y)^* \in \lambda(D') \otimes \lambda(D'') \tag{4.20}$$

under $\Psi_{\mathfrak{t}^*} \circ \widetilde{\mathcal{D}}_{D''} \otimes \widetilde{\mathcal{D}}_{D'} \circ R$ is

$$(-1)^{\epsilon_{\mathrm{L}}}\big(\lambda(\mathsf{j}_Y^*)\check{x} \wedge_{\kappa(D^*)} \check{u}\big) \otimes \big(\lambda(\mathsf{j}_X^*)\check{w} \wedge_{\mathfrak{c}(D^*)} \check{y}\big)^* \in \lambda(D^*).$$

By (4.10), (4.13), (4.17), and (4.19), the image of the element (4.20) under $\widetilde{\mathcal{D}}_D \circ \Psi_{\mathfrak{t}}$ is

$$(-1)^{\epsilon_{\mathrm{R}}}\big(\lambda(\mathsf{j}_Y^*)\check{x} \wedge_{\kappa(D^*)} \check{u}\big) \otimes \big(\lambda(\mathsf{j}_X^*)\check{w} \wedge_{\mathfrak{c}(D^*)} \check{y}\big)^* \in \lambda(D^*).$$

Since $2 | (\epsilon_{\mathrm{L}} - \epsilon_{\mathrm{R}})$, this establishes the claim. $\qquad\square$

For any $D_1 \in \mathcal{F}(X_1, X_2)$ and $D_2 \in \mathcal{F}(X_2, X_3)$, let

$$\widetilde{\mathcal{C}}_{D_1, D_2} \colon \lambda(D_1) \otimes \lambda(D_2) \longrightarrow \lambda(D_2 \circ D_1)$$

be the isomorphism $\Psi_{\mathfrak{t}}$ in (4.10) corresponding to the diagram (2.19). The exact sequence (4.6) in this case specializes to

$$0 \longrightarrow \kappa(D_1) \longrightarrow \kappa(D_2 \circ D_1) \xrightarrow{D_1} \kappa(D_2) \xrightarrow{\delta} \mathfrak{c}(D_1) \xrightarrow{D_2} \mathfrak{c}(D_2 \circ D_1) \longrightarrow \mathfrak{c}(D_2) \longrightarrow 0, \tag{4.21}$$

$$\delta(x_2) = -x_2 + \operatorname{Im} D_1.$$

Let
$$\epsilon_{D_1,D_2} = (\operatorname{ind} D_2)\mathfrak{d}(\mathfrak{c}(D_1)) + \big(\mathfrak{d}(\mathfrak{c}(D_1)) + \mathfrak{d}(\mathfrak{c}(D_2))\big)\mathfrak{d}(\operatorname{Im}\delta).$$

Then,

$$
\begin{aligned}
\widetilde{\mathcal{C}}_{D_1,D_2}\big(x_1 \otimes (v \wedge_{\mathfrak{c}(D_1)} w)^* \otimes (\lambda(D_1)u \wedge_{\kappa(D_2)} v)\otimes y_2^*\big) \\
= (-1)^{\epsilon_{D_1,D_2}}\big(x_1 \wedge_{\kappa(D_2 \circ D_1)} u\big) \otimes \big(\lambda(D_2)w \wedge_{\mathfrak{c}(D_2 \circ D_1)} y_2\big)^*,
\end{aligned}
\tag{4.22}
$$

whenever

$$x_1 \in \lambda(\kappa(D_1)) - 0, \quad y_2 \in \lambda(\mathfrak{c}(D_2)) - 0, \quad u \in \lambda\left(\frac{\kappa(D_2 \circ D_1)}{\kappa(D_1)}\right) - 0,$$

$$v \in \lambda\left(\frac{\kappa(D_2)}{\kappa(D_2) \cap (\operatorname{Im} D_1)}\right) - 0, \quad w \in \lambda\left(\frac{X_2}{\kappa(D_2) + (\operatorname{Im} D_1)}\right) - 0.$$

Proposition 4.8 (Compositions I, [10, Proposition D.2.6]). *For any triple of Fredholm operators $D_1 : X_1 \longrightarrow X_2$, $D_2 : X_2 \longrightarrow X_3$, and $D_3 : X_3 \longrightarrow X_4$, the diagram (2.30) induced by the isomorphisms (4.10) commutes.*

Proof. We denote by $D''D'$ the composition $D'' \circ D'$ of two maps D' and D'' and define

$$\epsilon_{\mathrm{L}} = \epsilon_{D_1,D_2} + \epsilon_{D_2 D_1, D_3}, \qquad \epsilon_{\mathrm{R}} = \epsilon_{D_2,D_3} + \epsilon_{D_1, D_3 D_2}.$$

For $i = 1, 2, 3$, let

$$x_i \in \lambda(\kappa(D_i)) - 0 \qquad \text{and} \qquad y_i \in \lambda(\mathfrak{c}(D_i)) - 0.$$

For $(i,j) \in \{(1,2), (2,3), (1,23), (12,3)\}$, let

$$u_{i,j} \in \lambda\left(\frac{\kappa(D_j D_i)}{\kappa(D_i)}\right) - 0, \quad v_{i,j} \in \lambda\left(\frac{\kappa(D_j)}{\kappa(D_j) \cap \operatorname{Im}(D_i)}\right) - 0, \quad w_{i,j} \in \lambda\left(\frac{X_j}{\kappa(D_j) + \operatorname{Im}(D_i)}\right) - 0,$$

where $D_{12} = D_2 D_1$, $D_{23} = D_3 D_2$, and $X_{23} = X_2$; see Figure 6. Below we choose these elements in a compatible way.

Applying Lemma 4.1 to the exact sequence (4.21) with D_1 and D_2 replaced by D_i and D_j with (i,j) as above, we obtain

$$
\begin{aligned}
\mathfrak{d}(\kappa(D_3)) &= \langle u_{12,3}\rangle + \langle v_{12,3}\rangle, & \mathfrak{d}(\mathfrak{c}(D_1)) &= \langle v_{1,23}\rangle + \langle w_{1,23}\rangle, \\
\mathfrak{d}(\mathfrak{c}(D_j D_i)) &= \mathfrak{d}(\mathfrak{c}(D_i)) + \mathfrak{d}(\mathfrak{c}(D_j)) - \langle v_{i,j}\rangle, & \operatorname{ind} D_j D_i &= \operatorname{ind} D_i + \operatorname{ind} D_j,
\end{aligned}
$$

where $(i,j) = (1,2), (2,3)$. From this, we find that

$$
\begin{aligned}
\epsilon_{\mathrm{L}} &= A + C(\langle v_{1,2}\rangle + \langle v_{12,3}\rangle) + \langle u_{12,3}\rangle\langle v_{1,2}\rangle \mod 2, \\
\epsilon_{\mathrm{R}} &= A + C(\langle v_{2,3}\rangle + \langle v_{1,23}\rangle) + \langle v_{2,3}\rangle\langle w_{1,23}\rangle \mod 2,
\end{aligned}
\tag{4.23}
$$

where

$$A = (\operatorname{ind} D_3 D_2) \cdot \mathfrak{d}(\mathfrak{c}(D_1)) + (\operatorname{ind} D_3) \cdot \mathfrak{d}(\mathfrak{c}(D_2)), \quad C = \mathfrak{d}(\mathfrak{c}(D_1)) + \mathfrak{d}(\mathfrak{c}(D_2)) + \mathfrak{d}(\mathfrak{c}(D_3)).$$

In light of the top row in the first diagram in Figure 6, the bottom row in the second diagram, and Lemma 4.1, we can take

$$u_{1,23} = u_{1,2} \wedge_{\frac{\kappa(D_3 D_2 D_1)}{\kappa(D_1)}} u_{12,3} \qquad \text{and} \qquad w_{12,3} = \lambda(D_2)w_{1,23} \wedge_{\frac{X_3}{\kappa(D_3)+\mathrm{Im}(D_2 D_1)}} w_{2,3}. \qquad (4.24)$$

Along with Corollary 4.4, these equalities insure that

$$\begin{aligned}
&\big((x_1 \wedge_{\kappa(D_2 D_1)} u_{1,2}) \wedge_{\kappa(D_3 D_2 D_1)} u_{12,3}\big) \otimes \big(\lambda(D_3)w_{12,3} \wedge_{\mathfrak{c}(D_3 D_2 D_1)} y_3\big)^* \\
&= \big(x_1 \wedge_{\kappa(D_3 D_2 D_1)} u_{1,23}\big) \otimes \big(\lambda(D_3 D_2)w_{1,23} \wedge_{\mathfrak{c}(D_3 D_2 D_1)} (\lambda(D_3)w_{2,3} \wedge_{\mathfrak{c}(D_3 D_2)} y_3)\big)^*
\end{aligned} \qquad (4.25)$$

in $\lambda(D_3 D_2 D_1)$. In light of the right column and bottom row in the first diagram in Figure 6, the top row and left column in the second diagram in Figure 6, and Lemma 4.1, we can take

$$\begin{aligned}
u_{2,3} &= \lambda(D_1)u_{12,3} \wedge_{\frac{\kappa(D_3 D_2)}{\kappa(D_2)}} \mu, & v_{1,23} &= v_{1,2} \wedge_{\frac{\kappa(D_3 D_2)}{\kappa(D_3 D_2)\cap \mathrm{Im}(D_1)}} \mu, \\
v_{12,3} &= \lambda(D_2)\mu \wedge_{\frac{\kappa(D_3)}{\kappa(D_3)\cap \mathrm{Im}(D_2 D_1)}} v_{2,3}, & w_{1,2} &= \mu \wedge_{\frac{X_2}{\kappa(D_2)+\mathrm{Im}(D_1)}} w_{1,23}
\end{aligned} \qquad (4.26)$$

for some

$$\mu \in \lambda\left(\frac{\kappa(D_3 D_2)}{\kappa(D_2) + \kappa(D_3 D_2) \cap \mathrm{Im}(D_1)}\right) - 0.$$

In light of the left column of the first diagram and the right column of the second diagram in Figure 6, (4.26), and Corollary 4.4, we can take

$$\begin{aligned}
x_2 &= \lambda(D_1)u_{1,2} \wedge_{\kappa(D_2)} v_{1,2}, & x_3 &= \lambda(D_2 D_1)u_{12,3} \wedge_{\kappa(D_3)} v_{12,3} = \lambda(D_2)u_{2,3} \wedge_{\kappa(D_3)} v_{2,3}, \\
y_2 &= v_{2,3} \wedge_{\mathfrak{c}(D_2)} w_{2,3}, & y_1 &= v_{1,2} \wedge_{\mathfrak{c}(D_1)} w_{1,2} = v_{1,23} \wedge_{\mathfrak{c}(D_1)} w_{1,23}.
\end{aligned} \qquad (4.27)$$

Combining the above definitions of x_2 and y_2 with (4.26) and applying Lemma 4.3 to the two diagrams in Figure 6, we find that

$$\begin{aligned}
\lambda(D_1)x_2 \wedge_{\kappa(D_3 D_2)} u_{2,3} &= (-1)^{\langle u_{12,3}\rangle\langle v_{1,2}\rangle} \lambda(D_1)u_{1,23} \wedge_{\kappa(D_3 D_2)} v_{1,23}, \\
\lambda(D_2)w_{1,2} \wedge_{\mathfrak{c}(D_2 D_1)} y_2 &= (-1)^{\langle v_{2,3}\rangle\langle w_{1,23}\rangle} v_{12,3} \wedge_{\mathfrak{c}(D_2 D_1)} w_{12,3}.
\end{aligned} \qquad (4.28)$$

By (4.22), (4.27), and (4.28), the images of

$$x_1 \otimes y_1^* \otimes x_2 \otimes y_2^* \otimes x_3 \otimes y_3^* \in \lambda(D_1) \otimes \lambda(D_2) \otimes \lambda(D_3)$$

under $\widetilde{C}_{D_2 \circ D_1, D_3} \circ \widetilde{C}_{D_1, D_2} \otimes \mathrm{id}$ and $\widetilde{C}_{D_1, D_3 \circ D_2} \circ \mathrm{id} \otimes \widetilde{C}_{D_2, D_3}$ are

$$(-1)^{\epsilon_{\mathrm{L}}+\langle v_{2,3}\rangle\langle w_{1,23}\rangle}\big((x_1 \wedge_{\kappa(D_2 D_1)} u_{1,2}) \wedge_{\kappa(D_3 D_2 D_1)} u_{12,3}\big) \otimes \big(\lambda(D_3)w_{12,3} \wedge_{\mathfrak{c}(D_3 D_2 D_1)} y_3\big)^* \quad \text{and}$$

$$(-1)^{\epsilon_{\mathrm{R}}+\langle u_{12,3}\rangle\langle v_{1,2}\rangle}\big(x_1 \wedge_{\kappa(D_3 D_2 D_1)} u_{1,23}\big) \otimes \big(\lambda(D_3 D_2)w_{1,23} \wedge_{\mathfrak{c}(D_3 D_2 D_1)} (\lambda(D_3)w_{2,3} \wedge_{\mathfrak{c}(D_3 D_2)} y_3)\big)^*,$$

respectively. By (4.23), the second and third identities in (4.26), and (4.25), these two elements of $\lambda(D_3 D_2 D_1)$ are the same, which establishes the claim. $\qquad \square$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \longrightarrow \frac{\kappa(D_2 D_1)}{\kappa(D_1)} \longrightarrow \frac{\kappa(D_3 D_2 D_1)}{\kappa(D_1)} \longrightarrow \frac{\kappa(D_3 D_2 D_1)}{\kappa(D_2 D_1)} \longrightarrow 0$$

$$u_{1,2} \qquad D_1 \qquad u_{1,23} \qquad D_1 \qquad u_{12,3} \qquad D_1$$

$$0 \longrightarrow \kappa(D_2) \longrightarrow \kappa(D_3 D_2) \longrightarrow \frac{\kappa(D_3 D_2)}{\kappa(D_2)} \longrightarrow 0$$

$$x_2 \qquad\qquad\qquad u_{2,3}$$

$$0 \longrightarrow \frac{\kappa(D_2)}{\kappa(D_2)\cap\mathrm{Im}(D_1)} \longrightarrow \frac{\kappa(D_3 D_2)}{\kappa(D_3 D_2)\cap\mathrm{Im}(D_1)} \longrightarrow \frac{\kappa(D_3 D_2)}{\kappa(D_2)+\kappa(D_3 D_2)\cap\mathrm{Im}(D_1)} \longrightarrow 0$$

$$v_{1,2} \qquad\qquad v_{1,23} \qquad\qquad \mu$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \longrightarrow \frac{\kappa(D_3 D_2)}{\kappa(D_2)+\kappa(D_3 D_2)\cap\mathrm{Im}(D_1)} \xrightarrow{D_2} \frac{\kappa(D_3)}{\kappa(D_3)\cap\mathrm{Im}(D_2 D_1)} \longrightarrow \frac{\kappa(D_3)}{\kappa(D_3)\cap\mathrm{Im}(D_2)} \longrightarrow 0$$

$$\mu \qquad\qquad v_{12,3} \qquad\qquad v_{2,3}$$

$$0 \longrightarrow \frac{X_2}{\kappa(D_2)+\mathrm{Im}(D_1)} \xrightarrow{D_2} \mathfrak{c}(D_2 D_1) \longrightarrow \mathfrak{c}(D_2) \longrightarrow 0$$

