Absolute vs. Relative Gromov-Witten Invariants

Mohammad F. Tehrani and Aleksey Zinger*

May 3, 2014. Updated: September 11, 2015

Abstract

We compare the absolute and relative Gromov-Witten invariants of compact symplectic manifolds when the symplectic hypersurface contains no relevant holomorphic curves. We show that these invariants are then the same, except in a narrow range of dimensions of the target and genera of the domains, and provide examples when they fail to be the same.

Contents

1 Introduction

Gromov-Witten invariants of a compact symplectic manifold (X,ω) are certain, often delicate, counts of J-holomorphic curves in X; they play prominent roles in symplectic topology, algebraic geometry, and string theory. For a symplectic hypersurface V in (X,ω), i.e. a closed symplectic submanifold of real codimension 2, relative Gromov-Witten invariants of (X,ω,V) count J-holomorphic curves in X with specified contacts with V. If V contains no (non-constant) J-holomorphic curves that could possibly contribute to a specific absolute invariant of X, one could hope that such an absolute invariant equals the corresponding relative invariant with the basic

*Partially supported by NSF grant 0846978

contact condition, divided by the number of orderings of the contact points. We show that this is indeed the case, except in a narrow range of dimensions of the target and genera of the domains; see Theorem 1 and Remarks 1.2-1.4. Examples 1-3 illustrate the three cases when the absolute and relative invariants can fail to be equal.

For $g,k\in\mathbb{Z}^{\ge 0}$, we denote by $\overline{\mathcal{M}}_{g,k}$ the Deligne-Mumford moduli space of stable k-marked genus g connected nodal curves. If $2g+k<3$, $\overline{\mathcal{M}}_{g,k}$ is empty with this definition, though it is often convenient to formally take it to be a point in these cases, as done when we set up notation for GW-invariants below. If $g,k\in\mathbb{Z}^{\ge 0}$, $A\in H_2(X;\mathbb{Z})$, and J is an an almost complex structure on X compatible with (or tamed by) ω, let $\overline{\mathfrak{M}}_{g,k}(X,A)$ denote the moduli spaces of stable J-holomorphic k-marked maps from connected nodal curves of genus g. If in addition $V\subset X$ is a symplectic hypersurface, $\mathbf{s}\equiv(s_1,\ldots,s_\ell)$ is an ℓ-tuple of positive integers such that

$$s_1+\ldots+s_\ell = A\cdot V, \tag{1.1}$$

and J is compatible with V in a suitable sense, let $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ denote the moduli spaces of stable J-holomorphic $(k+\ell)$-marked maps from connected nodal curves of genus g that have contact with V at the last ℓ marked points of orders $s_1,\ldots,s_\ell$, respectively. These moduli spaces are introduced in [17, 13, 18] under certain assumptions on J and reviewed in Section 2. The expected dimensions of these two moduli spaces are given by

$$\begin{aligned} \dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{g,k}(X,A) &= 2\big(\langle c_1(X),A\rangle + (n-3)(1-g) + k\big),\\ \dim^{\mathrm{vir}}\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A) &= 2\big(\langle c_1(X),A\rangle + (n-3)(1-g) + k + \ell(\mathbf{s}) - |\mathbf{s}|\big), \end{aligned} \tag{1.2}$$

where $\ell(\mathbf{s})\equiv\ell$ and $|\mathbf{s}|\equiv s_1+\ldots+s_\ell$. In particular, these dimensions are the same if

$$\mathbf{s} = \mathbf{1}_\ell \equiv \big(\underbrace{1,\ldots,1}_{\ell}\big)\,,$$

i.e. the tuple $\mathbf{s}$ imposes no contact conditions on degree A J-holomorphic curves, beyond what a generic such curve can be expected to satisfy.

For each $i=1,\ldots,k$, let

$$\mathrm{ev}_i\colon \overline{\mathfrak{M}}_{g,k}(X,A), \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A) \longrightarrow X \tag{1.3}$$

be the i-th evaluation map. It sends the equivalence class of a J-holomorphic map $u:\Sigma\longrightarrow X$ from a genus g nodal curve Σ to $u(x_i)\in X$, where $x_i\in\Sigma$ is the i-th marked point. Let

$$\mathrm{st}\colon \overline{\mathfrak{M}}_{g,k}(X,A), \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A) \longrightarrow \overline{\mathcal{M}}_{g,k} \tag{1.4}$$

denote the forgetful morphism to the Deligne-Mumford space. If $2g+k\ge 3$, it sends the equivalence class of a J-holomorphic map $u\colon\Sigma\longrightarrow X$ from a marked genus g nodal curve Σ to the equivalence class of the stable k-marked genus g nodal curve Σ' obtained from $(\Sigma,x_1,\ldots,x_k)$ by contracting the unstable components (spheres with one or two special, i.e. nodal or marked, points); see Figure 1.

Along with the virtual class for $\overline{\mathfrak{M}}_{g,k}(X,A)$, constructed in [30] in "semi-positive" cases, in [1] in the algebraic case, and in [7, 21] in the general case, the first morphisms in (1.3) and (1.4) give rise

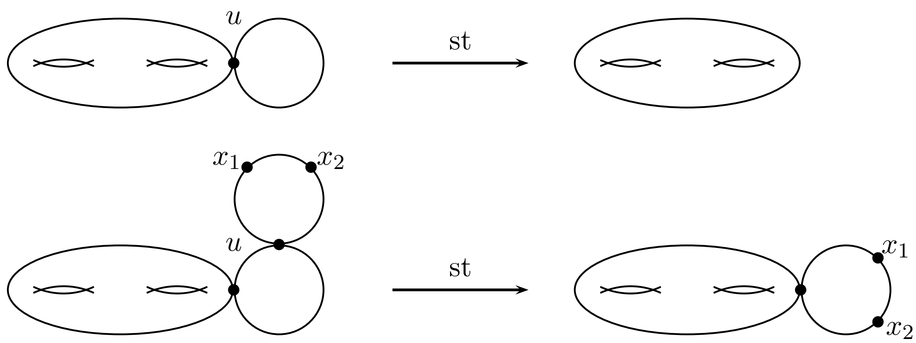

Figure 1: Examples of the stabilization morphism (1.4).

to the (absolute) GW-invariants of (X,ω):

$$\mathrm{GW}^X_{g,A}(\kappa;\alpha_1,\ldots,\alpha_k)\equiv\left\langle \mathrm{st}^*\kappa\prod_{i=1}^{k}\mathrm{ev}_i^*\alpha_i,[\overline{\mathfrak{M}}_{g,k}(X,A)]^{\mathrm{vir}}\right\rangle\quad\forall\,\kappa\in H^*(\overline{\mathcal{M}}_{g,k}),\ \alpha_i\in H^*(X),\tag{1.5}$$

where H^* denotes the cohomology with $\mathbb{Q}$-coefficients. The number above vanishes unless

$$\deg\kappa+\sum_{i=1}^{k}\deg\alpha_i=2\big(\langle c_1(X),A\rangle+(n-3)(1-g)+k\big)\,.\tag{1.6}$$

Along with the virtual class for $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$, the second morphisms in (1.3) and (1.4) give rise to the relative GW-invariants of (X,ω,V):

$$\mathrm{GW}^{X,V}_{g,A;\mathbf{s}}(\kappa;\alpha_1,\ldots,\alpha_k)\equiv\left\langle \mathrm{st}^*\kappa\prod_{i=1}^{k}\mathrm{ev}_i^*\alpha_i,[\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)]^{\mathrm{vir}}\right\rangle\quad\forall\,\kappa\in H^*(\overline{\mathcal{M}}_{g,k}),\ \alpha_i\in H^*(X).\tag{1.7}$$

Such a virtual class is constructed in [13] in "semi-positive" cases and in [18] in the algebraic case and is used in [17] in the general case; see Section 2 for more details. The number in (1.7) vanishes unless

$$\deg\kappa+\sum_{i=1}^{k}\deg\alpha_i=2\big(\langle c_1(X),A\rangle+(n-3)(1-g)+k+\ell(\mathbf{s})-|\mathbf{s}|\big)\,.$$

The numbers in (1.5) and (1.7) are (graded-) symmetric and linear in the inputs α_i. By the latter property, they give rise to well-defined numbers

$$\mathrm{GW}^X_{g,A}(\kappa;\alpha),\mathrm{GW}^{X,V}_{g,A;\mathbf{s}}(\kappa;\alpha)\in\mathbb{Q}\qquad\forall\,\kappa\in H^*(\overline{\mathcal{M}}_{g,k}),\ \alpha\in H^*(X)^{\otimes k}\,.$$

The numbers

$$\mathrm{GW}^X_{g,A}(\alpha)\equiv\mathrm{GW}^X_{g,A}(1;\alpha)\qquad\text{and}\qquad\mathrm{GW}^{X,V}_{g,A;\mathbf{s}}(\alpha)=\mathrm{GW}^{X,V}_{g,A;\mathbf{s}}(1;\alpha)$$

are called primary GW-invariants or GW-invariants with primary insertions. In some cases, the numbers (1.5) and (1.7) can be described as signed counts of concrete geometric objects, J-holomorphic or (J,ν)-holomorphic maps; see Sections 2 and 5.

