Transversality for J-Holomorphic Maps: a Complex-Geometric Perspective

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Abstract

We provide a treatment of transversality for J-holomorphic maps and the associated evaluation maps and derivatives of arbitrary order from the generally overlooked viewpoint of Ivashkovich-Shevchishin. In contrast to the usual approach, we establish these statements simultaneously through a single application of a universal moduli space setup.

Contents

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1 Introduction

Gromov's introduction [10] of J-holomorphic curves techniques into symplectic topology has revolutionized this field and led to its numerous connections with algebraic geometry. The ideas put forward in [10] have been further elucidated and developed in [16, 18, 22, 23, 15] and in many other works. Chapters 2 and 4 of [18] concern two of the three fundamental building blocks of the subject of J-holomorphic curves, the local structure of J-holomorphic maps and Gromov's convergence for sequences of J-holomorphic maps; an alternative systematic exposition of these two topics appears in [33]. Chapter 3 and Sections 6.2 and 6.3 of [18], Section 4 of [22], and Section 3 of [23] deal with the third fundamental building block of the subject, transversality issues for J-holomorphic maps that are relevant to constructing pseudocycles out of moduli spaces of these maps. The present paper provides a streamlined and more general treatment of these issues. We adapt this treatment to moduli spaces of real J-holomorphic maps in [34], providing a geometric interpretation of the positive-genera real Gromov-Witten invariants of [5] in semi-positive cases.

1.1 Preview of the main statements

We begin by formulating the main statements of this paper in the most basic case of J-holomorphic maps from connected domains. The $g = 0$ case of Theorem 1.1 and the $(g, \mathfrak{m}) = (0, 0)$ case of Theorem 1.2(2) below include the main conclusions of [18, Chapter 3]. The analogues of these statements for J-holomorphic maps from disconnected domains, provided by Theorems 1.3 and 1.4 in Section 1.2, imply the main conclusions of [18, Sections 6.2,6.3]. We illustrate the general approach behind the proofs of the main statements in this paper on the special cases of Theorems 1.1 and 1.2 in Section 1.3.

We call a subset $\widehat{\mathcal{J}}$ of a topological space $\mathcal{J}$ ubiquitous if $\widehat{\mathcal{J}}$ contains a countable intersection of open dense subsets of $\mathcal{J}$ (the term used in [18] is residual). We discuss the significance of this notion in the contexts such as those of the main theorems of this paper in Section 3.1.

For a manifold X, denote by $\mathcal{J}(X)$ the space of almost complex structures J on X with the C^∞-topology. If in addition ω is a symplectic form on X, let

$$\mathcal{J}_\omega(X) \subset \mathcal{J}'_\omega(X) \subset \mathcal{J}(X) \tag{1.1}$$

be the subspaces of ω-compatible and of ω-tamed almost complex structures. For $g, k \in \mathbb{Z}^{\geqslant 0}$, $A \in H_2(X;\mathbb{Z})$, and $J \in \mathcal{J}(X)$, denote by $\mathfrak{M}^*_{g,k}(A;J)$ the moduli space of equivalence classes of simple degree A J-holomorphic maps u from smooth connected compact genus g Riemann surfaces $(\Sigma, \mathfrak{j})$ with k (distinct) marked points to X. For each $i = 1, \ldots, k$, let

$$\mathsf{ev}_i\colon \mathfrak{M}^*_{g,k}(A;J) \longrightarrow X \qquad \text{and} \qquad L_i \longrightarrow \mathfrak{M}^*_{g,k}(A;J)$$

be the evaluation map and the universal tangent line bundle, respectively, for the i-th marked point; see Section 2.3.

Theorem 1.1. *Let X be a $2n$-manifold. There exists a ubiquitous subset $\widehat{\mathcal{J}} \subset \mathcal{J}(X)$ such that the moduli space $\mathfrak{M}^*_{g,k}(A;J)$ is a smooth oriented manifold of dimension*

$$\dim_{\mathbb{R}} \mathfrak{M}^*_{g,k}(A;J) = 2\big(\big\langle c_1(TX), A\big\rangle + (n-3)(1-g) + k\big) \tag{1.2}$$

and ev_i *is a smooth map for all*

$$g,k\in\mathbb{Z}^{\geqslant 0}, \quad i=1,\ldots,k, \quad A\in H_2(X;\mathbb{Z}), \quad J\in\widehat{\mathcal{J}}. \tag{1.3}$$

For every symplectic form ω *on* X*, the same statement holds with* $\mathcal{J}(X)$ *replaced by* $\mathcal{J}_\omega(X)$ *and* $\mathcal{J}'_\omega(X)$*.*

Let ∇ be a connection in TX and Σ be a smooth surface. Given $z\in\Sigma$ and $v\in T_z\Sigma$, choose a smooth curve

$$\alpha_v\colon(-\delta,\delta)\longrightarrow\Sigma, \quad \tau\longrightarrow\alpha_v(\tau), \qquad \text{s.t.} \quad \alpha_v(0)=z, \quad \alpha_v'(0)=v. \tag{1.4}$$

For a smooth map $f:\Sigma\longrightarrow X$, denote by D^{∇}_{f,α_v} the covariant derivative of sections of $\alpha_v^*f^*TX$ with respect to τ determined by ∇. For $m\in\mathbb{Z}^+$, let

$$\mathfrak{D}^m_f v = \underbrace{D^{\nabla}_{f,\alpha_v}\ldots D^{\nabla}_{f,\alpha_v}}_{m-1}\frac{\mathrm{d}}{\mathrm{d}\tau}(f\circ\alpha)\Big|_{\tau=0}\in T_{f(z)}X\,.$$

In particular, $\mathfrak{D}^1_f v=\mathrm{d}_x f(v)$ is independent of the choices of ∇ and α_v satisfying (1.4). If $\mathfrak{D}^1_f v,\ldots,\mathfrak{D}^{m-1}_f v$ vanish, then $\mathfrak{D}^m_f v$ is also independent of these choices. If in addition f is $(J,\mathfrak{j})$-holomorphic for some $J\in\mathcal{J}(X)$ and $\mathfrak{j}\in\mathcal{J}(\Sigma)$, then

$$\mathfrak{D}^m_f\big(c\cdot_{\mathfrak{j}} v\big)=c^m\cdot_J\big(\mathfrak{D}^m_f v\big) \qquad \forall\, c\in\mathbb{C}; \tag{1.5}$$

this follows from [33, Corollary 3.6].

Let $g,k\in\mathbb{Z}^{\geqslant 0}$, $A\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}(X)$. For a tuple $\mathfrak{m}\equiv(m_i)_{i=1,\ldots,k}$ in $(\mathbb{Z}^{\geqslant 0})^k$, define

$$\begin{split}\mathcal{Z}^*_{g,\mathfrak{m}}(A;J)=\big\{[z_1,\ldots,z_k,u\colon\Sigma\longrightarrow X]\in\mathfrak{M}^*_{g,k}(A;J)\colon \mathfrak{D}^m_u v=0\ \forall\ m\in\mathbb{Z}^+,\ m\leqslant m_i,\\ v\in T_{z_i}\Sigma,\ i=1,\ldots,k\big\}\,.\end{split} \tag{1.6}$$

By (1.5), the section

$$\begin{split}\mathfrak{D}^{m_i+1}_i\in\Gamma\big(\mathcal{Z}^*_{g,\mathfrak{m}}(A;J);L_i^{*\otimes_{\mathbb{C}}(m_i+1)}\otimes_{\mathbb{C}}\mathsf{ev}_i^*(TX,J)\big),\\ \mathfrak{D}^{m_i+1}_i v^{\otimes m}=\mathfrak{D}^{m_i+1}_u v \ \ \forall\ [u,v]\in L_i,\end{split} \tag{1.7}$$

is well-defined for every $i=1,\ldots,k$.

Theorem 1.2. *Let* X *be a* $2n$*-manifold.*

(1) There exists a ubiquitous subset $\widehat{\mathcal{J}}\subset\mathcal{J}(X)$ *such that the space* $\mathcal{Z}^*_{g,\mathfrak{m}}(A;J)$ *is a smooth oriented submanifold of* $\mathfrak{M}^*_{g,k}(A;J)$ *of codimension* $n|\mathfrak{m}|$ *and (1.7) is a smooth section transverse to the zero set for all* g,k,i*,* A*, and* J *as in (1.3) and* $\mathfrak{m}\in(\mathbb{Z}^{\geqslant 0})^k$*.*

(2) If $k\in\mathbb{Z}^{\geqslant 0}$ *and* $h:Y\longrightarrow X^k$ *is a smooth map from a smooth manifold, then* $\widehat{\mathcal{J}}$ *can be chosen in (1) so that in addition the restriction of the smooth map*

$$\mathsf{ev}\equiv\mathsf{ev}_1\times\ldots\times\mathsf{ev}_k\colon\mathfrak{M}^*_{g,k}(A;J)\longrightarrow X^k$$

to $\mathcal{Z}^*_{g,\mathfrak{m}}(A;J)$ *is transverse to* h *for all* g*,* A*,* J *and* $\mathfrak{m}$ *as in (1).*

For every symplectic form ω on X, the same statements hold with $\mathcal{J}(X)$ replaced by $\mathcal{J}_\omega(X)$ and $\mathcal{J}'_\omega(X)$.

Theorems 1.3 and 1.4 in Section 1.2 extend Theorems 1.1 and 1.2 to J-holomorphic maps from disconnected domains. They directly imply the same statements for J-holomorphic maps from nodal domains, as well as for the more general GU domains introduced in [22]. Other special cases of Theorems 1.3 and 1.4 of the closely related Theorems 2.5 and 2.6 include Lemma 7.5 in [11], Theorems 1.1 and 1.2 in [20], Proposition A.1 in [26], Proposition 1.8 in [29], and Theorem 1.5(1) in [31].

The crucial new ingredient for the purposes of constructing pseudocycles out of moduli spaces of stable J-holomorphic maps from positive-genus Riemann surfaces is the notion of inhomogeneous perturbation ν of the $\bar{\partial}_J$-operator introduced in [22]; see Section 2.4. It in particular leads to extensions of Theorems 1.3 and 1.4 to degree $A=0$ maps; see Theorems 2.5 and 2.6 in Section 2.5.

1.2 Transversality for J-holomorphic maps

Let X be a manifold and B be a manifold, possibly with boundary. Denote by

$$\pi_X : B\times X \longrightarrow X$$

the projection to the second component, by $\mathcal{J}(B;X)$ the space of fiberwise complex structures on the vector bundle π_X^*TX with the C^∞-topology, and by $\mathrm{Symp}(B;X)$ the space of smooth fiberwise symplectic structures on π_X^*TX. For $J\in\mathcal{J}(B;X)$, $\omega\in\mathrm{Symp}(B;X)$, and $b\in B$, let $J_b\in\mathcal{J}(X)$ and $\omega_b\in\mathrm{Symp}(X)$ be the associated almost complex and symplectic structures on X. For $J_\circ\in\mathcal{J}(\partial B;X)$, define

$$\mathcal{J}_{J_\circ}(B;X) = \big\{J\in\mathcal{J}(B;X) : J_b=(J_\circ)_b \ \forall\, b\in\partial B\big\}. \tag{1.8}$$

For $\omega\in\mathrm{Symp}(B;X)$, denote by

$$\mathcal{J}_\omega(B;X) \subset \mathcal{J}'_\omega(B;X) \subset \mathcal{J}(B;X)$$

the subspaces of ω-compatible and of ω-tamed almost complex structures. For $J_\circ\in\mathcal{J}_\omega(\partial B;X)$ and $J_\circ\in\mathcal{J}'_\omega(\partial B;X)$, define

$$\mathcal{J}_{\omega;J_\circ}(B;X) \subset \mathcal{J}_\omega(B;X) \qquad \text{and} \qquad \mathcal{J}'_{\omega;J_\circ}(B;X) \subset \mathcal{J}'_\omega(B;X),$$

respectively, similarly (1.8).

For $\chi\in\mathbb{Z}$, $k\in\mathbb{Z}^{\geqslant 0}$, $A\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}(X)$, denote by $\mathfrak{M}^{\bullet *}_{\chi,k}(A;J)$ the moduli space of equivalence classes of simple degree A J-holomorphic maps u from smooth, possibly disconnected, compact Riemann surfaces $(\Sigma,\mathfrak{j})$ of holomorphic Euler characteristic χ with k (distinct) marked points to X. For a manifold B, possibly with boundary, and $J\in\mathcal{J}(B;X)$, let

$$\mathfrak{M}^{\bullet *}_{\chi,k}(A;J) = \big\{(b,[\mathbf{u}]) : b\in B,\ [\mathbf{u}]\in\mathfrak{M}^{\bullet *}_{\chi,k}(A;J_b)\big\}.$$

This space inherits a topology from spaces of smooth maps from smooth domains. For each $i\in[k]$, denote by

$$\mathsf{ev}_i : \mathfrak{M}^{\bullet *}_{\chi,k}(A;J) \Longrightarrow X \qquad \text{and} \qquad L_i \longrightarrow \mathfrak{M}^{\bullet *}_{\chi,k}(A;J) \tag{1.9}$$

the natural evaluation map and the universal tangent line bundle, respectively, for the i-th marked point; these are pullbacks from one of the factors. For a tuple $\mathfrak{m}\equiv(m_i)_{i\in[k]}$ in $(\mathbb{Z}^{\geqslant0})^k$, define

$$\mathcal{Z}^{\bullet*}_{\chi,\mathfrak{m}}(A;J)\subset\mathfrak{M}^{\bullet*}_{\chi,k}(A;J) \quad\text{and}\quad \mathfrak{D}_i^{m_i+1}\in\Gamma\big(\mathcal{Z}^{\bullet*}_{\chi,\mathfrak{m}}(A;J);L_i^{*\otimes_{\mathbb{C}}(m_i+1)}\otimes_{\mathbb{C}}\mathsf{ev}_i^*(TX,J)\big) \tag{1.10}$$

as in (1.6) and (1.7) with $\mathfrak{M}^{\bullet*}_{\chi,k}(A;J)$ in place of $\mathfrak{M}^{*}_{g,k}(A;J)$.

Theorem 1.3. *Let X be a $2n$-manifold. For every manifold $B_\circ$, there exists a ubiquitous subset*

$$\widehat{\mathcal{J}}(B_\circ;X)\subset\mathcal{J}(B_\circ;X) \tag{1.11}$$

with the following properties.

(0) If $B_\circ^1,B_\circ^2,\ldots$ are the topological components of $B_\circ$, then

$$\widehat{\mathcal{J}}(B_\circ;X)=\widehat{\mathcal{J}}(B_\circ^1;X)\times\widehat{\mathcal{J}}(B_\circ^2;X)\times\ldots.$$

(1) For all $\chi\in\mathbb{Z}$, $k\in\mathbb{Z}^+$ $A\in H_2(X;\mathbb{Z})$, and $J\in\widehat{\mathcal{J}}(B_\circ;X)$,

(1a) $\mathfrak{M}^{\bullet*}_{\chi,k}(A;J)$ is a smooth manifold of dimension

$$\dim_{\mathbb{R}}\mathfrak{M}^{\bullet*}_{\chi,k}(A;J)=\dim_{\mathbb{R}}B_\circ+2\big(\langle c_1(TX),A\rangle+(n-3)\chi+k\big),$$

(1b) $\mathcal{Z}^{\bullet*}_{\chi,\mathfrak{m}}(A;J)$ is a smooth submanifold of $\mathfrak{M}^{\bullet*}_{\chi,k}(A;J)$ of codimension $2n|\mathfrak{m}|$, and the section $\mathfrak{D}^{m_i+1}$ in (1.10) is smooth and transverse to the zero set for all $\mathfrak{m}\in(\mathbb{Z}^{\geqslant0})^k$ and $i\in[k]$.

(2) For all $J_\circ\in\widehat{\mathcal{J}}(B_\circ;X)$ and manifolds B with boundary $\partial B=B_\circ$, there exists a ubiquitous subset

$$\widehat{\mathcal{J}}_{J_\circ}(B;X)\subset\mathcal{J}_{J_\circ}(B;X) \tag{1.12}$$

satisfying the properties in (1) with $B_\circ$ and manifold replaced by B and manifold with boundary so that

$$\partial\,\mathfrak{M}^{\bullet*}_{\chi,k}(A;J)=\mathfrak{M}^{\bullet*}_{\chi,k}(A;J_\circ),\quad \partial\mathcal{Z}^{\bullet*}_{\chi,\mathfrak{m}}(A;J)=\mathcal{Z}^{\bullet*}_{\chi,\mathfrak{m}}(A;J_\circ)\,. \tag{1.13}$$

(3) An orientation on $B_\circ$ determines orientations on all spaces in (1) so that (2) holds in the category of oriented manifolds.

For every $\omega\in\mathrm{Symp}(B;X)$, the same statements hold with $\mathcal{J}$ replaced by $\mathcal{J}_\omega$ and $\mathcal{J}'_\omega$.

Theorem 1.4. *Let X and $B_\circ$ be as in Theorem 1.3. If $k\in\mathbb{Z}^{\geqslant0}$ and $h\colon Y\longrightarrow X^k$ is a smooth map from a manifold, there exists a ubiquitous subset as in (1.11) satisfying (1) in Theorem 1.3 and the following properties.*

(1) For all χ, A, $\mathfrak{m}\in(\mathbb{Z}^{\geqslant0})^k$, and $J\in\widehat{\mathcal{J}}(B_\circ;X)$ as in Theorem 1.3(1), the properties (1a) and (1b) in Theorem 1.3 are satisfied and the map

$$\mathsf{ev}\equiv\mathsf{ev}_1\times\ldots\times\mathsf{ev}_k\colon\mathcal{Z}^{*}_{\chi,\mathfrak{m}}(J)\longrightarrow X^k \tag{1.14}$$

is transverse to h.

(2) For all $J_\circ$ and B as in Theorem 1.3(2), there exists a ubiquitous subset as in (1.12) satisfying the conditions in Theorem 1.3(2) and the additional condition in (1) above.

For every $\omega\in\mathrm{Symp}(B;X)$, the same statements hold with $\mathcal{J}$ replaced by $\mathcal{J}_\omega$ and $\mathcal{J}'_\omega$.

The ubiquitous subsets as in (1.11) and (1.12) provided by Theorem 1.4 depend on h. In typical applications of Theorems 1.3 and 1.4, $B_\circ$ is either a one-point set or a two-point set and $B=[0,1]$.

1.3 Preview of the proofs

Let X, g, k, and A be as in Theorem 1.1. Denote by $\widetilde{\mathfrak{B}}^*_g(A)$ the space of triples $(\Sigma,\mathfrak{j},u)$ consisting of a smooth connected compact Riemann surface $(\Sigma,\mathfrak{j})$ and a simple degree A map $\Sigma\longrightarrow X$ with the C^∞-topology. For $g=0$, we always take $(\Sigma,\mathfrak{j})$ to be $S^2=\mathbb{P}^1$ with its standard complex structure. For $g=1$, we allow $(\Sigma,\mathfrak{j})$ to vary in the space of pairs arising as the quotients $\mathbb{C}/\Lambda_\tau$ with $\Lambda_\tau\subset\mathbb{C}$ denoting the lattice spanned by 1 and $\tau\in\mathbb{C}$ such that $\operatorname{Im}\tau>0$. For $g=2$, we fix Σ and allow $\mathfrak{j}$ to vary in the Teichmüller space $\mathscr{T}_g$ determined by Σ. We denote by $\mathscr{T}_0$ and $\mathscr{T}_1$ the one-point space and the open upper-half plane $\mathbb{H}\subset\mathbb{C}$, respectively.

For $(\Sigma,\mathfrak{j},u)\in\widetilde{\mathfrak{B}}^*_g(A)$, let

$$\Gamma(u)\equiv\Gamma\big(\Sigma;u^*TX\big),\quad \Gamma^{0,1}_{J,\mathfrak{j}}(u)\equiv\Gamma\big(\Sigma;(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}u^*(TX,J)\big),$$
$$\bar\partial_{J,\mathfrak{j}}u=\frac{1}{2}\big(\mathrm{d}u+J\circ\mathrm{d}u\circ\mathfrak{j}\big)\in\Gamma^{0,1}_{J,\mathfrak{j}}(u)\,.$$

The moduli space $\mathfrak{M}^*_{g,k}(A;J)$ of Theorem 1.1 is a smooth manifold of the expected dimension (1.2) if the linearization

$$D_{J;(\mathfrak{j},u)}:T_{(\mathfrak{j},u)}\widetilde{\mathfrak{B}}^*_g(A)=T_{\mathfrak{j}}\mathscr{T}_g\oplus\Gamma(u)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u)\tag{1.15}$$

of the $\bar\partial_J$-operator on the space $\widetilde{\mathfrak{B}}^*_g(A)$ at $(\Sigma,\mathfrak{j},u)$ is surjective whenever $\bar\partial_{J,\mathfrak{j}}u=0$; see Proposition 4.2.

The standard way of establishing Theorem 1.1 is to show that the linearization

$$D_{J;(\mathfrak{j},u)}\bar\partial:T_{(J,\mathfrak{j},u)}\big(\mathcal{J}\times\widetilde{\mathfrak{B}}^*_g(A)\big)=T_J\mathcal{J}\oplus T_{(\mathfrak{j},u)}\widetilde{\mathfrak{B}}^*_g(A)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u)\tag{1.16}$$

of the $\bar\partial$-operator on the space $\mathcal{J}\times\widetilde{\mathfrak{B}}^*_g(A)$ at $(J;\mathfrak{j},u)$ is surjective for all elements of the universal moduli space

$$\mathfrak{U}\widetilde{\mathfrak{M}}^*_g(A)\equiv\big\{(J;\mathfrak{j},u)\in\mathcal{J}\times\widetilde{\mathfrak{B}}^*_g(A):\bar\partial_{J,\mathfrak{j}}u=0\big\}\,.\tag{1.17}$$

The restriction of (1.16) to $T_{(\mathfrak{j},u)}\widetilde{\mathfrak{B}}^*_g(A)$ is (1.15). The surjectivity of (1.16) for every element of (1.17) implies that

(S1) this subspace is an infinite-dimensional manifold,

(S2) the homomorphism (1.15) is onto for all $(\Sigma,\mathfrak{j},u)\in\widetilde{\mathfrak{B}}^*_g(A)$ with $\bar\partial_{J,\mathfrak{j}}u=0$ if and only if J is a regular value of the projection

$$\pi:\mathfrak{U}\widetilde{\mathfrak{M}}^*_g(A)\longrightarrow\mathcal{J},\qquad\pi(J;\mathfrak{j},u)=J,\tag{1.18}$$

(S3) the subspace $\widehat{\mathcal{J}}$ of regular values of (1.18) is ubiquitous;

see Section 4.3.

The surjectivity of (1.16), established in the proof of [18, Proposition 3.2.1], is a consequence of the ellipticity of the $\bar\partial_J$-operator and is obtained by explicitly showing that $D_{J;(\mathfrak{j},u)}\bar\partial(T_J\mathcal{J})$ covers the cokernel of (1.15); see Lemma 3.1. The general principle behind the argument summarized in the previous paragraph is captured by Proposition 4.2. This principle would have applied directly in the present situation if $\mathcal{J}$ and $\widetilde{\mathfrak{B}}^*_g(A)$ were Banach manifolds (they are instead infinite-dimensional manifolds locally modeled on Fréchet vector spaces of smooth maps). The standard approach to deal with this issue is to replace $\mathcal{J}$ and $\widetilde{\mathfrak{B}}^*_g(A)$ with their C^ℓ and W^p_k-analogues, respectively; see Section 3.4. The desired conclusion in the C^∞-category then follows via Taubes's argument, appearing in the proofs of [18, Theorems 3.1.6(ii),6.2.6(ii)] and captured by Proposition 4.5. An alternative approach to dealing with the above issue, which stays in the C^∞-category, is due to Floer; it is outlined in [18, Remark 3.2.7] and in the last two pages of [18, Section 3.4], which are not used for anything else in [18].

For the constructions of GW-pseudocycles in [18, Section 6.6], [22, Section 2], and [23, Section 2] and for many other purposes in GW-theory, it is useful to establish that the spaces $\mathfrak{M}^*_\gamma(J)$ of equivalence classes of simple J-holomorphic maps from nodal domains of a fixed combinatorial type γ are also smooth manifolds of the expected dimensions. The possible combinatorial types are the connected genus g graphs γ whose vertices v are decorated by the elements $g_v\in\mathbb{Z}^{\geqslant 0}$ and A_v of $H_2(X;\mathbb{Z})$; see Section 3.2. The vertices and edges of γ correspond to the irreducible components and the nodes of the domains of the elements of $\mathfrak{M}^*_\gamma(J)$. Each space $\mathfrak{M}^*_\gamma(J)$ is an open subset of the preimage of a submanifold Δ_γ of a Cartesian product X_γ under a map

$$\mathsf{ev}_\gamma\colon \prod_v \mathfrak{M}^*_{g_v,S_v}(A_v;J) \longrightarrow X_\gamma\,, \tag{1.19}$$

where $\mathfrak{M}^*_{g_v,S_v}(A_v;J)$ is the moduli space of equivalence classes of simple degree A_v J-holomorphic maps u_v from smooth connected compact genus g_v Riemann surfaces $(\Sigma_v,\mathfrak{j}_v)$ with (distinct) marked points indexed by the set S_v of the flags based at vertex v; see the beginning of Section 3.4. The expected dimension of $\mathfrak{M}^*_\gamma(J)$ is

$$\dim_{\mathbb{R}}\mathfrak{M}^*_\gamma(J) = \dim_{\mathbb{R}}\mathfrak{M}^*_{g,0}(A;J) - 2|\gamma|,$$

where g is the sum of the genus of the graph γ and of all the numbers g_v assigned to the vertices of γ, A is the sum of the homology classes A_v assigned to these vertices, $|\gamma|$ is the number of nodes of γ, and the first term on the right-hand side is as in (1.2).