$$w_{1,2} \qquad\qquad\qquad y_2$$

$$0 \longrightarrow \frac{X_2}{\kappa(D_3 D_2)+\mathrm{Im}(D_1)} \xrightarrow{D_2} \frac{X_3}{\kappa(D_3)+\mathrm{Im}(D_2 D_1)} \longrightarrow \frac{X_3}{\kappa(D_3)+\mathrm{Im}(D_2)} \longrightarrow 0$$

$$w_{1,23} \qquad\qquad w_{12,3} \qquad\qquad w_{2,3}$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$
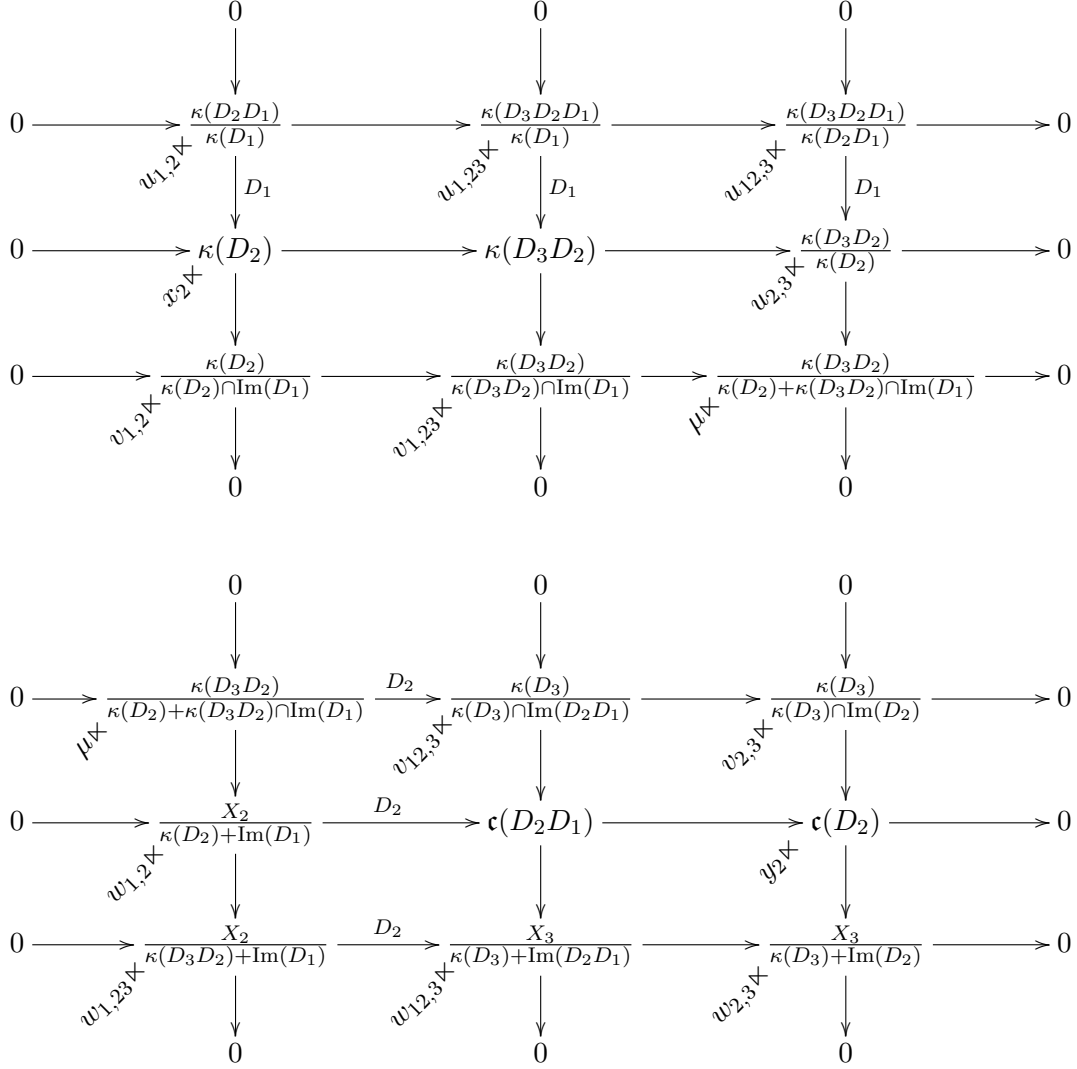
Figure 6: Commutative diagrams of exact sequences used in the proof of Proposition 4.8

Remark 4.9. The proof of this crucial proposition in [10, Appendix D.2] does not appear to establish anything. Up to notational differences, it describes an expression for

$$\big\{\widetilde{C}_{D_2\circ D_1,D_3} \circ \widetilde{C}_{D_1,D_2}\otimes\mathrm{id}\big\}\big(x_1\otimes y_1^* \otimes x_2\otimes y_2^* \otimes x_3\otimes y_3^*\big) \in \lambda(D_3\circ D_2\circ D_1)$$

without any signs and simply claims that

$$\big\{\widetilde{C}_{D_1,D_3\circ D_2} \circ \mathrm{id}\otimes\widetilde{C}_{D_2,D_3}\big\}\big(x_1\otimes y_1^* \otimes x_2\otimes y_2^* \otimes x_3\otimes y_3^*\big) \in \lambda(D_3\circ D_2\circ D_1)$$

is given by the same expression, without providing an explicit formula for $\widetilde{C}_{D_1,D_2}$, using the statement of Lemma 4.3, or indicating the significance of the grading of the lines $\lambda(V)$. As illustrated by the proof of Proposition 4.8 above, the two expressions require auxiliary terms from different

vectors spaces and it takes significant care to show that it is possible to choose them compatibly. Furthermore, there are two typos at the end of the proof of the closely related [10, Corollary D.2.4] with two subscripts that should be different being the same and resulting in the order of two factors switched between the statements of [10, Proposition D.2.3] and [10, Corollary D.2.4].

Proposition 4.10 (Compositions II). *For any pair* $(\mathfrak{t}_1, \mathfrak{t}_2)$ *of exact triples of Fredholm operators as in (2.16), the diagram (2.31) induced by the isomorphisms (4.10) commutes.*

Proof. We continue with the notation described in the first sentence of the proof of Proposition 4.8 and define

$$\mathfrak{t}_{12} = \mathcal{C}_{\mathcal{T}}(\mathfrak{t}_1, \mathfrak{t}_2), \quad \epsilon_{\mathrm{L}} = (\mathrm{ind}\, D_1'')(\mathrm{ind}\, D_2') + \epsilon_{D_1', D_2'} + \epsilon_{D_1'', D_2''} + \epsilon_{\mathfrak{t}_{12}}, \quad \epsilon_{\mathrm{R}} = \epsilon_{\mathfrak{t}_1} + \epsilon_{\mathfrak{t}_2} + \epsilon_{D_1, D_2}.$$

For $k=1, 2$ and $\star = ', ''$, let

$$x_k^\star \in \lambda(\kappa(D_k^\star)) - 0, \qquad y_k' \in \lambda(\mathfrak{c}(D_k')) - 0, \qquad y_k'' \in \lambda\left(\frac{X_{k+1}}{\mathrm{Im}(\mathfrak{i}_{k+1}) + \mathrm{Im}(D_k)}\right) - 0.$$

With $\star$ denoting $', ''$ or a blank, let

$$u^\star \in \lambda\left(\frac{\kappa(D_2^\star D_1^\star)}{\kappa(D_1^\star)}\right) - 0, \quad v^\star \in \lambda\left(\frac{\kappa(D_2^\star)}{\kappa(D_2^\star) \cap \mathrm{Im}(D_1^\star)}\right) - 0, \quad w^\star \in \lambda\left(\frac{X_2^\star}{\kappa(D_2^\star) + \mathrm{Im}(D_1^\star)}\right) - 0;$$

see Figures 7 and 8. For $k=1, 2, 12$, let

$$\delta_k \colon \kappa(D_k'') \longrightarrow \mathfrak{c}(D_k'),$$

where $D_{12}' = D_2' D_1'$ and $D_{12}'' = D_2'' D_1''$, be the connecting homomorphisms in the sequences (4.6) corresponding to $\mathfrak{t}_1$, $\mathfrak{t}_2$, and $\mathfrak{t}_{12}$, respectively, and

$$u_k \in \lambda\left(\frac{\kappa(D_k)}{\kappa(D_k) \cap \mathrm{Im}(\mathfrak{i}_k)}\right) - 0, \quad v_k \in \lambda\left(\frac{\kappa(D_k'')}{\mathfrak{j}_k(\kappa(D_k))}\right) - 0, \quad w_k \in \lambda\left(\frac{X_{k+1}'}{\mathfrak{i}_{k+1}^{-1}(\mathrm{Im}(D_k))}\right) - 0,$$

with $\mathfrak{i}_{12} = \mathfrak{i}_1$, $\mathfrak{j}_{12} = \mathfrak{j}_1$, and $12+1 \equiv 3$; see Figures 7 and 8. Define

$$\widetilde{w}'' \in \lambda\left(\frac{X_2}{\mathfrak{j}_2^{-1}(\kappa(D_2'')) + \mathrm{Im}(D_1)}\right) - 0 \qquad \text{by} \quad w'' = \lambda(\mathfrak{j}_2)\widetilde{w}''.$$

Below we choose these elements in a compatible way.

In order to describe the two relevant signs, we define

$$A = (\mathrm{ind}\, D_2'')(\mathfrak{c}_1' + \mathfrak{c}_1'' + \mathfrak{c}_2') + \left(\mathrm{ind}\, D_1'' + \mathrm{ind}\, D_2'\right)\mathfrak{c}_1' + \mathfrak{c}_1''\kappa_2', \qquad C = \mathfrak{c}_1' + \mathfrak{c}_1'' + \mathfrak{c}_2' + \mathfrak{c}_2'',$$
$$A_{\mathrm{L}} = \kappa_1''\kappa_2' + (\kappa_1'' + \kappa_2')\langle v' \rangle + (\mathfrak{c}_1' + \mathfrak{c}_2')\langle v'' \rangle + \left(\langle v' \rangle + \langle v'' \rangle\right)\langle v_{12} \rangle,$$
$$A_{\mathrm{R}} = \mathfrak{c}_1''\mathfrak{c}_2' + (\kappa_2' + \kappa_2'')\langle v_1 \rangle + (\mathfrak{c}_1' + \mathfrak{c}_1'')\langle v_2 \rangle + \left(\langle v_1 \rangle + \langle v_2 \rangle\right)\langle v \rangle,$$

where $\kappa_i^\star = \mathfrak{d}(\kappa(D_i^\star))$ and $\mathfrak{c}_i^\star = \mathfrak{d}(\mathfrak{c}(D_i^\star))$ with $i = 1, 2$ and $\star = ', ''$. Applying Lemma 4.1 to the exact sequences (4.6) with $D^\star$ replaced by $D_k^\star$, for $\star = ', ''$ and blank and $k = 1, 2, 12$, and (4.21) with D_k replaced by $D_k^\star$, for $\star = ', ''$ and blank and $k = 1, 2$, we obtain

$$\begin{aligned} \mathrm{ind}\, D_2 &= \mathrm{ind}\, D_2' + \mathrm{ind}\, D_2'', & \mathfrak{d}(\mathfrak{c}(D_k)) &= \mathfrak{d}(\mathfrak{c}(D_k')) + \mathfrak{d}(\mathfrak{c}(D_k'')) - \langle v_k \rangle, \\ \mathrm{ind}\, D_2'' D_1'' &= \mathrm{ind}\, D_1'' + \mathrm{ind}\, D_2'', & \mathfrak{d}(\mathfrak{c}(D_2^\star D_1^\star)) &= \mathfrak{d}(D_1^\star) + \mathfrak{d}(\mathfrak{c}(D_2^\star)) - \langle v^\star \rangle. \end{aligned} \tag{4.29}$$

From this, we find that

$$\begin{aligned}
\epsilon_{\mathrm{L}} &= A + C\big(\langle v'\rangle + \langle v''\rangle + \langle v_{12}\rangle\big) + A_{\mathrm{R}} + \langle v_{12}\rangle, \\
\epsilon_{\mathrm{R}} &= A + C\big(\langle v_1\rangle + \langle v_2\rangle + \langle v\rangle\big) + A_{\mathrm{L}} + \langle v_1\rangle + \langle v_2\rangle,
\end{aligned} \tag{4.30}$$

modulo 2. By the identities in the second column in (4.29),

$$\langle v\rangle + \langle v_1\rangle + \langle v_2\rangle = C - \mathfrak{d}(\mathfrak{c}(D_2 D_1)) = \langle v_{12}\rangle + \langle v'\rangle + \langle v''\rangle. \tag{4.31}$$

From the exact sequences (4.6) and (4.21), we also find

$$\kappa_i'' = \langle u_i\rangle + \langle v_i\rangle, \quad \mathfrak{c}_i' = \langle v_i\rangle + \langle w_i\rangle, \qquad \kappa_2^\star = \langle u^\star\rangle + \langle v^\star\rangle, \quad \mathfrak{c}_1^\star = \langle v^\star\rangle + \langle w^\star\rangle,$$

where $i = 1, 2$ and $\star = {}', {}''$. From this, we find that

$$\begin{aligned}
\langle u'\rangle\langle u_1\rangle + \langle w''\rangle\langle w_2\rangle &+ \langle w'\rangle\langle u_2\rangle + \langle u''\rangle\langle w_1\rangle + \langle v\rangle\langle v_{12}\rangle \\
&\cong A_{\mathrm{L}} + A_{\mathrm{R}} + \big(\langle v_1\rangle + \langle v_2\rangle + \langle v_{12}\rangle\big)\big(\langle v'\rangle + \langle v''\rangle + \langle v\rangle\big)
\end{aligned} \tag{4.32}$$

modulo 2.

In light of the bottom row and right column in the first diagram in Figure 7, the top row and left column in the second diagram in Figure 7, and Lemma 4.1, we can take

$$\begin{aligned}
u &= \lambda(\mathfrak{i}_1) u' \wedge_{\frac{\kappa(D_2 D_1)}{\kappa(D_1)}} \mu, & u_{12} &= u_1 \wedge_{\frac{\kappa(D_2 D_1)}{\kappa(D_2 D_1) \cap \mathrm{Im}(\mathfrak{i}_1)}} \mu, \\
w &= \eta \wedge_{\frac{X_2}{\kappa(D_2) + \mathrm{Im}(D_1)}} \widetilde{w}'', & w_{12} &= \lambda(\mathfrak{i}_3^{-1} \circ D_2)\eta \wedge_{\frac{X_3'}{\mathfrak{i}_3^{-1}(\mathrm{Im}(D_2 D_1))}} w_2
\end{aligned} \tag{4.33}$$

for some

$$\mu \in \lambda\left(\frac{\kappa(D_2 D_1)}{\kappa(D_1) + \kappa(D_2 D_1) \cap \mathrm{Im}(\mathfrak{i}_1)}\right) - 0, \qquad \eta \in \lambda\left(\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\kappa(D_2) + \mathfrak{j}_2^{-1}(\kappa(D_2'')) \cap \mathrm{Im}(D_1)}\right) - 0.$$

Along with Lemma 4.3 applied to the two diagrams in Figure 7, these equalities insure that

$$\begin{aligned}
\big((\lambda(\mathfrak{i}_1) x_1' &\wedge_{\kappa(D_1)} u_1) \wedge_{\kappa(D_2 D_1)} u\big) \otimes \big(\lambda(D_2) w \wedge_{\mathfrak{c}(D_2 D_1)} (\lambda(\mathfrak{i}_3) w_2 \wedge_{\mathfrak{c}(D_2)} y_2'')\big)^* \\
&= (-1)^{\langle u'\rangle\langle u_1\rangle + \langle \widetilde{w}''\rangle\langle w_2\rangle}\big(\lambda(\mathfrak{i}_1)(x_1' \wedge_{\kappa(D_2' D_1')} u') \wedge_{\kappa(D_2 D_1)} u_{12}\big) \\
&\qquad \otimes \big(\lambda(\mathfrak{i}_3) w_{12} \wedge_{\mathfrak{c}(D_2 D_1)} (\lambda(D_2)\widetilde{w}'' \wedge_{\frac{X_3}{\mathrm{Im}(\mathfrak{i}_3) + \mathrm{Im}(D_2 D_1)}} y_2'')\big)^*,
\end{aligned} \tag{4.34}$$

in $\lambda(D_2 D_1)$.