Remark 1.1. The numbers (1.5) and (1.7) do not cover GW-invariants that arise from natural classes on $\overline{\mathfrak{M}}_{g,k}(X,A)$ and $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$, such as ψ-classes (which are generally different from the ψ-classes on $\overline{\mathcal{M}}_{g,k}$ pulled back by the morphism (1.4)) and the euler classes of obstruction bundles of various kinds; both types of classes are central to GW-theory. The geometric constructions of the numbers (1.5) and (1.7) reviewed in Sections 2 and 5 are not compatible with such classes.

Definition 1. Let (X,ω) be a compact symplectic manifold, $g\in\mathbb{Z}^{\ge0}$, and $A\in H_2(X;\mathbb{Z})$. A symplectic hypersurface $V\subset X$ is (g,A)-hollow if there exists an $\omega|_V$-tame almost complex structure J_V on V such that every non-constant J_V-holomorphic map $u\colon\Sigma\longrightarrow V$ from a smooth connected Riemann surface Σ satisfies

$$g(\Sigma)>g, \qquad \text{or} \qquad \langle u^*\omega,\Sigma\rangle>\omega(A), \qquad \text{or} \qquad \langle u^*\omega,\Sigma\rangle=\omega(A),\ u_*[\Sigma]\neq A.$$

Theorem 1. *Suppose (X,ω) is a compact symplectic manifold of real dimension $2n$, $g,k\in\mathbb{Z}^{\ge0}$, $A\in H_2(X;\mathbb{Z})$, and $V\subset X$ is a (g,A)-hollow symplectic hypersurface such that $A\cdot V\ge0$. If*

$$(g,A)\neq(1,0) \qquad \text{and} \qquad (n-5)g(g-1)\ge0\,, \tag{1.8}$$

then the absolute GW-invariants (1.5) and the basic corresponding relative GW-invariants (1.7) agree:

$$\mathrm{GW}^X_{g,A}(\kappa;\alpha)=\frac{1}{(A\cdot V)!}\mathrm{GW}^{X,V}_{g,A;\mathbf{1}_{A\cdot V}}(\kappa;\alpha) \qquad \forall\ \kappa\in H^*(\overline{\mathcal{M}}_{g,k}),\ \alpha\in H^*(X)^{\otimes k}\,. \tag{1.9}$$

This identity also holds if $\kappa=1$, $A\neq0$, and either $g=2$ or $n\neq4$.

Remark 1.2. By Theorem 1, the absolute GW-invariants with primary insertions, i.e. $\kappa=1$, and the corresponding relative invariants in degree $A\neq0$ may fail to be equal only if $n=4$ and $g\ge3$ at the same time; the possibility of such a failure is illustrated by Example 3. With non-trivial constraints κ, the two invariants in degree $A\neq0$ may fail to be equal only if $1\le n\le4$ and $g\ge2$ at the same time; the possibility of such a failure is illustrated by Example 2. Example 1, which is motivated by [14, Example 12.5], illustrates the possibility of failure of (1.9) with $A=0$.

Remark 1.3. In Section 3, we give two versions of essentially the same proof of Theorem 1. The first version is a direct comparison of the two invariants. It is particularly suitable for considering the independence of the geometrically constructed curve counts of the chosen Donaldson divisor in [2, 8, 15]; see Section 5. The argument involves several cases; in all, but one of them, the conclusion is established by a dimension-counting argument. In the exceptional case, when $\kappa=1$, $n=3$, and $g\ge3$, we also use the fact that $\lambda_g^2=0$ on $\overline{\mathcal{M}}_g$; see [27, (5.3)]. The second version of the proof is a formal application of the symplectic sum formula for GW-invariants, as in the setup introduced in [23, Section 2.2], successfully applied in the genus 0 case in [12], and used in the attempted proof of [15, Theorem 11.1]. As indicated by [15], establishing (1.9) in this way leads to the analogue of (1.9) for virtual classes, at least in the algebraic category; see Section 3.3.

Remark 1.4. It is sufficient to verify the condition of Definition 1 for J-holomorphic maps $u\colon\Sigma\longrightarrow V$ that are simple in the sense of [26, Section 2.5]. By [26, Section 3.2], moduli spaces of such maps have the expected dimensions for a generic ω_V-tame (or compatible) almost complex structure J_V on V. Thus, by the first equation in (1.2), V is (g,A)-hollow if

$$A'\cdot V>\langle c_1(X),A'\rangle+(n-4)(1-g') \tag{1.10}$$

for all $g' \in \mathbb{Z}^{\geq 0}$ with $g' \leq g$ and $A' \in H_2(X;\mathbb{Z})$ with $\omega(A') \leq \omega(A)$ such that A' can be represented by a J_0-holomorphic curve for some fixed ω-tame almost complex structure J_0 on X. By Gromov's Compactness Theorem, the number of such classes A' is finite. Since $\omega(A') > 0$ for all such classes, (1.10) can be achieved by taking V to be Poincare dual to a sufficiently high multiple of a rational symplectic form close to ω. Such V, called Donaldson hypersurfaces, always exist by [3] and are central to the construction of genus 0 curve counts in [2] and its attempted extensions to positive genera in [8] and [15]; see Section 5.

Remark 1.5. For the purposes of the direct proof of Theorem 1 in Section 3.1, it is sufficient to assume that there exist an almost complex structure J_V on V and an arbitrarily small perturbation ν on V as in Section 2 so that every (J_V,ν)-holomorphic map $u: \Sigma \longrightarrow V$ from a smooth connected Riemann surface Σ satisfies

$$u_*[\Sigma] = 0, \quad \text{or} \quad g(\Sigma) > g, \quad \text{or} \quad \langle u^*\omega, \Sigma\rangle > \omega(A), \quad \text{or} \quad \langle u^*\omega, \Sigma\rangle = \omega(A),\ u_*[\Sigma] \neq A.$$

For the purposes of the proof of Theorem 1 via the symplectic sum formula in Section 3.2, it is sufficient to assume the GW-invariants of V of genus g' and in the class A' vanish whenever $A' \neq 0$, $g' \leq g$, and $\omega(A') \leq \omega(A)$.

The next three examples illustrate different cases when (1.9) fails to hold. They are justified in Section 4.

Example 1. Suppose (X,ω) is a compact symplectic manifold of real dimension $2n$ and $V \subset X$ is a symplectic hypersurface. Let $\mathsf{j} \in H^2(\overline{\mathcal{M}}_{1,1})$ be the Poincare dual of a generic point and $\alpha \in H^2(X)$. The genus 1 degree 0 GW-invariants of (X,ω) and (X,ω,V) satisfy

$$\mathrm{GW}^X_{1,0}(\mathsf{j};1) = \frac{\chi(X)}{2} = \frac{1}{0!}\mathrm{GW}^{X;V}_{1,0;()}(\mathsf{j};1) + \frac{\chi(V)}{2}\,, \tag{1.11}$$

$$\mathrm{GW}^X_{1,0}(\alpha) = -\frac{\langle \alpha\, c_{n-1}(X), X\rangle}{24} = \frac{1}{0!}\mathrm{GW}^{X,V}_{1,0;()}(\alpha) - \frac{\langle \alpha|_V\, c_{n-2}(V), V\rangle}{24}, \tag{1.12}$$

where $\chi(\cdot)$ is the euler characteristic and () in the subscript is the length 0 contact vector (and thus gives 0! in the denominators).

Example 2. Denote by $\mathbb{P}^1$ the one-dimensional complex projective space with the standard symplectic form and by $V_\delta \subset \mathbb{P}^1$ the symplectic hypersurface consisting of $\delta \in \mathbb{Z}^{\geq 0}$ distinct points. Let $\mathrm{pt} \in H^2(\mathbb{P}^1)$ be the Poincare dual of a point and $\kappa \in H^2(\overline{\mathcal{M}}_{2,2})$ be the Poincare dual of the divisor whose generic element consists of two components, one of genus 2 and the other of genus 0; see the bottom right diagram in Figure 1. The genus 2 degree 1 GW-invariants of $\mathbb{P}^1$ and $(\mathbb{P}^1, V_\delta)$ satisfy

$$\frac{1}{240} = \mathrm{GW}^{\mathbb{P}^1}_{2,1}(\kappa^4;\mathrm{pt},\mathrm{pt}) = \frac{1}{\delta!}\mathrm{GW}^{\mathbb{P}^1,V_\delta}_{2,1;\mathbf{1}_\delta}(\kappa^4;\mathrm{pt},\mathrm{pt}) + \frac{\delta}{1,152}\,. \tag{1.13}$$

Example 3. Denote by $\mathbb{P}^4$ the four-dimensional complex projective space with the standard symplectic form and by $V_\delta \subset \mathbb{P}^4$ a smooth complex hypersurface of degree δ. Let $\mathrm{pt} \in H^8(\mathbb{P}^4)$ be the Poincare dual of a point. The genus 3 degree 1 primary GW-invariants of $\mathbb{P}^4$ and $(\mathbb{P}^4, V_\delta)$ satisfy

$$-\frac{37}{82,944} = \mathrm{GW}^{\mathbb{P}^4}_{3,1}(\mathrm{pt}) = \frac{1}{\delta!}\mathrm{GW}^{\mathbb{P}^4,V_\delta}_{3,1;\mathbf{1}_\delta}(\mathrm{pt}) + \frac{\delta(\delta^2-5\delta+8)}{72,576}\,. \tag{1.14}$$

Remark 1.6. The proof in Section 4.3 of the second equality in (1.14) applies to primary GW-invariants of $\mathbb{P}^4$ and $(\mathbb{P}^4, V_\delta)$ in degree d as long as V_δ contains no curves of genus at most 3 and degree at most d that pass through the constraints. In these cases, the last term in (1.14) should be multiplied by the genus 0 degree d absolute GW-invariant with an extra point insertion. The condition on V_δ in particular excludes the $d=1$ GW-invariants with primary insertions $(\mathbb{P}^1, \mathbb{P}^2)$ if $\delta=1$.