For a generic J, the map ev_γ in (1.19) is smooth. It is thus sufficient to show that ev_γ is transverse to Δ_γ for a still generic J. By the proof of Proposition 4.2, this is implied by the transversality of the smooth map

$$\mathsf{ev}_\gamma\colon \big\{\big(J,(\mathfrak{j}_v,u_v)_v\big)\in\mathcal{J}\times\prod_v\widetilde{\mathfrak{B}}^*_{g_v,S_v}(A_v)\colon (J;\mathfrak{j}_v,u_v)\in\mathfrak{U}\widetilde{\mathfrak{M}}^*_{g_v,S_v}(A_v)\ \forall\,v\big\}\longrightarrow X_\gamma \tag{1.20}$$

to Δ_γ, where $\widetilde{\mathfrak{B}}^*_{g_v,S_v}(A_v)$ and $\mathfrak{U}\widetilde{\mathfrak{M}}^*_{g_v,S_v}(A_v)$ are the degree A_v genus g_v S_v-marked analogues of the configurations space $\widetilde{\mathfrak{B}}^*_g(A)$ and the universal moduli space $\mathfrak{U}\widetilde{\mathfrak{M}}^*_g(A)$ in (1.17), respectively. This in particular implies that the associated universal moduli space

$$\mathfrak{U}\widetilde{\mathfrak{M}}^*_\gamma\equiv\big\{\big(J,(\mathfrak{j}_v,u_v)_v\big)\in\mathcal{J}\times\prod_v\widetilde{\mathfrak{B}}^*_{g_v,S_v}(A_v)\colon \big[(\mathfrak{j}_v,u_v)_v\big]\in\mathfrak{M}^*_\gamma(J)\big\} \tag{1.21}$$

is a smooth manifold.

The transversality of (1.20) for the genus 0 nodal domains (thus, the genus of γ is 0 and $g_v=0$ for all v) is [18, Proposition 6.2.8]. Its proof is specific to the genus 0 case (though it is also applicable to maps that are constant on some irreducible components). It combines the reasoning as in the proof of [18, Proposition 3.2.1], which establishes the surjectivity of (1.16) in the $g=0$ case, with [18, Theorem 6.3.1], according to which the differential of the evaluation map

$$\mathsf{ev}_f\colon \mathfrak{U}\widetilde{\mathfrak{M}}^*_{g_v,S_v}(A_v)\longrightarrow X$$

is a submersion for every $f\in S_v$. The proof of the latter applies the technical conclusion of [18, Lemma 3.4.3]. Less detailed versions of this approach in more general settings appear in Section 4 of [22] and in Section 3 of [23].

Theorem 1.2(1) is essentially [26, Proposition A.1]. The proof of the latter applies the approach of [18] summarized above to the direct sum of the bundle sections $\mathfrak{D}_i^m$ with $m\leqslant m_i$ and $i=1,\ldots,k$ over the domain of the map ev_γ in (1.20) instead of the map ev_γ. The analogues of Theorem 6.3.1 and Lemma 3.4.3 of [18] in this situation are Lemma A.3 in [26] and Theorem 2.100 in [27], respectively.

Propositions 2.3 and 4.1 in [20] imply the $\mathfrak{m}=0$ case of Theorem 1.2 with h being the inclusion of the diagonal $\Delta\subset X^2$. Unlike the two-step proofs in [18, 22, 23, 26], the reasoning in [20] obtains the relevant analogues of the universal moduli space (1.21) in one step as the preimages of Banach submanifolds by transverse maps. However, this approach does not extend beyond the $\mathfrak{m}=0$ case of Theorem 1.2, as the relevant maps would no longer be transverse.

We follow a completely different approach in showing that the analogues $\mathfrak{U}\widetilde{\mathcal{Z}}^*_{\gamma;\mathfrak{m}}$ of the universal moduli spaces (1.21) relevant to Theorems 1.1-1.4 are cut out transversely. For $(\Sigma,\mathfrak{j},u)\in\widetilde{\mathfrak{B}}^*_g(A)$, a finite tuple $\mathbf{z}\equiv(z_f)_{f\in S}$ of points on Σ, and a tuple $\mathfrak{m}\equiv(m_f)_{f\in S}$ of nonnegative integers, let

$$\begin{aligned}\Gamma_{\mathfrak{m}}(u;\mathbf{z}) &= \big\{\xi\in\Gamma(u)\colon \xi(z_f)=0,\ \nabla^m\xi|_{z_i}=0\ \forall\ m=1,\ldots,m_f,\ f\in S\big\},\\ \Gamma^{0,1}_{J,\mathfrak{j};\mathfrak{m}}(u;\mathbf{z}) &= \big\{\eta\in\Gamma^{0,1}_{J,\mathfrak{j}}(u)\colon \nabla^{m-1}\eta|_{z_i}=0\ \forall\ m=1,\ldots,m_f,\ f\in S\big\}.\end{aligned}$$

It is immediate that the C^∞-analogues $\widetilde{\mathfrak{B}}^*_{\gamma;\mathfrak{m}}$ of the moduli spaces $\mathfrak{U}\widetilde{\mathcal{Z}}^*_{\gamma;\mathfrak{m}}$ (i.e. before imposing $\bar\partial$ condition) are infinite-dimensional manifolds; see Lemma 3.3. The smoothness of $\mathfrak{U}\widetilde{\mathcal{Z}}^*_{\gamma;\mathfrak{m}}$ then follows from the surjectivity of the analogue of (1.16) over $\mathfrak{U}\widetilde{\mathcal{Z}}^*_{\gamma;\mathfrak{m}}$ for each element $(J;(\mathfrak{j}_v,u_v)_{v\in S_v})$ of $\mathfrak{U}\widetilde{\mathcal{Z}}^*_{\gamma;\mathfrak{m}}$. The latter is in turn the case if the image under $D_{(J;\mathfrak{j}_v,u_v)}\bar\partial$ of the subspace of $T_J\mathcal{J}$ consisting of infinitesimal deformations of J supported in an arbitrarily small open subset of X intersecting the image $u_v(\Sigma_v)$ of each component u_v of u covers the cokernel of the restriction

$$D_{J;(\mathfrak{j}_v,u_v)}\colon \Gamma_{\mathfrak{m}_v}(u_v;\mathbf{z}_v)\longrightarrow \Gamma^{0,1}_{J,\mathfrak{j}_v;\mathfrak{m}_v}(u_v;\mathbf{z}_v) \tag{1.22}$$

of the homomorphism $D_{J;(\mathfrak{j}_v,u_v)}$ as in (1.15) for every v, where $\mathbf{z}_v$ is the tuple of marked points carried by u_v (which includes the nodes of the domain Σ of u carried by Σ_v) and $\mathfrak{m}_v$ is an associated tuple of nonnegative integers; see the proof of Proposition 3.5.

In complex geometry, a restriction as in (1.22) corresponds to an operator on the sections of the vector bundle

$$u_v^*(TX,J)\otimes_{\mathbb{C}}\mathcal{O}_{\Sigma_v}\bigg(-\sum_{f\in S}m_fz_f\bigg)\longrightarrow\Sigma_v, \tag{1.23}$$

where the $\mathcal{O}_{\Sigma_v}$ factor above is the holomorphic line bundle determined by the divisor $-\sum_{f\in S} m_f z_f$ in Σ_v; see [9, Section 1.1]. The twisting construction of [24, Lemma 2.4.1] extends this classical correspondence to generalized Cauchy-Riemann operators over Riemann surfaces as in (1.15). By Serre Duality [12, Lemma 2.3.2], the cokernel of a restriction as in (1.22) is then isomorphic to the dual of the kernel of the formal adjoint operator $D^*_{J;(\mathfrak{j}_v,u_v)}$ on the $(1,0)$-forms that may now have poles at the points z_f; see Proposition 4.9. This is immaterial for pairing such forms with the infinitesimal deformations of J, as done in the proof of Proposition 3.2.1 in [18], provided they are supported away from $u_v(z_f)$. Thus, the same argument applies to the restricted operator (1.15); see Lemma 3.1.

In summary, our approach to Theorems 1.1-1.4 involves the appearance of only one universal moduli space. Similarly to the arguments in [18, 20, 22, 23, 26], it runs off the "local universal transversality" of Lemma 3.1. The proof of the "local universal transversality" (in all cases) depends on the ellipticity of a generalized Cauchy-Riemann operator and on Serre Duality for such an operator. In contrast to [18, 20, 22, 23, 26], we obtain "local universal transversality" for generalized Cauchy-Riemann operators on all twistings of u^*TX as in (1.23), instead of just on u^*TX. This results in no additional complications and avoids the delicate arguments on local deformations of J-holomorphic maps that underpin the proofs of [18, Theorem 6.3.1] and [26, Lemma A.3].

The complications in constructing GW-pseudocycles in positive genera that arise from J-holomorphic maps that are constant maps on some irreducible components of the domain are avoided in [22, 23] by contracting such irreducible components and producing a so-called GU map. The domains of these maps are Riemann surfaces whose singular points may contain more than two smooth branches; see Sections 2.1 and 2.3. Our approach for establishing Theorems 1.1-1.4 readily extends to spaces of GU maps of a fixed combinatorial type via the "local universal transversality" of Lemma 3.2; see Theorems 2.5 and 2.6.

1.4 Outline and acknowledgments

The relevant notation for the moduli spaces of complex curves and for their covers is set up in Section 2.2. Terminology concerning maps from singular Riemann surfaces is defined Section 2.3. Section 2.4 introduces a version of Ruan-Tian perturbations. We define the spaces

$$\mathfrak{M}^*_{\gamma;\gamma_0,\varpi}(J,\nu)\subset\mathfrak{M}^{\dagger*}_{\gamma;\gamma_0,\varpi}(J,\nu)\subset\mathfrak{M}_{\gamma;\gamma_0,\varpi}(J,\nu)$$

of basic and reduced GU maps of a fixed combinatorial type and state analogues of Theorems 1.3 and 1.4 for them in Section 2.5. These analogues are proved in Sections 3.3 and 3.4. The first of these sections introduces suitable deformation-obstruction settings and then shows that the deformations of Ruan-Tian pairs (J,ν) suffice to cover the obstruction space in all relevant cases; see Lemmas 3.1 and 3.2. By Section 3.4, Lemmas 3.1 and 3.2 ensure the smoothness of the universal moduli space of basic (J,ν)-maps of a fixed combinatorial type. The latter implies the smoothness of the corresponding space of (J,ν)-maps for a fixed pair (J,ν); see Proposition 4.4. This then concludes the proof of Theorems 2.5 and 2.6.

The proof of Theorems 1.3 and 1.4, which we omit, is a simplified version of the proof of Theorems 2.5 and 2.6. In particular, it does not require Lemma 3.2. Theorems 1.3 and 1.4 are essentially the $\mathrm{Ver}_0=\varnothing$ case of Theorems 2.5 and 2.6 (this case is not formally admissible in the terminology of the two theorems).

The present paper grew out of the author's desire for a systematic treatment of transversality issues arising in various settings in the theory of J-holomorphic curves, including the common ones as in [18, 22, 23] and the more specialized one in [5, 11, 14, 20, 26, 29, 31]. He would like to thank P. Georgieva, J. Starr, and C. Wendl for enlightening discussions that influenced the preparation of the present paper.

2 Spaces of (J,ν)-maps

The Ruan-Tian perturbations ν used to regularize the Cauchy-Riemann equation $\bar\partial_{J,\mathfrak j}u=0$ for maps into an almost complex manifold (X,J) are sections of certain bundles over $\widetilde{\mathcal U}_{g,k}\times X$, where $\widetilde{\mathcal U}_{g,k}$ is the universal curve (2.8) over a finite cover $\widetilde{\mathcal M}_{g,k}$ of $\overline{\mathcal M}_{g,k}$ as in Definition 2.1 with $S=[k]$. Unfortunately, the total space of $\widetilde{\mathcal U}_{g,k}$ in general has singularities around the nodal points of the fibers of π of the from

$$\big\{(t,x,y)\in\mathbb C^3\colon xy=t^m\big\}\longrightarrow\mathbb C,\qquad (t,x,y)\longrightarrow t;$$

see the proof of [1, Proposition 1.4]. This causes some difficulty in defining notions of smoothness for bundle sections over $\widetilde{\mathcal U}_{g,k}\times X$. The approach of [23, Section 2] to deal with this issue is to embed the universal curve (2.8) into some $\mathbb P^N$. Following a suggestion of P. Georgieva, we bypass such an embedding by using perturbations supported away from the nodes as in [15].

The notions of marked nodal and GU Riemann surfaces are introduced in Section 2.1. The topology of the Deligne-Mumford moduli space $\overline{\mathcal M}_{g,S}$ of stable genus g S-marked nodal curves and covers $\widetilde{\mathcal M}_{g,k}$ of $\overline{\mathcal M}_{g,k}$ are described in Section 2.2. We introduce GU maps to a manifold in Section 2.3 and Ruan-Tian perturbations in the first half of Section 2.4. In the second half of Section 2.4, we define notions of a GU (J,ν)-map and moduli spaces of (J,ν)-maps in the spirit of [23, Section 3]. In Section 2.5, we define spaces of GU (J,ν)-maps of a fixed combinatorial type and relax the degree restriction in the statements of Theorems 1.3 and 1.4; see Theorems 2.5 and 2.6.

2.1 GU Riemann surfaces

A (smooth) Riemann surface or complex curve is a pair $(\Sigma,\mathfrak j)$ consisting of a compact smooth two-dimensional manifold Σ (without boundary) and a complex structure $\mathfrak j$ in the fibers of $T\Sigma$. A nodal Riemann surface is a pair $(\Sigma,\mathfrak j)$ obtained from a Riemann surface $(\widetilde\Sigma,\mathfrak j)$ by identifying pairs of distinct points in a finite subset $\widetilde S_\Sigma\subset\widetilde\Sigma$ (with each point of $\widetilde S_\Sigma$ identified with precisely one other point of $\widetilde S_\Sigma$); see Figure 1. A GU Riemann surface is a pair $(\Sigma,\mathfrak j)$ obtained from a Riemann surface $(\widetilde\Sigma,\mathfrak j)$ by identifying each point in a finite subset $\widetilde S_\Sigma\subset\widetilde\Sigma$ with at least one other point of $\widetilde S_\Sigma$. In both cases, the pair $(\widetilde\Sigma,\mathfrak j)$ is called the normalization of $(\Sigma,\mathfrak j)$.

An irreducible component of $(\Sigma,\mathfrak j)$ is the image of a topological component of $\widetilde\Sigma$ under the quotient projection

$$q_\Sigma\colon\widetilde\Sigma\longrightarrow\Sigma\,.\qquad(2.1)$$

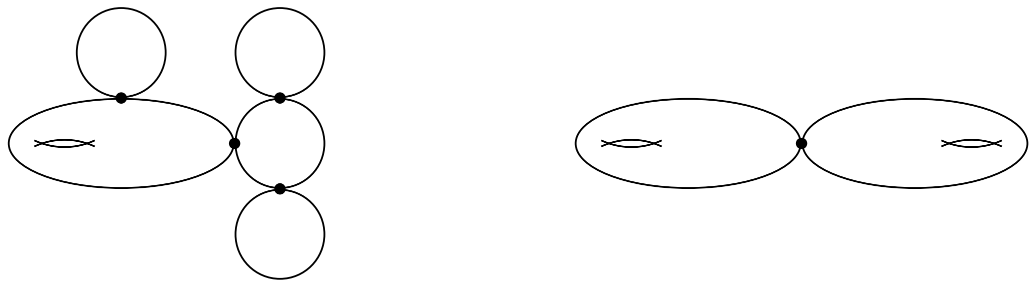


Figure 1: Nodal Riemann surfaces of (arithmetic) genera 1 and 2, respectively

We call the images of the points of $\widetilde{S}_\Sigma$ under this map the lumps of Σ. Each lump joins two or more smooth branches. A lump in a nodal surface joins precisely two smooth branches and is thus a node in the usual sense. We denote the set of lumps of a GU Riemann surface $(\Sigma, \mathfrak{j})$ by S_Σ and its complement by Σ^*. The (arithmetic) genus of a GU Riemann surface $(\Sigma, \mathfrak{j})$ is the number

$$\mathfrak{a}(\Sigma) = \frac{2-\chi(\widetilde{\Sigma})}{2} + \big|\widetilde{S}_\Sigma\big| - \big|S_\Sigma\big|, \tag{2.2}$$

where $\chi(\widetilde{\Sigma})$ is the Euler characteristic of $\widetilde{\Sigma}$.

An equivalence between GU Riemann surfaces $(\Sigma, \mathfrak{j})$ and $(\Sigma', \mathfrak{j}')$ is a homeomorphism $h\colon \Sigma \longrightarrow \Sigma'$ induced by a biholomorphic map $\widetilde{h}$ from $(\widetilde{\Sigma}, \mathfrak{j})$ to $(\widetilde{\Sigma}', \mathfrak{j}')$. A GU map u between GU Riemann surfaces $(\Sigma, \mathfrak{j})$ and $(\Sigma', \mathfrak{j}')$ is a holomorphic map $\widetilde{u}$ from $(\widetilde{\Sigma}_0, \mathfrak{j})$, where $\widetilde{\Sigma}_0$ is a union of topological components of $\widetilde{\Sigma}$, to $(\widetilde{\Sigma}', \mathfrak{j}')$ such that the restriction of $\widetilde{u}$ to every topological component of $\widetilde{\Sigma}_0$ is not constant. We call the topological components of the closure of $\widetilde{\Sigma}$–$\widetilde{\Sigma}_0$ the contracted components of u. Such a map is of degree 1 if $|\widetilde{u}^{-1}(z')|=1$ for every $z'\in\widetilde{\Sigma}'$ and $\widetilde{u}^{-1}(\widetilde{S}_{\Sigma'})\subset\widetilde{S}_\Sigma$. A GU morphism between GU Riemann surfaces $(\Sigma, \mathfrak{j})$ and $(\Sigma', \mathfrak{j}')$ is a continuous map $u\colon \Sigma \longrightarrow \Sigma'$ such that the restriction of u to the union Σ_0 of the irreducible components of Σ on which u is not constant is induced by a GU map. We say that such a morphism is of degree 1 if its restriction to Σ_0 is induced by a degree 1 GU map. We call it a contraction if in addition for every $z'\in\Sigma'$ the subset $u^{-1}(z')\subset\Sigma$ is either a point or a connected GU Riemann surface of genus 0.

Let S be a finite set. A genus g S-marked GU Riemann surface is a tuple

$$\mathcal{C} \equiv \big(\Sigma, \mathfrak{j}, (z_i)_{i\in S}\big), \tag{2.3}$$

where $(\Sigma, \mathfrak{j})$ is a GU Riemann surface of genus g and $z_i\in\widetilde{\Sigma}$. A genus g S-marked nodal Riemann surface is a genus g S-marked GU Riemann surface as in (2.3) such that $(\Sigma, \mathfrak{j})$ is a nodal Riemann surface and $z_i\notin\widetilde{S}_\Sigma$ are distinct points. Since the restriction of (2.1) to $\widetilde{\Sigma}-\widetilde{S}_\Sigma$ is a homeomorphism onto Σ^*, it is customary to view the marked points z_i of a nodal Riemann surface as distinct points of Σ^*. It is also common to index the marked points by the sets

$$[k] \equiv \big\{i\in\mathbb{Z}^+ : i\leqslant k\big\}, \qquad k\in\mathbb{Z}^{\geqslant 0},$$

but allowing arbitrary finite indexing sets is often more convenient.

An **equivalence** between a genus g S-marked GU Riemann surface $\mathcal{C}$ as in (2.3) and another genus g S-marked GU Riemann surface

$$\mathcal{C}' \equiv \big(\Sigma', \mathfrak{j}', (z_i')_{i\in S}\big) \tag{2.4}$$

is an equivalence h between the GU Riemann surfaces $(\Sigma,\mathfrak{j})$ and $(\Sigma',\mathfrak{j}')$ such that $\widetilde{h}(z_i)=z_i'$ for all $i\in S$. We denote by $\mathrm{Aut}(\mathcal{C})$ the group of **automorphisms**, i.e. self-equivalences, of a genus g S-marked GU Riemann surface $\mathcal{C}$. Such a Riemann surface is called **stable** if $\mathrm{Aut}(\mathcal{C})$ is a finite group.

An S-**marked GU map** u between S-marked GU Riemann surfaces $\mathcal{C}$ and $\mathcal{C}'$ as above (not necessarily of the same genus) is a GU map u between the GU Riemann surfaces $(\Sigma,\mathfrak{j})$ and $(\Sigma',\mathfrak{j}')$ so that $\widetilde{u}(z_i)=z_i'$ for all $i\in S$ such that $z_i\in\mathrm{Dom}(\widetilde{u})$ and $z_i'\in\widetilde{u}(\widetilde{S}_\Sigma\cap\mathrm{Dom}(\widetilde{u}))$ for all $i\in S$ such that $z_i\notin\mathrm{Dom}(\widetilde{u})$. An S-**marked GU morphism** between $\mathcal{C}$ and $\mathcal{C}'$ is a GU morphism u between $(\Sigma,\mathfrak{j})$ and $(\Sigma',\mathfrak{j}')$ such that $u(q_\Sigma(z_i))=q_{\Sigma'}(z_i')$ for all $i\in S$.

If $\widetilde{\Sigma}_0,\ldots,\widetilde{\Sigma}_N$ is a partition of $\widetilde{\Sigma}$ into unions of topological components so that $\Sigma_r\equiv q_\Sigma(\widetilde{\Sigma}_r)$ is disjoint from Σ_s for $r,s\in[N]$ distinct, then

$$\mathfrak{a}(\Sigma) = \sum_{r=0}^{N}\mathfrak{a}(\Sigma_r) + \sum_{r=1}^{N}\Big(\big|q_\Sigma^{-1}(\Sigma_0)\cap\widetilde{\Sigma}_r\big|-1\Big). \tag{2.5}$$

If u is a degree 1 GU morphism from a connected GU surface $(\Sigma,\mathfrak{j})$ to $(\Sigma',\mathfrak{j}')$, $\widetilde{\Sigma}_0\subset\widetilde{\Sigma}$ is the domain of the holomorphic map $\widetilde{u}$ as above,

$$S_{\Sigma_0}\equiv\big\{z\in\Sigma_0\colon |q_\Sigma^{-1}(z)\cap\widetilde{\Sigma}_0|\geqslant 2\big\}$$

are the lumps of Σ_0, and $\Sigma_1,\ldots,\Sigma_N\subset\Sigma$ are the topological components of $q_\Sigma(\widetilde{\Sigma}-\widetilde{\Sigma}_0)$, then

$$\begin{split}\mathfrak{a}(\Sigma') =& \mathfrak{a}(\Sigma_0) + \big(|S_{\Sigma_0}|-|u(S_{\Sigma_0})|\big)\\ &+\sum_{r=1}^{N}\big|q_\Sigma(\widetilde{\Sigma}_0)\cap q_\Sigma(\widetilde{\Sigma}_r)-S_{\Sigma_0}\big| - \big|u(\Sigma_1\cup\ldots\cup\Sigma_N)-u(S_{\Sigma_0})\big|.\end{split} \tag{2.6}$$

If for every $z'\in\Sigma'$ the subset $u^{-1}(z')\subset\Sigma$ is connected, then

$$\big|u(\Sigma_1\cup\ldots\cup\Sigma_N)-u(S_{\Sigma_0})\big| = \big|\big\{r\in[N]\colon q_\Sigma(\widetilde{\Sigma}_0)\cap q_\Sigma(\widetilde{\Sigma}_r)\cap S_{\Sigma_0}=\varnothing\big\}\big|.$$

If Σ' is nodal (in addition to u being a morphism), then

$$\big|u(S_{\Sigma_0})\big| = \big|S_{\Sigma_0}\big|, \qquad \big|q_\Sigma(\widetilde{\Sigma}_0)\cap q_\Sigma(\widetilde{\Sigma}_r)\cap S_{\Sigma_0}\big|\leqslant 1 \quad \forall\, r\in[N].$$

Combining the observations in the last two sentences with (2.5) and (2.6), we conclude that $\mathfrak{a}(\Sigma)=\mathfrak{a}(\Sigma')$ if u is a contraction to a nodal Riemann surface $(\Sigma',\mathfrak{j}')$.