We next make use of the three commutative diagrams in Figure 8. They can be viewed as the three coordinate planes in $\mathbb{Z}^3$, with all three diagrams sharing the center and any pair sharing an axis. We choose $v', w', u_2, v_2, v_1, u'', \mu$ arbitrarily, then find y_1', w_1, v_{12}, v so that

$$\begin{aligned}
v' \wedge_{\mathfrak{c}(D_1)} w' &= y_1' = \lambda(\delta_1) v_1 \wedge_{\mathfrak{c}(D_1)} w_1, & v_1 \wedge_{\frac{\kappa(D_2'' D_1'')}{\mathfrak{j}_1(\kappa(D_1))}} u'' &= (-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(\mathfrak{j}_1)\mu \wedge_{\frac{\kappa(D_2'' D_1'')}{\mathfrak{j}_1(\kappa(D_2))}} v_{12}, \\
\lambda(\mathfrak{i}_2) v' &\wedge_{\frac{\kappa(D_2)}{\kappa(D_2) \cap \mathrm{Im}(\mathfrak{i}_2 D_1)}} u_2 = (-1)^{\langle w'\rangle\langle u_2\rangle}\lambda(D_1)\mu \wedge_{\frac{\kappa(D_2)}{\kappa(D_2) \cap \mathrm{Im}(\mathfrak{i}_2 D_1)}} v,
\end{aligned} \tag{4.35}$$

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$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \longrightarrow \kappa(D_1') \xrightarrow{\ i_1\ } \kappa(D_1) \longrightarrow \frac{\kappa(D_1)}{\kappa(D_1)\cap\mathrm{Im}(i_1)} \longrightarrow 0$$

with vertical maps x_1', u_1

$$0 \longrightarrow \kappa(D_2'D_1') \xrightarrow{\ i_1\ } \kappa(D_2D_1) \longrightarrow \frac{\kappa(D_2D_1)}{\kappa(D_2D_1)\cap\mathrm{Im}(i_1)} \longrightarrow 0$$

with vertical maps u_{12}

$$0 \longrightarrow \frac{\kappa(D_2'D_1')}{\kappa(D_1')} \xrightarrow{\ i_1\ } \frac{\kappa(D_2D_1)}{\kappa(D_1)} \longrightarrow \frac{\kappa(D_2D_1)}{\kappa(D_1)+\kappa(D_2D_1)\cap\mathrm{Im}(i_1)} \longrightarrow 0$$

with vertical maps u', u, μ

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \longrightarrow \frac{j_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+j_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)} \longrightarrow \frac{X_2}{\kappa(D_2)+\mathrm{Im}(D_1)} \longrightarrow \frac{X_2}{j_2^{-1}(\kappa(D_2''))+\mathrm{Im}(D_1)} \longrightarrow 0$$

with vertical maps η, w and $i_3^{-1}\circ D_2$, $\widetilde{u}''$ and D_2, D_2

$$0 \longrightarrow \frac{X_3'}{i_3^{-1}(\mathrm{Im}(D_2D_1))} \xrightarrow{\ i_3\ } \mathfrak{c}(D_2D_1) \longrightarrow \frac{X_3}{\mathrm{Im}(i_3)+\mathrm{Im}(D_2D_1)} \longrightarrow 0$$

with vertical maps w_{12}

$$0 \longrightarrow \frac{X_3'}{i_3^{-1}(\mathrm{Im}(D_2))} \xrightarrow{\ i_3\ } \mathfrak{c}(D_2) \longrightarrow \frac{X_3}{\mathrm{Im}(i_3)+\mathrm{Im}(D_2)} \longrightarrow 0$$

with vertical maps w_2, y_2''

$$0 \qquad\qquad 0 \qquad\qquad 0$$

Figure 7: Commutative diagrams of exact sequences used in the proof of Proposition 4.10

and finally take η, x_2'', v'' so that

$$\lambda(i_2)w' \wedge_{\frac{j_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+\mathrm{Im}(i_2 D_1')}} v_2 = (-1)^{\langle v\rangle\langle v_{12}\rangle}\lambda(D_1\circ j_1^{-1})v_{12} \wedge_{\frac{j_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+\mathrm{Im}(i_2 D_1')}} \eta\,,$$
$$\lambda(j_2)u_2 \wedge_{\kappa(D_2'')} v_2 = x_2'' = \lambda(D_1'')u'' \wedge_{\kappa(D_2'')} v''\,. \tag{4.36}$$

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By Lemma 4.3 applied to the three commutative squares in Figure 8, (4.35), and (4.36),

$$\lambda(D_1{\circ}\mathsf{j}_1^{-1})\big(v_1 \wedge_{\substack{\kappa(D_2''D_1'')\\\mathsf{j}_1(\kappa(D_1))}} u''\big) \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathrm{Im}(\mathsf{i}_2 D_1')}} \big(\lambda(\mathsf{i}_2)w_1 \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathsf{j}_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)}} v''\big)$$

$$= (-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(\mathsf{i}_2)\big(\lambda(\delta_1)v_1 \wedge_{\mathfrak{c}(D_1)} w_1\big) \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathrm{Im}(\mathsf{i}_2 D_1')}} \big(\lambda(D_1'')u'' \wedge_{\kappa(D_2'')} v''\big)$$

$$= (-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(\mathsf{i}_2)\big(v' \wedge_{\mathfrak{c}(D_1)} w'\big) \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathrm{Im}(\mathsf{i}_2 D_1')}} \big(\lambda(\mathsf{j}_2)u_2 \wedge_{\kappa(D_2'')} v_2\big)$$

$$= (-1)^{\langle u''\rangle\langle w_1\rangle + \langle w'\rangle\langle u_2\rangle}\big(\lambda(\mathsf{i}_2)v' \wedge_{\substack{\kappa(D_2)\\\kappa(D_2)\cap\mathrm{Im}(\mathsf{i}_2 D_1)}} u_2\big) \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathrm{Im}(\mathsf{i}_2 D_1')}} \big(\lambda(\mathsf{i}_2)w' \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\kappa(D_2)+\mathrm{Im}(\mathsf{i}_2 D_1')}} v_2\big)$$

$$= (-1)^{\langle u''\rangle\langle w_1\rangle + \langle v\rangle\langle v_{12}\rangle}\big(\lambda(D_1)\mu \wedge_{\substack{\kappa(D_2)\\\kappa(D_2)\cap\mathrm{Im}(\mathsf{i}_2 D_1)}} v\big) \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathrm{Im}(\mathsf{i}_2 D_1')}} \big(\lambda(D_1{\circ}\mathsf{j}_1^{-1})v_{12} \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\kappa(D_2)+\mathrm{Im}(\mathsf{i}_2 D_1')}} \eta\big)$$

$$= (-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(D_1{\circ}\mathsf{j}_1^{-1})\big(\lambda(\mathsf{j}_1)\mu \wedge_{\substack{\kappa(D_2''D_1'')\\\mathsf{j}_1(\kappa(D_2))}} v_{12}\big) \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathrm{Im}(\mathsf{i}_2 D_1')}} \big(v \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathsf{j}_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)}} \eta\big).$$

Along with the second equation in (4.35), this gives

$$\lambda(\mathsf{i}_2)w_1 \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathsf{j}_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)}} v'' = \big(v \wedge_{\substack{\mathsf{j}_2^{-1}(\kappa(D_2''))\\\mathsf{j}_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)}} \eta\big). \tag{4.37}$$

In addition to the choices of y_1' and x_2'' specified in (4.35) and (4.36), we take

$$\begin{aligned} x_1'' &= \lambda(\mathsf{j}_1)u_1 \wedge_{\kappa(D_1'')} v_1, & \lambda(\mathsf{j}_2)y_1'' &= v'' \wedge_{\mathfrak{c}(D_2')} w'', \\ x_2' &= \lambda(D_1')u' \wedge_{\kappa(D_2')} v', & y_2' &= \lambda(\delta_2)v_2 \wedge_{\mathfrak{c}(D_2')} w_2. \end{aligned} \tag{4.38}$$

By (4.33), the last two equations in (4.35), the first equation in (4.36), (4.37), and Corollary 4.4,

$$\begin{aligned} \lambda(\mathsf{i}_2)w_1 \wedge_{\mathfrak{c}(D_1)} y_1'' = v \wedge_{\mathfrak{c}(D_1)} w, &\quad \lambda(\mathsf{i}_2)x_2' \wedge_{\lambda(D_2)} u_2 = (-1)^{\langle w'\rangle\langle u_2\rangle}\lambda(D_1)u \wedge_{\kappa(D_2)} v, \\ \lambda(D_2')w' \wedge_{\mathfrak{c}(D_2'D_1')} y_2' &= (-1)^{\langle v\rangle\langle v_{12}\rangle}\lambda(\delta_{12})v_{12} \wedge_{\mathfrak{c}(D_2'D_1')} w_{12}, \\ x_1'' \wedge_{\kappa(D_2''D_1'')} u'' &= (-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(\mathsf{j}_1)u_{12} \wedge_{\kappa(D_2''D_1'')} v_{12}; \end{aligned} \tag{4.39}$$

the third identity above also uses

$$\lambda(D_2') = \lambda(\mathsf{i}_3^{-1}{\circ}D_2) \circ \lambda(\mathsf{i}_2), \quad \lambda(\delta_2) = \lambda(\mathsf{i}_3^{-1}{\circ}D_2) \circ \lambda(\mathsf{j}_2)^{-1}, \quad \lambda(\delta_{12}) = \lambda(\mathsf{i}_3^{-1}{\circ}D_2) \circ \lambda(D_1{\circ}\mathsf{j}_1^{-1}).$$

By (4.10), the second equality in the first identity in (4.35), the first equality in the last identity in (4.36), the first and last equations in (4.38), (4.22), and the first two equations in (4.39), the image of

$$x_1' {\otimes} y_1'^* \otimes x_1'' {\otimes} (\lambda(\mathsf{j}_2)y_1'')^* \otimes x_2' {\otimes} y_2'^* \otimes x_2'' {\otimes} (\lambda(\mathsf{j}_3)y_2'')^* \in \lambda(D_1') \otimes \lambda(D_1'') \otimes \lambda(D_2') \otimes \lambda(D_2'')$$

under $\widetilde{C}_{D_1,D_2} \circ \Psi_{\mathfrak{t}_1} {\otimes} \Psi_{\mathfrak{t}_2}$ is

$$(-1)^{\epsilon_{\mathrm{R}} + \langle w'\rangle\langle u_2\rangle}\big((\lambda(\mathsf{i}_1)x_1' \wedge_{\kappa(D_1)} u_1) \wedge_{\kappa(D_2 D_1)} u\big) \otimes \big(\lambda(D_2)w \wedge_{\mathfrak{c}(D_2 D_1)} (\lambda(\mathsf{i}_3)w_2 \wedge_{\mathfrak{c}(D_2)} y_2'')\big)^*.$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \longrightarrow \frac{\kappa(D_2')}{\kappa(D_2')\cap\operatorname{Im}(D_1')} \xrightarrow{\;i_2\;} \frac{\kappa(D_2)}{\kappa(D_2)\cap\operatorname{Im}(i_2 D_1')} \longrightarrow \frac{\kappa(D_2)}{\kappa(D_2)\cap\operatorname{Im}(i_2)} \longrightarrow 0$$

$$v'\uparrow \qquad\qquad\qquad\qquad\qquad w_2\uparrow \;\; \downarrow j_2$$

$$0 \longrightarrow \mathfrak{c}(D_1') \xrightarrow{\;i_2\;} \frac{j_2^{-1}(\kappa(D_2''))}{\operatorname{Im}(i_2 D_1')} \xrightarrow{\;j_2\;} \kappa(D_2'') \longrightarrow 0$$

$$y_1'\uparrow \qquad\qquad\qquad\qquad\qquad x_2''\uparrow$$

$$0 \longrightarrow \frac{X_2'}{\kappa(D_2')+\operatorname{Im}(D_1')} \xrightarrow{\;i_2\;} \frac{j_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+\operatorname{Im}(i_2 D_1')} \xrightarrow{\;j_2\;} \frac{\kappa(D_2'')}{j_2(\kappa(D_2))} \longrightarrow 0$$

$$w'\uparrow \qquad\qquad\qquad\qquad\qquad v_2\uparrow$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \longrightarrow \frac{\kappa(D_1'')}{j_1(\kappa(D_1))} \longrightarrow \frac{\kappa(D_2'' D_1'')}{j_1(\kappa(D_1))} \longrightarrow \frac{\kappa(D_2'' D_1'')}{\kappa(D_1'')} \longrightarrow 0$$

$$v_1\uparrow \;\; \downarrow \delta_1 \qquad\qquad \downarrow D_1\circ j_1^{-1} \qquad u''\uparrow \;\; \downarrow D_1''$$

$$0 \longrightarrow \mathfrak{c}(D_1') \xrightarrow{\;i_2\;} \frac{j_2^{-1}(\kappa(D_2''))}{\operatorname{Im}(i_2 D_1')} \xrightarrow{\;j_2\;} \kappa(D_2'') \longrightarrow 0$$

$$y_1'\uparrow \qquad\qquad\qquad\qquad\qquad x_2''\uparrow$$

$$0 \longrightarrow \frac{X_2'}{i_2^{-1}(\operatorname{Im}(D_1'))} \xrightarrow{\;i_2\;} \frac{j_2^{-1}(\kappa(D_2''))}{j_2^{-1}(\kappa(D_2''))\cap\operatorname{Im}(D_1)} \xrightarrow{\;j_2\;} \frac{\kappa(D_2'')}{\kappa(D_2'')\cap\operatorname{Im}(D_1'')} \longrightarrow 0$$

$$w_1\uparrow \qquad\qquad\qquad\qquad\qquad v''\uparrow$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

$$0 \longrightarrow \frac{\kappa(D_2 D_1)}{\kappa(D_1)+\kappa(D_2 D_1)\cap\operatorname{Im}(i_1)} \xrightarrow{\;j_1\;} \frac{\kappa(D_2'' D_1'')}{j_1(\kappa(D_1))} \longrightarrow \frac{\kappa(D_2'' D_1'')}{j_1(\kappa(D_2 D_1))} \longrightarrow 0$$

$$\mu\uparrow \;\; \downarrow D_1 \qquad\qquad \downarrow D_1\circ j_1^{-1} \qquad v_{12}\uparrow \;\; \downarrow D_1\circ j_1^{-1}$$

$$0 \longrightarrow \frac{\kappa(D_2)}{\kappa(D_2)\cap\operatorname{Im}(i_2 D_1')} \longrightarrow \frac{j_2^{-1}(\kappa(D_2''))}{\operatorname{Im}(i_2 D_1')} \longrightarrow \frac{j_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+\operatorname{Im}(i_2 D_1')} \longrightarrow 0$$