We review the definitions of absolute and relative invariants in Section 2, focusing on the geometric differences for the requirements on generic parameters (J, ν) determining the two types of invariants. These differences are fundamental to establishing Theorem 1 in Section 3 and the claims of Examples 1-3 in Section 4. In Section 5, we review the Cieliebak-Mohnke approach to constructing GW-invariants and relate a key issue in this approach to Theorem 1 and Examples 1-3.

This note was inspired by the discussions regarding [15, Theorem 11.1] and the related aspects of [8] at and following the SCGP workshop on constructing the virtual cycle in GW-theory. We would like to thank the SCGP for organizing and hosting this very enlightening workshop and the authors of [15] and [8] for bringing up important questions concerning relative GW-invariants. We are also grateful to C.-C. Liu and D. Maulik for sharing invaluable insights on [15, Theorem 11.1] and C. Faber for providing intersection numbers for Deligne-Mumford moduli spaces.

2 Review of GW-invariants

Let $g, k \in \mathbb{Z}^{\geq 0}$ be such that $2g+k \geq 3$,

$$\widetilde{\mathcal{M}}_{g,k} \longrightarrow \overline{\mathcal{M}}_{g,k} \tag{2.1}$$

be the branched cover of the Deligne-Mumford space of stable k-marked genus g curves by the associated moduli space of Prym structures constructed in [22], and

$$\pi_{g,k}: \check{\mathcal{U}}_{g,k} \longrightarrow \widetilde{\mathcal{M}}_{g,k}$$

be the corresponding universal curve. A k-marked genus g nodal curve with a Prym structure is a connected compact nodal k-marked Riemann surface $(\Sigma, z_1, \ldots, z_k)$ of arithmetic genus g together with a holomorphic map $\mathrm{st}_\Sigma : \Sigma \longrightarrow \check{\mathcal{U}}_{g,k}$ which surjects on a fiber of $\pi_{g,k}$ and takes the marked points of Σ to the corresponding marked points of the fiber.

If J is an almost complex structure on a smooth manifold X, $A \in H_2(X; \mathbb{Z})$, and

$$\nu \in \Gamma_{g,k}(X, J) \equiv \Gamma\big(\check{\mathcal{U}}_{g,k} \times X, \pi_1^*(T^*\check{\mathcal{U}}_{g,k})^{0,1} \otimes_{\mathbb{C}} \pi_2^*(TX, J)\big), \tag{2.2}$$

a k-marked genus g degree A (J, ν)-map is a tuple $(\Sigma, z_1, \ldots, z_k, \mathrm{st}_\Sigma, u)$ such that $(\Sigma, z_1, \ldots, z_k, \mathrm{st}_\Sigma)$ is a genus g k-marked nodal curve with a Prym structure and $u: \Sigma \longrightarrow X$ is a smooth (or L_1^p, with $p>2$) map such that

$$u_*[\Sigma] = A \qquad \text{and} \qquad \bar{\partial}_{J,\mathfrak{j}} u\big|_z \equiv \frac{1}{2}\big(\mathrm{d}_z u + J \circ \mathrm{d}_z u \circ \mathfrak{j}\big) = \nu(\mathrm{st}_\Sigma(z), u(z)) \circ \mathrm{d}_z \mathrm{st}_\Sigma \quad \forall\, z \in \Sigma,$$

where $\mathfrak{j}$ is the complex structure on Σ. Two such tuples are equivalent if they differ by a reparametrization of the domain commuting with the maps to $\check{\mathcal{U}}_{g,k}$.

Suppose (X,ω) is a compact symplectic manifold and J is an ω-tame almost complex structure. By [30, Corollary 3.9], the space $\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$ of equivalence classes of k-marked genus g degree A (J,ν)-maps is Hausdorff and compact in Gromov's convergence topology. By [30, Theorem 3.16], for a generic ν each stratum of $\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$ consisting of simple (not multiply covered) maps of a fixed combinatorial type is a smooth manifold of the expected even dimension, which is less than the expected dimension of the subspace of simple maps with smooth domains (except for this subspace itself). By [30, Theorem 3.11], the last stratum has a canonical orientation. By [30, Proposition 3.21], the images of the strata of $\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$ consisting of multiply covered maps under the morphism

$$\mathrm{st}\times\mathrm{ev}_1\times\ldots\times\mathrm{ev}_k\colon \overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)\longrightarrow \overline{\mathcal{M}}_{g,k}\times X^k \tag{2.3}$$

are contained in images of maps from smooth even-dimensional manifolds of dimension less than this stratum if ν is generic and (X,ω) is semi-positive in the sense of [26, Definition 6.4.1]. Thus, (2.3) is a pseudocycle. Intersecting it with generic representatives for the Poincare duals of the classes κ and α_i and dividing by the order of the covering (2.1), we obtain the (absolute) GW-invariants (1.5) of a semi-positive symplectic manifold (X,ω) in the stable range, i.e. with (g,k) such that $2g+k\geq 3$. If $g=0$, the same reasoning applies with $\nu=0$ and yields the same conclusion if (X,ω) satisfies a slightly stronger condition ($c_1(A)>0$ instead of $c_1(A)\geq 0$ in [26, Definition 6.4.1]). For general symplectic manifolds (X,ω), the GW-invariants (1.5) are defined in [7, 21] using Kuranishi structures (or finite-dimensional approximations) and local perturbations ν as in (2.2).

Suppose in addition $V\subset X$ is a closed symplectic hypersurface and $J(TV)=TV$. Thus, J induces a complex structure $\mathfrak{i}_{X,V}$ on (the fibers of) the normal bundle

$$\pi_{X,V}\colon \mathcal{N}_XV\equiv TX|_V/TV\longrightarrow V.$$

A connection $\nabla^{\mathcal{N}_XV}$ in $(\mathcal{N}_XV,\mathfrak{i}_{X,V})$ induces a splitting of the exact sequence

$$0\longrightarrow \pi_{X,V}^*\mathcal{N}_XV\longrightarrow T(\mathcal{N}_XV)\overset{\mathrm{d}\pi_{X,V}}{\longrightarrow}\pi_{X,V}^*TV\longrightarrow 0 \tag{2.4}$$

of vector bundles over $\mathcal{N}_XV$ which restricts to the canonical splitting over the zero section and is preserved by the multiplication by $\mathbb{C}^*$; see [33, Lemma 1.1]. For each trivialization

$$\mathcal{N}_XV|_U\approx U\times\mathbb{C}$$

over an open subset U of V, there exists $\alpha\in\Gamma(U;T^*V\otimes_{\mathbb{R}}\mathbb{C})$ such that the image of $\pi_{X,V}^*TV$ corresponding to this splitting is given by

$$T^{\mathrm{hor}}_{(x,w)}(\mathcal{N}_XV)=\big\{(v,-\alpha_x(v)w)\colon v\in T_xV\big\}\qquad\forall\,(x,w)\in U\times\mathbb{C}.$$

The isomorphism $(x,w)\longrightarrow(x,w^{-1})$ of $U\times\mathbb{C}^*$ maps this vector space to

$$\begin{aligned}T^{\mathrm{hor}}_{(x,w^{-1})}\big((\mathcal{N}_XV)^*\big)&=\big\{(v,w^{-2}\alpha_x(v)w)\colon v\in T_xV\big\}\\&=\big\{(v,\alpha_x(v)w^{-1})\colon v\in T_xV\big\}\qquad\forall\,(x,w)\in U\times\mathbb{C}^*.\end{aligned}$$