From the previous sentence, we obtain the following. If u is an S-marked contraction from a connected S-marked nodal Riemann surface $\mathcal{C}$ to another S-marked nodal Riemann surface $\mathcal{C}'$, then $\mathfrak{a}(\mathcal{C})=\mathfrak{a}(\mathcal{C}')$ and every contracted topological component Σ_r of u is a genus 0 Riemann surface which shares one or two nodes with the non-contracted part Σ_0 of u and carries at most one marked point z_i. If Σ_r does carry a marked point, then it shares precisely one node with Σ_0.

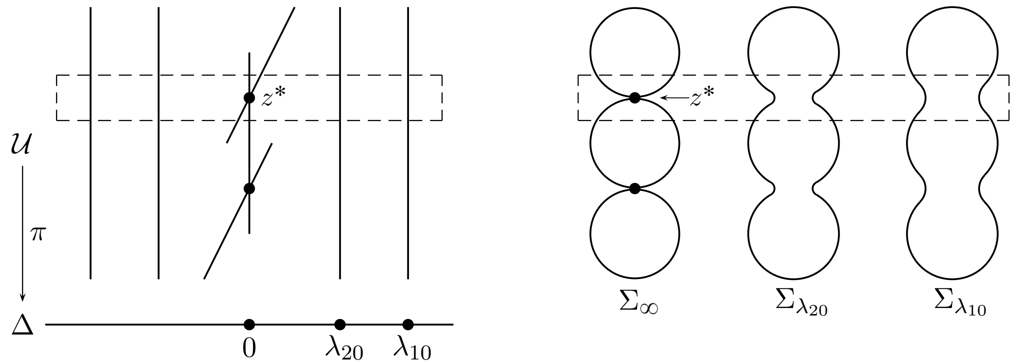


Figure 2: A complex-geometric presentation of a flat family of deformations of $\mathcal{C}_\infty = \pi^{-1}(0)$ and a differential-geometric presentation of these Riemann surfaces.

2.2 Moduli spaces of nodal Riemann surfaces

Let $\mathcal{C}$ be a genus g S-marked nodal Riemann surface as in (2.3). A flat family of deformations of $\mathcal{C}$ is a tuple $(\pi, (s_i)_{i\in S})$, where $\pi: \mathcal{U} \longrightarrow \Delta$ is a holomorphic map from a complex manifold to a neighborhood $\Delta \subset \mathbb{C}^N$ of 0 and $s_i: \Delta \longrightarrow \mathcal{U}$ are holomorphic sections of π, such that

- $\Sigma_\lambda \equiv \pi^{-1}(\lambda)$ is a nodal Riemann surface and $s_i(\lambda) \in \Sigma_\lambda^*$ are distinct points for each $\lambda \in \mathbb{C}^N$,
- $\pi^{-1}(0) = (\Sigma, \mathfrak{j})$ and $s_i(0) = z_i$ for each $i \in S$,
- π is a submersion outside of the nodes of the fibers of π,
- for every $\lambda^* \equiv (\lambda_1^*, \ldots, \lambda_N^*) \in \Delta$ and every node $z^* \in \Sigma_{\lambda^*}$, there exist $i \in [N]$ with $\lambda_i = 0$, neighborhoods Δ_{λ^*} of λ^* in Δ and $\mathcal{U}_{z^*}$ of z^* in $\mathcal{U}$, and a holomorphic map

$$\Psi: \mathcal{U}_{z^*} \longrightarrow \big\{\big((\lambda_1, \ldots, \lambda_N), x, y\big) \in \Delta_{\lambda^*} \times \mathbb{C}^2 : xy = \lambda_i\big\}$$

  such that Ψ is a homeomorphism onto a neighborhood of $(\lambda^*, 0, 0)$ and the composition of Ψ with the projection to Δ_{λ^*} equals $\pi|_{\mathcal{U}_{z^*}}$.

Figure 2 shows such a family from two perspectives.

A sequence of genus g S-marked nodal Riemann surfaces $\mathcal{C}_r$ converges to a genus g S-marked nodal Riemann surface $\mathcal{C}$ if there exist a flat family of deformations of $\mathcal{C}$ as above and $\lambda_r \in \Delta$ for all r sufficiently large such that $\lambda_r \longrightarrow 0$ as $r \longrightarrow \infty$ and the genus g S-marked nodal Riemann surface $(\pi^{-1}(\lambda_r), (s_i(\lambda_r))_{i\in S})$ is equivalent to $\mathcal{C}_r$. This in particular topologizes the set $\overline{\mathcal{M}}_{g,S}$ of the equivalence classes of stable connected genus g S-marked nodal Riemann surfaces. By [13, Theorem 2.7], $\overline{\mathcal{M}}_{g,S}$

- is compact and Hausdorff in the resulting topology,
- contains the subset $\mathcal{M}_{g,S}$ of equivalence classes of stable connected genus g S-marked smooth Riemann surfaces as an open subspace, and
- carries a natural complex orbifold structure of dimension

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,S} = 3g-3 + |S|.$$

If $2g+|S|\geqslant 3$, there is a forgetful morphism

$$\mathfrak{f}_{g,S}:\overline{\mathcal{U}}_{g,S}=\overline{\mathcal{M}}_{g,S\sqcup\{+\}}\longrightarrow\overline{\mathcal{M}}_{g,S}\,;$$

it drops the extra marked point and contracts the unstable irreducible component(s) of the resulting curve if necessary. This morphism determines the universal family over $\overline{\mathcal{M}}_{g,S}$. For $k\in\mathbb{Z}^{\geqslant 0}$, we denote by $\overline{\mathcal{M}}_{g,k}$ the moduli space $\overline{\mathcal{M}}_{g,[k]}$ and by $\overline{\mathcal{U}}_{g,k}$ its universal family $\overline{\mathcal{U}}_{g,[k]}$.

For a tuple $\mathscr{D}\equiv(g_1,S_1;g_2,S_2)$ consisting of $g_1,g_2\in\mathbb{Z}^{\geqslant 0}$ with $g=g_1+g_2$ and $S_1,S_2\subset S$ with $S=S_1\sqcup S_2$, denote by

$$\overline{\mathcal{M}}_{\mathscr{D}}\subset\overline{\mathcal{M}}_{g,S}$$

the closure of the subspace of marked curves with two irreducible components Σ_1 and Σ_2 of genera g_1 and g_2, respectively, and carrying the marked points indexed by S_1 and S_2, respectively. Let

$$\iota_{\mathscr{D}}:\overline{\mathcal{M}}_{g_1,S_1\sqcup\{+\}}\times\overline{\mathcal{M}}_{g_2,S_2\sqcup\{+\}}\longrightarrow\overline{\mathcal{M}}_{g,S}$$

be the natural immersion with image $\overline{\mathcal{M}}_{\mathscr{D}}$ (it identifies the two extra marked points into a node). We denote by $\mathrm{Div}_{g,S}$ the set of tuples $\mathscr{D}$ as above.

Definition 2.1. Suppose $g\in\mathbb{Z}^{\geqslant 0}$ and S is a finite set so that $2g+|S|\geqslant 3$. Let

$$\mathfrak{p}:\widetilde{\mathcal{M}}_{g,S}\longrightarrow\overline{\mathcal{M}}_{g,S} \tag{2.7}$$

be a finite branched cover in the orbifold category. A universal curve over $\widetilde{\mathcal{M}}_{g,S}$ is a tuple

$$\big(\pi:\widetilde{\mathcal{U}}_{g,S}\longrightarrow\widetilde{\mathcal{M}}_{g,S},(s_i)_{i\in S}\big), \tag{2.8}$$

where $\widetilde{\mathcal{U}}_{g,S}$ is a projective variety and π is a projective morphism with disjoint sections s_i, such that for each $\widetilde{\mathcal{C}}\in\widetilde{\mathcal{M}}_{g,S}$ the tuple $(\pi^{-1}(\widetilde{\mathcal{C}}),(s_i(\widetilde{\mathcal{C}}))_{i\in S})$ is a stable genus g S marked nodal Riemann surface whose equivalence class is $\mathfrak{p}(\widetilde{\mathcal{C}})$.

Definition 2.2. Suppose $g\in\mathbb{Z}^{\geqslant 0}$ and S is a finite set so that $2g+|S|\geqslant 3$. A cover (2.7) is regular if it admits a universal curve and for every element $\mathscr{D}\equiv(g_1,S_1;g_2,S_2)$ of $\mathrm{Div}_{g,S}$ there exist covers $\widetilde{\mathcal{M}}_{g_i,S_i\sqcup\{+\}}$ of $\overline{\mathcal{M}}_{g_i,S_i\sqcup\{+\}}$ such that

$$\big(\overline{\mathcal{M}}_{g_1,S_1\sqcup\{+\}}\times\overline{\mathcal{M}}_{g_2,S_2\sqcup\{+\}}\big)\times_{(\iota_{\mathscr{D}},\mathfrak{p})}\widetilde{\mathcal{M}}_{g,S}\approx\widetilde{\mathcal{M}}_{g_1,S_1\sqcup\{+\}}\times\widetilde{\mathcal{M}}_{g_2,S_2\sqcup\{+\}}\,.$$

The moduli space $\overline{\mathcal{M}}_{0,S}$ is a complex manifold isomorphic to a blowup of $(\mathbb{P}^1)^{|S|-3}$. It can be embedded into $(\mathbb{P}^1)^N$ for $N=N(|S|)$ sufficiently large; see [18, Appendix D]. The universal family over $\overline{\mathcal{M}}_{0,S}$ satisfies the requirement of Definition 2.1. For $g\geqslant 2$, [1, Theorems 2.2,3.9] provide covers (2.7) satisfying the last requirement of Definition 2.2 so that the orbifold fiber product

$$\pi:\widetilde{\mathcal{U}}_{g,k}\equiv\widetilde{\mathcal{M}}_{g,S}\otimes_{\overline{\mathcal{M}}_{g,S}}\overline{\mathcal{U}}_{g,S}\longrightarrow\widetilde{\mathcal{M}}_{g,S} \tag{2.9}$$

satisfies the requirement of Definition 2.1; see also [21, Section 2.2]. The same reasoning applies in the $g=1$ case if $S\neq\varnothing$.

2.3 Maps from GU domains

Let X be a manifold. If Σ is a connected smooth orientable surface, a C^1-map $u\colon \Sigma\longrightarrow X$ is

- somewhere injective if there exists $z\in\Sigma$ such that $u^{-1}(u(z))=\{z\}$ and $\mathrm{d}_z u\neq 0$,
- multiply covered if $u=u'\circ h$ for some connected smooth orientable surface Σ', branched cover $h\colon\Sigma\longrightarrow\Sigma'$ of degree different from ±1, and a C^1-map $u'\colon\Sigma'\longrightarrow X$,
- simple if it is not multiply covered.

By [33, Proposition 4.11], a simple J-holomorphic map is somewhere injective.

Let $\ell\in\mathbb{Z}^+\sqcup\{\infty\}$ and $(\Sigma,\mathfrak{j})$ be a GU Riemann surface $(\Sigma,\mathfrak{j})$. A continuous map $u\colon\Sigma\longrightarrow X$ is a C^ℓ-map if the induced map $\widetilde{u}:\widetilde{\Sigma}\longrightarrow X$ is C^ℓ. The degree of a C^ℓ-map $u\colon\Sigma\longrightarrow X$ is the homology class

$$A\equiv u_*[\Sigma]=\widetilde{u}_*[\widetilde{\Sigma}]\in H_2(X;\mathbb{Z}). \tag{2.10}$$

If in addition $J\in\mathcal{J}(X)$, we define

$$\bar\partial_{J,\mathfrak{j}}u=\frac{1}{2}\big(\mathrm{d}\widetilde{u}+J\circ\mathrm{d}\widetilde{u}\circ\mathfrak{j}\big)\colon\big(T\widetilde{\Sigma},-\mathfrak{j}\big)\longrightarrow\widetilde{u}^*(TX,J)\,.$$

A C^ℓ-map $u\colon\Sigma\longrightarrow X$ is J-holomorphic if $\bar\partial_{J,\mathfrak{j}}u=0$.

Let S be a finite set. An S-marked GU C^ℓ-map is a tuple $\mathbf{u}\equiv(\mathcal{C},u)$, where $\mathcal{C}$ is an S-marked GU Riemann surface as in (2.3) and u is a C^ℓ-map from $(\Sigma,\mathfrak{j})$. We call such a tuple

(1) reduced if the restriction of $\widetilde{u}$ to every topological component of $\widetilde{\Sigma}$ is simple and the images of any two such components under $\widetilde{u}$ are distinct;

(2) basic if $\mathbf{u}$ is reduced, $z_i\in\widetilde{\Sigma}-\widetilde{S}_\Sigma$ for every $i\in S$, and these points are distinct.

If $(\Sigma,\mathfrak{j})$ is an S-marked *nodal* Riemann surface, then a reduced C^ℓ-map is basic. An S-marked GU C^ℓ-map $\mathbf{u}$ as above is equivalent to another S-marked C^ℓ-map $(\mathcal{C}',u')$ if there exists an equivalence h between $\mathcal{C}$ and $\mathcal{C}'$ such that $u=u'\circ h$. An S-marked GU C^ℓ-map $\mathbf{u}$ is stable if its group of automorphisms is finite.

For $A\in H_2(X;\mathbb{Z})$, $g\in\mathbb{Z}^+$, and $J\in\mathcal{J}(X)$, let $\mathfrak{M}^\dagger_{g,S}(A;J)$ be the space of equivalence classes of stable J-holomorphic maps from connected smooth genus g S-marked GU Riemann surfaces. It inherits a topology from the space of smooth maps into X. Denote by

$$\mathfrak{M}^{\dagger*}_{g,S}(A;J),\mathfrak{M}_{g,S}(A;J)\subset\mathfrak{M}^\dagger_{g,S}(A;J)$$

the subspaces of reduced maps and of maps from domains with distinct marked points. Let

$$\mathfrak{M}^*_{g,S}(A;J)\equiv\mathfrak{M}^{\dagger*}_{g,S}(A;J)\cap\mathfrak{M}_{g,S}(A;J)\subset\mathfrak{M}^\dagger_{g,S}(A;J)$$

the subspaces of basic maps. For each $i\in S$, let

$$\mathsf{ev}_i\colon\mathfrak{M}^\dagger_{g,S}(A;J)\longrightarrow X,\qquad\big[\Sigma,\mathfrak{j},(z_j)_{j\in S},u\big]\longrightarrow u(z_i),$$

be the evaluation map for the i-th marked point.

For $A\neq 0$, the map

$$\mathfrak{f}\colon \mathfrak{M}^{\dagger}_{g,S\sqcup\{+\}}(A;J) \longrightarrow \mathfrak{M}^{\dagger}_{g,S}(A;J), \quad \big[\Sigma,\mathfrak{j},(z_i)_{i\in S\sqcup\{+\}},u\big] \longrightarrow \big[\Sigma,\mathfrak{j},(z_i)_{i\in S},u\big], \tag{2.11}$$

is a well-defined continuous orbi-bundle. The fiber of $\mathfrak{f}$ over $[\Sigma,\mathfrak{j},(z_i)_{i\in S},u]$ is the quotient of Σ by $\mathrm{Aut}(\Sigma,\mathfrak{j},(z_i)_{i\in S},u)$. The tangent spaces to the fibers determine a complex line orbi-bundle

$$\mathcal{T} \longrightarrow \mathfrak{M}^{\dagger}_{g,S\sqcup\{+\}}(A;J)\,.$$

For each $i\in S$, let

$$s_i\colon \mathfrak{M}^{\dagger}_{g,S}(A;J) \longrightarrow \mathfrak{M}^{\dagger}_{g,S\sqcup\{+\}}(A;J)$$

be the section such that

$$s_i\big([\Sigma,\mathfrak{j},(z_j)_{j\in S},u]\big) = \big[\Sigma,\mathfrak{j},(z'_j)_{j\in S\sqcup\{+\}},u\big], \quad z'_j = \begin{cases} z_j, & \text{if } j\in S;\\ z_i, & \text{if } j=+. \end{cases}$$

The complex line orbi-bundle

$$L_i\equiv s_i^*\mathcal{T} \longrightarrow \mathfrak{M}^{\dagger}_{g,S}(A;J) \tag{2.12}$$

is called the universal tangent line bundle for the i-th marked point.

The direct analogue of the line bundle (2.12) can be similarly defined over the space of equivalence classes of stable C^{ℓ}-maps from connected smooth genus g S-marked GU Riemann surfaces and for $A=0$ (after suitably restricting the domain of $\mathfrak{f}$ in (2.11)). The restriction of (2.12) to $\mathfrak{M}_{g,S}(A;J)$ naturally extends over Gromov's space of equivalence classes of stable nodal J-holomorphic maps into X, but this is not relevant for the purposes of the present paper.

2.4 Ruan-Tian perturbations

Let $g,k\in\mathbb{Z}^{\geqslant 0}$ with $2g+k\geqslant 3$ and $\mathfrak{p}$ as in (2.7) be a regular cover with $S=[k]$. We denote by $\widetilde{\mathcal{U}}^*_{g,k}\subset\widetilde{\mathcal{U}}_{g,k}$ the complement of the nodes of the fibers of π in (2.8) and by

$$\mathcal{T}\mathfrak{p}\equiv \ker \mathrm{d}\big(\pi|_{\widetilde{\mathcal{U}}^*_{g,k}}\big) \longrightarrow \widetilde{\mathcal{U}}^*_{g,k}$$

the vertical tangent bundle. The latter is a complex line bundle; let $\mathfrak{j}_{\mathcal{U}}$ denote its complex structure.

Let X be a manifold and

$$\pi_1,\pi_2\colon \widetilde{\mathcal{U}}^*_{g,k}\times X \longrightarrow \widetilde{\mathcal{U}}^*_{g,k}, X$$

be the projection maps. For a section ν of a bundle E over $\widetilde{\mathcal{U}}^*_{g,k}\times X$, we denote by $\mathrm{supp}(\nu)$ the closure of the set

$$\big\{(z,x)\in\widetilde{\mathcal{U}}^*_{g,k}\times X\colon \nu(z,x)\neq 0\big\} \subset \widetilde{\mathcal{U}}_{g,k}\times X$$

in $\widetilde{\mathcal{U}}_{g,k}\times X$. For $J\in\mathcal{J}(X)$, let

$$\begin{split}\Gamma^{0,1}_{\mathfrak{p}}(X;J) = \Big\{\nu\in\Gamma\big(\widetilde{\mathcal{U}}^*_{g,k}\times X; \pi_1^*(\mathcal{T}\mathfrak{p},-\mathfrak{j}_{\mathcal{U}})^*\otimes_{\mathbb{C}}\pi_2^*(TX,J)\big)\colon\\ \mathrm{supp}(\nu)\subset \big(\widetilde{\mathcal{U}}^*_{g,k}-\bigcup_{i=1}^{k}\mathrm{Im}(s_i)\big)\times X\Big\}.\end{split} \tag{2.13}$$

The condition that $\mathrm{supp}(\nu)$ be disjoint from the sections s_i is needed to define analogues of the bundle sections (1.7) in Section 2.5.

Define

$$\mathcal{H}_{\mathfrak{p}}(X) = \big\{(J,\nu)\colon J\in\mathcal{J}(X),\, \nu\in\Gamma_{\mathfrak{p}}^{0,1}(X;J)\big\}. \tag{2.14}$$

If in addition ω is a symplectic form on X, let

$$\mathcal{H}_{\mathfrak{p};\omega}(X) \subset \mathcal{H}'_{\mathfrak{p};\omega}(X) \subset \mathcal{H}_{\mathfrak{p}}(X) \tag{2.15}$$

be the subspaces of pairs (J,ν) so that $J\in\mathcal{J}_\omega(X)$ and $J\in\mathcal{J}'_\omega(X)$, respectively.

Definition 2.3. Suppose $g,g',k\in\mathbb{Z}^{\geqslant 0}$ with $2g+k\geqslant 3$, $\mathfrak{p}$ as in (2.7) is a regular cover, (X,J) is an almost complex manifold, and $\nu\in\Gamma_{\mathfrak{p}}^{0,1}(X;J)$. A genus g' k-marked GU (J,ν)-map is a tuple

$$\mathbf{u}\equiv\big(\mathcal{C},u_{\mathcal{M}}\colon\Sigma\dashrightarrow\widetilde{\mathcal{U}}_{g,k},u\colon\Sigma\longrightarrow X\big), \tag{2.16}$$

where $\mathcal{C}$ is a connected genus g' k-marked GU Riemann surface as in (2.3), $u_{\mathcal{M}}$ is a degree 1 k-marked GU map onto a fiber $\mathcal{C}'$ of (2.8), and u is a C^1-map such that

$$\bar{\partial}_{J,\mathfrak{j}}\widetilde{u}\big|_z=\begin{cases}\nu\big(\widetilde{u}_{\mathcal{M}}(z),\widetilde{u}(z)\big)\circ\mathrm{d}_z\widetilde{u}_{\mathcal{M}}, & \text{if } z\in\mathrm{Dom}(\widetilde{u}_{\mathcal{M}});\\ 0, & \text{otherwise.}\end{cases}$$

Definition 2.4. Suppose $g,g',k,\mathfrak{p}$, (X,J), and ν are as in Definition 2.3. A (J,ν)-map $\mathbf{u}$ as in (2.16) is

(1) reduced if the restriction of $\widetilde{u}$ to every contracted component of $u_{\mathcal{M}}$ is somewhere injective and the images of any two such components under $\widetilde{u}$ are distinct;

(2) basic if $\mathbf{u}$ is reduced and $z_i\in\widetilde{\Sigma}-\widetilde{S}_\Sigma$ are distinct points.

The degree of a GU (J,ν)-map (2.16) is the degree of u as in (2.10). We call a GU (J,ν)-map $\mathbf{u}$ as in (2.16) a nodal (J,ν)-morphism if $\mathcal{C}$ is a k-marked nodal Riemann surface and $u_{\mathcal{M}}$ is an S-marked contraction to a fiber $\mathcal{C}'$ of (2.8); see Section 2.1. This implies that $g'=g$ and that $\widetilde{u}_{\mathcal{M}}$ does not contract any loops of spheres. A reduced nodal (J,ν)-morphism is automatically basic.

A (J,ν)-map $\mathbf{u}$ as in (2.16) is equivalent to another (J,ν)-map

$$\mathbf{u}'\equiv\big(\mathcal{C}',u'_{\mathcal{M}}\colon\Sigma'\dashrightarrow\widetilde{\mathcal{U}}_{g,k},u'\colon\Sigma'\longrightarrow X\big)$$

if there exists an equivalence h between $\mathcal{C}$ and $\mathcal{C}'$ such that

$$\widetilde{u}_{\mathcal{M}}=\widetilde{u}'_{\mathcal{M}}\circ\widetilde{h}\big|_{\mathrm{Dom}(\widetilde{u}_{\mathcal{M}})} \qquad\text{and}\qquad u=u'\circ h\,.$$

A (J,ν)-map $\mathbf{u}$ is stable if its group of automorphisms is finite. This is the case if and only if the degree of the restriction of $\widetilde{u}$ to every contracted component of $u_{\mathcal{M}}$ containing only one or two special (nodal or marked) points is not zero.

For $A\in H_2(X;\mathbb{Z})$, let

$$\dim_{g,k}(A) = \big\langle c_1(TX), A\big\rangle + (n-3)(1-g) + k, \quad \text{where } \ 2n \equiv \dim_{\mathbb{R}} X\,. \tag{2.17}$$

For $(J,\nu)\in\mathcal{H}_{\mathfrak{p}}(X)$, we denote by $\overline{\mathfrak{M}}_{g,k}(A;J,\nu)$ the moduli space of equivalences classes of stable degree A genus g k-marked nodal (J,ν)-morphisms and by

$$\mathfrak{M}_{g,k}(A;J,\nu) \subset \overline{\mathfrak{M}}_{g,k}(A;J,\nu)$$

the subspace of maps from smooth domains; all of these maps are basic in the sense of Definition 2.4. This subspace inherits a topology from the space of smooth maps into X. The map

$$\mathsf{st}\times\mathsf{ev}\colon \mathfrak{M}_{g,k}(A;J,\nu) \longrightarrow \overline{\mathcal{M}}_{g,k}\times X^k, \quad \big[\mathcal{C}, u_{\mathcal{M}}, u\big] \longrightarrow \big(\mathfrak{p}(\pi(u_{\mathcal{M}}(\Sigma))), (u(z_i))_{i\in[k]}\big), \tag{2.18}$$

is continuous with respect to this topology.