$$0 \longrightarrow \frac{\kappa(D_2)}{\kappa(D_2)\cap\operatorname{Im}(D_1)} \longrightarrow \frac{j_2^{-1}(\kappa(D_2''))}{j_2^{-1}(\kappa(D_2''))\cap\operatorname{Im}(D_1)} \longrightarrow \frac{j_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+j_2^{-1}(\kappa(D_2''))\cap\operatorname{Im}(D_1)} \longrightarrow 0$$

$$v\uparrow \qquad\qquad\qquad\qquad\qquad \eta\uparrow$$
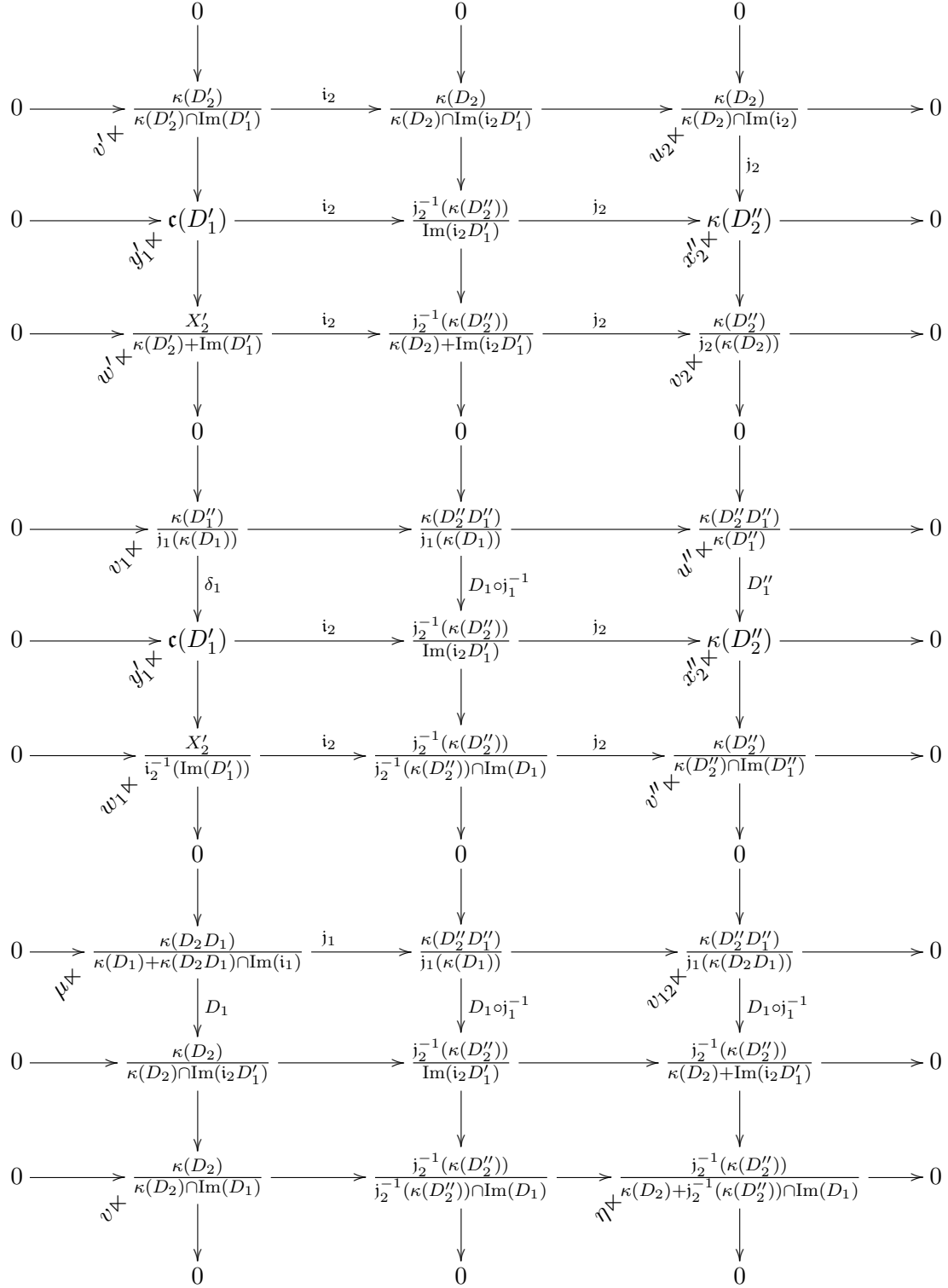
$$0 \qquad\qquad 0 \qquad\qquad 0$$

Figure 8: Commutative diagrams of exact sequences used in the proof of Proposition 4.10

By (4.22), the first equality in the first identity in (4.35), the second equality in the last identity in (4.36), the second and third equations in (4.38), (4.10), and the last two equations in (4.39), the image of this element under $\Psi_{\mathfrak{t}_{12}} \circ \widetilde{\mathcal{C}}_{D'_1, D'_2} \otimes \widetilde{\mathcal{C}}_{D''_1, D''_2} \circ \mathrm{id} \otimes R \otimes \mathrm{id}$ is

$$(-1)^{\epsilon_{\mathrm{L}} + \langle v \rangle \langle v_{12} \rangle + \langle u'' \rangle \langle w_1 \rangle} \big(\lambda(\mathfrak{i}_1)(x'_1 \wedge_{\kappa(D'_2 D'_1)} u') \wedge_{\kappa(D_2 D_1)} u_{12} \big)$$
$$\otimes \big(\lambda(\mathfrak{i}_3) w_{12} \wedge_{\mathfrak{c}(D_2 D_1)} (\lambda(D_2) \widetilde{w}'' \wedge_{\frac{X_3}{\mathrm{Im}(\mathfrak{i}_3) + \mathrm{Im}(D_2 D_1)}} y''_2) \big)^* .$$

By (4.34) and (4.30)-(4.32), these two elements of $\lambda(D_2 D_1)$ are the same, which establishes the claim. □

4.3 Stabilizations of Fredholm operators

We now describe stabilizations of Fredholm operators which are used to topologize determinant line bundles in the next section. In this subsection, we use them to deduce the Exact Squares property on page 17 for the isomorphisms (2.27) from the Naturality II, Normalization II, and Compositions II properties via Lemmas 4.3 and 4.12.

For any Banach vector space X, $N \in \mathbb{Z}^{\geq 0}$, and homomorphism $\Theta \colon \mathbb{R}^N \longrightarrow Y$, let

$$\iota_{X;N} \colon X \longrightarrow X \oplus \mathbb{R}^N, \qquad D_\Theta \colon X \oplus \mathbb{R}^N \longrightarrow Y, \quad \text{and} \quad \hat{\mathcal{I}}_{\Theta;D} \colon \lambda(D) \longrightarrow \lambda(D_\Theta)$$

be as in Section 3.1. Since $D = D_\Theta \circ \iota_{X;N}$ and the projection $\pi_2 \colon X \oplus \mathbb{R}^N \longrightarrow \mathbb{R}^N$ identifies $\mathfrak{c}(\iota_{X;N})$ with $\mathbb{R}^N$, the corresponding exact triple (2.19) gives rise to the isomorphism

$$\mathcal{I}_{\Theta;D} \colon \lambda(D_\Theta) \longrightarrow \lambda(D), \qquad \mathcal{I}_{\Theta;D}(\sigma) = \tilde{\mathcal{C}}_{\iota_{X;N}, D_\Theta} \big(1 \otimes (\Omega_N^* \circ \lambda(\pi_2)) \otimes \sigma \big), \tag{4.40}$$

where Ω_N is the standard volume tensor on $\mathbb{R}^N$ as before.

Lemma 4.11. *For any collection of exact triple isomorphisms $\Psi_{\mathfrak{t}}$ as in (2.27) satisfying the Normalization III and Compositions II properties and $N \in \mathbb{Z}^{\geq 0}$, there exists $A_N \in \mathbb{R}^*$ with the following property. For any Banach vector spaces X and Y, homomorphism $\Theta \colon \mathbb{R}^N \longrightarrow Y$, and $D \in \mathcal{F}(X, Y)$, the isomorphisms (3.2) and (4.40) induced by the isomorphisms $\Psi_{\mathfrak{t}}$ satisfy*

$$\mathcal{I}_{\Theta;D} \circ \hat{\mathcal{I}}_{\Theta;D} = (-1)^{(\mathrm{ind}\, D)N} A_N \, \mathrm{id}_{\lambda(D)}. \tag{4.41}$$

Proof. Let $A_N \in \mathbb{R}^*$ be such that

$$\tilde{\mathcal{C}}_{\iota_{0;N}, j_N} \colon \lambda(\iota_{0;N}) \otimes \lambda(j_N) \longrightarrow \lambda\big(\mathrm{id}_0 = j_N \circ \iota_{0;N}\big), \quad \tilde{\mathcal{C}}_{\iota_{0;N}, j_N}\big((1 \otimes \Omega_N^*) \otimes (\Omega_N \otimes 1^*)\big) = A_N \, 1 \otimes 1^*.$$

By the Compositions II and Normalization III properties applied to the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \overset{\mathrm{id}}{\longrightarrow} & X & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota_{X;N}} & & \downarrow{\scriptstyle \iota_{0;N}} & & \\
0 & \longrightarrow & X & \overset{\iota_{X;N}}{\longrightarrow} & X \oplus \mathbb{R}^N & \overset{\pi_2}{\longrightarrow} & \mathbb{R}^N & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D_\Theta} & & \downarrow{\scriptstyle j_N} & & \\
0 & \longrightarrow & Y & \overset{\mathrm{id}}{\longrightarrow} & Y & \longrightarrow & 0 & \longrightarrow & 0,
\end{array}
$$

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the diagram

$$\begin{array}{ccc}
\lambda(D) & \xrightarrow{\hat{\mathcal{I}}_{\Theta;D}} & \lambda(D_\Theta) \\
{\scriptstyle (-1)^{(\mathrm{ind}\,D)N}A_N}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{I}_{\Theta;D}} \\
\lambda(D) & \xrightarrow{\quad\mathrm{id}\quad} & \lambda(D)
\end{array}$$

commutes. This gives (4.41). $\qquad\qquad\Box$

For a short exact sequence

$$0 \longrightarrow \mathbb{R}^{N'} \xrightarrow{\;\mathfrak{i}\;} \mathbb{R}^N \xrightarrow{\;\mathfrak{j}\;} \mathbb{R}^{N''} \longrightarrow 0$$

of vector spaces, we denote by $\mathfrak{c}_{\mathfrak{i},\mathfrak{j}}$ the exact triple

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R}^{N'} & \xrightarrow{\;\mathfrak{i}\;} & \mathbb{R}^N & \xrightarrow{\;\mathfrak{j}\;} & \mathbb{R}^{N''} & \longrightarrow & 0
\end{array}$$

of Fredholm operators. Given a collection of exact triple isomorphisms as in (2.27), define $A_{\mathfrak{i},\mathfrak{j}} \in \mathbb{R}^*$ by

$$\Psi_{\mathfrak{c}_{\mathfrak{i},\mathfrak{j}}}\big(1 \otimes \Omega_{N'}^* \otimes 1 \otimes \Omega_{N''}^*\big) = A_{\mathfrak{i},\mathfrak{j}}^{-1}(-1)^{N'N''}\big(1 \otimes \Omega_N^*\big). \tag{4.42}$$

In the case of the collection of exact triple isomorphisms given by (4.10),

$$\Omega_{N'} \wedge_{\mathbb{R}^N} \Omega_{N''} = A_{\mathfrak{i},\mathfrak{j}}\,\Omega_N\,,$$

where $\wedge_{\mathbb{R}^N}$ is the isomorphism provided by Lemma 4.1 for the $(\mathfrak{i},\mathfrak{j})$ short exact sequence above.

For an exact triple $\mathfrak{s}$ of vector-space homomorphisms of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}^{N'} & \xrightarrow{\;\mathfrak{i}\;} & \mathbb{R}^N & \xrightarrow{\;\mathfrak{j}\;} & \mathbb{R}^{N''} & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \Theta'} & & \Big\downarrow{\scriptstyle \Theta} & & \Big\downarrow{\scriptstyle \Theta''} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\;\mathfrak{i}_Y\;} & Y & \xrightarrow{\;\mathfrak{j}_Y\;} & Y'' & \longrightarrow & 0\,,
\end{array} \tag{4.43}$$

let $A_{\mathfrak{s}} = A_{\mathfrak{i},\mathfrak{j}}$. If $\mathfrak{t}$ is an exact triple of Fredholm operators as in (2.6), we denote by $\mathfrak{t}_{\mathfrak{s}}$ the exact triple

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X' \oplus \mathbb{R}^{N'} & \xrightarrow{\;\mathfrak{i}_X \oplus \mathfrak{i}\;} & X \oplus \mathbb{R}^N & \xrightarrow{\;\mathfrak{j}_X \oplus \mathfrak{j}\;} & X'' \oplus \mathbb{R}^{N''} & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle D'_{\Theta'}} & & \Big\downarrow{\scriptstyle D_\Theta} & & \Big\downarrow{\scriptstyle D''_{\Theta''}} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\;\mathfrak{i}_Y\;} & Y & \xrightarrow{\;\mathfrak{j}_Y\;} & Y' & \longrightarrow & 0
\end{array}$$

of Fredholm operators.

Lemma 4.12. *Let $\{\Psi_{\mathfrak{t}}\}_{\mathfrak{t}}$ be a family of exact triple isomorphisms as in (2.27) satisfying the Naturality II and Compositions II properties. For every exact triple $\mathfrak{t}$ of Fredholm operators as in (2.6) and for every exact triple $\mathfrak{s}$ of homomorphisms as in (4.43), the diagram*

$$\begin{array}{ccc}
\lambda(D'_{\Theta'}) \otimes \lambda(D''_{\Theta''}) & \xrightarrow{\quad\Psi_{\mathfrak{t}_{\mathfrak{s}}}\quad} & \lambda(D_\Theta) \\
{\scriptstyle \mathcal{I}_{\Theta';D'} \otimes \mathcal{I}_{\Theta'';D''}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{I}_{\Theta;D}} \\
\lambda(D') \otimes \lambda(D'') & \xrightarrow{(-1)^{(\mathrm{ind}\,D')N''}A_{\mathfrak{s}}\Psi_{\mathfrak{t}}} & \lambda(D)
\end{array}$$

of isomorphisms induced by the isomorphisms (2.27) commutes.