Thus, the splitting of (2.4) induced by a connection in $(\mathcal{N}_XV,\mathfrak{i}_{X,V})$ extends to a splitting of the exact sequence

$$0\longrightarrow T^{\mathrm{vrt}}(\mathbb{P}_XV)\longrightarrow T(\mathbb{P}_XV)\overset{\mathrm{d}\pi_{X,V}}{\longrightarrow}\pi_{X,V}^*TV\longrightarrow 0,$$

where

$$\pi_{X,V}\colon \mathbb{P}_X V \equiv \mathbb{P}\big(\mathcal{N}_X V \oplus V\times\mathbb{C}\big) \longrightarrow V\,; \tag{2.5}$$

this splitting restricts to the canonical splittings over

$$\mathbb{P}_{X,\infty}V \equiv \mathbb{P}(\mathcal{N}_X V\oplus 0) \qquad\text{and}\qquad \mathbb{P}_{X,0}V \equiv \mathbb{P}(0\oplus X\times\mathbb{C}) \tag{2.6}$$

and is preserved by the multiplication by $\mathbb{C}^*$. Via this splitting, the almost complex structure $J_V\equiv J_X|_V$ and the complex structure $\mathfrak{i}_{X,V}$ in the fibers of $\pi_{X,V}$ induce an almost complex structure $J_{X,V}$ on $\mathbb{P}_X V$ which restricts to almost complex structures on $\mathbb{P}_{X,\infty}V$ and $\mathbb{P}_{X,0}V$ and is preserved by the $\mathbb{C}^*$-action. Furthermore, the projection $\pi_{X,V}\colon\mathbb{P}_X V\longrightarrow V$ is $(J_V,J_{X,V})$-holomorphic. By [33, Lemma 2.2], $\xi\in\Gamma(V,\mathcal{N}_X V)$ is $(J_{X,V},J|_V)$-holomorphic if and only if ξ lies in the kernel of the $\bar\partial$-operator on $(\mathcal{N}_X V,\mathfrak{i}_{X,V})$ corresponding to the connection used above.

For each $m\in\mathbb{Z}^{\ge 0}$, let

$$\begin{gathered} X_m^V = \big(X\sqcup\{1\}\times\mathbb{P}_X V\sqcup\ldots\sqcup\{m\}\times\mathbb{P}_X V\big)/\sim\,, \qquad\text{where} \\ x\sim 1\times\mathbb{P}_{X,\infty}V|_x\,,\quad r\times\mathbb{P}_{X,0}V|_x\sim(r+1)\times\mathbb{P}_{X,\infty}V|_x \quad\forall\,x\in V,\ r=1,\ldots,m-1; \end{gathered} \tag{2.7}$$

see Figure 2. Define

$$q_m\colon X_m^V\longrightarrow X \qquad\text{by}\qquad q_m(x)=\begin{cases} x, & \text{if } x\in X;\\ \pi_{X,V}([v,w]), & \text{if } x=(r,[v,w])\in r\times\mathbb{P}_X V\,.\end{cases}$$

We denote by J_m the almost complex structure on X_m^V so that

$$J_m|_X=J \qquad\text{and}\qquad J_m|_{\{r\}\times\mathbb{P}_X V}=J_{X,V}\quad\forall\ r=1,\ldots,m.$$

For each $(c_1,\ldots,c_m)\in\mathbb{C}^*$, define

$$\Theta_{c_1,\ldots,c_m}\colon X_m^V\longrightarrow X_m^V \qquad\text{by}\quad \Theta_{c_1,\ldots,c_m}(x)=\begin{cases} x, & \text{if } x\in X;\\ (r,[c_r v,w]), & \text{if } x=(r,[v,w])\in r\times\mathbb{P}_X V.\end{cases} \tag{2.8}$$

This diffeomorphism is biholomorphic with respect to J_m and preserves the fibers of the projection $\mathbb{P}_X V\longrightarrow V$ and the sections $\mathbb{P}_{X,0}V$ and $\mathbb{P}_{X,\infty}V$.

Suppose $J(TV)=TV$ and J is ω-tame. We denote by ∇ the Levi-Civita connection of the metric g_J on X determined by (ω,J) as in [26, (2.1.1)], by $\widetilde\nabla$ the corresponding J_X-linear connection, as above [26, (3.1.3)], and by $\widehat\nabla$ the connection given by

$$\widehat\nabla_v\zeta=\widetilde\nabla_v\zeta-\frac14\big\{\nabla_{J\zeta}J+J\nabla_\zeta J\big\}(v) \qquad\forall\ \zeta\in\Gamma(X;TX),\ v\in TX.$$

By the next paragraph, the $\bar\partial$-operator

$$\widehat\nabla^{0,1}\colon\Gamma(X;TX)\longrightarrow\Gamma\big(X;T^*X^{0,1}\otimes_{\mathbb{C}}TX\big), \qquad \zeta\longrightarrow\frac12\big(\nabla_\cdot\zeta+J\nabla_{J\cdot}\zeta\big),$$

restricts to an operator

$$\widehat\nabla^{0,1}\colon\Gamma(V;TV)\longrightarrow\Gamma\big(V;T^*V^{0,1}\otimes_{\mathbb{C}}TV\big),$$

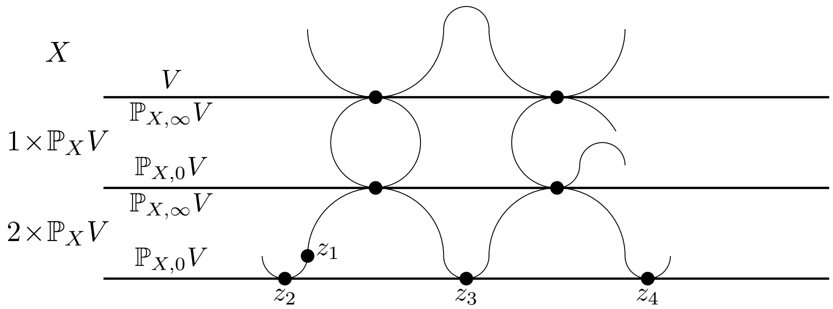

Figure 2: The image of a relative map with $k=1$ and $\mathbf{s}=(2,2,2)$ to the space X_2^V.

and thus descends to a $\bar{\partial}$-operator

$$\Gamma(V;\mathcal{N}_XV) \longrightarrow \Gamma\big(V;T^*V^{0,1}\otimes_{\mathbb{C}}\mathcal{N}_XV\big)$$

corresponding to some connection $\nabla^{\mathcal{N}_XV}$ in $(\mathcal{N}_XV,\mathfrak{i}_{X,V})$; see [33, Section 2.3]. Let $J_{X,V}$ denote the complex structure on $\mathbb{P}_XV$ induced by J_V and $\nabla^{\mathcal{N}_XV}$ as in the paragraph above the previous one; it depends only on the above $\bar{\partial}$-operator and not on the connection $\nabla^{\mathcal{N}_XV}$ realizing it.

If in addition $u:(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ is $(J,\mathfrak{j})$-holomorphic, i.e. $\bar{\partial}_{J,\mathfrak{j}}u=0$, the linearization of the $\bar{\partial}_{J,\mathfrak{j}}$-operator at u is given by

$$\begin{aligned} D_u\colon \Gamma(\Sigma,u^*TX) \longrightarrow \Gamma^{0,1}_{J,\mathfrak{j}}(\Sigma;u^*TX) &\equiv \Gamma\big(\Sigma,(T^*\Sigma,j)^{0,1}\otimes_{\mathbb{C}}u^*(TX,J)\big),\\ D_u\xi = \frac{1}{2}\big(\widehat{\nabla}^u\xi+\{u^*J\}\circ\widehat{\nabla}^u\xi\circ\mathfrak{j}\big)&+\frac{1}{4}N_J^u(\xi,\mathrm{d}u), \end{aligned} \qquad (2.9)$$

where $\widehat{\nabla}^u$ and N_J^u are the pull-backs of the connection $\widehat{\nabla}$ and of the Nijenhuis tensor N_J of J normalized as in [26, p18], respectively, by u; see [26, (3.1.7)]. If in addition $u(\Sigma)\subset V$,

$$D_u\big(\Gamma(\Sigma,u^*TV)\big)\subset\Gamma^{0,1}_{J,\mathfrak{j}}(\Sigma,u^*TV),$$

because the restriction of D_u to $\Gamma(\Sigma;u^*TV)$ is the linearization of the $\bar{\partial}_{J,\mathfrak{j}}$-operator at u for the space of maps to V. Thus, D_u descends to a first-order differential operator

$$D_u^{\mathcal{N}_XV}\colon\Gamma(\Sigma,u^*\mathcal{N}_XV)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(\Sigma,u^*\mathcal{N}_XV). \qquad (2.10)$$

By (2.9), this operator is $\mathbb{C}$-linear if

$$N_J(v,w)\in T_xV \qquad \forall\ v,w\in T_xX,\ x\in V. \qquad (2.11)$$

Under this assumption, $\xi\in\Gamma(\Sigma,u^*\mathcal{N}_XV)$ is a $(J_{X,V},\mathfrak{j})$-holomorphic map if and only if $\xi\in\ker D_u^{\mathcal{N}_XV}$.