Let B be a manifold, possibly with boundary. Denote by

$$\pi_B, \pi_{\mathcal{U}}, \pi_X\colon B\times\widetilde{\mathcal{U}}^*_{g,k}\times X \longrightarrow B, \widetilde{\mathcal{U}}^*_{g,k}, X$$

the projection maps. For

$$\nu \in \Gamma\big(B\times\widetilde{\mathcal{U}}^*_{g,k}\times X; \pi_{\mathcal{U}}^*\mathcal{T}\mathfrak{p}^*\otimes_{\mathbb{R}}\pi_X^*TX\big)$$

and $b\in B$, let

$$\nu_b \in \Gamma\big(\widetilde{\mathcal{U}}^*_{g,k}\times X; \pi_1^*\mathcal{T}\mathfrak{p}^*\otimes_{\mathbb{R}}\pi_2^*TX\big)$$

be the associated bundle section. Define

$$\mathcal{H}_{\mathfrak{p}}(B;X) = \big\{(J,\nu)\in\mathcal{J}(B;X)\times\Gamma\big(B\times\widetilde{\mathcal{U}}^*_{g,k}\times X; \pi_{\mathcal{U}}^*\mathcal{T}\mathfrak{p}^*\otimes_{\mathbb{R}}\pi_X^*TX\big)\colon (J_b,\nu_b)\in\mathcal{H}_p(X)\ \forall\, b\in B\big\}.$$

For $\omega\in\mathrm{Symp}(B;X)$, denote by

$$\mathcal{H}_{\mathfrak{p};\omega}(B;X) \subset \mathcal{H}'_{\mathfrak{p};\omega}(B;X) \subset \mathcal{H}_{\mathfrak{p}}(B;X)$$

the subspaces of pairs (J,ν) so that $J\in\mathcal{J}_\omega(X)$ and $J\in\mathcal{J}'_\omega(X)$, respectively. For $(J_\circ,\nu_\circ)$ in $\mathcal{H}_{\mathfrak{p};\omega}(\partial B;X)$, $\mathcal{H}'_{\mathfrak{p};\omega}(\partial B;X)$, or $\mathcal{H}_{\mathfrak{p}}(\partial B;X)$, define

$$\mathcal{H}_{\omega;J_\circ,\nu_\circ}(B;X) \subset \mathcal{H}_{\mathfrak{p};\omega}(B;X), \quad \mathcal{H}'_{\omega;J_\circ,\nu_\circ}(B;X) \subset \mathcal{H}'_{\mathfrak{p};\omega}(B;X), \quad \mathcal{H}_{J_\circ,\nu_\circ}(B;X) \subset \mathcal{H}_{\mathfrak{p}}(B;X),$$

respectively, analogously to (1.8).

2.5 Transversality for (J,ν)-maps

The general structure of GU Riemann surfaces and GU J-holomorphic maps are described by graph-like combinatorial objects. An edge of a graph is an element of the two-fold symmetric product of the set of vertices or equivalently a two-element subset of the set of flags (an edge either joins two different vertices or goes from a vertex back to itself). However, the topological types of GU Riemann surfaces correspond to more complicated objects, which we call GU graphs below. An edge of a GU graph is an m-element subset of the set of flags for some $m\geqslant 2$. All these objects are defined below.

Let S be a finite set. An S-marked GU graph is a tuple

$$\gamma \equiv \big(\mathfrak{g}\!:\mathrm{Ver}\longrightarrow\mathbb{Z}^{\geqslant 0}, \varepsilon\!: S\sqcup\mathrm{Fl}\longrightarrow\mathrm{Ver}, \mathrm{Edg}\big), \tag{2.19}$$

where Ver and Fl are finite sets (of vertices and flags, respectively) and Edg is a partition of Fl into subsets e with $|e|\geqslant 2$. An S-marked graph is an S-marked GU graph as in (2.19) such that $|e|=2$ for every $e\in\mathrm{Edg}$. An S-marked graph can be depicted as in the left and middle diagrams of Figure 3 on page 21, where $S=\{1,2\}$ and a line segment connects each label $i\in S$ with $\varepsilon(i)\in\mathrm{Ver}$. An example of a GU graph which is not a graph is represented by the right diagram of Figure 3, along with the specifications in the two lines above its caption. Let

$$\mathfrak{a}(\gamma) \equiv 1+\sum_{v\in\mathrm{Ver}}\mathfrak{g}(v) - |\mathrm{Ver}|+|\mathrm{Fl}|-|\mathrm{Edg}| \tag{2.20}$$

be the arithmetic genus of γ.

For $f\in\mathrm{Fl}$, we denote by $e_f\in\mathrm{Edg}$ the unique element of Edg containing f. For each $v\in\mathrm{Ver}$, let

$$S_v(\gamma) = \varepsilon^{-1}(v) \subset S\sqcup\mathrm{Fl}. \tag{2.21}$$

A vertex $v\in\mathrm{Ver}$ of γ is trivalent if

$$2\mathfrak{g}(v)+\big|S_v(\gamma)\big| \geqslant 3. \tag{2.22}$$

The GU graph γ is trivalent if all its vertices are trivalent. The GU graph γ is connected if for all $v,v'\in\mathrm{Ver}$ distinct there exist

$$\begin{gathered} m\in\mathbb{Z}^+,\; f_1^-,f_1^+,\ldots,f_m^-,f_m^+\in\mathrm{Fl} \qquad \text{s.t.}\\ \varepsilon\big(f_1^-\big)=v,\; \varepsilon\big(f_m^+\big)=v',\; \varepsilon\big(f_i^+\big)=\varepsilon\big(f_{i+1}^-\big)\;\forall\,i\in[m-1],\; e_{f_i^-}=e_{f_i^+}\;\forall\,i\in[m]. \end{gathered}$$

An equivalence between an S-marked GU graph as in (2.19) and another S-marked GU graph

$$\gamma' \equiv \big(\mathfrak{g}'\!:\mathrm{Ver}'\longrightarrow\mathbb{Z}^{\geqslant 0}, \varepsilon'\!: S\sqcup\mathrm{Fl}'\longrightarrow\mathrm{Ver}', \mathrm{Edg}'\big) \tag{2.23}$$

is a pair of bijections $h_{\mathrm{Ver}}\!: \mathrm{Ver}\longrightarrow\mathrm{Ver}'$ and $h_{\mathrm{Fl}}\!: \mathrm{Fl}\longrightarrow\mathrm{Fl}'$ such that

$$\mathfrak{g}=\mathfrak{g}'\circ h_{\mathrm{Ver}}, \quad h_{\mathrm{Ver}}\circ\varepsilon|_S=\varepsilon'|_S, \quad h_{\mathrm{Ver}}\circ\varepsilon|_{\mathrm{Fl}}=\varepsilon'\circ h_{\mathrm{Fl}}, \quad h_{\mathrm{Fl}}(e)\in\mathrm{Edg}'\;\forall\,e\in\mathrm{Edg}.$$

For $g,k\in\mathbb{Z}^{\geqslant 0}$, let $\mathcal{A}_{g,k}$ denote the (finite) set of (equivalence classes of) connected trivalent *graphs* γ as in (2.19) with $S=[k]$ and $\mathfrak{a}(\gamma)=g$. This set is empty unless $2g+k\geqslant 3$.

Let γ be as in (2.19). An S-marked GU Riemann surface $\mathcal{C}$ as in (2.3) is of combinatorial type γ if the set of the topological components of $\widetilde{\Sigma}$ and the set $\widetilde{S}_\Sigma$ of the lump branches of Σ can be identified with the sets Ver and Fl, respectively, so that

- the genus of the topological component $\widetilde{\Sigma}_v$ of $\widetilde{\Sigma}$ corresponding to $v\in\mathrm{Ver}$ is $\mathfrak{g}(v)$,
- $z_i\in\widetilde{\Sigma}_{\varepsilon(i)}$ for each $i\in S$ and $z_f\in\widetilde{\Sigma}_{\varepsilon(f)}$ for each $f\in\mathrm{Fl}$, where $z_f\in\widetilde{S}_\Sigma$ is the point corresponding to f,
- for $f,f'\in\mathrm{Fl}$, $q_\Sigma(z_f)=q_\Sigma(z_{f'})$ if and only if $f,f'\in e$ for some $e\in\mathrm{Edg}$.

Let $A\in H_2(X;\mathbb{Z})$. A degree A k-marked GU graph is a tuple

$$\gamma \equiv \big((\mathfrak{g},\mathfrak{d})\colon \mathrm{Ver}\longrightarrow \mathbb{Z}^{\geqslant 0}\oplus H_2(X;\mathbb{Z}), \varepsilon\colon [k]\sqcup \mathrm{Fl}\longrightarrow \mathrm{Ver}, \mathrm{Edg}\big) \tag{2.24}$$

such that the tuple

$$\gamma_{\mathcal{M}} \equiv \big(\mathfrak{g}\colon \mathrm{Ver}\longrightarrow \mathbb{Z}^{\geqslant 0}, \varepsilon\colon [k]\sqcup \mathrm{Fl}\longrightarrow \mathrm{Ver}, \mathrm{Edg}\big)$$

is a k-marked GU graph and

$$\sum_{v\in \mathrm{Ver}}\mathfrak{d}(v) = A, \qquad \big\langle \omega, \mathfrak{d}(v)\big\rangle \geqslant 0 \;\; \forall\; v\in \mathrm{Ver}\,. \tag{2.25}$$

Let $\mathfrak{a}(\gamma)$ denote the arithmetic genus $\mathfrak{a}(\gamma_{\mathcal{M}})$ of $\gamma_{\mathcal{M}}$ as in (2.20) and

$$\dim(\gamma) = \big\langle c_1(TX), A\big\rangle + (n-3)\big(1-\mathfrak{a}(\gamma)\big) + k - 2|\mathrm{Fl}| + 3|\mathrm{Edg}|. \tag{2.26}$$

We say that a k-marked C^1-map $\mathbf{u}\equiv(\mathcal{C},u)$ is of combinatorial type γ if the k-marked GU Riemann surface $\mathcal{C}$ is combinatorial type $\gamma_{\mathcal{M}}$ and for every $v\in \mathrm{Ver}$ the degree of the restriction of u to the irreducible component $\Sigma_v\subset\Sigma$ corresponding to v is $\mathfrak{d}(v)$.

The general structure of a GU (J,ν)-map $\mathbf{u}$ as in (2.16) is specified by triples $(\gamma;\gamma',\varpi)$, with

- γ as in (2.24) describing the X-component u of $\mathbf{u}$,
- γ' as in (2.23) describing the fiber $\mathcal{C}'$ of (2.8) containing the image of $u_{\mathcal{M}}$, and
- ϖ describing the GU map from $\mathcal{C}$ to $\mathcal{C}'$.

This is made precise below.

Let γ be as in (2.19). Denote by $\mathcal{A}(\gamma)$ the collection of pairs (γ_0,ϖ), where

$$\gamma_0 \equiv \big(\mathfrak{g}_0\colon \mathrm{Ver}_0\longrightarrow \mathbb{Z}^{\geqslant 0}, \varepsilon_0\colon [k]\sqcup \mathrm{Fl}_0\longrightarrow \mathrm{Ver}_0, \mathrm{Edg}_0\big) \tag{2.27}$$

is a connected k-marked GU graph with $\mathrm{Ver}_0\subset\mathrm{Ver}$ and $\mathrm{Fl}_0\subset\mathrm{Fl}\cap\varepsilon^{-1}(\mathrm{Ver}_0)$ and

$$\varpi\colon [k]-\varepsilon^{-1}\big(\mathrm{Ver}_0\big) \longrightarrow \mathrm{Fl}\cap\varepsilon^{-1}(\mathrm{Ver}_0)-\mathrm{Fl}_0 \tag{2.28}$$

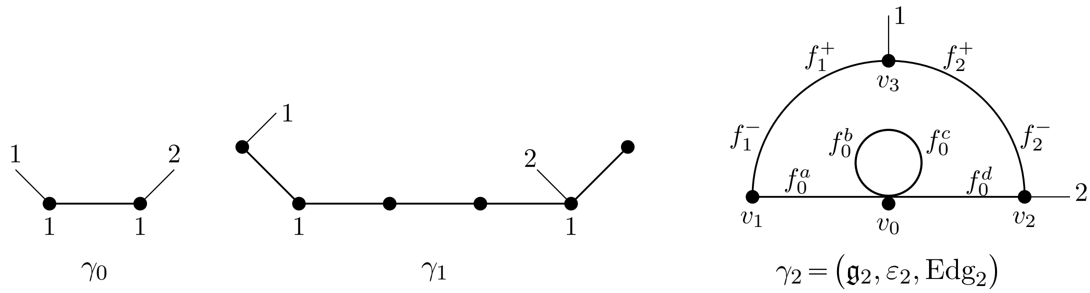
is an injective map, such that

$$\mathfrak{g}_0 = \mathfrak{g}|_{\mathrm{Ver}_0}, \quad \varepsilon_0\big|_{([k]\cap\varepsilon^{-1}(\mathrm{Ver}_0))\sqcup\mathrm{Fl}_0} = \varepsilon\big|_{([k]\cap\varepsilon_0^{-1}(\mathrm{Ver}))\sqcup\mathrm{Fl}_0}, \tag{2.29}$$

$$\varepsilon_0\big|_{[k]-\varepsilon^{-1}(\mathrm{Ver}_0)} = \varepsilon\circ\varpi\colon [k]-\varepsilon^{-1}(\mathrm{Ver}_0) \longrightarrow \mathrm{Ver}_0. \tag{2.30}$$

Thus, γ_0 is obtained from γ by

- dropping every vertex $v\in\mathrm{Ver}-\mathrm{Ver}_0$,
- combining some of the flags in $\mathrm{Fl}\cap\varepsilon^{-1}(\mathrm{Ver}_0)$ into the elements of Edg_0,
- attaching each marked point $i\in[k]-\varepsilon^{-1}(\mathrm{Ver}_0)$ in place of one of the remaining flags $f=\varpi(i)$ in $\mathrm{Fl}\cap\varepsilon^{-1}(\mathrm{Ver}_0)$.



$$\varepsilon_2(f_0^b),\varepsilon_2(f_0^c)=v_0,\ \varepsilon_2(f_0^a),\varepsilon_2(f_1^-)=v_1,\ \varepsilon_2(f_0^d),\varepsilon_2(f_2^-)=v_2,\ \varepsilon_2(f_1^+),\varepsilon_2(f_2^+)=v_3$$
$$\mathfrak{g}_2(v_1),\mathfrak{g}_2(v_2)=1,\ \mathfrak{g}_2(v_0),\mathfrak{g}_2(v_3)=0,\ \mathrm{Edg}_2=\big\{\{f_0^a,f_0^b,f_0^c,f_0^d\},\{f_1^\pm\},\{f_2^\pm\}\big\}$$

Figure 3: A graph γ_0 as in (2.27), a graph $\gamma=\gamma_1$ as in (2.19), and a GU graph $\gamma=\gamma_2$ as in (2.19) such that $(\gamma_0,\varpi_1)\in\mathcal{A}(\gamma_1)$ and $(\gamma_0,\varpi_2)\in\mathcal{A}(\gamma_2)$ for some $\varpi_1(1)$ and $\varpi_2(1)$. The value of $\mathfrak{g}$ on the vertices with the number 1 next to them is 1; its value on the remaining vertices of γ_0 and γ_1 is 0.

Examples of pairs $(\gamma_0,\varpi)\in\mathcal{A}(\gamma)$ appear in Figure 3.

For $g,k\in\mathbb{Z}^{\geqslant 0}$ and a k-marked GU graph γ as in (2.19), let $\mathcal{A}_g(\gamma)$ denote the subset of pairs $(\gamma_0,\varpi)\in\mathcal{A}(\gamma)$ so that $\gamma_0\in\mathcal{A}_{g,k}$. For $A\in H_2(X;\mathbb{Z})$ and a degree A k-marked GU graph γ as in (2.24), denote by $\mathcal{A}_g(\gamma)\subset\mathcal{A}_g(\gamma_{\mathcal{M}})$ the subset of pairs (γ_0,ϖ) with γ_0 as in (2.27) so that $\mathfrak{d}(v)\neq 0$ for all $v\in\mathrm{Ver}-\mathrm{Ver}_0$.

Suppose γ is as in (2.19) with $S=[k]$, γ_0 is as in (2.27), $\mathcal{C}$ is a k-marked GU Riemann surface of type γ as in (2.3), and $\mathcal{C}'$ is an S-marked nodal Riemann surface of type γ_0 as in (2.4). If $u_{\mathcal{M}}$ is a degree 1 k-marked map from $\mathcal{C}$ to $\mathcal{C}'$, we can identify Ver_0 and Fl_0 with subsets of Ver and Fl, respectively, so that

$$\widetilde{u}_{\mathcal{M}}(\widetilde{\Sigma}_v)=\widetilde{\Sigma}'_v\quad\forall\,v\in\mathrm{Ver}_0,\qquad \widetilde{u}_{\mathcal{M}}(z_f)=z'_f\quad\forall\,f\in\mathrm{Fl}_0\,,$$

and the assumptions in (2.29) are satisfied; see Section 2.1. Since $\mathcal{C}'$ is nodal, there also exists a unique injective map ϖ as in (2.28) satisfying (2.30); if $\mathcal{C}'$ were not nodal, ϖ might not have been injective and might have taken values in Fl_0. We define the combinatorial type of $u_{\mathcal{M}}$ to be the pair $(\gamma_0,\varpi)\in\mathcal{A}(\gamma)$.

Suppose γ is as in (2.24), $(\gamma_0,\varpi)\in\mathcal{A}_g(\gamma_{\mathcal{M}})$, and $\mathfrak{p}$ as in (2.7) is a regular cover. For $(J,\nu)\in\mathcal{H}_{\mathfrak{p}}(X)$, let $\mathfrak{M}^\dagger_{\gamma;\gamma_0,\varpi}(J,\nu)$ denote the space of equivalence classes of stable GU (J,ν)-maps $\mathbf{u}$ as in (2.16) so that the degree 1 S-marked map $u_{\mathcal{M}}$ is of combinatorial type (γ_0,ϖ) and the k-marked C^1-map $(\mathcal{C},\mathbf{u})$ is of combinatorial type γ. For a manifold B, possibly with boundary, and $(J,\nu)\in\mathcal{H}_{\mathfrak{p}}(B;X)$, let

$$\mathfrak{M}^\dagger_{\gamma;\gamma_0,\varpi}(J,\nu)=\big\{(b,[\mathbf{u}])\colon b\in B,\ [\mathbf{u}]\in\mathfrak{M}^\dagger_{\gamma;\gamma_0,\varpi}(J_b,\nu_b)\big\}.$$

These spaces inherit topologies from spaces of smooth maps from smooth domains.

For each $i\in[k]$, let

$$\mathsf{ev}_i\colon\mathfrak{M}^\dagger_{\gamma;\gamma_0,\varpi}(J,\nu)\longrightarrow X\qquad\text{and}\qquad L_i\longrightarrow\mathfrak{M}^\dagger_{\gamma;\gamma_0,\varpi}(J,\nu)\tag{2.31}$$

be the natural evaluation map and the universal tangent line bundle, respectively, for the i-th marked point; these are pullbacks from one of the factors. Denote by

$$\mathsf{st}\colon \mathfrak{M}^{\dagger}_{\gamma;\gamma_0,\varpi}(J,\nu) \longrightarrow \overline{\mathcal{M}}_{g,k}, \quad [\mathcal{C},u_{\mathcal{M}},u] \longrightarrow \mathfrak{p}\big(\pi(u_{\mathcal{M}}(\Sigma))\big), \tag{2.32}$$

the stabilization map. Let

$$\mathfrak{M}^{*}_{\gamma;\gamma_0,\varpi}(J,\nu) \subset \mathfrak{M}^{\dagger *}_{\gamma;\gamma_0,\varpi}(J,\nu) \subset \mathfrak{M}_{\gamma;\gamma_0,\varpi}(J,\nu)$$

be the subspaces of basic and reduced maps, respectively.

For a tuple $\mathfrak{m}\equiv(m_i)_{i\in[k]}$ in $(\mathbb{Z}^{\geqslant 0})^k$, define

$$\begin{gathered}\mathcal{Z}^{*}_{\gamma;\gamma_0,\varpi;\mathfrak{m}}(J,\nu) \subset \mathfrak{M}^{*}_{\gamma;\gamma_0,\varpi}(J,\nu) \qquad \text{and} \\ \mathfrak{D}^{m_i+1} \in \Gamma\big(\mathcal{Z}^{*}_{\gamma;\gamma_0,\varpi;\mathfrak{m}}(J,\nu); L_i^{*\otimes_{\mathbb{C}}(m_i+1)}\otimes_{\mathbb{C}}\mathsf{ev}_i^*(TX,J)\big)\end{gathered} \tag{2.33}$$

as in (1.6) and (1.7) with $\mathfrak{M}^{*}_{\gamma;\gamma_0,\varpi}(J,\nu)$ in place of $\mathfrak{M}^{*}_{0,k}(A;J)$. These are well-defined because the X component u of a (J,ν)-map $\mathbf{u}$ as in (2.16) is J-holomorphic on a neighborhood of every marked point z_i of $\mathcal{C}$ by (2.13).

Theorem 2.5. *Let $g,k,\mathfrak{p}$ be as in Definition 2.3. If $B_\circ$ and X are manifolds (without boundary), then there exists a ubiquitous subset*

$$\widehat{\mathcal{H}}_{\mathfrak{p}}(B_\circ;X)\subset\mathcal{H}_{\mathfrak{p}}(B_\circ;X) \tag{2.34}$$

with the following properties.

(1) If $B_\circ^1, B_\circ^2, \ldots$ are the topological components of $B_\circ$, then

$$\widehat{\mathcal{H}}_{\mathfrak{p}}(B_\circ;X) = \widehat{\mathcal{H}}_{\mathfrak{p}}(B_\circ^1;X)\times\widehat{\mathcal{H}}_{\mathfrak{p}}(B_\circ^2;X)\times\ldots.$$

(2) For all $(J,\nu)\in\widehat{\mathcal{H}}_{\mathfrak{p}}(B_\circ;X)$, $A\in H_2(X;\mathbb{Z})$, degree A k-marked GU graphs γ, and $(\gamma_0,\varpi)\in\mathcal{A}_g(\gamma)$,

(2a) $\mathfrak{M}^{\dagger *}_{\gamma;\gamma_0,\varpi}(J,\nu)$ is a smooth manifold of dimension $\dim_{\mathbb{R}}B_\circ+2\dim(\gamma)$, and the maps ev_i in (2.31) and st in (2.32) are smooth,

(2b) $\mathcal{Z}^{*}_{\gamma;\gamma_0,\varpi;\mathfrak{m}}(J,\nu)$ is a smooth submanifold of $\mathfrak{M}^{\dagger *}_{\gamma;\gamma_0,\varpi}(J,\nu)$ of codimension $(\dim_{\mathbb{R}}X)|\mathfrak{m}|$, and the section $\mathfrak{D}^{m_i+1}$ in (2.33) is smooth and transverse to the zero set for all $\mathfrak{m}\in(\mathbb{Z}^{\geqslant 0})^k$ and $i\in[k]$.

(3) For all $(J,\nu)\in\widehat{\mathcal{H}}_{\mathfrak{p}}(B_\circ;X)$ and manifolds B with boundary $\partial B=B_\circ$, there exists a ubiquitous subset

$$\widehat{\mathcal{H}}_{J_\circ,\nu_\circ}(B;X)\subset\mathcal{H}_{J_\circ,\nu_\circ}(B;X) \tag{2.35}$$

satisfying the properties in (2) with $B_\circ$ and manifold replaced by B and manifold with boundary so that

$$\partial\,\mathfrak{M}^{\dagger *}_{\gamma;\gamma_0,\varpi}(J,\nu) = \mathfrak{M}^{\dagger *}_{\gamma;\gamma_0,\varpi}(J_\circ,\nu_\circ), \quad \partial\mathcal{Z}^{*}_{\gamma;\gamma_0,\varpi;\mathfrak{m}}(J,\nu) = \mathcal{Z}^{*}_{\gamma;\gamma_0,\varpi;\mathfrak{m}}(J_\circ,\nu_\circ)\,. \tag{2.36}$$

(4) An orientation on $B_\circ$ determines orientations on all spaces in (2) so that (3) holds in the category of oriented manifolds.

For every $\omega \in \mathrm{Symp}(B;X)$, the same statements hold with $\mathcal{H}_{\mathfrak{p}}$ replaced by $\mathcal{H}_{\mathfrak{p};\omega}$ and $\mathcal{H}'_{\mathfrak{p};\omega}$.

Theorem 2.6. *If $B_\circ$, X, $g,k \in \mathbb{Z}^{\geqslant 0}$, and $\mathfrak{p}$ are as in Theorem 2.5 and $h\colon Y \longrightarrow X^k$ is a smooth map from a manifold, then there exists a ubiquitous subset as in (2.34) satisfying (1) in Theorem 2.5 and the following properties.*

(1) For all $(J,\nu) \in \widehat{\mathcal{H}}_{\mathfrak{p}}(B_\circ;X)$, γ, (γ_0,ϖ), and $\mathfrak{m} \in (\mathbb{Z}^{\geqslant 0})^k$ as Theorem 2.5(2), the properties (2a) and (2b) in Theorem 2.5 are satisfied and the map

$$\mathsf{ev} \equiv \mathsf{ev}_1 \times \ldots \times \mathsf{ev}_k \colon \mathcal{Z}^*_{\gamma;\gamma_0,\varpi;\mathfrak{m}}(J,\nu) \longrightarrow X^k \tag{2.37}$$

is transverse to h.

(2) For all $(J_\circ,\nu_\circ)$ and B as in Theorem 2.5(3), there exists a ubiquitous subset as in (2.35) satisfying the conditions in Theorem 2.5(3) and the additional condition in (1) above.

For every $\omega \in \mathrm{Symp}(B;X)$, the same statements hold with $\mathcal{H}_{\mathfrak{p}}$ replaced by $\mathcal{H}_{\mathfrak{p};\omega}$ and $\mathcal{H}'_{\mathfrak{p};\omega}$.