Proof. By our assumptions, the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \xrightarrow{\;\mathsf{i}_X\;} & X & \xrightarrow{\;\mathsf{j}_X\;} & X'' & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \iota_{X';N'}} & & \Big\downarrow{\scriptstyle \iota_{X;N}} & & \Big\downarrow{\scriptstyle \iota_{X'';N''}} & & \\
0 & \longrightarrow & X'\oplus\mathbb{R}^{N'} & \xrightarrow{\;\mathsf{i}_X\oplus\mathsf{i}\;} & X\oplus\mathbb{R}^{N} & \xrightarrow{\;\mathsf{j}_X\oplus\mathsf{j}\;} & X''\oplus\mathbb{R}^{N''} & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle D'_{\Theta'}} & & \Big\downarrow{\scriptstyle D_{\Theta}} & & \Big\downarrow{\scriptstyle D''_{\Theta''}} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\;\mathsf{i}_Y\;} & Y & \xrightarrow{\;\mathsf{j}_Y\;} & Y' & \longrightarrow & 0
\end{array}
$$

commutes. By Naturality II applied to the exact triple $\mathfrak{t}_{\mathrm{T}}$ in the top half of this diagram and the exact triple $\mathfrak{c}_{\mathsf{i},\mathsf{j}}$ and by (4.42),

$$
\Psi_{\mathfrak{t}_{\mathrm{T}}}\big(1\otimes(\Omega^*_{N'}\circ\lambda(\pi'_2))\otimes 1\otimes(\Omega^*_{N''}\circ\lambda(\pi''_2))\big) = A^{-1}_{\mathfrak{s}}(-1)^{N'N''}\,1\otimes(\Omega^*_N\circ\lambda(\pi_2)),
$$

where $\pi^\star_2 \colon X^\star\oplus\mathbb{R}^{N^\star} \longrightarrow \mathbb{R}^{N^\star}$ is the projection on the second component and $\star =','' $ or blank. Thus, the claim follows from the Compositions II property applied to the above diagram, along with (4.40). $\qquad\square$

Corollary 4.13 (Exact Squares). *A family of exact triple isomorphisms $\Psi_{\mathfrak{t}}$ as in (2.27) satisfying the Naturality II, Normalization II, and Compositions II properties also satisfies the Exact Squares property on page 17.*

Proof. We augment the domains in (2.32) by a commuting square of homomorphisms between finite-dimensional vector spaces, obtaining a version of the commutative diagram (2.32) with surjective Fredholm operators. The conclusion of this corollary holds for such a diagram by the Normalization II property and Lemma 4.3. The diagrams (2.33) corresponding to the original and new diagrams (2.32) are related by Lemma 4.12. This gives rise to a cube of commuting diagrams; see Figure 9. We put the new version of (2.33) on the back face and the diagrams arising from Lemma 4.12 on the top, right, bottom, and left faces. This forces some coefficients $A_\star$ on each edge of the front face in order to make the last four diagrams commute. The resulting coefficient distribution on the edges of the front face may be different from the coefficients distribution (all $A_\star=1$) on the original version of (2.33). However, by Lemma 4.3, the two coefficient distributions are equivalent at least if the original diagram consists of surjective Fredholm operators. Since the coefficients depend only on the supplementary commuting square of homomorphisms between finite-dimensional vector spaces and on the parities of the indices of the Fredholm operators (not the dimensions of their kernels or cokernels), it follows that the two coefficient distributions are equivalent in all cases; this establishes Corollary 4.13.

We denote the range of the operator $D_{\star\star}$ by $Y_{\star\star}$. Let

$$
\Theta_{\mathrm{TL}}\colon\mathbb{R}^{N_{\mathrm{TL}}}\longrightarrow Y_{\mathrm{TL}},\quad \tilde{\Theta}_{\mathrm{TR}}\colon\mathbb{R}^{N_{\mathrm{TR}}}\longrightarrow Y_{\mathrm{TM}},\quad \tilde{\Theta}_{\mathrm{BL}}\colon\mathbb{R}^{N_{\mathrm{BL}}}\longrightarrow Y_{\mathrm{CL}},\quad \tilde{\Theta}_{\mathrm{BR}}\colon\mathbb{R}^{N_{\mathrm{BR}}}\longrightarrow Y_{\mathrm{CM}},
$$

be homomorphisms such that

$$
\mathfrak{c}\big((D_{\mathrm{TL}})_{\Theta_{\mathrm{TL}}}\big),\mathfrak{c}\big((D_{\mathrm{TR}})_{\mathsf{j}_{\mathrm{T}}\circ\tilde{\Theta}_{\mathrm{TR}}}\big),\mathfrak{c}\big((D_{\mathrm{BL}})_{\mathsf{j}_{\mathrm{L}}\circ\tilde{\Theta}_{\mathrm{BL}}}\big),\mathfrak{c}\big((D_{\mathrm{BR}})_{\mathsf{j}_{\mathrm{R}}\circ\mathsf{j}_{\mathrm{C}}\circ\tilde{\Theta}_{\mathrm{BR}}}\big)=0. \tag{4.44}
$$
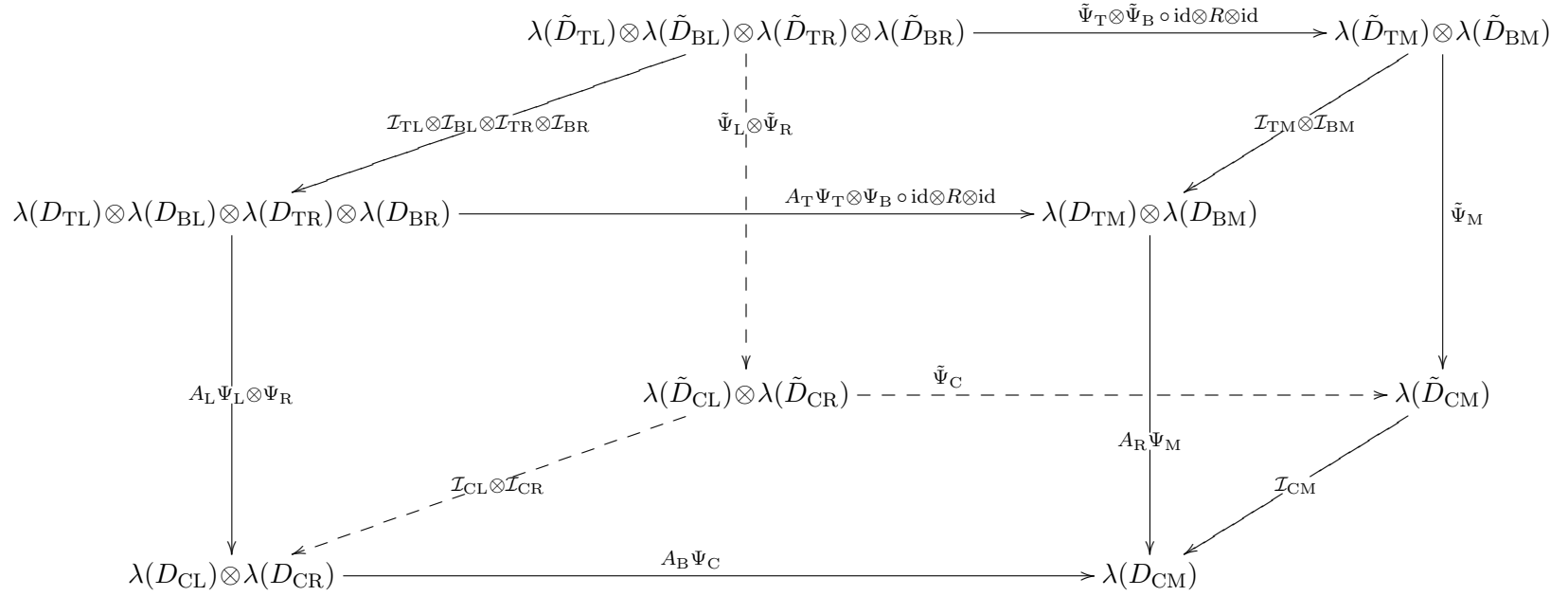
Figure 9: The cube of commutative diagrams used in the proof of Corollary 4.13, where $\tilde{D}_{\star\star} = (D_{\star\star})_{\Theta_{\star\star}}$, $\mathcal{I}_{\star\star} = \mathcal{I}_{\Theta_{\star\star}; D_{\star\star}}$, and $\tilde{\Psi}_\star$ are the isomorphisms (2.27) corresponding to the top, center, and bottom rows and left, middle, and right columns of the regularized version of the diagram (2.32) described in the proof

Let

$$N_{\mathrm{TM}} = N_{\mathrm{TL}} + N_{\mathrm{TR}}, \quad N_{\mathrm{CL}} = N_{\mathrm{TL}} + N_{\mathrm{BL}}, \quad N_{\mathrm{CR}} = N_{\mathrm{TR}} + N_{\mathrm{BR}}, \quad N_{\mathrm{BM}} = N_{\mathrm{BL}} + N_{\mathrm{BR}},$$
$$N_{\mathrm{CM}} = N_{\mathrm{CL}} + N_{\mathrm{CR}} = N_{\mathrm{TM}} + N_{\mathrm{BM}}.$$

We define $\Theta_{\star *} \colon \mathbb{R}^{N_{\star *}} \longrightarrow Y_{\star *}$ for $(\star, *) \in \{\mathrm{T}, \mathrm{C}, \mathrm{B}\} \times \{\mathrm{L}, \mathrm{M}, \mathrm{R}\} - \{(\mathrm{T}, \mathrm{L})\}$ by

$$\Theta_{\mathrm{TR}} = \mathfrak{j}_{\mathrm{T}} \circ \tilde{\Theta}_{\mathrm{TR}}, \quad \Theta_{\mathrm{BL}} = \mathfrak{j}_{\mathrm{L}} \circ \tilde{\Theta}_{\mathrm{BL}}, \quad \Theta_{\mathrm{BR}} = \mathfrak{j}_{\mathrm{R}} \circ \mathfrak{j}_{\mathrm{C}} \circ \tilde{\Theta}_{\mathrm{BR}} = \mathfrak{j}_{\mathrm{B}} \circ \mathfrak{j}_{\mathrm{M}} \circ \tilde{\Theta}_{\mathrm{BR}},$$
$$\Theta_{\mathrm{TM}}(x_{\mathrm{TL}}, x_{\mathrm{TR}}) = \mathfrak{i}_{\mathrm{T}}\big(\Theta_{\mathrm{TL}}(x_{\mathrm{TL}})\big) + \tilde{\Theta}_{\mathrm{TR}}(x_{\mathrm{TR}}), \quad \Theta_{\mathrm{CL}}(x_{\mathrm{TL}}, x_{\mathrm{BL}}) = \mathfrak{i}_{\mathrm{L}}\big(\Theta_{\mathrm{TL}}(x_{\mathrm{TL}})\big) + \tilde{\Theta}_{\mathrm{BL}}(x_{\mathrm{BL}}),$$
$$\Theta_{\mathrm{CR}}(x_{\mathrm{TR}}, x_{\mathrm{BR}}) = \mathfrak{i}_{\mathrm{R}}\big(\Theta_{\mathrm{TR}}(x_{\mathrm{TR}})\big) + \mathfrak{j}_{\mathrm{C}}(\tilde{\Theta}_{\mathrm{BR}}(x_{\mathrm{BR}})),$$
$$\Theta_{\mathrm{BM}}(x_{\mathrm{BL}}, x_{\mathrm{BR}}) = \mathfrak{i}_{\mathrm{B}}\big(\Theta_{\mathrm{BL}}(x_{\mathrm{BL}})\big) + \mathfrak{j}_{\mathrm{M}}(\tilde{\Theta}_{\mathrm{BR}}(x_{\mathrm{BR}})),$$
$$\Theta_{\mathrm{CM}}(x_{\mathrm{TL}}, x_{\mathrm{TR}}, x_{\mathrm{BL}}, x_{\mathrm{BR}}) = \mathfrak{i}_{\mathrm{M}}(\Theta_{\mathrm{TM}}(x_{\mathrm{TL}}, x_{\mathrm{TR}})) + \mathfrak{i}_{\mathrm{C}}(\tilde{\Theta}_{\mathrm{BL}}(x_{\mathrm{BL}})) + \tilde{\Theta}_{\mathrm{BR}}(x_{\mathrm{BR}})$$

for all $x_{\star *} \in \mathbb{R}^{N_{\star *}}$ with $(\star, *) \in \{\mathrm{T}, \mathrm{B}\} \times \{\mathrm{L}, \mathrm{R}\}$. For any $N' \le N$, we denote by $\mathfrak{i} \colon \mathbb{R}^{N'} \longrightarrow \mathbb{R}^{N}$ and $\mathfrak{j} \colon \mathbb{R}^{N} \longrightarrow \mathbb{R}^{N'}$ the inclusion as $\mathbb{R}^{N'} \oplus 0^{N-N'}$ and the projection onto the last N' coordinates, respectively. We also define

$$\mathfrak{i}' \colon \mathbb{R}^{N_{\mathrm{CL}}} \longrightarrow \mathbb{R}^{N_{\mathrm{CM}}}, \qquad \mathfrak{i}'(x_{\mathrm{TL}}, x_{\mathrm{BL}}) = (x_{\mathrm{TL}}, 0, x_{\mathrm{BL}}, 0),$$
$$\mathfrak{j}' \colon \mathbb{R}^{N_{\mathrm{CM}}} \longrightarrow \mathbb{R}^{N_{\mathrm{CR}}}, \qquad \mathfrak{i}'(x_{\mathrm{TL}}, x_{\mathrm{TR}}, x_{\mathrm{BL}}, x_{\mathrm{BR}}) = (x_{\mathrm{TR}}, x_{\mathrm{BR}}),$$

for all $x_{\star *} \in \mathbb{R}^{N_{\star *}}$. In particular, the diagram in Figure 10 commutes and its 6 rows and 6 columns are exact.

By the commutativity and exactness properties of the diagram in Figure 10, the diagram (2.32) with $D_{\star *}$ replaced by $\tilde{D}_{\star *} \equiv (D_{\star *})_{\Theta_{\star *}}$, $\mathfrak{i}_{\mathrm{C}} \colon X_{\mathrm{CL}} \longrightarrow X_{\mathrm{CM}}$ and $\mathfrak{j}_{\mathrm{C}} \colon X_{\mathrm{CM}} \longrightarrow X_{\mathrm{CR}}$ replaced by

$$\mathfrak{i}_{\mathrm{C}} \oplus \mathfrak{i}' \colon X_{\mathrm{CL}} \oplus \mathbb{R}^{N_{\mathrm{CL}}} \longrightarrow X_{\mathrm{CM}} \oplus \mathbb{R}^{N_{\mathrm{CM}}} \qquad \text{and} \qquad \mathfrak{j}_{\mathrm{C}} \oplus \mathfrak{j}' \colon X_{\mathrm{CM}} \oplus \mathbb{R}^{N_{\mathrm{CM}}} \longrightarrow X_{\mathrm{CR}} \oplus \mathbb{R}^{N_{\mathrm{CR}}},$$

respectively, and the remaining homomorphisms $\mathfrak{i}_{\star}$ and $\mathfrak{j}_{\star}$ on $X_{* \circ}$ by $\mathfrak{i}_{\star} \oplus \mathfrak{i}$ and $\mathfrak{j}_{\star} \oplus \mathfrak{j}$ on $X_{* \circ} \oplus \mathbb{R}^{N_{* \circ}}$, respectively, still commutes and its 3 rows and 3 columns are still exact. Thus, by (4.44), the Normalization II property, and Lemma 4.3, the diagram on the back face of the cube in Figure 9 commutes.