If $J(TV)\subset V$, Σ is a smooth connected Riemann surface, and $u\colon\Sigma\longrightarrow X$ is a J-holomorphic map such that $u(\Sigma)\not\subset V$, then $u^{-1}(V)$ is an isolated set of points z_i; see the beginning of [4, Section 5.1]. Furthermore, u has a well-defined order of contact with V at each $z_i\in u^{-1}(V)$, $\mathrm{ord}^V_{z_i}u\in\mathbb{Z}^+$; if Σ is compact,

$$\sum_{z_i\in u^{-1}(V)}\mathrm{ord}^V_{z_i}u = u_*[\Sigma]\cdot V\,.$$

If $A\in H_2(X;\mathbb{Z})$, $g,k,\ell\in\mathbb{Z}^{\ge0}$, and $\mathbf{s}=(s_1,\ldots,s_\ell)\in(\mathbb{Z}^+)^\ell$ is a tuple satisfying (1.1), let

$$\mathfrak{M}^V_{g,k;\mathbf{s}}(X,A)\subset\overline{\mathfrak{M}}_{g,k+\ell}(X,A) \tag{2.12}$$

denote the subset of equivalence of stable J-holomorphic maps u from marked genus g nodal curves $(\Sigma,z_1,\ldots,z_{k+\ell})$ such that

$$u^{-1}(V)=\big\{z_{k+1},\ldots,z_{k+\ell}\big\} \qquad\text{and}\qquad \mathrm{ord}^V_{z_{k+i}}u=s_i \quad \forall\, i=1,\ldots,\ell.$$

If J satisfies (2.11), we denote by

$$\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)\supset\mathfrak{M}^V_{g,k;\mathbf{s}}(X,A) \tag{2.13}$$

the space of equivalence classes of stable $J_{X,V}$-holomorphic maps $u:\Sigma\longrightarrow X^V_m$, with $m\in\mathbb{Z}^{\ge0}$, from connected marked genus g nodal curves $(\Sigma,z_1,\ldots,z_{k+\ell})$ such that the restriction of u to each irreducible component of X^V_m is contained in either X or in $\{r\}\times\mathbb{P}_XV$ for some $r=1,\ldots,m$, but not in V or $\{r\}\times\mathbb{P}_{X,0}V$ for any r,

$$q_{m*}u_*[\Sigma]=A, \qquad \mathrm{ord}^{\{m\}\times\mathbb{P}_{X,0}V}_{z_{k+i}}u=s_i \quad \forall\; i=1,\ldots,\ell,$$

and the orders of contacts of the two branches at each node on V, $\{r\}\times\mathbb{P}_{X,0}V$, or $\{r\}\times\mathbb{P}_{X,0}V$ agree; see Figure 2. Two maps u as above are equivalent if they differ by an isomorphism of marked domains and a composition with an isomorphism (2.8); see [4, Section 4.2] for more details.

The relative moduli spaces $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ are introduced in [17] in a somewhat different formulation and under a stronger assumption on J than (2.11), which essentially requires it to be given via the Symplectic Neighborhood Theorem [25, Theorem 3.30] and makes the setup very amenable for the gluing needed to construct a virtual class. In [13], the relative moduli spaces are re-introduced, again in a somewhat different formulation from the previous paragraph, with ω-compatible J satisfying (2.11). The relative non-amenability of this setup with the gluing is not material in cases when the relative invariants (1.7) can be defined geometrically, as in the next paragraph. By [17, Section 3.2] and [13, Section 6], the spaces $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ are compact; they are also Hausdorff.

With notation as in (2.2) and J as in the previous two paragraphs, let

$$\Gamma^V_{g,k}(X,J)\subset\Gamma_{g,k}(X,J)$$

denote the subspace of elements ν such that

$$\nu|_{\check{\mathcal{U}}_{g,k}\times V}\in\Gamma_{g,k}(V,J|_V), \quad \widetilde{\nabla}_w\nu+J\widetilde{\nabla}_{Jw}\nu\in(T^*\check{\mathcal{U}}_{g,k})^{0,1}\otimes_{\mathbb{C}}T_xV \quad \forall\, w\in T_xX,\ x\in V. \tag{2.14}$$

The first condition in (2.14) insures that every (J,ν)-holomorphic map $u\colon\Sigma\longrightarrow X$ has well-defined order of contact with V at all points of $u^{-1}(V)$ not contained in an irreducible component of Σ mapped into V. The second condition in (2.14) implies that the linearization of the $\bar\partial_{J,\mathfrak{j}}-\nu$ operator at $u\colon\Sigma\longrightarrow V$ induces a $\mathbb{C}$-linear map

$$D^{\mathcal{N}_XV}_u\colon\Gamma(\Sigma,u^*\mathcal{N}_XV)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(\Sigma,u^*\mathcal{N}_XV)$$

for every (J,ν)-holomorphic map $u\colon \Sigma \longrightarrow V$. The moduli spaces

$$\mathfrak{M}^V_{g,k;\mathbf{s}}(X,A;J,\nu) \subset \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu)$$

can then be defined analogously to (2.12) and (2.13). The component maps into the rubber layers $\{r\}\times\mathbb{P}_X V$ are then $(J_{X,V},\nu')$-holomorphic, with

$$\begin{gathered}\nu' \in \Gamma_{g',k'}(\mathbb{P}_X V, J),\\ \{\nu'|_w\}(v) = \big(\{\widetilde{\nabla}_w \nu\}(v), \nu(v)\big) \in T^{\mathrm{vrt}}_w \mathcal{N}_X V \oplus T^{\mathrm{hor}}_w \mathcal{N}_X V \qquad \forall\, w\in \mathcal{N}_X V,\ v\in T\check{\mathcal{U}}_{g',k'}\,.\end{gathered}$$

By the same reasoning as for $J_{X,V}$, ν' given by the second line above extends over $\mathbb{P}_{X,\infty}V$, is $\mathbb{C}^*$-equivariant, and satisfies (2.14) with (X,V) replaced by $(\mathbb{P}_X V, \mathbb{P}_{X,0}V)$ and $(\mathbb{P}_X V, \mathbb{P}_{X,\infty}V)$.

By [13, Proposition 7.3], the space $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu)$ is compact. By [13, Lemma 7.5], if ν is generic each stratum of $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu)$ consisting of simple maps of a fixed combinatorial type is a smooth manifold of the expected even dimension, which is less than the expected dimension of the subspace of simple maps with smooth domains (except for this subspace itself). By [13, Theorem 7.4], the last stratum has a canonical orientation. As explained in [4, Section 4.3], the images of the strata of $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ consisting of multiply covered maps under the morphism

$$\mathrm{st}\times\mathrm{ev}_1\ldots\times\mathrm{ev}_k\times\mathrm{ev}_{k+1}\ldots\times\mathrm{ev}_{k+\ell}\colon \overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A;J,\nu) \longrightarrow \overline{\mathcal{M}}_{g,k+\ell}\times X^k\times V^\ell \tag{2.15}$$

are contained in images of maps from smooth even-dimensional manifolds of dimension less than the main stratum if ν is generic, subject to the conditions (2.11) and (2.14), $(V,\omega|_V)$ is semi-positive, and (X,ω,V) is semi-positive in the sense of [4, Definition 4.7(1)]. Such strata do not even exist if the domains of all elements of $\overline{\mathfrak{M}}^V_{g,k;\mathbf{s}}(X,A)$ possibly contributing to the number (1.7) are stable for some J, as happens in Section 5. By the proof of [15, Proposition 8.2], all relevant domains are stable for a generic J if

$$A'\cdot V \ge \langle c_1(X), A'\rangle + \frac{1}{2}\dim_{\mathbb{R}} X + 2g \tag{2.16}$$

for all $A'\in H_2(X;\mathbb{Z})$ with $\omega(A')\le\omega(A)$ such that A' can be represented by a J-holomorphic curve. In the above cases, (2.15) is thus a pseudocycle. Intersecting it with generic representatives for the Poincare duals of the cohomology classes κ on $\overline{\mathcal{M}}_{g,k+\ell}$, $\alpha_1,\ldots,\alpha_k$ on X, and $\alpha_{k+1},\ldots,\alpha_{k+\ell}$ on V and dividing by the order of the covering (2.1), we obtain the relative GW-invariant

$$\mathrm{GW}^{X,V}_{g,k;\mathbf{s}}\big(\kappa;\alpha_1,\ldots,\alpha_k;\alpha_{k+1},\ldots,\alpha_{k+\ell}\big) = \mathrm{GW}^{X,V}_{g,k;\mathbf{s}}\big(\kappa;\alpha_1\otimes\ldots\otimes\alpha_k;\alpha_{k+1}\otimes\ldots\otimes\alpha_{k+\ell}\big).$$

The relative GW-invariant (1.7) is the above invariant with κ pulled back from $\overline{\mathcal{M}}_{g,k}$ by the forgetful morphism from $\overline{\mathcal{M}}_{g,k+\ell}$ and $\alpha_{k+i}=1$ for all $i=1,\ldots,\ell$. If $g=0$, the same reasoning applies with $\nu=0$ and yields the same conclusion if (X,ω,V) satisfies the slightly stronger condition of [4, Definition 4.7(2)]. For general triple (X,ω,V), the relative GW-invariants (1.7) are defined similarly to [7, 21] using Kuranishi structures (or finite-dimensional approximations) and local perturbations ν as in (2.14).