3 Proof of Theorems 1.3-2.6

In light of Propositions 4.4 and 4.5, Theorems 2.5 and 2.6 come down to Proposition 3.5. It is in turn a consequence of Lemmas 3.1 and 3.2, which concern simple J-holomorphic maps from a smooth connected domain and from components Σ_v of (J,ν)-maps $\mathbf{u}$ as in (2.16) not contracted by $u_{\mathcal{M}}$, respectively. The substance of these lemmas is that the admissible deformations of J in the first case and of ν in the second supported in an open set W intersecting the image of a map cover the cokernel of the linearization of the $\bar{\partial}_J$-operator, in the first case, and of the $\bar{\partial}_J - \nu|_{\Sigma_v}$-operator in the second.

For the remainder of Section 3, we fix $g,k,\mathfrak{p}$, X, $B_\circ$, B, and h as in Theorems 2.5 and 2.6. We denote by n half the real dimension of X, as before. Since the collection of tuples $(A,\gamma,\gamma_0,\varpi)$ as in the two theorems is countable, it is sufficient to find ubiquitous subsets satisfying the required properties for each such tuple $(A,\gamma,\gamma_0,\varpi)$. We thus also fix $A \in H_2(X;\mathbb{Z})$, a degree A k-marked GU graphs γ as in (2.24), and $(\gamma_0,\varpi) \in \mathcal{A}_g(\gamma)$ with γ_0 as in (2.27) and ϖ as in (2.28). Let

$$\mathrm{Ver}_0^c = \mathrm{Ver} - \mathrm{Ver}_0.$$

We denote by $\mathrm{Aut}(\gamma)$ the group of automorphisms of γ and by $\mathrm{Aut}(\mathbb{P}^1)$ the group of holomorphic automorphisms of $\mathbb{P}^1$.

3.1 Baire spaces and ubiquitousness

We first discuss the significance of the ubiquitous property in the contexts such as those of Theorems 1.1-1.4, 2.5, and 2.6. A Baire space is a topological space $\mathcal{J}$ such that every ubiquitous subset $\widehat{\mathcal{J}}$ of $\mathcal{J}$ is dense in $\mathcal{J}$. By Baire Category Theorem [19, Theorem 48.2], every complete metric is a Baire space. Along with [19, Theorem 43.6], this implies that the three spaces in (1.1) and the three spaces in (2.15) are Baire spaces. A less direct, but more structural geometric, reasoning for this appears below.

A Fréchet vector space is a vector space V with a topology induced by a complete translation-invariant metric d, i.e.

$$d\colon \mathcal{J}\times\mathcal{J} \longrightarrow \mathbb{R}, \qquad d\big(v_1+w, v_2+w\big) = d\big(v_1, v_2\big) \quad \forall\, v_1, v_2, w\in V.$$

For $\ell\in\mathbb{Z}^{\geqslant 0}$, a C^ℓ Fréchet manifold $\mathcal{J}$ is a Hausdorff topological space locally modeled on Fréchet vector spaces with C^ℓ-overlaps between the charts. In other words, $\mathcal{J}$ comes with an atlas $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha\in\mathcal{A}}$ of charts, where each $U_\alpha\subset\mathcal{J}$ is an open subset and $\varphi_\alpha\colon U_\alpha\longrightarrow V_\alpha$ is a homeomorphism onto an open subset of Fréchet vector space, such that the overlap maps

$$\varphi_{\alpha\beta}\equiv\varphi_\alpha\circ\varphi_\beta^{-1}\colon \varphi_\beta\big(U_\alpha\cap U_\beta\big) \longrightarrow \varphi_\alpha\big(U_\alpha\cap U_\beta\big), \qquad \alpha,\beta\in\mathcal{A},$$

are C^ℓ-diffeomorphisms between open subspaces of Fréchet vector spaces. Since a C^ℓ Fréchet manifold is locally modeled on Fréchet vector spaces, it is a Baire space.

Let (X, g, ∇) be a (smooth finite-dimensional) manifold with a metric and connection on TX and $(E, |\cdot|^E, \nabla^E)$ be a (smooth finite-rank) vector bundle over X with a norm and a connection. For each ℓ in $\mathbb{Z}^{\geqslant 0}\sqcup\{\infty\}$, the space $\Gamma^\ell(X;V)$ of C^ℓ-sections of E is a Fréchet vector space with respect to the metric d_E^ℓ given by

$$d_E^\ell(s_1,s_2) = \sup_{x\in X}\sum_{m=0}^{\ell} 2^{-m}\min\Big(|\underbrace{\nabla^E\ldots\nabla^E}_{m}(s_1-s_2)|^E, 1\Big) \qquad \forall\ s_1,s_2\in\Gamma^\ell(X;V);$$

this follows from [19, Theorem 43.6]. If X is compact, the topology induced by this norm is independent of the choices of $g, \nabla, |\cdot|^E, \nabla^E$.

The space $\mathcal{J}^\ell(X)$ of C^ℓ almost complex structures on X is a (smooth) Fréchet manifold with the tangent bundle described by

$$T_J\mathcal{J}^\ell(X) = \big\{A\in\Gamma^\ell\big(X;\mathrm{End}(TX)\big)\colon JA=-AJ\big\} \quad \forall\, J\in\mathcal{J}^\ell(X). \tag{3.1}$$

The charts $(U_\alpha, \varphi_\alpha)$ are the inverses of the maps

$$T_J\mathcal{J}^\ell(X) \longrightarrow \mathcal{J}^\ell(X), \qquad A\longrightarrow J\mathrm{e}^{JA}, \tag{3.2}$$

restricted to sufficiently small neighborhoods of 0 in each $T_J\mathcal{J}$. Since the space $\mathcal{J}'^\ell_\omega(X)$ of C^ℓ almost complex structures on X tamed by a symplectic form ω on X is an open subset of $\mathcal{J}^\ell(X)$, $\mathcal{J}'^\ell_\omega(X)$ is also a Fréchet manifold with the tangent bundle described by (3.1) with $\mathcal{J}^\ell(X)$ replaced by $\mathcal{J}'^\ell_\omega(X)$. The space $\mathcal{J}^\ell_\omega(X)$ of C^ℓ ω-compatible almost complex structures on X is a Fréchet manifold as well. Its tangent bundle is described by

$$T_J\mathcal{J}^\ell_\omega(X) = \big\{A\in T_J\mathcal{J}^\ell(X)\colon \omega(A\cdot,\cdot)=-\omega(\cdot,A\cdot)\big\} \quad \forall\, J\in\mathcal{J}^\ell_\omega(X). \tag{3.3}$$

Local charts on $\mathcal{J}^\ell_\omega(X)$ are obtained by restricting (3.2) to $T_J\mathcal{J}^\ell_\omega(X)$.

Let B be a manifold, possibly with boundary, and X and ℓ be as above. Denote by

$$\pi_X\colon B\times X \longrightarrow X$$

the projection to the second component, by $\mathcal{J}^\ell(B;X)$ the space of C^ℓ fiberwise complex structures on the vector bundle π_X^*TX, and by $\mathrm{Symp}(B;X)$ the space of smooth fiberwise symplectic structures on π_X^*TX. For $J\in\mathcal{J}^\ell(B;X)$, $\omega\in\mathrm{Symp}(B;X)$, and $b\in B$, let $J_b\in\mathcal{J}^\ell(X)$ and $\omega_b\in\mathrm{Symp}(X)$ be the associated almost complex and symplectic structures on X. For $J_\circ\in\mathcal{J}^\ell(\partial B;X)$, define

$$\mathcal{J}^\ell_{J_\circ}(B;X)\subset\mathcal{J}^\ell(B;X)$$

as in (1.8).

For $\omega\in\mathrm{Symp}(B;X)$, denote by

$$\mathcal{J}^\ell_\omega(B;X)\subset\mathcal{J}'^\ell_\omega(B;X)\subset\mathcal{J}^\ell(B;X)$$

the subspaces of ω-compatible and of ω-tamed almost complex structures. For $J_\circ\in\mathcal{J}^\ell_\omega(\partial B;X)$ and $J_\circ\in\mathcal{J}'^\ell_\omega(\partial B;X)$, define

$$\mathcal{J}^\ell_{\omega;J_\circ}(B;X)\subset\mathcal{J}^\ell_\omega(B;X)\qquad\text{and}\qquad\mathcal{J}'^\ell_{\omega;J_\circ}(B;X)\subset\mathcal{J}'^\ell_\omega(B;X),$$

respectively, as in (1.8). Let

$$\begin{gathered}\mathcal{J}_\omega(B;X)=\mathcal{J}^\infty_\omega(B;X),\qquad\mathcal{J}'_\omega(B;X)=\mathcal{J}'^\infty_\omega(B;X),\qquad\mathcal{J}^\ell(B;X)=\mathcal{J}^\infty(B;X),\\ \mathcal{J}_{\omega;J_\circ}(B;X)=\mathcal{J}^\infty_{\omega;J_\circ}(B;X),\quad\mathcal{J}'_{\omega;J_\circ}(B;X)=\mathcal{J}'^\infty_{\omega;J_\circ}(B;X),\quad\mathcal{J}^\ell_{J_\circ}(B;X)=\mathcal{J}^\infty_{J_\circ}(B;X)\,.\end{gathered}$$

3.2 Configuration spaces

The set $\mathcal{M}_{\gamma_0}$ of equivalence classes of connected genus g k-marked nodal Riemann surfaces of combinatorial type $\gamma_0\in\mathcal{A}_{g,k}$ is a subspace of $\overline{\mathcal{M}}_{g,k}$. Let

$$\mathcal{M}_{\gamma_0;v}=\mathcal{M}_{\mathfrak{g}(v),S_v(\gamma_0)}\quad\forall\,v\in V\,.$$

The image of the immersion

$$\iota_{\gamma_0}\colon\prod_{v\in\mathrm{Ver}_0}\mathcal{M}_{\gamma_0;v}\longrightarrow\overline{\mathcal{M}}_{g,k}\tag{3.4}$$

identifying the marked points z_f with $f\in e$ for each $e\in\mathrm{Edg}$ into a node is $\mathcal{M}_{\gamma_0}$. This immersion descends to an isomorphism from the quotient of its domain by the natural $\mathrm{Aut}(\gamma_0)$ action to $\mathcal{M}_{\gamma_0}$.

By the last requirement in Definition 2.2, there exist covers $\widetilde{\mathcal{M}}_{\gamma_0;v}\longrightarrow\mathcal{M}_{\gamma_0;v}$ with $v\in\mathrm{Ver}_0$, universal curves

$$\big(\pi_{\gamma_0;v}\colon\widetilde{\mathcal{U}}_{\gamma_0;v}\longrightarrow\widetilde{\mathcal{M}}_{\gamma_0;v},(s_{\gamma_0;f})_{f\in S_v(\gamma_0)}\big),$$

and an immersion

$$\widetilde{\iota}_{\gamma_0}\colon\prod_{v\in\mathrm{Ver}_0}\widetilde{\mathcal{M}}_{\gamma_0;v}\longrightarrow\widetilde{\mathcal{M}}_{g,k}\tag{3.5}$$

lifting (3.4). For each $v\in\mathrm{Ver}_0$, let

$$\mathrm{pr}_{\gamma_0;v}\colon\prod_{v\in\mathrm{Ver}_0}\widetilde{\mathcal{M}}_{\gamma_0;v}\longrightarrow\widetilde{\mathcal{M}}_{\gamma_0;v}\qquad\text{and}\qquad\widetilde{\iota}_{\gamma_0;v}\colon\mathrm{pr}^*_{\gamma_0;v}\widetilde{\mathcal{U}}_{\gamma_0;v}\longrightarrow\widetilde{\mathcal{U}}_{g,k}$$

be the component projection map and the natural bundle lifting (3.5), respectively; the restriction of the latter to each fiber is the normalization of an irreducible component of a fiber of π.

For $v\in\mathrm{Ver}_0$, let

$$S_v^0(\gamma)=S_v(\gamma)\cap\big(S_v(\gamma_0)\cup\mathrm{Im}\,\varpi\big)\subset[k]\sqcup\mathrm{Fl},\quad S_v^c(\gamma)=S_v(\gamma)-S_v^0(\gamma)\subset\mathrm{Fl}\,.$$

If $S_v^c(\gamma)\neq\emptyset$, we define

$$\begin{aligned}\widetilde{\mathcal{M}}_{\gamma;v}=\big\{(z_f)_{f\in S_v^c(\gamma)}\in(\widetilde{\mathcal{U}}_{\gamma_0;v})^{S_v^c(\gamma)}:\ & z_f\notin\mathrm{Im}\,s_{f'}\ \forall\,f\in S_v^c(\gamma),\ f'\in S_v^0(\gamma),\\ & \pi_{\gamma_0;v}(z_f)=\pi_{\gamma_0;v}(z_{f'}),\ z_f\neq z_{f'}\ \forall\,f,f'\in S_v^c(\gamma),\ f\neq f'\big\}\end{aligned}$$

and take

$$\pi_{\gamma;v}:\widetilde{\mathcal{U}}_{\gamma;v}\equiv\widetilde{\mathcal{M}}_{\gamma;v}\times_{\widetilde{\mathcal{M}}_{\gamma_0;v}}\widetilde{\mathcal{U}}_{\gamma_0;v}\longrightarrow\widetilde{\mathcal{M}}_{\gamma;v}$$

to be the projection to the first component. For $f\in S_v^c(\gamma)$, define

$$s_{\gamma;f}:\widetilde{\mathcal{M}}_{\gamma;v}\longrightarrow\widetilde{\mathcal{U}}_{\gamma;v},\quad s_{\gamma;f}\big((z_{f'})_{f'\in S_v^c(\gamma)}\big)=\big((z_{f'})_{f'\in S_v^c(\gamma)},z_f\big).$$

For $f\in S_v(\gamma)\cap S_v(\gamma_0)$ and $f=\varpi(i)$ with $i\in\varpi^{-1}(v)$, we take

$$s_{\gamma;f}:\widetilde{\mathcal{M}}_{\gamma;v}\longrightarrow\widetilde{\mathcal{U}}_{\gamma;v}$$

to be the pullback of $s_{\gamma_0;f}$ and $s_{\gamma_0;i}$, respectively, by the natural projection $\widetilde{\mathcal{M}}_{\gamma;v}\longrightarrow\widetilde{\mathcal{M}}_{\gamma_0;v}$.

If $S_v(\gamma)=S_v^0(\gamma)$, let

$$\begin{gathered}\pi_{\gamma;v}=\pi_{\gamma_0;v}:\widetilde{\mathcal{U}}_{\gamma;v}\equiv\widetilde{\mathcal{U}}_{\gamma_0;v}\longrightarrow\widetilde{\mathcal{M}}_{\gamma;v}\equiv\widetilde{\mathcal{M}}_{\gamma_0;v},\\ s_{\gamma;f}=s_{\gamma_0;f}\ \forall\,f\in S_v(\gamma)\cap S_v(\gamma_0),\ s_{\gamma;\varpi(i)}=s_{\gamma_0;i}\ \forall\,i\in\varpi^{-1}(v).\end{gathered}$$

In both cases,

$$\big(\pi_{\gamma;v}:\widetilde{\mathcal{U}}_{\gamma;v}\longrightarrow\widetilde{\mathcal{M}}_{\gamma;v},(s_{\gamma;f})_{f\in S_v(\gamma)}\big)\tag{3.6}$$

is the universal curve.

For $v\in\mathrm{Ver}_0^c$, define (3.6) by

$$\begin{gathered}\widetilde{\mathcal{M}}_{\gamma;v}=\big\{(z_f)_{f\in S_v(\gamma)}:z_f\neq z_{f'}\ \forall\,f,f'\in S_v(\gamma)\cap\mathrm{Fl},\ f\neq f'\big\},\quad\widetilde{\mathcal{U}}_{\gamma;v}=\widetilde{\mathcal{M}}_{\gamma;v}\times\mathbb{P}^1,\\ s_{\gamma;f}:\widetilde{\mathcal{M}}_{\gamma;v}\longrightarrow\widetilde{\mathcal{U}}_{\gamma;v},\quad s_{\gamma;f}\big((z_{f'})_{f'\in S_v(\gamma)}\big)=\big((z_{f'})_{f'\in S_v(\gamma)},z_f\big).\end{gathered}$$

There is a natural action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\widetilde{\mathcal{U}}_{\gamma;v}$ that sends a fiber of $\pi_{\gamma;v}$ to a fiber holomorphically and commutes with the sections.

In both cases, denote by

$$\mathrm{pr}_{\gamma;v}:\widetilde{\mathcal{M}}_\gamma\equiv\prod_{v'\in\mathrm{Ver}}\widetilde{\mathcal{M}}_{\gamma;v'}\longrightarrow\widetilde{\mathcal{M}}_{\gamma;v}\qquad\text{and}\qquad\widetilde{\pi}_{\gamma;v}:\widehat{\mathcal{U}}_{\gamma;v}\equiv\mathrm{pr}_{\gamma;v}^*\widetilde{\mathcal{U}}_{\gamma;v}\longrightarrow\widetilde{\mathcal{M}}_\gamma\tag{3.7}$$

the component projection map and the induced bundle projection, respectively. For $v\in\mathrm{Ver}_0$, the composition

$$\widetilde{\mathcal{M}}_\gamma \longrightarrow \prod_{v\in\mathrm{Ver}_0}\widetilde{\mathcal{M}}_{\gamma_0;v} \longrightarrow \widetilde{\mathcal{M}}_{g,k}$$

of the natural projection with the immersion $\widetilde{\iota}_{\gamma_0}$ lifts to a bundle map

$$\widehat{\iota}_{\gamma;v}\colon \widehat{\mathcal{U}}_{\gamma;v} \longrightarrow \widetilde{\mathcal{U}}_{g,k}$$

which restricts to each fiber of $\widetilde{\pi}_{\gamma;v}$ as the normalization of an irreducible component of a fiber of π. It sends the marked points z_f with $f\in S_v(\gamma)\cap S_v(\gamma_0)$ and $z_{\varpi(i)}$ with $i\in\varpi^{-1}(v)$ of the former to the marked and nodal points z_f and z_i of the latter. The marked points $z_f\in S_0^c(\gamma)$ of the former are disregarded by this map.

Fix $\ell,p\in\mathbb{Z}$ with $\ell\geqslant 2+|\mathfrak{m}|$ and $p>2$. We denote by $\mathcal{H}_{\mathfrak{p}}^\ell(X)$, $\mathcal{H}_{\mathfrak{p}}^\ell(B_\circ;X)$, and $\mathcal{H}_{\mathfrak{p}}^\ell(B;X)$ the $C^{2\ell}$ completions of the corresponding spaces of smooth pairs (J,ν) as in Section 2.4 and use the analogous notation for the other $\mathcal{H}$ spaces appearing in Theorems 2.5 and 2.6. By the assumption $p>2$ and the Sobolev Embedding Theorem [30, Corollary 4.3], every $W^{\ell,p}$-map from a Riemann surface Σ to X is $C^{\ell-1}$.

For each $v\in\mathrm{Ver}$, denote by $\widetilde{\mathfrak{B}}_v$ the space of tuples

$$\mathbf{u}_v \equiv \big(\Sigma_v, \mathfrak{j}_v, (z_f)_{f\in S_v(\gamma)}, u_v\big) \tag{3.8}$$

so that $\big(\Sigma_v, \mathfrak{j}_v, (z_f)_{f\in S_v(\gamma)}\big)$ is a fiber of $\widetilde{\pi}_{\gamma;v}$ and $u_v:\Sigma_v\longrightarrow X$ is a $W^{\ell,p}$ map. Analogously to Section 1.2, let

$$\widetilde{\mathfrak{B}}_v^* \subset \widetilde{\mathfrak{B}}_v^{\dagger *} \subset \widetilde{\mathfrak{B}}_v$$

be the entire spaces $\widetilde{\mathfrak{B}}_v$ if $v\in\mathrm{Ver}_0$ and the subspaces of basic and reduced, respectively, maps in the sense of Section 2.3 if $v\in\mathrm{Ver}_0^c$. For $f\in S_v(\gamma)$, let

$$\mathsf{ev}_f\colon \widetilde{\mathfrak{B}}_v \longrightarrow X, \qquad \mathsf{ev}_f(\mathbf{u}_v) = u(z_f),$$

be the evaluation map at the marked point z_f corresponding to f.

Denote by

$$\mathcal{T}_{\gamma;v} \equiv \ker \mathrm{d}\widetilde{\pi}_{\gamma;v} \longrightarrow \widehat{\mathcal{U}}_{\gamma;v}$$

the vertical tangent bundle of the projection $\widetilde{\pi}_{\gamma;v}$ in (3.7) and by $\mathfrak{j}_{\gamma;v}$ its complex structure. For each $i\in S_v(\gamma)\cap[k]$, denote by $L_{v;i}\longrightarrow\widetilde{\mathfrak{B}}_v$ the tautological tangent line bundle for the i-th marked point, i.e. the pullback of $\mathcal{T}_{\gamma;v}$ by the map

$$\widetilde{\mathfrak{B}}_v \longrightarrow \widehat{\mathcal{U}}_{\gamma;v}, \qquad \big(\Sigma_v, \mathfrak{j}_v, (z_f)_{f\in S_v(\gamma)}, u_v\big) \longrightarrow z_i\,.$$

Define

$$F_{\mathfrak{m};i} = \bigoplus_{m=1}^{m_i}\mathrm{Hom}_{\mathbb{R}}\big(\mathrm{Sym}_{\mathbb{R}}^{m_i}(L_i), \mathsf{ev}_i^*TX\big) \longrightarrow \widetilde{\mathfrak{B}}_v, \quad \mathfrak{D}_{\mathfrak{m};i} = \bigoplus_{m=1}^{m_i}\mathfrak{D}^{m_i} \in \Gamma\big(\widetilde{\mathfrak{B}}_v; F_{\mathfrak{m};i}\big) \quad \forall\, i\in S_v(\gamma)\cap[k],$$

$$F_{\mathfrak{m};v} = \bigoplus_{i\in S_v(\gamma)\cap[k]} F_{\mathfrak{m};i}, \quad \mathfrak{D}_{\mathfrak{m};v} = \bigoplus_{i\in S_v(\gamma)\cap[k]}\mathfrak{D}_{\mathfrak{m};i}, \quad \widetilde{\mathfrak{B}}_{v;\mathfrak{m}} = \mathfrak{D}_{\mathfrak{m};v}^{-1}(0),$$

$$\widetilde{\mathfrak{B}}_{\mathfrak{m};v}^* = \widetilde{\mathfrak{B}}_{v;\mathfrak{m}} \cap \widetilde{\mathfrak{B}}_v^*, \quad \widetilde{\mathfrak{B}}_{\mathfrak{m};v}^{\dagger *} = \widetilde{\mathfrak{B}}_{v;\mathfrak{m}} \cap \widetilde{\mathfrak{B}}_v^{\dagger *}\,.$$

3.3 Spaces of deformations and obstructions

Let $v\in\mathrm{Ver}$. For each $\mathbf{u}_v\in\widetilde{\mathfrak{B}}_v$ as in (3.8), define

$$\begin{aligned}\Gamma(\mathbf{u}_v)&=L^p_\ell(\Sigma_v;u_v), \quad \Gamma_0(\mathbf{u}_v)=\big\{\xi\in\Gamma(\mathbf{u}_v)\colon \xi(z_f)=0\ \forall\, f\in S_v(\gamma)\big\},\\ \Gamma_{\mathfrak{m}}(\mathbf{u}_v)&=\big\{\xi\in\Gamma_0(\mathbf{u}_v)\colon \nabla^m\xi|_{z_i}=0\ \forall\ m\in[m_i],\ i\in[k]\cap S_v(\gamma)\big\}.\end{aligned}\tag{3.9}$$

If in addition $J\in\mathcal{J}^\ell(X)$, let

$$\begin{aligned}\Gamma^{0,1}_J(\mathbf{u}_v)&=L^p_{\ell-1}\big(\Sigma_v;(T^*\Sigma_v,-\mathfrak{j}_v)^*\otimes_{\mathbb{C}}u_v^*(TX,J)\big),\\ \Gamma^{0,1}_{J;\mathfrak{m}}(\mathbf{u}_v)&=\big\{\eta\in\Gamma^{0,1}_J(\mathbf{u}_v)\colon \nabla^{m-1}\eta|_{z_i}=0\ \forall\ m\in[m_i],\ i\in[k]\cap S_v(\gamma)\big\}.\end{aligned}\tag{3.10}$$

Suppose $v\in\mathrm{Ver}_0^c$. For $J\in\mathcal{J}^\ell(X)$, let

$$\widetilde{\mathfrak{M}}^{\dagger*}_v(J)\subset\widetilde{\mathfrak{B}}^{\dagger*}_v, \qquad \widetilde{\mathfrak{M}}^{*}_{\mathfrak{m};v}(J)\subset\widetilde{\mathfrak{B}}^{*}_{\mathfrak{m};v}, \qquad \text{and} \quad \widetilde{\mathfrak{M}}^{\dagger*}_{\mathfrak{m};v}(J)\subset\widetilde{\mathfrak{B}}^{\dagger*}_{\mathfrak{m};v}\tag{3.11}$$

be the subspaces of J-holomorphic maps. For $\mathbf{u}_v\in\widetilde{\mathfrak{M}}^{\dagger*}_v(J)$ as in (3.8), let

$$D_{J;\mathbf{u}_v}\colon\Gamma(\mathbf{u}_v)\longrightarrow\Gamma^{0,1}_J(\mathbf{u}_v),\quad D^0_{J;\mathbf{u}_v}\colon\Gamma_0(\mathbf{u}_v)\longrightarrow\Gamma^{0,1}_J(\mathbf{u}_v),\quad D^{\mathfrak{m}}_{J;\mathbf{u}_v}\colon\Gamma_{\mathfrak{m}}(\mathbf{u}_v)\longrightarrow\Gamma^{0,1}_{J;\mathfrak{m}}(\mathbf{u}_v)\tag{3.12}$$

be the linearization of the $\bar\partial$-operator at $\mathbf{u}_v$ and its restrictions; see Section 4.2.