Let $\mathcal{I}_{\star *} = \mathcal{I}_{\Theta_{\star *}; D_{\star *}}$ be the isomorphisms defined by (4.40). By the commutativity of the 3 pairs of exact rows and 3 pairs of exact columns in Figure 10 and Lemma 4.12, the diagrams on the top, right, bottom, and left faces in Figure 9 commute for some $A_{\mathrm{T}}, A_{\mathrm{R}}, A_{\mathrm{B}}, A_{\mathrm{L}} \in \mathbb{R}^*$ determined by $N_{\mathrm{TL}}, N_{\mathrm{TR}}, N_{\mathrm{BL}}, N_{\mathrm{BR}}$ and the indices of the Fredholm operators $D_{\star *}$. By Lemma 4.3,

$$A_{\mathrm{T}} A_{\mathrm{R}} = A_{\mathrm{B}} A_{\mathrm{L}} \tag{4.45}$$

if $\mathfrak{c}(D_{\star *}) = \{0\}$ for $(\star, *) \in \{\mathrm{T}, \mathrm{B}\} \times \{\mathrm{L}, \mathrm{R}\}$. Thus, (4.45) always holds, which establishes Corollary 4.13 in all cases. $\qquad\square$

Remark 4.14. For the collection of exact triple isomorphisms (2.27) given by (4.10), $A_{\star} = (-1)^{\epsilon_{\star}}$ with

$$\epsilon_{\mathrm{T}} = N_{\mathrm{TR}} N_{\mathrm{BL}} + (\mathrm{ind}\, D_{\mathrm{CL}}) N_{\mathrm{TR}} + (\mathrm{ind}\, D_{\mathrm{TR}}) N_{\mathrm{BL}} + (\mathrm{ind}\, D_{\mathrm{BL}}) N_{\mathrm{BR}}, \quad \epsilon_{\mathrm{R}} = (\mathrm{ind}\, D_{\mathrm{TM}}) N_{\mathrm{BM}},$$
$$\epsilon_{\mathrm{B}} = N_{\mathrm{TR}} N_{\mathrm{BL}} + (\mathrm{ind}\, D_{\mathrm{CL}}) N_{\mathrm{CR}}, \quad \epsilon_{\mathrm{L}} = (\mathrm{ind}\, D_{\mathrm{TL}}) N_{\mathrm{BL}} + (\mathrm{ind}\, D_{\mathrm{TR}}) N_{\mathrm{BR}}.$$

In this case, it can be checked directly that $\epsilon_{\mathrm{T}} + \epsilon_{\mathrm{R}} + \epsilon_{\mathrm{B}} + \epsilon_{\mathrm{L}} \in 2\mathbb{Z}$.
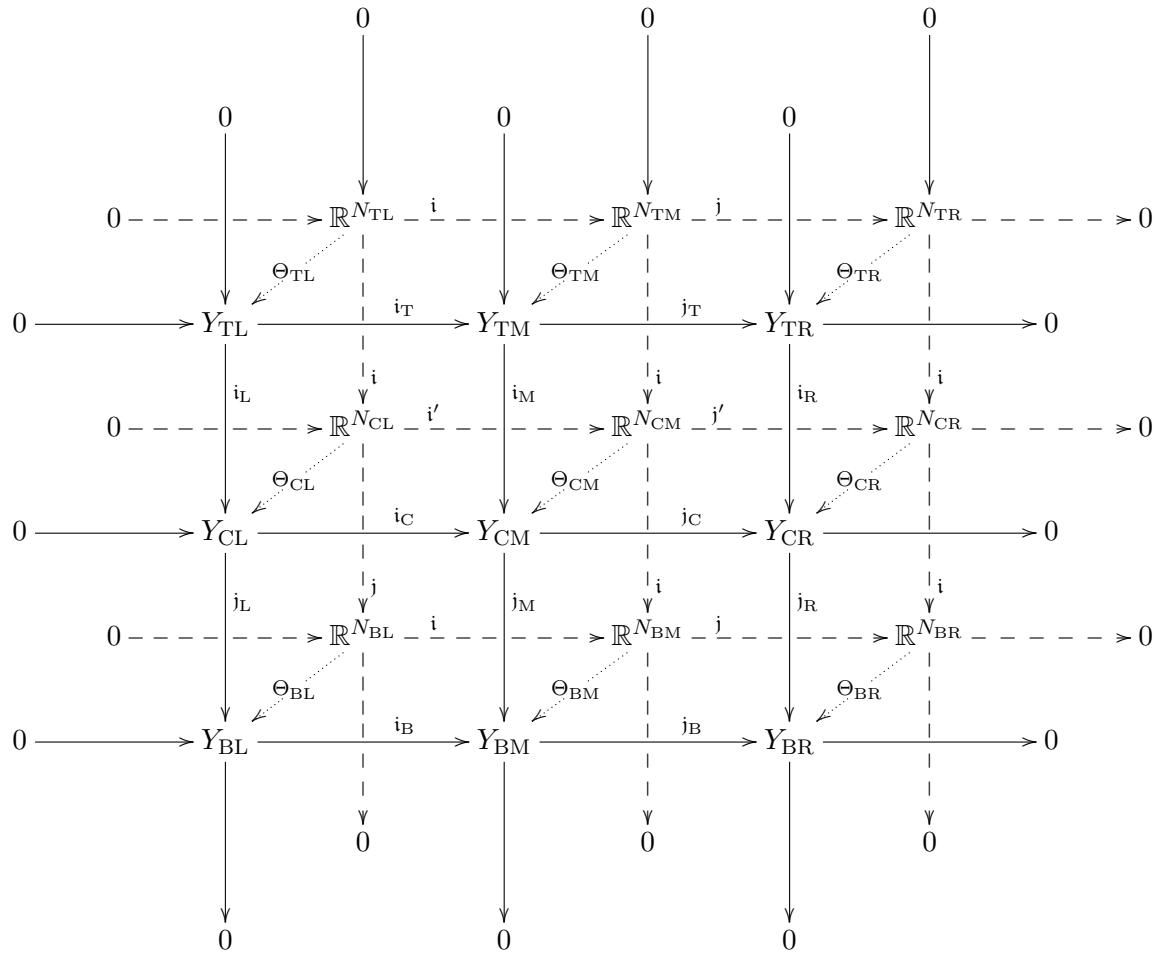
Figure 10: The panel of commutative diagrams, with exact rows and columns, used in the proof of Corollary 4.13 to regularize the square grid (2.32)

Remark 4.15. For the collection of exact triple isomorphisms (2.27) given by (4.10), $A_N = 1$ in Lemma 4.11. Thus, $A_N = A_{-N,N}$ in the case of the collection of isomorphisms (2.27) of Theorem 2 on page 35 corresponding to the collection $(A_{i,c})_{i,c}$.

5 Topology

It remains to topologize each set $\det_{X,Y}$ as a line bundle over $\mathcal{F}(X,Y)$ with good properties. By Proposition 5.3, the Normalization I property on page 12 and the isomorphisms (3.2) determine a topology on $\det_{X,Y}$ if the collection of exact triple isomorphisms $\Psi_{\mathfrak{t}}$ as in (2.27) satisfies the Normalization II,III and Compositions I,II properties. By Corollary 5.4, all isomorphisms $\Psi_{\mathfrak{t}}$ are continuous with respect to these topologies if this collection also satisfies the Naturality II property. By Corollary 5.8, a collection of dualization isomorphisms as in (2.36) satisfying the Normalization IV* and Dual Exact Triples properties consists of continuous isomorphisms with respect to these topologies.

5.1 Continuity of overlap and exact triple maps

For Banach vector spaces X, Y, X', Y', X'', Y'', let

$$\mathcal{T}^*(X,Y;X',Y';X'',Y'') \subset \mathcal{T}(X,Y;X',Y';X'',Y'')$$

denote the subspace of short exact sequences as in (2.6) with surjective Fredholm operators D, D', D''.

Lemma 5.1. *Let X, Y, X', Y', X'', Y'' be Banach vector spaces. A family of exact triple isomorphisms $\Psi_{\mathfrak{t}}$ as in (2.27) satisfying the Normalization II property induces a continuous bundle map*

$$\Psi \colon \pi_{\mathrm{L}}^* \det_{X',Y'} \otimes \pi_{\mathrm{R}}^* \det_{X'',Y''} \longrightarrow \pi_{\mathrm{C}}^* \det_{X,Y}$$

over $\mathcal{T}^*(X,Y;X',Y';X'',Y'')$ with respect to the topologies determined by the Normalization I property.

Proof. We abbreviate $\mathcal{T}^*(X,Y;X',Y';X'',Y'')$ as $\mathcal{T}^*$. Let $\mathfrak{t}_0 \in \mathcal{T}^*$ be as in (2.6), with all seven homomorphisms carrying subscript 0, and $T \colon Y \longrightarrow X$, $T' \colon Y' \longrightarrow X'$, and $T'' \colon Y'' \longrightarrow X''$ be right inverses for D_0, D_0', and D_0'', respectively. For each $\mathfrak{t}$ as in (2.6) and $\star = ', ''$ or blank, let

$$\Phi_{D_0^\star;\mathfrak{t}} \colon \kappa(D^\star) \longrightarrow \kappa(D_0^\star), \qquad \Phi_{D_0^\star;\mathfrak{t}}(x) = x + T^\star\big(\{D^\star - D_0^\star\}(x)\big),$$

be as in (2.25); this map is an isomorphism if $\mathfrak{t}$ is sufficiently close to $\mathfrak{t}_0$. We need to show that the map

$$\Psi_{\mathfrak{t}_0;\mathfrak{t}} \colon \lambda(\kappa(D_0')) \otimes \lambda(\kappa(D_0'')) \longrightarrow \lambda(\kappa(D_0)) \tag{5.1}$$

described by

$$\Psi_{\mathfrak{t}}\big(\lambda(\Phi_{D_0';\mathfrak{t}}^{-1})x' \otimes 1^* \otimes \lambda(\Phi_{D_0'';\mathfrak{t}}^{-1})x'' \otimes 1^*\big) = \big(\lambda(\Phi_{D_0;\mathfrak{t}}^{-1})\Psi_{\mathfrak{t}_0;\mathfrak{t}}(x' \otimes x'')\big) \otimes 1^*$$

depends continuously on $\mathfrak{t} \in \mathcal{T}^*$ near $\mathfrak{t}_0$.

Let $x'_1, \ldots, x'_k$ be a basis for $\kappa(D'_0)$ and $x_1, \ldots, x_\ell \in \kappa(D_0)$ be such that $\mathsf{j}_{X;0}(x_1), \ldots, \mathsf{j}_{X;0}(x_\ell)$ is a basis for $\kappa(D''_0)$. If $\mathfrak{t}$ as in (2.6) is sufficiently close to $\mathfrak{t}_0$, then $\Phi^{-1}_{D'_0;\mathfrak{t}}(x'_1), \ldots, \Phi^{-1}_{D'_0;\mathfrak{t}}(x'_k)$ is a basis for $\kappa(D')$ and

$$\mathsf{j}_X\big(\Phi^{-1}_{D_0;\mathfrak{t}}(x_1)\big), \ldots, \mathsf{j}_X\big(\Phi^{-1}_{D_0;\mathfrak{t}}(x_\ell)\big) \in X''$$

is a basis for $\kappa(D'')$. In particular,

$$\mathsf{j}_X\big(\Phi^{-1}_{D_0;\mathfrak{t}}(x_1)\big) \wedge \ldots \wedge \mathsf{j}_X\big(\Phi^{-1}_{D_0;\mathfrak{t}}(x_\ell)\big) = f(\mathfrak{t})\, \Phi^{-1}_{D''_0;\mathfrak{t}}\big(\mathsf{j}_{X;0}(x_1)\big) \wedge \ldots \wedge \Phi^{-1}_{D''_0;\mathfrak{t}}\big(\mathsf{j}_{X;0}(x_\ell)\big),$$
$$\mathsf{i}_X\big(\Phi^{-1}_{D'_0;\mathfrak{t}}(x'_1)\big) \wedge \ldots \wedge \mathsf{i}_X\big(\Phi^{-1}_{D'_0;\mathfrak{t}}(x'_k)\big) \wedge \Phi^{-1}_{D_0;\mathfrak{t}}(x_1) \wedge \ldots \wedge \Phi^{-1}_{D_0;\mathfrak{t}}(x_\ell)$$
$$= g(\mathfrak{t})\, \Phi^{-1}_{D_0;\mathfrak{t}}\big(\mathsf{i}_{X;0}(x'_1)\big) \wedge \ldots \wedge \Phi^{-1}_{D_0;\mathfrak{t}}\big(\mathsf{i}_{X;0}(x'_k)\big) \wedge \Phi^{-1}_{D_0;\mathfrak{t}}(x_1) \wedge \ldots \wedge \Phi^{-1}_{D_0;\mathfrak{t}}(x_\ell)$$

for some $\mathbb{R}^+$-valued continuous functions f and g. By the Normalization II property, the homomorphism (5.1) is then given by

$$\Psi_{\mathfrak{t}_0;\mathfrak{t}}\big((x'_1 \wedge \ldots \wedge x'_k) \otimes (\mathsf{j}_{X;0}(x_1) \wedge \ldots \wedge \mathsf{j}_{X;0}(x_\ell))\big) = \frac{g(\mathfrak{t})}{f(\mathfrak{t})}\, \mathsf{i}_{X;0}(x'_1) \wedge \ldots \wedge \mathsf{i}_{X;0}(x'_k) \wedge x_1 \wedge \ldots \wedge x_\ell$$

and thus depends continuously on $\mathfrak{t}$. $\qquad\square$

Corollary 5.2. *Let $(\Psi_\mathfrak{t})_\mathfrak{t}$ be a collection of exact triple isomorphisms as in (2.27) satisfying the Normalization II,III and Compositions II properties. For any Banach vector spaces X and Y and homomorphism $\Theta \colon \mathbb{R}^N \longrightarrow Y$, the induced map*

$$\mathcal{I}_\Theta \colon \iota_\Theta^* \det{}_{X \oplus \mathbb{R}^N, Y} \longrightarrow \det{}_{X,Y}$$

as in (4.40) is continuous over $\mathcal{F}^*(X, Y)$ with respect to the topologies determined by the Normalization I property.

Proof. By Lemma 4.11, $\mathcal{I}_{\Theta;D} = (-1)^{(\mathrm{ind}\, D)N} A_N \hat{\mathcal{I}}^{-1}_{\Theta;D}$, with $\hat{\mathcal{I}}_{\Theta;D}$ given by (3.2). By Lemma 5.1, the family of isomorphisms $\hat{\mathcal{I}}_{\Theta;D}$ induce a continuous bundle map

$$\hat{\mathcal{I}}_\Theta \colon \det{}_{X,Y} \longrightarrow \iota_\Theta^* \det{}_{X \oplus \mathbb{R}^N, Y}$$

over $\mathcal{F}^*(X, Y)$. This implies the claim. $\qquad\square$

Let X and Y be as above. The subsets

$$U_{X;\Theta} \equiv \big\{ D \in \mathcal{F}(X, Y) \colon \mathfrak{c}(D_\Theta) = 0 \big\}$$

form an open cover of $\mathcal{F}(X, Y)$ as Θ ranges over all homomorphisms $\mathbb{R}^N \longrightarrow Y$ and N ranges over all nonnegative integers. We topologize $\det_{X,Y}|_{U_{X;\Theta}}$ by requiring the bundle isomorphism

$$\iota_\Theta^* \det{}_{X \oplus \mathbb{R}^N, Y} \longrightarrow \det{}_{X,Y}, \qquad \sigma \longrightarrow \mathcal{I}_{\Theta;D}(\sigma) \ \ \forall\, \sigma \in \lambda(D_\Theta), \ D \in \mathcal{F}(X, Y),$$

to be a homeomorphism over $U_{X;\Theta}$ with respect to the topology on the domain induced by the topology on $\det_{X \oplus \mathbb{R}^N, Y}|_{\mathcal{F}^*(X \oplus \mathbb{R}^N, Y)}$ described at the beginning of this section. We next show that this topology is well-defined.