3 Proof of Theorem 1

A generic (J,ν)-holomorphic map contributing to the absolute GW-invariant (1.5) has intersection number $A\cdot V$ with V. One would thus expect it to meet V at $A\cdot V$ distinct points. The different orderings of these points would ideally give rise to $(A\cdot V)!$ distinct relative maps contributing to the relative GW-invariant (1.7). However, a regular pair (J,ν) determining the number (1.5) may not satisfy the conditions (2.11) and (2.14) required of the pairs (J,ν) determining the number (1.7), while a generic pair satisfying (2.11) and (2.14) may not be regular for the purposes of determining the number (1.5). Thus, there is no à priori reason for the identity (1.9) to hold in general. Below we give two versions of nearly the same proof of Theorem 1: first by a direct comparison and then by formally applying the symplectic sum formula.

3.1 By direct comparison

The restriction (2.11) on J (or even the stronger one in [17]) is not material, as we can simply fix one admissible J and then choose a suitable ν to compute the GW-invariants (1.5) and (1.7). We start by choosing a generic $\nu|_V\in\Gamma_{g',k'}(V,J)$ and then extend it to X so that it satisfies the second condition in (2.14). A generic such extension ν determines the *relative* GW-invariant (1.7). It counts the (J,ν)-maps that pass through generic representatives of the Poincare duals of κ and α_i have images in X with no components mapped into V. Dropping the contact marked points, we obtain a regular element of $\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$ which contributes to the *absolute* GW-invariant (1.5). However, because ν may not be generic as far as the absolute invariants are concerned, $\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$ may contain other elements u which meet generic representatives of the Poincare duals of κ and α_i. Any such u must have at least some components mapped into V, as all other components can be regularized with ν subject to the condition (2.14).

Spaces $\mathfrak{M}_\Gamma(\nu)$ of maps as at the end of the previous paragraph can be represented by decorated connected bipartite graphs Γ with vertices v

- alternating between those representing the topological components Σ_v of the domain of the maps into V and into X (without being contained in V),
- labeled by pairs indicating the genus g_v of Σ_v and the degree A_v of the map on Σ_v, and
- decorated by disjoint subsets of $\{1,\ldots,k\}$, indicating the marked points carried by Σ_v;

see Figure 3. Since $\mathfrak{M}_\Gamma(\nu)$ is contained in $\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$,

$$g_\Gamma+\sum_{v\in\Gamma}g_v=g,\qquad \sum_{v\in\Gamma}A_v=A\in H_2(X;\mathbb{Z})\,,\quad\text{and}\quad \sum_{v\in\Gamma}k_v=k,$$

where $v\in\Gamma$ means that v is a vertex in Γ, g_Γ is the genus of the graph Γ (number of edges minus the number vertices plus 1), and k_v is the number of original marked points attached to a vertex $v\in\Gamma$ (the number of the original marked points carried by the topological component Σ_v of Σ). We denote by Γ_V the set of vertices of Γ corresponding to the components mapped into V and by Γ_X the set of remaining vertices. For each $v\in\Gamma$, let $\ell_v\in\mathbb{Z}^{\ge0}$ denote the number of edges leaving v (the number of nodes joining Σ_v to other topological components of Σ). The stability condition on the elements of $\overline{\mathfrak{M}}_{g,k}(X,A)$ implies that $k_v+\ell_v\ge3$ for each $v\in\Gamma$ with $(g_v,A_v)=(0,0)$.

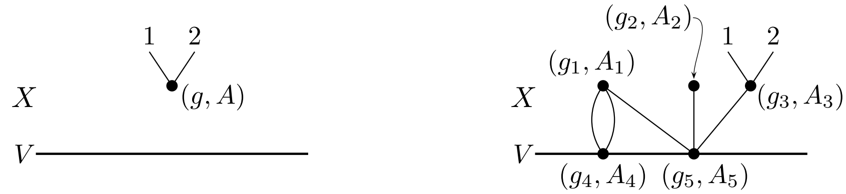

Figure 3: Bipartite graphs Γ representing elements of $\overline{\mathfrak{M}}_{g,2}(X,A;J,\nu)$.

If the domains of all relevant elements of $\overline{\mathfrak{M}}_{g,k}(X,A)$ are stable, as is the case in Section 5, the above perturbations ν can be chosen globally as elements of $\Gamma^V_{g,k}(X,J)$. Otherwise, the same general principle applies by using compatible Kuranishi structures for maps to X and to V. Theorem 1 is established by showing that the subspace

$$\mathfrak{M}_\Gamma(\kappa;\alpha;\nu)\subset\mathfrak{M}_\Gamma(\nu)\subset\overline{\mathfrak{M}}_{g,k}(X,A;J,\nu)$$

of the elements that are of type Γ and meet generic representatives of the Poincare duals of κ and α is empty for a generic ν satisfying (2.14) unless Γ is the one-vertex graph of maps to X, as in the first diagram in Figure 3. We can assume that κ and α satisfy (1.6).

Since V is assumed to be (g,A)-hollow in Theorem 1, we can use the Symplectic Neighborhood Theorem [25, Theorem 3.30] to choose an ω-tame almost complex structure J on X so that $J(TV)\subset TV$, $J_V\equiv J|_V$ satisfies the conditions of Definition 1, and J satisfies (2.11). Thus, the degree A_v of the restriction of any element of $\mathfrak{M}_\Gamma(\nu)$ to a topological component Σ_v of the domain mapped into V is zero. If the genus g_v of such Σ_v is zero, the restriction of any element u of $\mathfrak{M}_\Gamma(0)$ to Σ_v is regular as a map into X and stays so after a small generic deformation ν as in the previous paragraph. If $g_v=0$ for all $v\in\Gamma_V$, $\mathfrak{M}_\Gamma(\nu)$ consists of regular maps into X for a generic ν satisfying (2.14) and thus has the expected dimension. Since this dimension is smaller than the virtual dimension of $\overline{\mathfrak{M}}_{g,k}(X,A)$, unless $\Gamma_V=\emptyset$, $\mathfrak{M}_\Gamma(\kappa;\alpha;\nu)=\emptyset$. In particular, if $g=0$, all (J,ν)-maps for a generic ν satisfying (2.14) are regular as maps to X and transverse to V. Thus, the sets of stable maps contributing to the numbers on the two sides of (1.9) are the same in this case, up to the orderings of the $A\cdot V$ intersection points with V. This establishes the $g=0$ case of (1.9).

If $n\ge5$,

$$\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{g',0}(V,0)=2(n-4)(1-g')<0\qquad\forall\ g'\ge2.$$

In these cases, we can choose deformations ν satisfying (2.14) so that $\mathfrak{M}_\Gamma(\nu)=\emptyset$ if $g_v\ge2$ for any $v\in\Gamma_V$. For the purposes of establishing the $g\ge1$ cases of (1.9), it thus remains to consider the spaces $\mathfrak{M}_\Gamma$ so that $g_v\in\{0,1\}$ for all $v\in\Gamma_V$. Denote by $\Gamma_{V;1}\subset\Gamma_V$ the subset of vertices so that $g_v=1$. In the next paragraph, we show that

$$\dim\mathfrak{M}_\Gamma(\nu)\le\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{g,k}(X,A)-2\sum_{v\in\Gamma_{V;1}}\ell_v \tag{3.1}$$

for a generic ν satisfying (2.14), if either $n\ge5$ or $g_v\le1$ for all $v\in\Gamma_V$ (in particular, if $g=1$). Thus, $\mathfrak{M}_\Gamma(\kappa;\alpha;\nu)=\emptyset$ in these cases if Γ is not the basic one-vertex graph as in the first diagram in Figure 3, and so (1.9) again holds.

Removing the vertices of $\Gamma_{V;1}$ from Γ and replacing the edges leading to them by the marked points on the remaining vertices, we obtain graphs Γ_i, with $i=1,\ldots,N$ for some $N\in\mathbb{Z}^+$, representing subspaces $\mathfrak{M}_{\Gamma_i}(0)$ of the moduli spaces $\overline{\mathfrak{M}}_{g_i,k_i+\ell_i}(X,A_i)$ with

$$\sum_{i=1}^{N}(g_i-1)+\sum_{v\in\Gamma_{V;1}}\ell_v=g-1,\quad \sum_{i=1}^{N}A_i=A,\quad \sum_{i=1}^{N}k_i+\sum_{v\in\Gamma_{V;1}}k_v=k\,,\quad \sum_{i=1}^{N}\ell_i=\sum_{v\in\Gamma_{V;1}}\ell_v\,,$$

where $k_i\in\mathbb{Z}^{\ge0}$ is the number of the original marked points carried by the component Γ_i. The moduli spaces $\overline{\mathfrak{M}}_{1,k_v+\ell_v}(V,0;J,\nu)$ corresponding to $v\in\Gamma_{V;1}$ are of dimension $2(k_v+\ell_v)\in\mathbb{Z}^+$ for a generic choice of $\nu|_V$. Since $\mathfrak{M}_{\Gamma_i}(\nu)$ contains no component of positive genus mapped into V, it has the expected dimension for a generic extension of $\nu|_V$ satisfying (2.14). Taking into account the matching conditions at the nodes joining elements of $\mathfrak{M}_{\Gamma_i}(\nu)$ to elements of $\overline{\mathfrak{M}}_{1,k_v+\ell_v}(V,0;J,\nu)$, we find that

$$\begin{aligned}\dim\mathfrak{M}_{\Gamma}(\nu)&\le\sum_{i=1}^{N}\dim\mathfrak{M}_{\Gamma_i}(\nu)+\sum_{v\in\Gamma_{V;1}}\dim\overline{\mathfrak{M}}_{1,k_v+\ell_v}(V,0;J,\nu)-2n\sum_{v\in\Gamma_{V;1}}\ell_v\\&\le 2\sum_{i=1}^{N}\big(\langle c_1(X),A_i\rangle+(n-3)(1-g_i)+k_i+\ell_i\big)+2\sum_{v\in\Gamma_{V;1}}(k_v+\ell_v)-2n\sum_{v\in\Gamma_{V;1}}\ell_v\\&=2\big(\langle c_1(X),A\rangle+(n-3)(1-g)+k\big)+2(n-3+1+1-n)\sum_{v\in\Gamma_{V;1}}\ell_v\,.\end{aligned}$$

Along with the first equation in (1.2), this establishes (3.1) and concludes the proof of the first claim of Theorem 1.