Lemma 3.1. *Suppose $v\in\mathrm{Ver}_0^c$, $J\in\mathcal{J}^\ell(X)$, and $\mathbf{u}_v\in\widetilde{\mathfrak{M}}^{\dagger*}_v(J)$ is as in (3.8). If $W\subset X$ is an open subset intersecting $u_v(\Sigma_v)$, then*

$$\begin{aligned}\Gamma^{0,1}_J(\mathbf{u}_v)&=\mathrm{Im}\,D^0_{J;\mathbf{u}_v}+\big\{A\circ\mathrm{d}u_v\circ\mathfrak{j}_v\colon A\in T_J\mathcal{J}^\ell(X),\ \mathrm{supp}(A)\subset W\big\},\\ \Gamma^{0,1}_{J;\mathfrak{m}}(\mathbf{u}_v)&\subset\mathrm{Im}\,D^{\mathfrak{m}}_{J;\mathbf{u}_v}+\big\{A\circ\mathrm{d}u_v\circ\mathfrak{j}_v\colon A\in T_J\mathcal{J}^\ell(X),\ \mathrm{supp}(A)\subset W\big\}.\end{aligned}\tag{3.13}$$

For every $\omega\in\mathrm{Symp}(X)$, the same statements hold with $\mathcal{J}$ replaced by $\mathcal{J}_\omega$ and $\mathcal{J}'_\omega$.

Proof. The first claim is the $\mathfrak{m}=0$ case of the second claim. We can assume that $u_v(z_f)\notin W$ for all $f\in S_v(\gamma)$; this implies that the right-hand side of (3.13) is contained in the left-hand side (this is also the case if $\mathbf{u}_v\in\widetilde{\mathfrak{M}}^{\dagger*}_{\mathfrak{m};v}(J)$). We can also assume that $\mathbf{u}_v\in\widetilde{\mathfrak{M}}^{*}_v(J)$ and thus that $z_f\neq z_{f'}$ for all $f,f'\in S_v(\gamma)$ distinct. Denote by $S\subset\Sigma_v$ the subset of the marked points and by $D^*_{J;\mathbf{u}_v}$ the formal adjoint of $D_{J;\mathbf{u}_v}$. Let

$$\Gamma^{1,0}_{J;\mathfrak{m}}(\mathbf{u}_v)\subset L^p_1\big(\Sigma_v-S;(T^*\Sigma_v,\mathfrak{j}_v)^*\otimes_{\mathbb{C}}u_v^*(T^*X,J)\big)\tag{3.14}$$

be the subspace of $(1,0)$-forms on $(\Sigma_v,\mathfrak{j}_v)$ with values in $u_v^*(T^*X,J)$ that have poles of order at most 1 at z_f for $f\in S_v(\gamma)\cap\mathrm{Fl}$ and at most $\max(1,m_i)$ at z_i for $i\in S_v(\gamma)\cap[k]$; see Section 4.5. Let

$$\ker\,D^{\mathfrak{m}*}_{J;\mathbf{u}_v}=\big\{\mu\in\Gamma^{1,0}_{J;\mathfrak{m}}(\mathbf{u}_v)\colon D^*_{J;\mathbf{u}_v}\mu=0\big\}.$$

By Proposition 4.9, the homomorphism

$$L^{\mathfrak{m}}_{J;\mathbf{u}_v}\colon\mathrm{cok}\,D^{\mathfrak{m}}_{J;\mathbf{u}_v}\longrightarrow\mathrm{Hom}_{\mathbb{R}}\big(\ker D^{\mathfrak{m}*}_{J;\mathbf{u}_v},\mathbb{R}\big), \qquad \big\{L^{\mathfrak{m}}_{J;\mathbf{u}_v}([\eta])\big\}(\mu)=\Re\Big(\int_{\Sigma_v}\mu\wedge\eta\Big),$$

is a well-defined isomorphism. Thus, it is sufficient to show that for every $\mu\in\ker D^{\mathfrak{m}*}_{J;\mathbf{u}_v}$ nonzero, there exists

$$A\in T_J\mathcal{J}^\ell(X) \qquad \text{s.t.} \quad \text{supp}(A)\subset W, \;\; \Re\Big(\int_{\Sigma_v}\mu\wedge\big(A\circ\mathrm{d}u_v\circ\mathrm{j}_v\big)\Big)\neq 0. \tag{3.15}$$

If $\omega\in\text{Symp}(X)$ and $J\in\mathcal{J}'^\ell_\omega(X)$, then $T_J\mathcal{J}'^\ell_\omega(X)=T_J\mathcal{J}^\ell(X)$. If $J\in\mathcal{J}^\ell_\omega(X)$, then A as above lies in $T_J\mathcal{J}^\ell_\omega(X)$ if A satisfies the additional condition in (3.3).

Let $\mu\in\ker D^{\mathfrak{m}*}_{J;\mathbf{u}_v}$ with $\mu\neq 0$ and ω_Σ be an orientation form on Σ. By shrinking W if necessary, we can assume that W is contained in a coordinate chart on X. Since u_v is a somewhere injective J-holomorphic map, there exists a non-empty open subset $U\subset u_v^{-1}(W)-S$ such that

$$\mathrm{d}_z u_v\neq 0, \;\; u_v^{-1}\big(u_v(z)\big)=\{z\} \quad \forall\, z\in U; \tag{3.16}$$

see [33, Corollary 3.14]. Since $\mu\neq 0$, there exists $z\in U$ such that $\mu_z\neq 0$; see Lemma 4.8. Let $w\in T_z\Sigma$ be a nonzero vector. Since

$$\mu_z(w)\neq 0\in T^*_{u_v(z)}X \qquad \text{and} \qquad \mathrm{d}_z u_v(\mathrm{j}_v w)\neq 0\in T_{u_v(z)}X,$$

there exists $A_{u_v(z)}\in\text{End}(T_{u_v(z)}X)$ such that

$$\big\{\mu_z(w)\big\}\big(A_{u_v(z)}\big(\mathrm{d}_z u_v(\mathrm{j}_v w)\big)\big)=1 \qquad \text{and} \qquad J_{u_v(z)}A_{u_v(z)}=-A_{u_v(z)}J_{u_v(z)}. \tag{3.17}$$

Let $A\in\Gamma(W;\text{End}(TX))$ be an extension of $A_{u_v(z)}$ such that $J_xA_x=-A_xJ_x$ for every $x\in W$.

Let $\kappa\colon U\longrightarrow\mathbb{C}$ be the function defined by

$$\kappa(z')\omega_\Sigma\big|_{z'}=\mu_{z'}\wedge\big(A_{u_v(z')}\circ\mathrm{d}_{z'}u_v\circ\mathrm{j}_v\big) \qquad \forall\; z'\in U.$$

By the first condition in (3.17), $\kappa(z)\in\mathbb{R}^+$. By the continuity of κ an (3.16), there thus exists a neighborhood of $W'\subset W$ of $u_v(z)$ such that

$$u_v^{-1}(W')\subset U, \qquad \Re\big(\kappa(z')\big)\in\mathbb{R}^+ \;\; \forall\, z'\in u_v^{-1}(W'). \tag{3.18}$$

Let $\beta\colon X\longrightarrow\mathbb{R}^+$ be a smooth function such that $\beta(u_v(z))=1$ and $\text{supp}(\beta)\subset u_v^{-1}(W')$. By (3.18),

$$\Re\Big(\int_{\Sigma_v}\mu\wedge\big(\beta A\circ\mathrm{d}u_v\circ\mathrm{j}_v\big)\Big)=\Re\Big(\int_{u_v^{-1}(W')}\mu\wedge\big(\beta A\circ\mathrm{d}u_v\circ\mathrm{j}_v\big)\Big)=\int_{u_v^{-1}(W')}\big(\beta\Re(\kappa)\big)\omega_\Sigma>0.$$

Along with the condition that $J_xA_x=-A_xJ_x$ for every $x\in X$, this implies that $\beta A\in T_J\mathcal{J}^\ell(X)$ satisfies both requirements in (3.15) with A replaced by βA.

Suppose $\omega\in\text{Symp}(X)$ and $J\in\mathcal{J}^\ell_\omega(X)$. By shrinking W if necessary, we can assume that W is contained in a Darboux coordinate chart on (X,ω); see Theorem 3.15 in [18]. By Lemma 3.2.2 in [18], we can then choose $A_{u_v(z)}\in\text{End}(T_{u_v(z)}X)$ so that it satisfies

$$\omega_{u_v(z)}\big(A_{u_v(z)}\cdot,\cdot\big)=-\omega_{u_v(z)}\big(\cdot,A_{u_v(z)}\cdot\big)$$

in addition (3.17) and extend it to $A\in\Gamma(W;\text{End}(TX))$ so that

$$J_xA_x=-A_xJ_x, \;\; \omega_x(A_x\cdot,\cdot)=-\omega_x(\cdot,A_x\cdot) \quad \forall\, x\in W.$$

The last condition implies that the element $\beta A\in T_J\mathcal{J}^\ell(X)$ constructed above satisfies the additional condition in (3.3) and thus lies in $T_J\mathcal{J}^\ell_\omega(X)$. $\square$

Suppose $v\in\mathrm{Ver}_0$. For $(J,\nu)\in\mathcal{H}_{\mathfrak p}^{\ell}(X)$, define

$$\nu_{\gamma;v}=\big\{\widehat\iota_{\gamma;v}\times\mathrm{id}_X\big\}^*\nu\in\Gamma\big(\widetilde{\mathcal U}_{\gamma;v}\times X;\pi_1^*(\mathcal T_{\gamma;v},-\mathfrak j_{\gamma;v})^*\otimes_{\mathbb C}\pi_2^*(TX,J)\big).$$

Denote by

$$\widetilde{\mathfrak M}_v^{\dagger *}(J;\nu)\subset\widetilde{\mathfrak B}_v^{\dagger *}\qquad\text{and}\qquad\widetilde{\mathfrak M}_{\mathfrak m;v}^{\dagger *}(J;\nu)\subset\widetilde{\mathfrak B}_{\mathfrak m;v}^{\dagger *}$$

the subspaces of tuples as in (3.8) so that

$$\bar\partial_{J,\mathfrak j}u_v\big|_z=\nu_{\gamma;v}\big(z,u_v(z)\big)\qquad\forall\,z\in\Sigma_v\,. \tag{3.19}$$

For $\mathbf u_v\in\widetilde{\mathfrak M}_v^{\dagger *}(J;\nu)$, let

$$D_{J,\nu;\mathbf u_v}:\Gamma(\mathbf u_v)\longrightarrow\Gamma_J^{0,1}(\mathbf u_v),\quad D^0_{J,\nu;\mathbf u_v}:\Gamma_0(\mathbf u_v)\longrightarrow\Gamma_J^{0,1}(\mathbf u_v),\quad D^{\mathfrak m}_{J,\nu;\mathbf u_v}:\Gamma_{\mathfrak m}(\mathbf u_v)\longrightarrow\Gamma_{J;\mathfrak m}^{0,1}(\mathbf u_v)$$

be the linearization of the $\{\bar\partial-\nu_{\gamma;v}\}$-operator at $\mathbf u_v$ and its restrictions; see Section 4.2.

Lemma 3.2. *Suppose $v\in\mathrm{Ver}_0$, $(J,\nu)\in\mathcal H_{\mathfrak p}^{\ell}(X)$, and $\mathbf u_v\in\widetilde{\mathfrak M}_{\mathfrak m;v}^{\dagger *}(J,\nu)$ is as in (3.8). If $W\subset\widetilde{\mathcal U}_{g,k}$ is an open subset intersecting $\widehat\iota_{\gamma;v}(\Sigma_v)$, then*

$$\begin{aligned}\Gamma_J^{0,1}(\mathbf u_v)&=\mathrm{Im}\,D^0_{J,\nu;\mathbf u_v}+\big\{\{\widehat\iota_{\gamma;v}\times u_v\}^*\nu':\nu'\in\Gamma_{\mathfrak p}^{0,1}(X;J),\ \mathrm{supp}(\nu')\subset W\times X\big\},\\ \Gamma_{J;\mathfrak m}^{0,1}(\mathbf u_v)&=\mathrm{Im}\,D^{\mathfrak m}_{J,\nu;\mathbf u_v}+\big\{\{\widehat\iota_{\gamma;v}\times u_v\}^*\nu':\nu'\in\Gamma_{\mathfrak p}^{0,1}(X;J),\ \mathrm{supp}(\nu')\subset W\times X\big\}.\end{aligned}$$

Proof. The first claim is the $\mathfrak m=0$ case of the second claim. We can assume that $\widehat\iota_{\gamma;v}(z_f)\notin W$ for all $f\in S_v(\gamma)$. Denote by $D^*_{J,\nu;\mathbf u_v}$ the formal adjoint of $D_{J,\nu;\mathbf u_v}$. With $\Gamma_{J;\mathfrak m}^{1,0}(\mathbf u_v)$ as in (3.14), let

$$\ker\,D^{\mathfrak m *}_{J,\nu;\mathbf u_v}=\big\{\mu\in\Gamma_{J;\mathfrak m}^{1,0}(\mathbf u_v):D^*_{J,\nu;\mathbf u_v}\mu=0\big\}.$$

By the same reasoning as in the proof of Lemma 3.1, it is sufficient to show that for every $\mu\in\ker D^{\mathfrak m *}_{J,\nu;\mathbf u_v}$ nonzero, there exists

$$\nu'\in\Gamma_{\mathfrak p}^{0,1}(X;J)\qquad\text{s.t.}\quad\mathrm{supp}(\nu')\subset W\times X,\quad\Re\Big(\int_{\Sigma_v}\mu\wedge\big(\{\widehat\iota_{\gamma;v}\times u_v\}^*\nu'\big)\Big)\neq0. \tag{3.20}$$

Let $\mu\in\ker D^{\mathfrak m *}_{J,\nu;\mathbf u_v}$ with $\mu\neq0$ and ω_Σ be an orientation form on Σ. By shrinking W if necessary, we can assume that W is contained in a coordinate chart on $\widetilde{\mathcal U}_{g,k}$. Since $\mu\neq0$, there exists $z\in\widehat\iota_{\gamma;v}^{-1}(W)$ such that $\mu_z\neq0$; see Lemma 4.8. Let $w\in T_z\Sigma$ be a nonzero vector. Since

$$\mu_z(w)\neq0\in T^*_{u_v(z)}X\qquad\text{and}\qquad\mathrm d_z\widehat\iota_{\gamma;v}(w)\neq0\in\mathcal T_{\widehat\iota_{\gamma;v}(z)}\mathfrak p,$$

there exists $\nu'_z\in\mathrm{Hom}(\mathcal T_{\widehat\iota_{\gamma;v}(z)}\mathfrak p;T_{u_v(z)}X)$ such that

$$\big\{\mu_z(w)\big\}\big(\nu'_z(\mathrm d_z\widehat\iota_{\gamma;v}(w))\big)=1\qquad\text{and}\qquad J_{u_v(z)}\nu'_z=-\nu'_z\mathfrak j_{\mathcal U}|_{\widehat\iota_{\gamma;v}(z)}. \tag{3.21}$$

Let $\nu'\in\Gamma(\mathcal U^*_{g,k}\times X;\mathrm{Hom}(\pi_1^*\mathcal T\mathfrak p,\pi_2^*TX))$ be an extension of ν'_z such that

$$J_x\nu'_{z',x}=-\nu'_{z',x}\mathfrak j_{\mathcal U}|_{z'}\qquad\forall\ (z',x)\in\widetilde{\mathcal U}^*_{g,k}\times X. \tag{3.22}$$

By the continuity of the function

$$\kappa\colon U\longrightarrow\mathbb{R},\qquad \kappa(z')\omega_\Sigma\big|_{z'}=\mu_{z'}\wedge\big(\nu'_{z',u_v(z')}(\mathrm{d}_{z'}\widehat{\iota}_{\gamma;v}|_{\Sigma_v})\big),$$

the first condition in (3.21), and the injectivity of $\widehat{\iota}_{\gamma;v}$, there exists a neighborhood of $W'\subset W$ of $\widehat{\iota}_{\gamma;v}(z)$ such that

$$\Re\big(\kappa(z')\big)\in\mathbb{R}^+\quad\forall\,z'\in\widehat{\iota}_{\gamma;v}^{-1}(W'). \tag{3.23}$$

Let $\beta\colon\widetilde{\mathcal{U}}_{g,k}\longrightarrow\mathbb{R}^+$ be a smooth function such that $\beta(\widehat{\iota}_{\gamma;v}(z))=1$ and $\mathrm{supp}(\beta)\subset W'$. By (3.23),

$$\begin{aligned}\Re\Big(\int_{\Sigma_v}\mu\wedge\big(\{\widehat{\iota}_{\gamma;v}\times u_v\}^*(\beta\nu')\big)\Big)&=\Re\Big(\int_{\widehat{\iota}_{\gamma;v}^{-1}(W')}\mu\wedge\big(\{\widehat{\iota}_{\gamma;v}\times u_v\}^*(\beta\nu')\big)\Big)\\&=\int_{\widehat{\iota}_{\gamma;v}^{-1}(W')}\big(\beta\Re(\kappa)\big)\omega_\Sigma>0.\end{aligned}$$

Along with (3.22), this implies that $\beta\nu'\in\Gamma^{0,1}_{\mathfrak{p}}(X;J)$ satisfies both requirements in (3.20) with ν' replaced by $\beta\nu'$. $\square$

3.4 Universal moduli spaces

For $e\in\mathrm{Edg}$, let

$$X_{\gamma;e}\equiv\prod_{f\in e}X,\qquad \Delta_{\gamma;e}=\big\{(x_f)_{f\in e}\colon x_f=x_{f'}\ \forall\,f,f'\in e\big\}.$$

Define

$$\Delta_\gamma\equiv\prod_{e\in\mathrm{Edg}}\Delta_{\gamma;e}\subset X_\gamma\equiv\prod_{e\in\mathrm{Edg}}X_{\gamma;e}. \tag{3.24}$$

The evaluation maps ev_f induce maps

$$\mathsf{ev}_\gamma\equiv\prod_{e\in\mathrm{Edg}}\prod_{f\in e}\mathsf{ev}_f\colon\prod_{v\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_v^{\dagger*}\longrightarrow X_\gamma,\quad \mathsf{ev}\equiv\prod_{i\in[k]}\mathsf{ev}_i\colon\prod_{v\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_v^{\dagger*}\longrightarrow X^k. \tag{3.25}$$

Let $\widetilde{\mathfrak{B}}^\star=\mathsf{ev}_\gamma^{-1}(\Delta_\gamma)$, $\widetilde{\mathfrak{B}}^{\dagger*}\subset\widetilde{\mathfrak{B}}^\star$ be the subspace of tuples $(\mathbf{u}_v)_{v\in\mathrm{Ver}}$, with $\mathbf{u}_v$ as in (3.8), so that $\mathrm{Im}\,u_{v_1}\neq\mathrm{Im}\,u_{v_2}$ for all $v_1,v_2\in\mathrm{Ver}_0^c$ distinct, and

$$\widetilde{\mathfrak{B}}^*=\widetilde{\mathfrak{B}}^{\dagger*}\cap\prod_{v\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_v^*\,.$$

A tuple in $\widetilde{\mathfrak{B}}^{\dagger*}$ (resp. in $\widetilde{\mathfrak{B}}^*$) corresponds to a reduced (resp. basic) k-marked GU map as in (2.16). With

$$\pi_v\colon\prod_{v'\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_{v'}^{\dagger*}\longrightarrow\widetilde{\mathfrak{B}}_v$$

denoting the projection map, let

$$\begin{gathered}F_{\mathfrak{m}}=\bigoplus_{v\in\mathrm{Ver}}\pi_v^*F_{\mathfrak{m};v},\quad \mathfrak{D}_{\mathfrak{m}}=\bigoplus_{v\in\mathrm{Ver}}\pi_v^*\mathfrak{D}_{\mathfrak{m};v},\quad \widetilde{\mathfrak{B}}_{\mathfrak{m}}^*=\mathfrak{D}_{\mathfrak{m}}^{-1}(0)\cap\widetilde{\mathfrak{B}}^*,\\ \widetilde{\mathfrak{B}}_{\mathfrak{m};h}^*=\big\{(\mathbf{u},y)\in\widetilde{\mathfrak{B}}_{\mathfrak{m}}^*\times Y\colon\mathsf{ev}(\mathbf{u})=h(y)\big\}.\end{gathered} \tag{3.26}$$

By the reasoning at the top of [18, p47], the spaces $\widetilde{\mathfrak{B}}_v$ are separable smooth Banach manifolds. By the next lemma, the subspaces

$$\widetilde{\mathfrak{B}}^{\dagger *} \subset \prod_{v\in \mathrm{Ver}} \widetilde{\mathfrak{B}}_v, \qquad \widetilde{\mathfrak{B}}^*_{\mathfrak{m}} \subset \widetilde{\mathfrak{B}}^{\dagger *}, \quad \text{and} \quad \widetilde{\mathfrak{B}}^*_{\mathfrak{m};h} \subset \widetilde{\mathfrak{B}}^*_{\mathfrak{m}} \times Y$$

are smooth Banach submanifolds of codimensions

$$\begin{gathered}\mathrm{codim}_{\mathbb{R}} \widetilde{\mathfrak{B}}^{\dagger *} = 2n\big(|\mathrm{Fl}|-|\mathrm{Edg}|\big), \\ \mathrm{codim}_{\mathbb{R}}\big(\widetilde{\mathfrak{B}}^*_{\mathfrak{m}}, \widetilde{\mathfrak{B}}^{\dagger *}\big) = n\sum_{i\in[k]} m_i(m_i+3), \quad \mathrm{codim}_{\mathbb{R}}\big(\widetilde{\mathfrak{B}}^*_{\mathfrak{m};h}, \widetilde{\mathfrak{B}}^*_{\mathfrak{m}}\times Y\big) = 2nk,\end{gathered} \tag{3.27}$$

respectively.

Lemma 3.3. *The subspace $\widetilde{\mathfrak{B}}^{\dagger *}$ is a smooth Banach submanifold of $\prod_{v\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_v$ of codimension (3.27). The restriction of the bundle section $\mathfrak{D}_{\mathfrak{m}}$ to $\widetilde{\mathfrak{B}}^*$ is transverse to the zero set. The restriction of the map ev to $\widetilde{\mathfrak{B}}^*_{\mathfrak{m}}$ is transverse to h.*

Proof. For each $v\in\mathrm{Ver}$, $f\in S_v(\gamma)$, and $\mathbf{u}_v\in\widetilde{\mathfrak{B}}_v$ as in (3.8), define

$$L_f\colon \Gamma(\mathbf{u}_v) \longrightarrow T_{\mathsf{ev}_f(\mathbf{u}_v)}X, \qquad L_f(\xi_v) = \xi_v(z_f).$$

Since $z_f \neq z_{f'}$ for all $f,f'\in S_v(\gamma)\cap\mathrm{Fl}$ distinct, the homomorphism

$$L_v\colon \Gamma(\mathbf{u}_v) \longrightarrow \bigoplus_{f\in S_v(\gamma)\cap \mathrm{Fl}} T_{\mathsf{ev}_f(\mathbf{u}_v)}X, \qquad L_v(\xi_v) = \big(L_f(\xi_v)\big)_{f\in S_v(\gamma)\cap\mathrm{Fl}}, \tag{3.28}$$

is surjective for every $\mathbf{u}_v\in\widetilde{\mathfrak{B}}_v$.

Since $\widetilde{\mathfrak{B}}^{\dagger *}_v \subset \widetilde{\mathfrak{B}}_v$ is an open subset, $\widetilde{\mathfrak{B}}^{\dagger *}_v$ is a separable smooth Banach manifold of codimension 0. Furthermore, the maps in (3.25) and the bundle section in (3.26) are smooth. The restriction of the differential of ev_γ at $\mathbf{u}\equiv(\mathbf{u}_v)_{v\in\mathrm{Ver}}$ to

$$\Gamma(\mathbf{u}) \equiv \bigoplus_{v\in\mathrm{Ver}} \Gamma(\mathbf{u}_v) \subset \bigoplus_{v\in\mathrm{Ver}} T_{\mathbf{u}_v}\widetilde{\mathfrak{B}}^{\dagger *}_v = T_{\mathbf{u}}\Big(\prod_{\in\mathrm{Ver}} \widetilde{\mathfrak{B}}^{\dagger *}_v\Big)$$

is given by

$$\mathrm{d}_{\mathbf{u}}\mathsf{ev}_\gamma\big((\xi_v)_{v\in\mathrm{Ver}}\big) = \big(L_v(\xi_v)\big)_{v\in\mathrm{Ver}}.$$

By the surjectivity of (3.28), this restriction is surjective. Thus, the map ev_γ is transverse to Δ_γ. In light of the Implicit Function Theorem for Banach manifolds, it follows that

$$\widetilde{\mathfrak{B}}^\star \subset \prod_{\in\mathrm{Ver}} \widetilde{\mathfrak{B}}^{\dagger *}_v$$

is a smooth Banach submanifold of codimension (3.27). Since $\widetilde{\mathfrak{B}}^{\dagger *} \subset \widetilde{\mathfrak{B}}^\star$ is an open subset, this implies the first claim of the lemma.