Proposition 5.3 (Continuity of transition maps)**.** *Let* $(\Psi_t)_t$ *be a collection of exact triple isomorphisms as in (2.27) satisfying the Normalization II,III and Compositions I,II properties. For any Banach vector spaces X and Y and homomorphisms $\Theta_i\colon \mathbb{R}^{N_i}\longrightarrow Y$ with $i=1,2$, the induced overlap map*

$$\mathcal{I}_{\Theta_2;D}^{-1}\circ\mathcal{I}_{\Theta_1;D}\colon \iota_{\Theta_1}^*\mathrm{det}_{X\oplus\mathbb{R}^{N_1},Y}\longrightarrow \iota_{\Theta_2}^*\mathrm{det}_{X\oplus\mathbb{R}^{N_2},Y}$$

is continuous over $U_{X;\Theta_1}\cap U_{X;\Theta_2}$ with respect to the topologies determined by the Normalization I property.

Proof. With $N\equiv N_1+N_2$, let

$$0 \longrightarrow \mathbb{R}^{N_1} \xrightarrow{\iota_1} \mathbb{R}^N = \mathbb{R}^{N_1}\oplus\mathbb{R}^{N_2} \xrightarrow{\pi_{\mathrm{R};N_2}} \mathbb{R}^{N_2} \longrightarrow 0 \qquad \text{and}$$

$$0 \longrightarrow \mathbb{R}^{N_2} \xrightarrow{\iota_2} \mathbb{R}^N = \mathbb{R}^{N_1}\oplus\mathbb{R}^{N_2} \xrightarrow{\pi_{\mathrm{L};N_1}} \mathbb{R}^{N_1} \longrightarrow 0$$

be the natural exact sequences of vector spaces and

$$\iota_{k;X} = \mathrm{id}_X\oplus\iota_k\colon\ X\oplus\mathbb{R}^{N_k} \longrightarrow X\oplus\mathbb{R}^N$$

for $k=1,2$. Define

$$\Theta\colon \mathbb{R}^N \longrightarrow Y \qquad \text{by} \quad \Theta(u_1,u_2) = \Theta_1(u_1) + \Theta_2(u_2).$$

Thus, the diagram

of Fredholm operators commutes.

By the Commutativity I property, the diagram

$$(5.2)$$

also commutes (excluding the dotted arrows). We define the isomorphisms A, A_1, A_2 in this diagram by

$$A(\sigma) = 1 \otimes (\Omega_N^* \circ \lambda(\pi_2)) \otimes \sigma, \qquad A_k(\sigma_k) = 1 \otimes (\Omega_{N_k}^* \circ \lambda(\pi_2)) \otimes \sigma_k, \tag{5.3}$$

where $\pi_2 : \mathfrak{c}(\iota_{X;N_k}) \longrightarrow \mathbb{R}^{N_k}$ is the isomorphism induced by the projection $X \oplus \mathbb{R}^{N_k} \longrightarrow \mathbb{R}^{N_k}$. By (4.40),

$$\mathcal{I}_{\Theta;D} = \widetilde{\mathcal{C}}_{\iota_{X;N}, D_\Theta} \circ A, \quad \mathcal{I}_{\Theta_k;D} = \widetilde{\mathcal{C}}_{\iota_{X;N_k}, D_{\Theta_k}} \circ A_k, \quad k = 1, 2. \tag{5.4}$$

Let $R \colon X \oplus \mathbb{R}^N \longrightarrow X \oplus \mathbb{R}^N$ be the isomorphism given by

$$R(x, u_1, u_2) = (x, u_2, u_1) \qquad \forall\, (x, u_1, u_2) \in X \oplus \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$$

and

$$\widetilde{\mathcal{I}}_{R;D} \equiv \widetilde{\mathcal{I}}_{R, \mathrm{id}_Y; D_\Theta} : \lambda(D_\Theta) \longrightarrow \lambda(D_\Theta \circ R^{-1}) = \lambda\big(\mathcal{I}_{R, \mathrm{id}_Y}(D_\Theta)\big),$$

$$\widetilde{\mathcal{I}}_{R;N_1} \equiv \widetilde{\mathcal{I}}_{\mathrm{id}_{X \oplus \mathbb{R}^{N_2}}, R^{-1}; \iota_{X \oplus \mathbb{R}^{N_2}; N_1}} : \lambda(\iota_{X \oplus \mathbb{R}^{N_2}; N_1}) \longrightarrow \lambda\big(R^{-1} \circ \iota_{X \oplus \mathbb{R}^{N_2}; N_1}\big) = \lambda(\iota_{2;X})$$

be the corresponding isomorphisms as in (2.5).

Let $C_1, C_2 \in \mathbb{R}^*$ be such that

$$\widetilde{\mathcal{C}}_{\iota_{0;N_1}, \iota_1}\big(1 \otimes \Omega_{N_1}^* \otimes 1 \otimes (\Omega_{N_2}^* \circ \lambda(\pi_{R;N_2}))\big) = C_1 1 \otimes \Omega_N^*,$$

$$\widetilde{\mathcal{C}}_{\iota_{0;N_2}, \iota_2}\big(1 \otimes \Omega_{N_2}^* \otimes 1 \otimes (\Omega_{N_1}^* \circ \lambda(\pi_{L;N_1}))\big) = C_2 1 \otimes \Omega_N^*,$$

where $\pi_{R;N_2} : \mathfrak{c}(\iota_1) \longrightarrow \mathbb{R}^{N_2}$ and $\pi_{L;N_1} : \mathfrak{c}(\iota_2) \longrightarrow \mathbb{R}^{N_1}$ are are the isomorphisms induced by the projections $\pi_{R;N_2}$ and $\pi_{L;N_1}$. By the Compositions II and Normalization III properties applied to the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ \mathrm{id}\ } & X & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota_{X;N_1}} & & \downarrow{\scriptstyle \iota_{0;N_1}} & & \\
0 & \longrightarrow & X & \xrightarrow{\ \iota_{X;N_1}\ } & X \oplus \mathbb{R}^{N_1} & \xrightarrow{\ \pi_2\ } & \mathbb{R}^{N_1} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota_{1;X}} & & \downarrow{\scriptstyle \iota_1} & & \\
0 & \longrightarrow & X & \xrightarrow{\ \iota_{X;N}\ } & X \oplus \mathbb{R}^N & \xrightarrow{\ \pi_2\ } & \mathbb{R}^N & \longrightarrow & 0 \, ,
\end{array}
$$

the diagram

$$
\begin{array}{ccc}
\lambda(\iota_{0;N_1}) \otimes \lambda(\iota_1) & \xrightarrow{\ \widetilde{\mathcal{I}}_{\pi_2, \pi_2; \iota_{X;N_1}}^{-1} \otimes \widetilde{\mathcal{I}}_{\pi_2, \pi_2; \iota_{1;X}}^{-1}\ } & \lambda(\iota_{X;N_1}) \otimes \lambda(\iota_{1;X}) \\
\downarrow{\scriptstyle \widetilde{\mathcal{C}}_{\iota_{0;N_1}, \iota_1}} & & \downarrow{\scriptstyle \widetilde{\mathcal{C}}_{\iota_{X;N_1}, \iota_{1:X}}} \\
\lambda(\iota_{0;N}) & \xrightarrow{\ \widetilde{\mathcal{I}}_{\pi_2, \pi_2; \iota_{X;N}}^{-1}\ } & \lambda(\iota_{X;N})
\end{array}
$$

commutes. Thus,

$$\widetilde{\mathcal{C}}_{\iota_{X;N_1}, \iota_{1;X}}\big(1 \otimes (\Omega_{N_1}^* \circ \lambda(\pi_2)) \otimes 1 \otimes (\Omega_{N_2}^* \circ \lambda(\pi_{R;N_2;X}))\big) = C_1 1 \otimes (\Omega_N^* \circ \lambda(\pi_2)), \tag{5.5}$$

where $\pi_{R;N_2;X} : \mathfrak{c}(\iota_{1;X}) \longrightarrow \mathbb{R}^{N_2}$ is the isomorphism induced by the projection $X \oplus \mathbb{R}^N \longrightarrow \mathbb{R}^{N_2}$ onto the last N_2 Euclidean coordinates. Similarly,

$$\widetilde{\mathcal{C}}_{\iota_{X;N_2}, \iota_{2;X}}\big(1 \otimes (\Omega_{N_2}^* \circ \lambda(\pi_2)) \otimes 1 \otimes (\Omega_{N_1}^* \circ \lambda(\pi_{L;N_1;X}))\big) = C_2 1 \otimes (\Omega_N^* \circ \lambda(\pi_2)), \tag{5.6}$$

where $\pi_{\mathrm{L};N_1;X}\colon \mathfrak{c}(\iota_{2;X})\longrightarrow \mathbb{R}^{N_1}$ is the isomorphism induced by the projection $X\oplus\mathbb{R}^N\longrightarrow\mathbb{R}^{N_1}$ onto the first N_1 Euclidean coordinates.

Since $(D_{\Theta_1})_{\Theta_2}=D_\Theta$ and $\iota_{X\oplus\mathbb{R}^{N_1};N_2}=\iota_{1;X}$,

$$\big\{\mathrm{id}\otimes\widetilde{\mathcal{C}}_{\iota_{1;X};,D_\Theta}\big\}^{-1}\big(A_1(\mathcal{I}_{\Theta_2;D_{\Theta_1}}(\sigma)))\big)$$
$$= \big\{\mathrm{id}\otimes\widetilde{\mathcal{C}}_{\iota_{1;X};,D_\Theta}\big\}^{-1}\big(1\otimes(\Omega_{N_1}^*\circ\lambda(\pi_2))\otimes\widetilde{\mathcal{C}}_{\iota_{1;X};,D_\Theta}(1\otimes(\Omega_{N_2}^*\circ\lambda(\pi_{\mathrm{R};N_2}))\otimes\sigma)\big)$$
$$= 1\otimes(\Omega_{N_1}^*\circ\lambda(\pi_2))\otimes 1\otimes(\Omega_{N_2}^*\circ\lambda(\pi_{\mathrm{R};N_2}))\otimes\sigma.$$

Combining this with (5.4), the commutativity of the upper rectangle in (5.2), and (5.5), we obtain

$$\mathcal{I}_{\Theta;D}^{-1}\circ\mathcal{I}_{\Theta_1;D} = C_1\,\mathcal{I}_{\Theta_2;D_{\Theta_1}}^{-1}. \tag{5.7}$$

On the other hand, $(D_{\Theta_2})_{\Theta_1}=D_\Theta\circ R^{-1}$ and $\iota_{X\oplus\mathbb{R}^{N_2};N_1}=R\circ\iota_{2;X}$. By the Composition I property applied to $D_\Theta\circ R^{-1}\circ\iota_{X\oplus\mathbb{R}^{N_2};N_1}$ and the Normalization III property, the diagram

$$\begin{array}{ccc}
\lambda(\iota_{X\oplus\mathbb{R}^{N_2};N_1})\otimes\lambda(D_\Theta) & \xrightarrow{\ \widetilde{\mathcal{I}}_{R;N_1}\otimes\mathrm{id}\ } & \lambda(\iota_{2;X})\otimes\lambda(D_\Theta) \\
\Big\downarrow{\scriptstyle\mathrm{id}\otimes\widetilde{\mathcal{I}}_{R;D}} & & \Big\downarrow{\scriptstyle\widetilde{\mathcal{C}}_{\iota_{2;X},D_\Theta}} \\
\lambda(\iota_{X\oplus\mathbb{R}^{N_2};N_1})\otimes\lambda(D_\Theta\circ R^{-1}) & \xrightarrow[\ \widetilde{\mathcal{C}}_{\iota_{X\oplus\mathbb{R}^{N_2};N_1},D_\Theta\circ R^{-1}}\ ]{} & \lambda(D_{\Theta_2})
\end{array}$$

commutes. Since

$$\widetilde{\mathcal{I}}_{R;N_1}\big(1\otimes(\Omega_{N_1}^*\circ\lambda(\pi_{\mathrm{R};N_1}))\big) = 1\otimes(\Omega_{N_1}^*\circ\lambda(\pi_{\mathrm{L};N_1})),$$

the last commutative diagram gives

$$\big\{\mathrm{id}\otimes\widetilde{\mathcal{C}}_{\iota_{2;X};,D_\Theta}\big\}^{-1}\big(A_2\big(\mathcal{I}_{\Theta_1;D_{\Theta_2}}(\widetilde{\mathcal{I}}_{R;D}(\sigma))\big)\big) = 1\otimes(\Omega_{N_2}^*\circ\lambda(\pi_2))\otimes 1\otimes(\Omega_{N_1}^*\circ\lambda(\pi_{\mathrm{L};N_1}))\otimes\sigma.$$

Combining this with (5.4), the commutativity of the lower rectangle in (5.2), and (5.6), we obtain

$$\mathcal{I}_{\Theta;D}^{-1}\circ\mathcal{I}_{\Theta_2;D} = C_2\,\widetilde{\mathcal{I}}_{R;D}^{-1}\circ\mathcal{I}_{\Theta_1;D_{\Theta_2}}^{-1}. \tag{5.8}$$

From (5.7) and (5.8), we conclude that

$$\mathcal{I}_{\Theta_2;D}^{-1}\circ\mathcal{I}_{\Theta_1;D} = (C_1/C_2)\,\mathcal{I}_{\Theta_1;D_{\Theta_2}}\circ\widetilde{\mathcal{I}}_{R;D}\circ\mathcal{I}_{\Theta_2;D_{\Theta_1}}^{-1}.$$

The outer maps on the right-hand side above are continuous over $U_{X;\Theta_1}\cap U_{X;\Theta_2}$ by Corollary 5.2, while the middle map is continuous over $U_{X;\Theta_1}\cap U_{X;\Theta_2}$ by Lemma 5.1. $\qquad\square$

Corollary 5.4 (Continuity of (2.27)). *Let* $(\Psi_\mathfrak{t})_\mathfrak{t}$ *be a collection of exact triple isomorphisms as in (2.27) satisfying the Naturality II, Normalization II,III, and Compositions I,II properties. For any Banach vector spaces* X,Y,X',Y',X'',Y'', *the bundle map*

$$\Psi\colon \pi_{\mathrm{L}}^*\mathrm{det}_{X',Y'}\otimes\pi_{\mathrm{R}}^*\mathrm{det}_{X'',Y''}\longrightarrow\pi_{\mathrm{C}}^*\mathrm{det}_{X,Y}$$

over $\mathcal{T}(X,Y;X',Y';X'',Y'')$ *is continuous with respect to the topologies provided by Proposition 5.3.*

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Proof. We abbreviate $\mathcal{T}(X,Y;X',Y';X'',Y'')$ as $\mathcal{T}$. Let $\mathfrak{t}_0 \in \mathcal{T}$ be as in (2.6), with all seven homomorphisms carrying subscript 0, and

$$\Theta'\colon \mathbb{R}^{N'} \longrightarrow Y' \qquad \text{and} \qquad \tilde{\Theta}''\colon \mathbb{R}^{N''} \longrightarrow Y$$

be homomorphisms such that $D'_0 \in U_{X';\Theta'}$ and $D''_0 \in U_{X'';\mathfrak{j}_{Y;0}\circ\tilde{\Theta}''}$. Let $N = N'+N''$, $\mathfrak{i}\colon \mathbb{R}^{N'} \longrightarrow \mathbb{R}^N$ be the inclusion as $\mathbb{R}^{N'}\times 0^{N''}$, and $\mathfrak{j}\colon \mathbb{R}^N \longrightarrow \mathbb{R}^{N''}$ be the projection onto the last N'' coordinates, and $A_{i,j}\in\mathbb{R}^*$ be as in (4.42). For each $\mathfrak{t}\in\mathcal{T}$ as in (2.6), define

$$\begin{aligned}
\Theta_{\mathfrak{t}}\colon \mathbb{R}^N \longrightarrow X, &\qquad \Theta_{\mathfrak{t}}(x',x'') = \mathfrak{i}_Y\big(\Theta'(x')\big) + \tilde{\Theta}''(x'') &&\forall\,(x',x'') \in \mathbb{R}^{N'}\oplus\mathbb{R}^{N''}, \\
\Theta''_{\mathfrak{t}}\colon \mathbb{R}^{N''} \longrightarrow X'', &\qquad \Theta''_{\mathfrak{t}}(x'') = \mathfrak{j}_Y\big(\tilde{\Theta}''(x'')\big) &&\forall\,x'' \in \mathbb{R}^{N''}.
\end{aligned}$$