Remark 3.1. A regular genus 1 degree 0 (J,ν)-map into V may not be regular as a (J,ν)-map into X. However, the space of such maps has the expected dimension for the target X because this dimension is *the same* as the expected dimension for the target V in the $g=1$ case. Thus, a boundary stratum of (J,ν)-maps with only $g=0,1$ components contained in V is of smaller dimension than the main stratum of maps into X. However, the space of (J,ν)-maps from smooth genus 1 domains into V has the same dimension as the main stratum; this is precisely what makes Example 1 possible.

Suppose next that $\kappa=1$ and $g\ge2$ in (1.9), i.e. only the primary insertions are considered. Given a bipartite graph Γ describing a subspace $\mathfrak{M}_{\Gamma}(0)$ of $\overline{\mathfrak{M}}_{g,k}(X,A)$ as in Figure 3, let Γ_0 be the decorated bipartite graph obtained by replacing the genus labels of all vertices $v\in\Gamma_V$ with 0. Thus, $\mathfrak{M}_{\Gamma_0}(0)$ is a subspace of $\overline{\mathfrak{M}}_{g_0,k}(X,A)$ for some $g_0<g$, unless $g_v=0$ for all $v\in\Gamma_V$ (in which case $\Gamma_0=\Gamma$ and thus $g_0=g$). If $n=1,2$ and $g_0<g$,

$$\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{g_0,k}(X,A)<\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{g,k}(X,A)$$

by the first equation in (1.2). Thus, for a generic $\nu\in\Gamma^V_{g_0,k}(X,J)$, $\mathfrak{M}_{\Gamma_0}(1;\alpha;\nu)=\emptyset$ in this case, and so $\nu\in\Gamma^V_{g,k}(X,J)$ can be chosen so that $\mathfrak{M}_{\Gamma}(1;\alpha;\nu)=\emptyset$ whenever $g'_v>0$ for any $v\in\Gamma_V$. This establishes the $n=1,2$ cases of the last claim of Theorem 1.

If $g\ge2$ in (1.9) and $n=3$,

$$\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{g_0,k}(X,A)=\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{g,k}(X,A)\,. \tag{3.2}$$

For any $v \in \Gamma_V$ with $g_v \geq 1$,

$$\overline{\mathfrak{M}}_{g_v,k_v+\ell_v}(V,0) = \overline{\mathcal{M}}_{g_v,k_v+\ell_v} \times V\,;$$

the obstruction bundle for this moduli space is

$$\pi_1^*\mathbb{E}^* \otimes \pi_2^* TV \longrightarrow \overline{\mathcal{M}}_{g_v,k_v+\ell_v} \times V\,, \tag{3.3}$$

where $\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{g_v,k_v+\ell_v}$ is the rank g_v Hodge vector bundle of holomorphic differentials; it has chern classes $\lambda_i \equiv c_i(\mathbb{E})$. For $g_v \geq 2$, it is the pull-back of the Hodge vector bundle over $\overline{\mathcal{M}}_g$ by the forgetful morphism; if $g_v = 1$, it is the pull-back of the Hodge line bundle over $\overline{\mathcal{M}}_{1,1}$. By [27, (5.3)] in the first case and for dimensional reasons in the second case,

$$\lambda_{g_v}^2 = 0 \in H^{4g_v}\big(\overline{\mathcal{M}}_{g_v,k_v+\ell_v}\big). \tag{3.4}$$

Since the obstruction bundle is given by (3.3),

$$\big[\overline{\mathfrak{M}}_{g_v,k_v+\ell_v}(V,0;J,\nu)\big] = e\big(\pi_1^*\mathbb{E}^* \otimes \pi_2^* TV\big) \cap \big[\overline{\mathcal{M}}_{g_v,k_v+\ell_v} \times V\big] \tag{3.5}$$

for a generic $\nu \in \Gamma^V_{g_v,k_v}(X,J)$. By (3.2), $\mathfrak{M}_{\Gamma_0}(1;\alpha;\nu)$ consists of isolated maps meeting V transversality at finitely many points p_j for such a choice of ν (if $\mathfrak{M}_{\Gamma_0}(1;\alpha;\nu)$ is not empty). These points include the nodes where irreducible components of elements of $\mathfrak{M}_{\Gamma_0}(1;\alpha;\nu)$ meet the elements of $\overline{\mathfrak{M}}_{g_v,k_v+\ell_v}(V,0;J,\nu)$ with $v \in \Gamma_V$. By (3.5) and (3.4), the homology class represented by the subspace of the latter passing through p_j is

$$e\big(\mathbb{E}^* \otimes T_{p_j}V\big) \cap \big[\overline{\mathcal{M}}_{g_v,k_v+\ell_v}\big] = \lambda_{g_v}^2 \cap \big[\overline{\mathcal{M}}_{g_v,k_v+\ell_v}\big] = 0.$$

Thus, the contribution of $\mathfrak{M}_\Gamma(1;\alpha;\nu)$ to the left-hand side of (1.9) is the degree of a zero-cycle which vanishes in the homology and thus is 0, if $g_v \geq 1$ for any $v \in \Gamma_V$. This establishes the $\kappa = 1$, $n = 3$, and $g \geq 2$ case of (1.9).

The remaining case of Theorem 1 is $\kappa = 1$, $n = 4$, $g = 2$, and $A \neq 0$ (otherwise both sides of (1.9) vanish for dimensional reasons). By the previous discussion, it is sufficient to show that $\mathfrak{M}_\Gamma(1;\alpha;\nu) = \emptyset$ for a generic ν satisfying (2.14) if $g_v = 2$ for some $v \in \Gamma_V$. This assumption implies that $g_{v'} = 0$ for all $v' \in \Gamma_V - v$ and

$$\dim^{\mathrm{vir}} \overline{\mathfrak{M}}_{g_0,k}(X,A) = \dim^{\mathrm{vir}} \overline{\mathfrak{M}}_{g,k}(X,A) + 4\,. \tag{3.6}$$

By the first equation in (1.2), the virtual dimension of $\overline{\mathfrak{M}}_{2,0}(V,0)$ is 0. Thus, we can choose a deformation ν satisfying (2.14) so that the image of all elements of $\overline{\mathfrak{M}}_{g_0,k_v+\ell_v}(X,A;J,\nu)$ is contained in arbitrary small neighborhoods of finitely many points of V. By (3.6), for a generic such ν there are no elements of $\mathfrak{M}_{\Gamma_0}(1;\alpha;\nu)$ that pass through these images, since each point in $V \subset X$ imposes a condition of real codimension 6 on maps to X. Thus, $\mathfrak{M}_\Gamma(1;\alpha;\nu) = \emptyset$ for a generic ν satisfying (2.14) in this case as well.

3.2 Via the symplectic sum formula

We next give a proof of Theorem 1 by applying the symplectic sum formula to the symplectic decomposition

$$X = X \underset{V = \mathbb{P}_{X,\infty}V}{\#} \mathbb{P}_X V\,, \tag{3.7}$$

with $\mathbb{P}_{X,\infty}V \subset \mathbb{P}_X V$ as in (2.5) and (2.6). The $\mathbb{P}^1$-bundle $\mathbb{P}_X V \longrightarrow V$ carries a symplectic form induced from $\omega|_V$ in a way well-defined up to symplectic deformation equivalence; see the beginning of Section 3.3.

According to the symplectic sum formula, the left-hand side of (1.9) is a weighted count of k-marked genus g degree A (J,ν)-maps u into

$$X_1^V \equiv X \underset{V=\mathbb{P}_{X,\infty}V}{\cup} \mathbb{P}_X V \tag{3.8}$$

that have the same contact order with the common hypersurface V at the two branches of each node, take no smooth point of the domain to V, and meet generic representatives of the Poincare duals of κ and α_i. The degree of such u is the class in X represented by the composition of u with the natural projection

$$q\colon X \underset{V=\mathbb{P}_{X,\infty}V}{\cup} \mathbb{P}_X V \longrightarrow X\,; \tag{3.9}$$

its weight is the product of the contacts with the common hypersurface (counted once for each pair of contacts from the two sides).