For each $v\in\mathrm{Ver}$, $i\in S_v(\gamma)\cap[k]$, and $\mathbf{u}_v\in\widetilde{\mathfrak{B}}_{\mathfrak{m};v}$ as in (3.8), define

$$L_{\mathfrak{m};i}\colon \Gamma(\mathbf{u}_v) \longrightarrow F_{\mathfrak{m};i}, \qquad L_{\mathfrak{m};i}(\xi_v) = \big(\nabla^m\xi_v|_{z_i}\big)_{m\in[m_i]}.$$

Since $\mathfrak{D}^m_{u_v}|_{z_i}=0$ for all $m\in[m_i]$, this homomorphism is independent of the choice of ∇. If $\mathbf{u}_v\in\widetilde{\mathfrak{B}}^*_{\mathfrak{m};v}$, then $z_f\neq z_{f'}$ for all $f,f'\in S_v(\gamma)$ distinct. Thus, the homomorphism

$$L_v\oplus\bigoplus_{i\in S_v(\gamma)\cap[k]}L_i\oplus L_{\mathfrak{m};v}\colon\Gamma(\mathbf{u}_v)\longrightarrow\bigoplus_{f\in S_v(\gamma)}T_{\mathsf{ev}_f(\mathbf{u}_v)}X\oplus F_{\mathfrak{m};v},\tag{3.29}$$
$$L_{\mathfrak{m};v}(\xi_v)=\big(L_v(\xi_v),(L_i(\xi_v))_{i\in S_v(\gamma)\cap[k]},\big(L_{\mathfrak{m};i}(\xi_v)\big)_{i\in S_v(\gamma)\cap[k]}\big)\,,$$

is surjective for every $\mathbf{u}_v\in\widetilde{\mathfrak{B}}^*_{\mathfrak{m};v}$.

The restriction of the linearization of $\mathfrak{D}_{\mathfrak{m}}$ at an element $\mathbf{u}\equiv(\mathbf{u}_v)_{v\in\mathrm{Ver}}$ of $\widetilde{\mathfrak{B}}^*_{\mathfrak{m}}$ to

$$\Gamma_0(\mathbf{u})\equiv\bigoplus_{v\in\mathrm{Ver}}\Gamma_0(\mathbf{u}_v)\equiv\big\{(\xi_v)_{v\in\mathrm{Ver}}\colon L_f(\xi_v)=0\ \forall\,f\in S_v(\gamma)\cap\mathrm{Fl},\ v\in\mathrm{Ver}\big\}\subset T_{\mathbf{u}}\widetilde{\mathfrak{B}}^*$$

is given by

$$\mathrm{d}_{\mathbf{u}}\mathfrak{D}_{\mathfrak{m}}\big((\xi_v)_{v\in\mathrm{Ver}}\big)=\big(L_{\mathfrak{m};v}(\xi_v)\big)_{v\in\mathrm{Ver}}\,.$$

By the surjectivity of (3.29), this restriction is surjective. This implies the second claim of the lemma.

The restriction of the differential of ev at an element $\mathbf{u}\equiv(\mathbf{u}_v)_{v\in\mathrm{Ver}}$ of $\widetilde{\mathfrak{B}}^*_{\mathfrak{m}}$ to

$$\big\{(\xi_v)_{v\in\mathrm{Ver}}\in\Gamma_0(\mathbf{u})\colon L_{\mathfrak{m};i}(\xi_v)=0\ \forall\,i\in S_v(\gamma)\cap[k],\ v\in\mathrm{Ver}\big\}\subset T_{\mathbf{u}}\widetilde{\mathfrak{B}}^*_{\mathfrak{m}}$$

is given by

$$\mathrm{d}_{\mathbf{u}}\mathsf{ev}\big((\xi_v)_{v\in\mathrm{Ver}}\big)=\big(L_i(\xi_v)\big)_{i\in S_v(\gamma)\cap[k],v\in\mathrm{Ver}}\,.$$

By the surjectivity of (3.29), this restriction is surjective. This establishes the last claim of the lemma. □

For $B'=B_\circ,B$ and $v\in\mathrm{Ver}$, let

$$\mathfrak{F}_{B';v}\longrightarrow\mathcal{H}^\ell_{\mathfrak{p}}(B';X)\times B'\times\widetilde{\mathfrak{B}}_v\quad\text{and}\quad\mathfrak{F}_{B';\mathfrak{m};v}\subset\mathfrak{F}_{B';v}$$

be the bundle with the fibers $\mathfrak{F}_{(J,\nu;b,\mathbf{u}_v)}=\Gamma^{0,1}_{J_b}(\mathbf{u}_v)$ and its subbundle with the fibers $\Gamma^{0,1}_{J_b;\mathfrak{m}}(\mathbf{u}_v)$. We define a section of $\mathfrak{F}_{B';v}$ by

$$\bar\partial_{B';v}(J,\nu;b,\mathbf{u}_v)\big|_z=\bar\partial_{J,\mathrm{j}_v}u_v\big|_z-\begin{cases}\nu_{\gamma;v}(z,u_v(z)), & \text{if }v\in\mathrm{Ver}_0;\\ 0, & \text{if }v\in\mathrm{Ver}_0^c;\end{cases}\qquad\forall\,z\in\Sigma_v.$$

The restriction of $\bar\partial_{B';v}$ to $\mathcal{H}^\ell_{\mathfrak{p}}(B';X)\times B'\times\widetilde{\mathfrak{B}}^{\dagger*}_{\mathfrak{m};v}$ takes values in $\mathfrak{F}_{B';\mathfrak{m};v}$. Let

$$\widetilde\pi_v\colon\mathcal{H}^\ell_{\mathfrak{p}}(B';X)\times B'\times\prod_{v'\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_{v'}\longrightarrow\mathcal{H}^\ell_{\mathfrak{p}}(B';X)\times B'\times\widetilde{\mathfrak{B}}_v\quad\text{and}$$
$$\widetilde\pi^c_Y\colon\mathcal{H}^\ell_{\mathfrak{p}}(B';X)\times B'\times\prod_{v\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_v\times Y\longrightarrow\mathcal{H}^\ell_{\mathfrak{p}}(B';X)\times B'\times\prod_{v\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_v$$

denote the projection maps.

Define

$$\mathfrak{F}_{B'}=\bigoplus_{v\in\mathrm{Ver}}\widetilde{\pi}_v^*\mathfrak{F}_{B';v}\longrightarrow\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\prod_{v\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_v\,,\quad \mathfrak{F}_{B';\mathfrak{m}}=\bigoplus_{v\in\mathrm{Ver}}\widetilde{\pi}_v^*\mathfrak{F}_{B';\mathfrak{m};v}\subset\mathfrak{F}_{B'},\tag{3.30}$$

$$\mathfrak{F}_{B';Y}=\widetilde{\pi}_Y^{c*}\mathfrak{F}_{B';\mathfrak{m}}\longrightarrow\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\prod_{v'\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_{v'}\times Y.$$

The restriction of the bundle section

$$\bar{\partial}_{B'}\equiv(\widetilde{\pi}_v^*\bar{\partial}_{B';v})_{v\in\mathrm{Ver}}:\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\prod_{v\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_v\longrightarrow\mathfrak{F}_{B'}\tag{3.31}$$

of the first bundle in (3.30) to $\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\widetilde{\mathfrak{B}}_{\mathfrak{m}}^*$ takes values in $\mathfrak{F}_{B';\mathfrak{m}}$. Thus, the restriction of the section

$$\bar{\partial}_{B';Y}\equiv\widetilde{\pi}_Y^{c*}\bar{\partial}_{B'}:\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\prod_{v'\in\mathrm{Ver}}\widetilde{\mathfrak{B}}_{v'}\times Y\longrightarrow\widetilde{\pi}_Y^{c*}\mathfrak{F}_{B'}$$

to $\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\widetilde{\mathfrak{B}}_{\mathfrak{m};h}^*$ takes values in $\mathfrak{F}_{B';Y}$. Let

$$\mathfrak{U}\widetilde{\mathfrak{M}}^{\dagger*}(B')=\bar{\partial}_{B'}^{-1}(0)\cap\big(\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\widetilde{\mathfrak{B}}^{\dagger*}\big),\quad \mathfrak{U}\widetilde{\mathcal{Z}}_{\mathfrak{m}}^*(B')=\bar{\partial}_{B'}^{-1}(0)\cap\big(\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\widetilde{\mathfrak{B}}_{\mathfrak{m}}^*\big),$$

$$\mathfrak{U}\widetilde{\mathcal{Z}}_{\mathfrak{m};h}^*(B')=\bar{\partial}_{B';Y}^{-1}(0)\cap\big(\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\widetilde{\mathfrak{B}}_{\mathfrak{m};h}^*\big)\,.$$

These are the universal moduli spaces associated with the moduli spaces appearing with Theorems 2.5 and 2.6.

By the reasoning at the bottom of [18, p49], the space $\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)$ is a separable smooth Banach manifold. By the reasoning at the bottom of [18, p50], (3.31) is a C^ℓ section of the C^ℓ Banach bundle (3.30). Along with Lemma 3.3, this implies that the restrictions

$$\bar{\partial}_{B'}^{\dagger}:\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\widetilde{\mathfrak{B}}^{\dagger*}\longrightarrow\mathfrak{F}_{B'},\quad \bar{\partial}_{B';\mathfrak{m}}:\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\widetilde{\mathfrak{B}}_{\mathfrak{m}}^*\longrightarrow\mathfrak{F}_{B';\mathfrak{m}},$$

$$\bar{\partial}_{B';Y;\mathfrak{m}}:\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)\times B'\times\widetilde{\mathfrak{B}}_{\mathfrak{m};h}^*\longrightarrow\mathfrak{F}_{B';Y}$$

of $\bar{\partial}_{B'}$ in the first two cases and of $\bar{\partial}_{B';Y}$ in the last case are also C^ℓ sections of C^ℓ Banach bundles.

Lemma 3.4. *For every* $(J,\nu)\in\mathcal{H}_{\mathfrak{p}}^{\ell}(B';X)$*, the restrictions*

$$\bar{\partial}_{B'}^{\dagger}:\{(J,\nu)\}\times B'\times\widetilde{\mathfrak{B}}^{\dagger*}\longrightarrow\mathfrak{F}_{B'},\quad \bar{\partial}_{B';\mathfrak{m}}:\{(J,\nu)\}\times B'\times\widetilde{\mathfrak{B}}_{\mathfrak{m}}^*\longrightarrow\mathfrak{F}_{B';\mathfrak{m}},$$

$$\text{and}\qquad \bar{\partial}_{B';Y;\mathfrak{m}}:\{(J,\nu)\}\times B'\times\widetilde{\mathfrak{B}}_{\mathfrak{m};h}^*\longrightarrow\mathfrak{F}_{B';Y}$$

are Fredholm sections of indices

$$\mathrm{ind}_{\mathbb{R}}^c\bar{\partial}_{B'}^{\dagger}=\dim_{\mathbb{R}}B_\circ+2\dim(\gamma)+6|\mathrm{Ver}_0^c|,\quad \mathrm{ind}_{\mathbb{R}}\bar{\partial}_{B';\mathfrak{m}}^{\dagger}=\dim_{\mathbb{R}}^cB_\circ+2\big(\dim(\gamma)-n|\mathfrak{m}|\big)+6|\mathrm{Ver}_0^c|,$$

$$\text{and}\qquad \mathrm{ind}_{\mathbb{R}}\bar{\partial}_{B';Y;\mathfrak{m}}=\dim_{\mathbb{R}}^cB_\circ+2\big(\dim(\gamma)-n|\mathfrak{m}|\big)-\big(2nk-\dim_{\mathbb{R}}Y\big)+6|\mathrm{Ver}_0^c|,$$

respectively.

Proof. For $v\in\mathrm{Ver}_0^c$, $b\in B'$, and $\mathbf{u}_v\in\widetilde{\mathfrak{B}}_v$ such that

$$\bar\partial_{B';v}(J,\nu;b,\mathbf{u}_v)=0\in\mathfrak{F}_{B';v}\,, \tag{3.32}$$

denote by $D_{J,\nu;\mathbf{u}_v}$ the operator $D_{J;\mathbf{u}_v}$ in (3.12). For all $v\in\mathrm{Ver}$, $b\in B'$, and $\mathbf{u}_v\in\widetilde{\mathfrak{B}}_v$, the operator $D_{J;\mathbf{u}_v}$ is then the restriction of the vertical differential $D\bar\partial_{B';v}$ of the section $\bar\partial_{B';v}$ of $\mathfrak{F}_{B';v}$ at $(J,\nu;b,\mathbf{u}_v)$ to the subspace

$$\Gamma(\mathbf{u}_v)\subset T_{\mathbf{u}_v}\widetilde{\mathfrak{B}}_v\subset T_{(J,\nu;b,\mathbf{u}_v)}\big(\mathcal{H}^\ell_{\mathfrak{p}}(B';X)\times B'\times\widetilde{\mathfrak{B}}_v\big)$$

consisting of the infinitesimal deformations of the map component u_v of $\mathbf{u}_v$ as in (3.8).

Since $D_{J;\mathbf{u}_v}$ is an elliptic operator and Σ_v is a compact manifold, $D_{J;\mathbf{u}_v}$ is a Fredholm operator. By Riemann-Roch,

$$\mathrm{ind}_{\mathbb{R}}D_{J,\nu;\mathbf{u}_v}=2\big(\langle c_1(TX),\mathfrak{d}(v)\rangle+n\big(1-\mathfrak{g}(v)\big)\big)\qquad\forall\,v\in\mathrm{Ver}.$$

Thus, the restriction

$$\bar\partial_{B';v}\colon\big\{(J,\nu)\big\}\times\{b\}\times\widetilde{\mathfrak{B}}_v\longrightarrow\mathfrak{F}_{B';v}$$

is a Fredholm section of index

$$\begin{aligned}\mathrm{ind}^{cc}_{\mathbb{R}}\bar\partial_{B';v}&=\mathrm{ind}_{\mathbb{R}}D_{J,\nu;\mathbf{u}_v}+\dim_{\mathbb{R}}\widetilde{\mathcal{M}}_{\gamma;v}\\&=2\big(\langle c_1(TX),\mathfrak{d}(v)\rangle+(n-3)\big(1-\mathfrak{g}(v)\big)+|S_v(\gamma)|\big)+\begin{cases}0,&\text{if }v\in\mathrm{Ver}_0;\\6,&\text{if }v\in\mathrm{Ver}_0^c.\end{cases}\end{aligned}\tag{3.33}$$

By (3.31), the first statement of Lemma 3.3, and (3.33), the restriction of $\bar\partial^\dagger_{B'}$ in the statement of the present lemma is a Fredholm section of index

$$\begin{aligned}\mathrm{ind}^{cc}_{\mathbb{R}}\bar\partial^\dagger_{B'}&=\dim_{\mathbb{R}}B_\circ+\sum_{v\in\mathrm{Ver}}\mathrm{ind}^{c}_{\mathbb{R}}\bar\partial_{B';v}-\mathrm{codim}_{\mathbb{R}}\widetilde{\mathfrak{B}}^{\dagger*}\\&=\dim_{\mathbb{R}}B_\circ+2\big(\langle c_1(TX),A\rangle+\sum_{v\in\mathrm{Ver}}(n-3)\big(1-\mathfrak{g}(v)\big)+k+|\mathrm{Fl}|\big)+6|\mathrm{Ver}_0^c|-2n\big(|\mathrm{Fl}|-|\mathrm{Edg}|\big).\end{aligned}$$

Combining this with (2.20) and (2.26), we obtain the first index statement.

The corank of the subbundle $\mathfrak{F}_{B';\mathfrak{m}}$ of $\mathfrak{F}_{B'}$ is $n\sum_{i\in[k]}m_i(m_i{+}1)$. Along with the previous paragraph and the second statement of Lemma 3.3, this implies that the restriction of $\bar\partial_{B';\mathfrak{m}}$ in the statement of the present lemma is a Fredholm section of index

$$\mathrm{ind}_{\mathbb{R}}\bar\partial^\dagger_{B';\mathfrak{m}}=\mathrm{ind}^c_{\mathbb{R}}\bar\partial^\dagger_{B'}-\mathrm{codim}_{\mathbb{R}}\big(\widetilde{\mathfrak{B}}^*_{\mathfrak{m}},\widetilde{\mathfrak{B}}^{\dagger*}\big)+n\sum_{i\in[k]}m_i(m_i+1)=\mathrm{ind}^c_{\mathbb{R}}\bar\partial^\dagger_{B'}-2n|\mathfrak{m}|.$$

This gives the second index statement.

By the previous paragraph and the third statement of Lemma 3.3, the restriction of $\bar\partial_{B';Y;\mathfrak{m}}$ in the statement of the present lemma is a Fredholm section of index

$$\mathrm{ind}_{\mathbb{R}}\bar\partial_{B';Y;\mathfrak{m}}=\mathrm{ind}_{\mathbb{R}}\bar\partial^\dagger_{B';\mathfrak{m}}+\dim_{\mathbb{R}}Y-\mathrm{codim}_{\mathbb{R}}\big(\widetilde{\mathfrak{B}}^*_{\mathfrak{m};h},\widetilde{\mathfrak{B}}^*_{\mathfrak{m}}\times Y\big)=\mathrm{ind}_{\mathbb{R}}\bar\partial^\dagger_{B';\mathfrak{m}}-\big(2nk-\dim_{\mathbb{R}}Y\big).$$

This completes the proof. □

Proposition 3.5. *The bundle sections*

$$\bar\partial^\dagger_{B_\circ}\colon \mathcal{H}^\ell_{\mathfrak p}(B_\circ;X)\times B_\circ\times\widetilde{\mathfrak B}^{\dagger *}\longrightarrow \mathfrak F_{B_\circ}, \quad \bar\partial_{B_\circ;\mathfrak m}\colon \mathcal{H}^\ell_{\mathfrak p}(B_\circ;X)\times B_\circ\times\widetilde{\mathfrak B}^{*}_{\mathfrak m}\longrightarrow \mathfrak F_{B_\circ;\mathfrak m},$$
$$\text{and}\qquad \bar\partial_{B_\circ;Y;\mathfrak m}\colon \mathcal{H}^\ell_{\mathfrak p}(B_\circ;X)\times B_\circ\times\widetilde{\mathfrak B}^{*}_{\mathfrak m;h}\longrightarrow \mathfrak F_{B_\circ;Y}$$

are transverse to the zero set. If $(J_\circ,\nu_\circ)\in\mathcal{H}^\ell_{\mathfrak p}(B_\circ;X)$ *is regular value of one of the projections*

$$\mathfrak{U}\widetilde{\mathfrak M}^{\dagger *}(B_\circ), \mathfrak{U}\widetilde{\mathcal Z}^*_{\mathfrak m}(B_\circ), \mathfrak{U}\widetilde{\mathcal Z}^*_{\mathfrak m;h}(B_\circ)\longrightarrow \mathcal{H}^\ell_{\mathfrak p}(B_\circ;X), \tag{3.34}$$

then the corresponding bundle section

$$\bar\partial^\dagger_{B}\colon \mathcal{H}^\ell_{J_\circ,\nu_\circ}(B;X)\times B\times\widetilde{\mathfrak B}^{\dagger *}\longrightarrow \mathfrak F_{B}, \quad \bar\partial_{B;\mathfrak m}\colon \mathcal{H}^\ell_{J_\circ,\nu_\circ}(B;X)\times B\times\widetilde{\mathfrak B}^{*}_{\mathfrak m}\longrightarrow \mathfrak F_{B;\mathfrak m},$$
$$\text{and}\qquad \bar\partial_{B;Y;\mathfrak m}\colon \mathcal{H}^\ell_{J_\circ,\nu_\circ}(B;X)\times B\times\widetilde{\mathfrak B}^{*}_{\mathfrak m;h}\longrightarrow \mathfrak F_{B;Y}$$

is also transverse to the zero set. For every $\omega\in\mathrm{Symp}(B_\circ;X)$ *(resp. in* $\omega\in\mathrm{Symp}(B;X)$*), the first (resp. second) statement holds with* $\mathcal{H}^\ell_{\mathfrak p}$ *replaced by* $\mathcal{H}^\ell_{\mathfrak p;\omega}$ *and* $\mathcal{H}'^\ell_{\mathfrak p;\omega}$.

Proof. For $v\in\mathrm{Ver}_0$, $(J,\nu)\in\mathcal{H}^\ell_{\mathfrak p}(B_\circ;X)$, $b\in B_\circ$, and $\mathbf u_v\in\widetilde{\mathfrak B}_v$ as in (3.8) satisfying (3.32), define

$$\begin{gathered}D^0_{J,\nu;b,\mathbf u_v}\bar\partial\colon T_{(J,\nu)}\mathcal{H}_{\mathfrak p}(B_\circ;X)\oplus\Gamma_0(\mathbf u_v)\longrightarrow \Gamma^{0,1}_{J_b}(\mathbf u_v),\\ D^0_{J,\nu;b,\mathbf u_v}\bar\partial(A,\nu';\xi_v)=D^0_{J_b,\nu_b;\mathbf u_v}\xi_v+\frac12 A_b\circ \mathrm d u_v\circ \mathfrak j_v-\{\widehat\iota_{\gamma;v}\times u_v\}^*\nu'_b.\end{gathered}$$

For $v\in\mathrm{Ver}_0^c$, we define $D^0_{J,\nu;b,\mathbf u_v}\bar\partial$ in the same way dropping $\{\widehat\iota_{\gamma;v}\times u_v\}^*\nu'_b$ above. Note that

$$\begin{aligned} A_b\circ\mathrm d u_v\circ\mathfrak j_v&=0 &&\text{if}\quad u_v(\Sigma_v)\cap\mathrm{supp}(A_b)=\varnothing,\\ \{\widehat\iota_{\gamma;v}\times u_v\}^*\nu'_b&=0 &&\text{if}\quad v\in\mathrm{Ver}_0^c \text{ or } \big(\widehat\iota_{\gamma;v}(\Sigma_v)\times X\big)\cap\mathrm{supp}(\nu'_b)=\varnothing.\end{aligned}\tag{3.35}$$

The homomorphism $D^0_{J,\nu;b,\mathbf u_v}\bar\partial$ is the restriction of the linearization of $\bar\partial_{B_\circ;v}$ at $(J,\nu;b,\mathbf u_v)$ to

$$T_{(J,\nu)}\mathcal{H}_{\mathfrak p}(B_\circ;X)\oplus\Gamma_0(\mathbf u_v)\subset T_{(J,\nu)}\mathcal{H}_{\mathfrak p}(B_\circ;X)\oplus T_{\mathbf u_v}\widetilde{\mathfrak B}_v\subset T_{(J,\nu;b,\mathbf u_v)}\big(\mathcal{H}_{\mathfrak p}(B_\circ;X)\times B_\circ\times\widetilde{\mathfrak B}_v\big).$$

Let $(J,\nu)\in\mathcal{H}^\ell_{\mathfrak p}(B_\circ;X)$, $b\in B_\circ$, and $\mathbf u\equiv(\mathbf u_v)_{v\in\mathrm{Ver}}\in\widetilde{\mathfrak B}^{\dagger *}$ be such that

$$\bar\partial^\dagger_{B_\circ}\big(J,\nu;b,\mathbf u\big)=0\in\mathfrak F_{B^\circ}\big|_{(J,\nu;b,\mathbf u)}\equiv\Gamma^{0,1}_{J_b}(\mathbf u)\equiv\bigoplus_{v\in\mathrm{Ver}}\Gamma^{0,1}_{J_b}(\mathbf u_v).$$

We show that the homomorphism

$$\begin{gathered}D^0_{J,\nu;b,\mathbf u}\bar\partial\colon T_{(J,\nu)}\mathcal{H}_{\mathfrak p}(B_\circ;X)\oplus\Gamma_0(\mathbf u)\longrightarrow\bigoplus_{v\in\mathrm{Ver}}\Gamma^{0,1}_{J_b}(\mathbf u_v),\\ D^0_{J,\nu;b,\mathbf u}\bar\partial\big(A,\nu';(\xi_v)_{v\in\mathrm{Ver}}\big)=\big(D^0_{J,\nu;b,\mathbf u_v}\bar\partial(A,\nu';\xi_v)\big)_{v\in\mathrm{Ver}},\end{gathered}\tag{3.36}$$

is surjective. Since $\Gamma_0(\mathbf u)\subset T_{\mathbf u}\widetilde{\mathfrak B}^{\dagger *}$, this implies that the bundle section $\bar\partial^\dagger_{B_\circ}$ is transverse to the zero set.