Thus, the diagram $\mathfrak{s}(\mathfrak{t})$ given by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}^{N'} & \xrightarrow{\;\mathfrak{i}\;} & \mathbb{R}^N & \xrightarrow{\;\mathfrak{j}\;} & \mathbb{R}^{N''} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\Theta'} & & \downarrow{\scriptstyle\Theta_{\mathfrak{t}}} & & \downarrow{\scriptstyle\Theta''_{\mathfrak{t}}} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\;\mathfrak{i}_Y\;} & Y & \xrightarrow{\;\mathfrak{j}_Y\;} & Y' & \longrightarrow & 0
\end{array}$$

commutes for every exact triple $\mathfrak{t}$ as in (2.6), and we obtain an embedding

$$\mathcal{T} \longrightarrow \mathcal{T}(X\oplus\mathbb{R}^N,Y;X'\oplus\mathbb{R}^{N'},Y';X''\oplus\mathbb{R}^{N''},Y''), \qquad \mathfrak{t} \longrightarrow \mathfrak{t}_{\mathfrak{s}(\mathfrak{t})}.$$

By Lemma 4.12, the diagram

$$\begin{array}{ccc}
\lambda(D'_{\Theta'}) \otimes \lambda(D''_{\Theta''_{\mathfrak{t}}}) & \xrightarrow{\quad\Psi_{\mathfrak{t}_{\mathfrak{s}(\mathfrak{t})}}\quad} & \lambda(D_{\Theta_{\mathfrak{t}}}) \\
{\scriptstyle\mathcal{I}_{\Theta';D'}\otimes\mathcal{I}_{\Theta''_{\mathfrak{t}};D''}}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathcal{I}_{\Theta_{\mathfrak{t}};D}} \\
\lambda(D') \otimes \lambda(D'') & \xrightarrow{(-1)^{(\operatorname{ind} D')N''}A_{i,j}\Psi_{\mathfrak{t}}} & \lambda(D)
\end{array}$$

commutes. By the definition of the topologies on the determinant line bundles, the vertical arrows in the above diagram induce continuous line-bundle isomorphisms over the open subset of $\mathcal{T}$ consisting of the exact triples $\mathfrak{t}$ as in (2.6) so that $D'\in U_{X';\Theta'}$ and $D''\in U_{X'';\Theta''_{\mathfrak{t}}}$. By Lemma 5.1, the top arrow induces a continuous line-bundle isomorphism over the same open subset. Thus, the bottom arrow in this diagram induces a continuous line-bundle isomorphism as well. $\qquad\square$

Remark 5.5. For the collection of exact triple isomorphisms (2.27) given by (4.10), $C_1 = (-1)^{N_1 N_2}$ and $C_2 = 1$ in the proof of Proposition 5.3, as can be seen from (4.22).

5.2   Continuity of dualization isomorphisms

We begin by verifying the Normalization I$'$ property on page 17; see Lemma 5.6. This allows us to confirm the continuity of (2.38) over $\mathcal{F}^*(X,Y)$; see Lemma 5.7. The continuity of (2.36) over $\mathcal{F}(X,Y)$ then follows from the Dual Exact Triples property on page 19; see the proof of Corollary 5.8. For each $D\in\mathcal{F}(X,Y)$, let

$$q_D\colon Y \longrightarrow \mathfrak{c}(D), \qquad q_D(y) = y + \operatorname{Im} D,$$

be the projection map as before.

Lemma 5.6 (Normalization I′). *Let $(\Psi_t)_t$ be a collection of exact triple isomorphisms as in (2.27) satisfying the Naturality II, Normalization II,III, and Compositions I,II properties. For any Banach vector spaces X and Y, $D_0 \in \mathcal{F}'(X,Y)$, and right inverse $S \colon \mathfrak{c}(D_0) \longrightarrow Y$ for q_{D_0}, there exists a neighborhood $U_{D_0,S}$ of D_0 in $\mathcal{F}'(X,Y)$ so that the bundle isomorphism (2.35) is well-defined and continuous with respect to the topology provided by Proposition 5.3.*

Proof. By the Open Mapping Theorem for Banach vector spaces,

$$U_{D_0,S} \equiv \big\{ D \in \mathcal{F}'(X,Y) \colon Y = \operatorname{Im} D \oplus \operatorname{Im} S \big\}$$

is an open neighborhood of D_0 in $\mathcal{F}'(X,Y)$. Let

$$\pi_X, \pi_S \colon Y = \operatorname{Im} D_0 \oplus \operatorname{Im} S \longrightarrow \operatorname{Im} D_0,\ \operatorname{Im} S$$

be the projection maps and $D_0^{-1} \colon \operatorname{Im} D_0 \longrightarrow X$ be the inverse of the isomorphism

$$D_0 \colon X \longrightarrow \operatorname{Im} D_0, \qquad x \longrightarrow D_0 x.$$

For each $D \in U_{D_0,S}$, the map

$$\psi_{D_0;D} \colon Y \longrightarrow Y, \qquad \psi_{D_0;D}(y) = D\big(D_0^{-1}(\pi_X(y))\big) + \pi_S(y),$$

is an isomorphism so that

$$D = \psi_{D_0;D} \circ D_0 \circ \operatorname{id}_X^{-1} \qquad \text{and} \qquad \psi_{D_0;D}(y) - S(q_{D_0}(y)) \in \operatorname{Im} D \quad \forall\, y \in Y.$$

By the last property,

$$\widetilde{\mathcal{I}}_{D_0,S;D} = \widetilde{\mathcal{I}}_{\operatorname{id}_X,\psi_{D_0;D}} \colon \lambda(D_0) \longrightarrow \lambda(D).$$

Since $\psi_{D_0;D}$ depends continuously on D, the claim follows from the continuity of (2.27) provided by Corollary 5.4, along with the Normalization III property. $\qquad\square$

Lemma 5.7. *Let $(\Psi_t)_t$ be a collection of exact triple isomorphisms as in (2.27) satisfying the Naturality II, Normalization II,III, and Compositions I,II properties. For any Banach vector spaces X and Y, the family of maps $\widetilde{\mathcal{D}}_D$ given by (2.38) induces a continuous bundle map*

$$\widetilde{\mathcal{D}} \colon \det{}_{X,Y} \longrightarrow \mathcal{D}^* \det{}_{Y^*,X^*}$$

over $\mathcal{F}^*(X,Y)$ with respect to the topologies provided by Proposition 5.3.

Proof. Let $D_0 \in \mathcal{F}^*(X,Y)$, $T \colon Y \longrightarrow X$ be a right inverse for D_0, and

$$\pi_T \colon X = \kappa(D_0) \oplus \operatorname{Im} T \longrightarrow \kappa(D_0), \qquad x \longrightarrow x - T D_0 x \quad \forall\, x \in X,$$

be the projection map. Thus, the homomorphism

$$S \colon \mathfrak{c}(D_0^*) \longrightarrow X^*, \qquad \alpha + \operatorname{Im} D_0^* \longrightarrow \alpha|_{\kappa(D_0)} \circ \pi_T,$$

is a right inverse for $q_{D_0^*}$. By the Normalization I property on page 12 and Lemma 5.6, it is sufficient to show that the map

$$\widetilde{\mathcal{I}}_{D_0^*,S;D^*}^{-1} \circ \widetilde{\mathcal{D}}_D \circ \widetilde{\mathcal{I}}_{D_0,T;D} \colon \lambda(D_0) \longrightarrow \lambda(D) \longrightarrow \lambda(D^*) \longrightarrow \lambda(D_0^*)$$

depends continuously on $D \in U_{D_0,S}$. By (2.25), (2.38), and (2.34), this map is given by

$$x \otimes 1^* \longrightarrow 1 \otimes \mathcal{P}\big(\lambda(\mathcal{D}_{D_0})x\big),$$

which establishes the claim. $\qquad\square$

Corollary 5.8 (Continuity of (2.36))**.** *Let $(\Psi_t)_t$ be a collection of exact triple isomorphisms as in (2.27) satisfying the Naturality II, Normalization II,III, and Compositions I,II properties. If a collection of bundle maps*

$$\widetilde{\mathcal{D}} \colon \det{}_{X,Y} \longrightarrow \mathcal{D}^*\det{}_{Y^*,X^*}$$

over $\mathcal{F}(X,Y)$ satisfies the Normalization IV* and Dual Exact Triples properties, then the maps in this collection are continuous with respect to the topologies provided by Proposition 5.3.

Proof. Let X, Y be Banach vector spaces. Let $D \in \mathcal{F}(X,Y)$ and $\Theta \colon \mathbb{R}^N \longrightarrow Y$ be a homomorphism so that $D \in U_{X;\Theta}$. By the Dual Exact Triples property for the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\iota_{X;N}} & X \oplus \mathbb{R}^N & \longrightarrow & \mathbb{R}^N & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D_\Theta} & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & Y & \xrightarrow{\mathrm{id}_Y} & Y & \longrightarrow & 0 & \longrightarrow & 0 \, ,
\end{array}
$$

the diagram

$$
\begin{array}{ccc}
\lambda(D) \otimes \lambda(j) & \xrightarrow{\ \Psi_t\ } & \lambda\big(D_\Theta\big) \\
{\scriptstyle \widetilde{\mathcal{D}}_j \otimes \widetilde{\mathcal{D}}_D \circ R}\Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{\mathcal{D}}_{D_\Theta}} \\
\lambda(j^*) \otimes \lambda(D^*) & \xrightarrow{\ \Psi_{t^*}\ } & \lambda\big(D_\Theta^*\big)
\end{array}
$$

commutes. By Corollary 5.4, the horizontal arrows in this diagram induce bundle maps that are continuous with respect to the topologies provided by Proposition 5.3. The right vertical arrow induces a continuous bundle map over $U_{X;\Theta}$ by Lemma 5.7. The isomorphisms R and $\widetilde{\mathcal{D}}_j$ on the left-hand side of this diagram do not depend on D. Thus, the isomorphisms $\widetilde{\mathcal{D}}_D$ also induce continuous bundle maps over $U_{X;\Theta}$. $\qquad\square$

5.3 Orientations along paths

Let X, Y be Banach vector spaces. We denote by

$$\mathcal{F}^\star(X,Y) \subset \mathcal{F}^*(X,Y) \subset \mathcal{F}(X,Y)$$

the subspace of isomorphisms between X and Y. If $D \in \mathcal{F}^\star(X,Y)$ is an isomorphism, the element $1 \otimes 1^*$ of $\lambda(D)$ determines an orientation on this line, which we will call the canonical orientation of $\lambda(D)$. Below we determine whether the extension of this orientation over a generic path in $\mathcal{F}(X,Y)$ ending in $\mathcal{F}^\star(X,Y)$ restricts to the canonical orientation over the endpoint as well; see Proposition 5.9.

Let $D_t \in \mathcal{F}(X,Y)$ with $t \in (-\delta, \delta)$ be a continuous path so that $D_t \in \mathcal{F}^\star(X,Y)$ for $t \neq 0$ and $\mathfrak{d}(\mathfrak{c}(D_0)) = 1$. By the continuity of the index, this implies that $\mathfrak{d}(\kappa(D_0)) = 1$ as well. Choose

$$x_0 \in \kappa(D) - \{0\}, \qquad y_0 \in Y - \mathrm{Im}\, D_0,$$

and a closed linear subspace $\dot{X} \subset X$ such that the operator

$$\dot{D}_0 \colon \dot{X} \longrightarrow \mathrm{Im}\, D_0, \qquad \dot{D}_0(x) = D_0 x,$$

is an isomorphism. Shrinking δ if necessary, we can assume that

$$Y = \operatorname{Im} D_t \oplus \mathbb{R}y_0 \qquad \forall\, t \in (-\delta, \delta).$$

Let $B\colon (-\delta, \delta) \longrightarrow \dot{X}$ and $f_t\colon (-\delta, \delta) \longrightarrow \mathbb{R}$ be continuous maps so that

$$D_t x_0 = D_t(B(t)) + f(t)y_0.$$

By our assumptions, $B(0) = 0$ and $f^{-1}(0) = \{0\}$. We will call $0 \in (-\delta, \delta)$ a transverse degeneration of the path $(D_t)_{t \in (-\delta, \delta)}$ if f changes sign at $t = 0$. This notion is independent of the choices of $x_0, y_0, \dot{X}$.

Continuing with the setup above, define

$$\Theta\colon \mathbb{R} \longrightarrow Y, \qquad \Theta(s) = sy_0.$$

Thus, $\kappa((D_t)_\Theta)$ is generated by the element $(x_0 - B(t), -f(t))$ of $X \oplus \mathbb{R}$. Since the operators $(D_t)_\Theta$ are surjective, the map

$$\tilde{\mathcal{I}}\colon (-\delta, \delta) \times \mathbb{R} \longrightarrow \bigsqcup_{t \in (-\delta, \delta)} \lambda\big((D_t)_\Theta\big), \qquad \tilde{\mathcal{I}}(t, s) = \big(x_0 - B(t), -f(t)\big)s \in \lambda\big((D_t)_\Theta\big),$$

is a continuous line bundle isomorphism over $(-\delta, \delta)$ by the Normalization I property on page 12. By the Normalization II property on page 12, the isomorphisms (3.2) are given by

$$\hat{\mathcal{I}}_{\Theta;D_t}\colon \lambda(D_t) \longrightarrow \lambda\big((D_t)_\Theta\big),$$
$$\hat{\mathcal{I}}_{\Theta;D_t}\big(f(t)1 \otimes 1^*\big) = \big(B(t) - x_0, f(t)\big) \otimes 1^* = \tilde{\mathcal{I}}(t, -1) \quad \text{if } t \neq 0.$$

Since the isomorphisms $\hat{\mathcal{I}}_{\Theta;D_t}$ extend over $t = 0$ by the Exact Triples property on page 13, it follows that the canonical orientations of $\lambda(D_t)$ with $t \neq 0$ do not extend across $t = 0$ if $0 \in (-\delta, \delta)$ is a transverse degeneration of the path $(D_t)_{t \in (-\delta, \delta)}$. This establishes the following.

Proposition 5.9 (Wall Crossing). *Suppose X, Y are Banach vector spaces and $(D_t)_{t \in [0,1]}$ is a continuous path in $\mathcal{F}(X, Y)$ so that D_t is an isomorphism except for finitely many values of t in $(0, 1)$. If all degenerations of the path $(D_t)_{t \in [0,1]}$ are transverse, then the canonical orientations of $\lambda(D_0)$ and $\lambda(D_1)$ extend continuously to orientations of $\lambda(D_t)$ over $[0, 1]$ if and only if the number of the degenerations is even.*

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