Since $\mathbf{u}\in\widetilde{\mathfrak{B}}^{\dagger *}$, $u_{v_1}(\Sigma_{v_1})\neq u_{v_2}(\Sigma_{v_2})$ for all $v_1,v_2\in\mathrm{Ver}_0^c$ distinct. It follows that $u_{v_1}^{-1}(X-u_{v_2}(\Sigma_{v_2}))$ is a dense open subset of Σ_{v_1} whenever $v_1,v_2\in\mathrm{Ver}_0^c$ are distinct; see [33, Corollary 3.9]. Thus, there exist open subsets $W_v\subset X$ with $v\in\mathrm{Ver}_0^c$ such that

$$u_v(\Sigma_v)\cap W_v\neq\varnothing \qquad \forall\, v\in\mathrm{Ver}_0^c, \tag{3.37}$$

$$u_{v_1}(\Sigma_{v_1})\cap W_{v_2}=\varnothing,\quad W_{v_1}\cap W_{v_2}=\varnothing \qquad \forall\, v_1,v_2\in\mathrm{Ver}_0^c,\ v_1\neq v_2. \tag{3.38}$$

The subsets $\widehat{\iota}_{\gamma;v}(\Sigma_v)\subset\widetilde{\mathcal{U}}_{g,k}$ with $v\in\mathrm{Ver}_0$ are also distinct. Thus, there exist open subsets $W_v\subset\widetilde{\mathcal{U}}_{g,k}$ with $v\in\mathrm{Ver}_0$ such that

$$\widehat{\iota}_{\gamma;v}(\Sigma_v)\cap W_v\neq\varnothing \qquad \forall\, v\in\mathrm{Ver}_0, \tag{3.39}$$

$$\widehat{\iota}_{\gamma;v}(\Sigma_{v_1})\cap W_{v_2}=\varnothing,\quad W_{v_1}\cap W_{v_2}=\varnothing \qquad \forall\, v_1,v_2\in\mathrm{Ver}_0,\ v_1\neq v_2. \tag{3.40}$$

Define

$$T_v\mathcal{H}=\begin{cases}\big\{(A,0)\in T_{(J,\nu)}\mathcal{H}_{\mathfrak{p}}(B_\circ;X)\colon \mathrm{supp}(A_b)\subset W_v\big\}, & \text{if } v\in\mathrm{Ver}_0^c,\\ \big\{(0,\nu')\in T_{(J,\nu)}\mathcal{H}_{\mathfrak{p}}(B_\circ;X)\colon \mathrm{supp}(\nu_b')\subset W_v\times X\big\} & \text{if } v\in\mathrm{Ver}_0\,.\end{cases}$$

By the first statements of Lemmas 3.1 and 3.2, (3.37), and (3.39),

$$D^0_{J,\nu;b,\mathbf{u}_v}\bar\partial\big(T_v\mathcal{H}\oplus\Gamma_0(\mathbf{u}_v)\big)=\Gamma^{0,1}_{J_b}(\mathbf{u}_v)\subset\Gamma^{0,1}_{J_b}(\mathbf{u}) \qquad \forall\ v\in\mathrm{Ver}.$$

By (3.35) and the first statement in (3.38),

$$D^0_{J,\nu;b,\mathbf{u}_v}\bar\partial\big(T_{v'}\mathcal{H}\big)=\{0\} \qquad \forall\ v\in\mathrm{Ver}_0^c,\ v'\in\mathrm{Ver}-\{v\}.$$

By the second statement in (3.35) and the first statement in (3.40),

$$D^0_{J,\nu;b,\mathbf{u}_v}\bar\partial\big(T_{v'}\mathcal{H}\big)=\{0\} \qquad \forall\ v\in\mathrm{Ver}_0,\ v'\in\mathrm{Ver}_0-\{v\}.$$

By the last statements in (3.38) and (3.40),

$$\bigoplus_{v\in\mathrm{Ver}}\big(T_v\mathcal{H}\oplus\Gamma_0(\mathbf{u}_v)\big)\subset T_{(J,\nu)}\mathcal{H}_{\mathfrak{p}}(B_\circ;X)\oplus\Gamma_0(\mathbf{u}).$$

By the last four statements, the homomorphism (3.36) is surjective. This establishes the transversality of the bundle section $\bar\partial^{\dagger}_{B_\circ}$.

Suppose in addition that $\mathbf{u}\in\widetilde{\mathfrak{B}}^*_{\mathfrak{m}}$. Let

$$\Gamma_{\mathfrak{m}}(\mathbf{u})=\bigoplus_{v\in\mathrm{Ver}}\Gamma_{\mathfrak{m}}(\mathbf{u}_v).$$

By the second statements of Lemmas 3.1 and 3.2, (3.37), and (3.39),

$$D^0_{J,\nu;b,\mathbf{u}_v}\bar\partial\big(T_v\mathcal{H}\oplus\Gamma_{\mathfrak{m}}(\mathbf{u}_v)\big)=\Gamma^{0,1}_{J_b;\mathfrak{m}}(\mathbf{u}_v)\subset\Gamma^{0,1}_{J_b;\mathfrak{m}}(\mathbf{u})\equiv\bigoplus_{v'\in\mathrm{Ver}}\Gamma^{0,1}_{J_b;\mathfrak{m}}(\mathbf{u}_{v'}) \qquad \forall\ v\in\mathrm{Ver}.$$

By the reasoning in the previous paragraph, this implies that the restriction

$$D^0_{J,\nu;b,\mathbf{u}}\bar\partial\colon T_{(J,\nu)}\mathcal{H}_{\mathfrak{p}}(B_\circ;X)\oplus\Gamma_{\mathfrak{m}}(\mathbf{u})\longrightarrow\Gamma^{0,1}_{J_b;\mathfrak{m}}(\mathbf{u})\equiv\mathfrak{F}_{B_\circ;\mathfrak{m}}\big|_{(J,\nu;b,\mathbf{u})} \tag{3.41}$$

the homomorphism (3.36) is surjective. Since $\Gamma_{\mathfrak m}(\mathbf u)\subset T_{\mathbf u}\widetilde{\mathfrak B}^*_{\mathfrak m}$, it follows that the bundle section $\bar\partial_{B_\circ;\mathfrak m}$ is transverse to the zero set. If in addition $(\mathbf u,y)\in\widetilde{\mathfrak B}^*_{\mathfrak m;h}$, then

$$\Gamma_{\mathfrak m}(\mathbf u)\subset T_{\mathbf u}\widetilde{\mathfrak B}^*_{\mathfrak m;h}\subset T_{\mathbf u}\widetilde{\mathfrak B}^*_{\mathfrak m}\oplus T_yY=T_{(\mathbf u,y)}\big(\widetilde{\mathfrak B}^*_{\mathfrak m}\times Y\big).$$

The surjectivity of (3.41) then implies that the bundle section $\bar\partial_{B_\circ;Y;\mathfrak m}$ is transverse to the zero set.

Suppose $(J_\circ,\nu_\circ)\in\mathcal H^\ell_{\mathfrak p}(B_\circ;X)$ is a regular value of a projection in (3.34). Denote by $\widetilde{\mathfrak B}$ the last component of the domain of the corresponding bundle section $\bar\partial$. Suppose $\bar\partial(J,\nu;b,\mathbf u)=0$. In particular, $(J,\nu)|_{B_\circ}=(J_\circ,\nu_\circ)$. If $b\in B_\circ$, then the restriction of the linearization of $\bar\partial$ at $(J,\nu;b,\mathbf u)$ to

$$T_{(b,\mathbf u)}\big(B\times\widetilde{\mathfrak B}\big)\subset T_{(J,\nu;b,\mathbf u)}\big(\mathcal H^\ell_{J_\circ,\nu_\circ}(B;X)\times B\times\widetilde{\mathfrak B}\big)$$

is surjective; see Lemma 4.3. If $b\notin B_\circ$, then the argument above with $B_\circ$ and $\mathcal H_{\mathfrak p}(B_\circ;X)$ replaced by B and $\mathcal H^\ell_{J_\circ,\nu_\circ}(B;X)$, respectively, shows that the linearization of $\bar\partial$ at $(J,\nu;b,\mathbf u)$ is surjective. We conclude that the bundle section $\bar\partial$ in question is transverse to the zero set.

The proof for $\mathcal H^\ell_{\mathfrak p;\omega}$ and $\mathcal H'^\ell_{\mathfrak p;\omega}$ in place of $\mathcal H^\ell_{\mathfrak p}$ is the same. □

Let $\widehat{\mathcal H}^\ell_{\mathfrak p}(B_\circ;X)\subset\mathcal H^\ell_{\mathfrak p}(B_\circ;X)$ be the intersection of the sets of regular values for the three projections in (3.34). By Sard-Smale Theorem (Proposition 4.1), $\widehat{\mathcal H}^\ell_{\mathfrak p}(B_\circ;X)$ is a ubiquitous subset of $\mathcal H^\ell_{\mathfrak p}(B_\circ;X)$ if $\ell\in\mathbb Z^+$ is sufficiently large.

4 Analytic preliminaries

4.1 Classical statements

Banach vector space, manifold, separable

Sard-Smale theorem, elliptic bootstrapping, elliptic implies Fredholm over compact domains

Proposition 4.1.

4.2 Fredholm bundle sections

Banach bundle, Fredholm section

Proposition 4.2. *if D is onto, then the moduli space is smooth*

4.3 Ubiquitous regularity

Lemma 4.3. *map to a manifold, transverse to a submanifold; regular value of projections v.s restricted to a fiber*

Proposition 4.4. *if universal section is surjective, then so is a generic restriction and each moduli space is smooth*

4.4 Taubes's argument

Taubes's argument moving from the ubiquitousness of a subset of the space C^ℓ-parameters to the ubiquitousness of its C^∞-analogue is applied explicitly in the proofs of Theorems 3.1.6(ii) and 6.2.6(ii) in [18] and implicitly in many other settings of similar nature. Proposition 4.5 below formalizes Taubes's argument in order to capture its substance and make it easier to apply.

If $\mathfrak{M}$ is a topological space, a map $f\colon\mathfrak{M}\longrightarrow\mathbb{R}$ is upper semi-continuous if the set $f^{-1}((-\infty,a))$ is an open subset of $\mathfrak{M}$ for every $a\in\mathbb{R}$. For example, the map

$$\mathfrak{U}\widetilde{\mathfrak{M}}^*_{0,0}(A)\longrightarrow\mathbb{Z}^{\geqslant 0}\subset\mathbb{R},\qquad (J,u)\longrightarrow \dim\operatorname{cok}D_{J;u},$$

is upper semi-continuous. If $\mathcal{J}$ is another topological space, a continuous map $\pi\colon\mathfrak{M}\longrightarrow\mathcal{J}$ is proper if $\pi^{-1}(K)$ is a compact subset of $\mathfrak{M}$ for every compact subset K of $\mathcal{J}$. For example, the map

$$\pi\colon\mathfrak{M}_r\equiv\Big\{(J,u)\in\mathfrak{U}\widetilde{\mathfrak{M}}^*_{0,0}(A)\colon \|\mathrm{d}u\|_{C^0}\leqslant r,\ \exists\, z\in\mathbb{P}^1 \text{ s.t. } \sup_{z'\in\mathbb{P}^1-\{z\}}\frac{d_X(u(z),u(z'))}{d_{\mathbb{P}^1}(z,z')}\geqslant 1/r\Big\}\longrightarrow\mathcal{J},$$
$$\pi(J,u)=J,$$

is proper for every $r\in\mathbb{R}^+$. This follows from elliptic bootstrapping for J-holomorphic maps; see [18, Theorem B.4.2].

Let $(\mathcal{J},\mathcal{T})$ be a topological space as in [19, §12], i.e. a set $\mathcal{J}$ together with a collection $\mathcal{T}$ of subsets of $\mathcal{J}$ satisfying certain properties. We call a sequence $(\mathcal{J}^\ell,\mathcal{T}^\ell)_{\ell\in\mathbb{Z}^+}$ of topological spaces an expansion of $(\mathcal{J},\mathcal{T})$ if

$$\mathcal{J}^\ell\supset\mathcal{J}^{\ell+1}\supset\mathcal{J}\quad\forall\,\ell\in\mathbb{Z}^+,\qquad \big\{U\cap\mathcal{J}^{\ell+1}\colon U\in\mathcal{T}^\ell\big\}\subset\mathcal{T}^{\ell+1}\quad\forall\,\ell\in\mathbb{Z}^+,$$

and $\mathcal{T}$ is the topology on $\mathcal{J}$ generated by the collections $\{U\cap\mathcal{J}\colon U\in\mathcal{T}^\ell\}$ with $\ell\in\mathbb{Z}^+$; see [19, §13]. This in particular implies that the inclusions

$$\big(\mathcal{J}^{\ell+1},\mathcal{T}^{\ell+1}\big)\longrightarrow\big(\mathcal{J}^\ell,\mathcal{T}^\ell\big),\quad \big(\mathcal{J},\mathcal{T}\big)\longrightarrow\big(\mathcal{J}^\ell,\mathcal{T}^\ell\big) \tag{4.1}$$

are continuous for every $\ell\in\mathbb{Z}^+$.

We call an expansion as above proper (resp. tight) if all inclusions in (4.1) are proper (resp. $\mathcal{T}$ is dense in $(\mathcal{J}^\ell,\mathcal{T}^\ell)$ for all $\ell\in\mathbb{Z}^+$). We call it first countable if every topological space $(\mathcal{J}^\ell,\mathcal{T}^\ell)$ is first countable. The existence of a first countable expansion implies that $(\mathcal{J},\mathcal{T})$ itself is first countable. For example, the spaces $(\mathcal{J}^\ell(X),\mathcal{T}^\ell(X))$ of C^ℓ almost complex structures on a compact manifold X form a proper tight first countable expansion of the space $(\mathcal{J}(X),\mathcal{T}(X))$ of smooth almost complex structures on X. The spaces $W^p_\ell(\mathbb{P}^1;X)$ of W^p_ℓ-maps $f\colon\mathbb{P}^1\longrightarrow X$ form a proper tight first countable expansion of the space $C^\infty(\mathbb{P}^1;X)$ of smooth maps $f\colon\mathbb{P}^1\longrightarrow X$.

For topological spaces $\mathcal{J}$ and $\mathfrak{B}$, $\mathfrak{M}\subset\mathcal{J}\times\mathfrak{B}$, and an upper semi-continuous function $f\colon\mathfrak{M}\longrightarrow\mathbb{Z}^{\geqslant 0}$, we define

$$\widehat{\mathcal{J}}(f)=\big\{J\in\mathcal{J}\colon f(J,u)=0\ \forall\,(J,u)\in\mathfrak{M}\big\}.$$

Proposition 4.5. *Suppose $(\mathcal{J}^\ell,\mathcal{T}^\ell)_{\ell\in\mathbb{Z}^+}$ is a proper tight first countable expansion of a topological space $(\mathcal{J},\mathcal{T})$, $(\mathfrak{B}^\ell,\mathfrak{T}^\ell)_{\ell\in\mathbb{Z}^+}$ is a proper expansion of a topological space $(\mathfrak{B},\mathfrak{T})$, $\mathfrak{M}^1\subset\mathcal{J}^1\times\mathfrak{B}^1$, and $f\colon\mathfrak{M}^1\longrightarrow\mathbb{Z}^{\geqslant0}$ is an upper semi-continuous function such that*

$$\widehat{\mathcal{J}}^\ell(f)\equiv\widehat{\mathcal{J}}^\ell\big(f|\mathfrak{M}^1\cap(\mathcal{J}^\ell\times\mathfrak{B}^\ell)\big)\subset\mathcal{J}^\ell$$

is a dense subset for every $\ell\in\mathbb{Z}^+$. If the spaces $(\mathcal{J}^1,\mathcal{T}^1)$ and $(\mathfrak{B}^1,\mathfrak{T}^1)$ are Hausdorff and there exists a sequence $(\mathfrak{M}^1_r)_{r\in\mathbb{Z}^+}$ of subspaces of $\mathfrak{M}^1$ such that

$$\mathfrak{M}^1_r\subset\mathfrak{M}^1_{r+1}\quad\forall\,r\in\mathbb{Z}^+,\qquad \mathfrak{M}^1=\bigcup_{r=1}^{\infty}\mathfrak{M}^1_r\,,$$

and the restriction of the projection $\mathcal{J}^1\times\mathfrak{B}^1\longrightarrow\mathcal{J}^1$ to $\mathfrak{M}^1_r$ is proper for every $r\in\mathbb{Z}^+$, then

$$\widehat{\mathcal{J}}(f)\equiv\widehat{\mathcal{J}}\big(f|\mathfrak{M}^1\cap(\mathcal{J}\times\mathfrak{B})\big)\subset\mathcal{J}$$

is a ubiquitous subset.

Proof. For each $r\in\mathbb{Z}^+$, let

$$\widehat{\mathcal{J}}^\ell_r(f)=\widehat{\mathcal{J}}^\ell\big(f|\mathfrak{M}^1_r\cap(\mathcal{J}^\ell\times\mathfrak{B}^\ell)\big)\subset\mathcal{J}^\ell\quad\forall\,\ell\in\mathbb{Z}^+,\qquad \widehat{\mathcal{J}}_r(f)\equiv\widehat{\mathcal{J}}\big(f|\mathfrak{M}^1_r\cap(\mathcal{J}\times\mathfrak{B})\big)\subset\mathcal{J}\,.$$

We show below that each $\widehat{\mathcal{J}}_r(f)$ is open and dense in $(\mathcal{J},\mathcal{T})$. This implies that

$$\widehat{\mathcal{J}}(f)\equiv\bigcap_{r=1}^{\infty}\widehat{\mathcal{J}}_r(f)$$

is a countable intersection of open dense subsets of $(\mathcal{J},\mathcal{T})$ and is thus ubiquitous.

Since the spaces $(\mathcal{J}^1,\mathcal{T}^1)$ and $(\mathfrak{B}^1,\mathfrak{T}^1)$ are Hausdorff, so are the spaces $(\mathcal{J}^\ell,\mathcal{T}^\ell)$, $(\mathfrak{B}^\ell,\mathfrak{T}^\ell)$, $(\mathcal{J},\mathcal{T})$, and $(\mathfrak{B},\mathfrak{T})$. By Lemma 4.6 below, all the inclusions

$$\begin{aligned}\big(\mathcal{J}^{\ell+1},\mathcal{T}^{\ell+1}\big)\times\big(\mathfrak{B}^{\ell+1},\mathfrak{T}^{\ell+1}\big)&\longrightarrow\big(\mathcal{J}^\ell,\mathcal{T}^\ell\big)\times\big(\mathfrak{B}^\ell,\mathfrak{T}^\ell\big),\\ (\mathcal{J},\mathcal{T})\times(\mathfrak{B},\mathfrak{T})&\longrightarrow\big(\mathcal{J}^\ell,\mathcal{T}^\ell\big)\times\big(\mathfrak{B}^\ell,\mathfrak{T}^\ell\big)\end{aligned}$$

are thus proper. Since the restriction of the projection $\mathcal{J}^1\times\mathfrak{B}^1\longrightarrow\mathcal{J}^1$ to $\mathfrak{M}^1_r$ is proper for every $r\in\mathbb{Z}^+$, it follows that the restrictions of the projections

$$\mathcal{J}^\ell\times\mathfrak{B}^\ell\longrightarrow\mathcal{J}^\ell\quad\text{and}\quad\mathcal{J}\times\mathfrak{B}\longrightarrow\mathcal{J}$$

to $\mathfrak{M}^1_r\cap(\mathcal{J}^\ell\times\mathfrak{B}^\ell)$ and $\mathfrak{M}^1_r\cap(\mathcal{J}\times\mathfrak{B})$ are proper as well. From Lemma 4.7 below, we then conclude that the subsets $\widehat{\mathcal{J}}^\ell_r(f)\subset\mathcal{J}^\ell$ and $\widehat{\mathcal{J}}_r(f)\subset\mathcal{J}$ are open.

Let $J\in\mathcal{J}$ and $U^\ell\in\mathcal{T}^\ell$ be a sequence such that $U^\ell\supset U^{\ell+1}$ for all $\ell\in\mathbb{Z}^+$ and $\{U^\ell\cap\mathcal{J}\}$ is a basis for $\mathcal{T}$ at J (i.e. every open neighborhood $U\in\mathcal{T}$ of J contains some $U^\ell\cap\mathcal{J}$). Since $\widehat{\mathcal{J}}^\ell(f)\subset\widehat{\mathcal{J}}^\ell_r(f)$ is dense in $\mathcal{J}^\ell$, there exists $J_\ell\in U_\ell\cap\widehat{\mathcal{J}}^\ell_r(f)$. Since $\widehat{\mathcal{J}}^\ell_r(f)$ is open in $\mathcal{J}^\ell$ and $\mathcal{J}$ is dense in $\mathcal{J}^\ell$, there also exists

$$J'_\ell\in U_\ell\cap\widehat{\mathcal{J}}^\ell_r(f)\cap\mathcal{J}\subset U_\ell\cap\widehat{\mathcal{J}}_r(f)\,.$$

The sequence $J'_\ell\in\widehat{\mathcal{J}}_r(f)$ then converges to J. We conclude that each $\mathcal{J}_r(f)\subset\mathcal{J}$ is dense. □

Lemma 4.6. *Let $f: \mathcal{J} \longrightarrow \mathcal{J}'$ and $g: \mathfrak{B} \longrightarrow \mathfrak{B}'$ be proper maps. If $\mathcal{J}'$ and $\mathfrak{B}'$ are Hausdorff, then*

$$f \times g: \mathcal{J} \times \mathfrak{B} \longrightarrow \mathcal{J}' \times \mathfrak{B}'$$

is also a proper map.

Proof. Let $K \subset \mathcal{J}' \times \mathfrak{B}'$ be a compact subset. Since the projections $\mathcal{J}' \times \mathfrak{B} \longrightarrow \mathcal{J}', \mathfrak{B}'$ are continuous, they map K to compact subsets $K_1 \subset \mathcal{J}'$ and $K_2 \subset \mathfrak{B}'$. Since the maps f and g are proper, the subset

$$\{f \times g\}^{-1}(K_1 \times K_2) = f^{-1}(K_1) \times g^{-1}(K_2) \subset \mathcal{J} \times \mathfrak{B}$$

is compact. Since K is a compact subset of a Hausdorff space, it is closed. Since $f \times g$ is a continuous map, the subspace

$$\{f \times g\}^{-1}(K) \subset \{f \times g\}^{-1}(K_1 \times K_2)$$

is closed and thus compact. □

Lemma 4.7. *Suppose $\mathcal{J}$ and $\mathfrak{B}$ are topological spaces, $\mathfrak{M} \subset \mathcal{J} \times \mathfrak{B}$, and $f: \mathfrak{M} \longrightarrow \mathbb{Z}^{\geqslant 0}$ is an upper semi-continuous function. If $\mathcal{J}$ is first countable and Hausdorff and the restriction of the projection $\mathcal{J} \times \mathfrak{B} \longrightarrow \mathcal{J}$ to $\mathfrak{M}$ is proper, then the set $\widehat{\mathcal{J}}(f)$ is open in $\mathcal{J}$.*

Proof. We show that the complement $\widehat{\mathcal{J}}^c \equiv \mathcal{J} - \widehat{\mathcal{J}}(f)$ of $\widehat{\mathcal{J}}(f)$ in $\mathcal{J}$ is closed. Suppose $J \in \mathcal{J}$, $J_i \in \widehat{\mathcal{J}}^c$ is a sequence converging to J such that $J_i \neq J$ for all $i \in \mathbb{Z}^+$, and $(J_i, u_i) \in \mathfrak{M}$ is a sequence such that $f(J_i, u_i) \geqslant 1$ for all $i \in \mathbb{Z}^+$. Since the set

$$K \equiv \{J\} \cup \{J_i : i \in \mathbb{Z}^+\} \subset \mathcal{J}$$

is compact and the restriction of the projection $\mathcal{J} \times \mathfrak{B} \longrightarrow \mathcal{J}$ to $\mathfrak{M}$ is proper, the infinite set

$$\{(J_i, u_i) : i \in \mathbb{Z}^+\} \subset \mathfrak{M} \cap (K \times \mathfrak{B})$$

has a limit point (J', u). Since $\mathcal{J}$ is Hausdorff, $J' = J$. Since f is upper semi-continuous and $f(J_i, u_i) \geqslant 1$ for all $i \in \mathbb{Z}^+$, $f(J, u) \geqslant 1$. Thus, $J \in \widehat{\mathcal{J}}^c$. □

4.5 Generalized Cauchy-Riemann operators

Lemma 4.8 (Unique Continuation)**.** *Let $(\Sigma, \mathfrak{j})$ be a connected, but possibly non-compact, Riemann surface D be a generalized Cauchy-Riemann operator over $(\Sigma, \mathfrak{j})$, and $\mu \in \ker D$. If there exists an non-empty open subset $U \subset \Sigma$ such that $\mu|_U = 0$, then $\mu = 0$.*

Proof. [33, Proposition 3.1]. □

Ivashkovich-Shevchishin twisting construction

Proposition 4.9. *Serre Duality*

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