Spaces $\mathfrak{M}_\Gamma(\kappa;\alpha)$ of such maps to X_1^V can be represented by the same kind of connected bipartite graphs Γ as in Section 3.1 with an additional decoration $d_e \in \mathbb{Z}^+$ for each edge e; see Figure 4, where edge labels 1 are not explicitly indicated. The subset Γ_V of vertices now describes the topological components Σ_v of the domain Σ that are mapped to $\mathbb{P}_X V$; the additional decorations d_e specify the orders of contacts with V of the branches of the nodes associated with the edges. The stability condition on Γ described before now applies only to the vertices $v \in \Gamma_X$. The composition of an element u in such a space $\mathfrak{M}_\Gamma(\kappa;\alpha)$ with q produces an element of the space $\mathfrak{M}_{\bar{\Gamma}}(\kappa;\alpha)$ considered above with $\bar{\Gamma}$ obtained from Γ by dropping the edge labels and contracting off the unstable vertices $v \in \Gamma_V$ and the edges leaving from them.

Breaking a graph Γ as in the previous paragraph at the mid-point of each edge, we obtain the relative moduli spaces

$$\overline{\mathfrak{M}}^V_{g_v,k_v;\mathbf{s}_v}(X,A_v) \qquad \text{and} \qquad \overline{\mathfrak{M}}^{\mathbb{P}_{X,\infty}V}_{g_v,k_v;\mathbf{s}_v}(\mathbb{P}_X V, A_v(\mathbf{s}_v))$$

with $v \in \Gamma_X$ and $v \in \Gamma_V$, respectively, where $\mathbf{s}_v$ is the tuple given by the labels on the edges and $A_v(\mathbf{s}_v)$ is the sum of the push-forward of A_v under the inclusion $\mathbb{P}_{X,0}V \longrightarrow \mathbb{P}_X V$ and $|\mathbf{s}_v|$ fiber classes. The left-hand side of (1.9) is the sum over all admissible graphs Γ of the weighted products of the corresponding relative invariants with the relative primary insertions given by the usual Kunneth decomposition of the diagonal in V^2 at each node; see the second-to-last equation on page 201 in [19] and equations (5.4), (5.7), and (5.8) in [17]. Since the intersection points of elements of $\mathfrak{M}_\Gamma(\kappa;\alpha)$ are unordered, while the contact points of the corresponding relative invariants are ordered, the contribution from each graph Γ should be divided by the number of orderings of the intersection points.

Some care is needed in translating the constraints κ and α_i in (1.5) into constraints for the relative invariants of (X,V) and $(\mathbb{P}_X V, \mathbb{P}_{X,\infty}V)$. If $v \in \Gamma_X$, the corresponding relative invariant of (X,V) keeps the insertion α_i at the absolute marked point corresponding to i, if it is carried by Σ_v. If $v \in \Gamma_V$, the corresponding relative invariant of $(\mathbb{P}_X V, \mathbb{P}_{X,\infty}V)$ gets the insertion $\pi^*_{X,V}(\alpha_i|_V)$ at

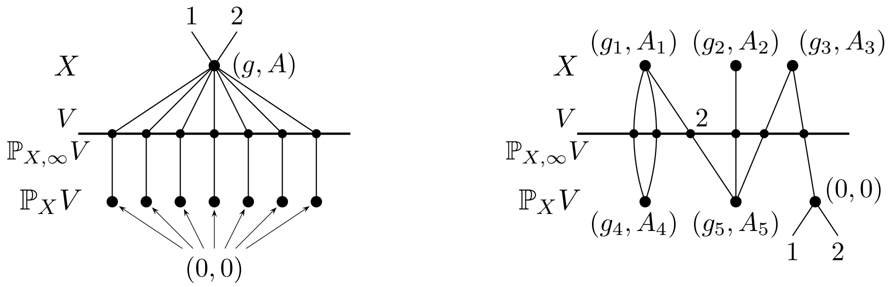

Figure 4: Bipartite graphs Γ representing elements of $\overline{\mathfrak{M}}_{g,2}(X_1^V, A; J, \nu)$ with $A\cdot V=7$.

the absolute marked point corresponding to i, where $\pi_{X,V} : \mathbb{P}_X V \longrightarrow V$ is the projection map. Denote by

$$\mathrm{st}_v : \overline{\mathfrak{M}}^V_{g_v,k_v;\mathbf{s}_v}(X, A_v) \longrightarrow \overline{\mathcal{M}}_{g_v,k_v+\ell(\mathbf{s}_v)} \quad \text{or} \quad \mathrm{st}_v : \overline{\mathfrak{M}}^{\mathbb{P}_{X,\infty}V}_{g_v,k_v;\mathbf{s}_v}(\mathbb{P}_X V, A_v(\mathbf{s}_v)) \longrightarrow \overline{\mathcal{M}}_{g_v,k_v+\ell(\mathbf{s}_v)}$$

the stabilization map, depending on whether $v\in\Gamma_X$ or $v\in\Gamma_V$, respectively; in the unstable range, the target of this map is one point. Let

$$\mathrm{gl}_\Gamma : \prod_{v\in\Gamma}\overline{\mathcal{M}}_{g_v,k_v+\ell(\mathbf{s}_v)} \longrightarrow \overline{\mathcal{M}}_{g,k}$$

be the morphism given by identifying pairs of points corresponding to the same edge in Γ. In particular,

$$\mathrm{gl}_\Gamma \circ \prod_{v\in\Gamma}\mathrm{st}_v = \mathrm{st}\circ\iota_\Gamma : \overline{\mathfrak{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}_{g,k}\,,$$

where $\iota_\Gamma : \overline{\mathfrak{M}}_\Gamma \longrightarrow \overline{\mathfrak{M}}_{g,k}(X_1^V, A)$ is the inclusion map. By the Kunneth formula,

$$\mathrm{gl}_\Gamma^*\kappa = \sum_j \bigotimes_{v\in\Gamma}\kappa_{j;v} \in \bigotimes_{v\in\Gamma} H^*\big(\overline{\mathcal{M}}_{g_v,k_v+\ell(\mathbf{s}_v)}\big) = H^*\bigg(\prod_{v\in\Gamma}\overline{\mathcal{M}}_{g_v,k_v+\ell(\mathbf{s}_v)}\bigg)$$

for some $\kappa_{j;v} \in H^*(\overline{\mathcal{M}}_{g_v,k_v+\ell(\mathbf{s}_v)})$. In the Γ-summand in the symplectic sum decomposition for the absolute GW-invariant (1.5), the insertion κ is replaced by the insertion $\kappa_{j;v}$ in the relative invariant corresponding to the vertex v and the resulting products are summed over all j. This is carried out in a specific case in Section 4.2.

Since V is assumed to be (g, A)-hollow in Theorem 1, we can choose an almost complex structure J_V on V so that it satisfies the conditions of Definition 1. Using a connection in $\mathcal{N}_X V$ as in Section 2, we can extend J_V to an almost complex structure J on $\mathbb{P}_X V$ so that the condition (2.11) is satisfied and the projection $\pi_{X,V} : \mathbb{P}_X V \longrightarrow V$ is (J_V, J)-holomorphic. Using the same connection, we can extend any $\nu\in\Gamma_{g,k}(V, J_V)$ to

$$\pi^*_{X,V}\nu \in \Gamma^{\mathbb{P}_{X,\infty}V}_{g,k}(\mathbb{P}_X V, J)$$

so that $\pi_{X,V}\circ u : \Sigma \longrightarrow V$ is (J_V, ν)-holomorphic whenever $u : \Sigma \longrightarrow \mathbb{P}_X V$ is $(J, \pi^*_{X,V}\nu)$-holomorphic.

By the previous paragraph, we can assume that the degree A_v of the composition of the restriction of any element of $\mathfrak{M}_\Gamma$ to a topological component Σ_v of the domain mapped into $\mathbb{P}_X V$ with $\pi_{X,V}$ is zero, i.e. all relevant relative invariants of $(\mathbb{P}_X V, \mathbb{P}_{X,\infty}V)$ lie in the fiber classes $d_v F$ with $d_v\in\mathbb{Z}^{\ge 0}$. A key point of the paragraph above the previous one is that the class integrated over the relative moduli space $\overline{\mathfrak{M}}^{\mathbb{P}_{X,\infty}V}_{g_v,k_v;\mathbf{s}_v}(\mathbb{P}_X V, dF)$ corresponding to the vertex v is pulled back by the projection map

$$\varphi \equiv \mathrm{st}\times\pi_{X,V} : \overline{\mathfrak{M}}^{\mathbb{P}_{X,\infty}V}_{g_v,k_v;\mathbf{s}_v}(\mathbb{P}_X V, d_vF) \longrightarrow \overline{\mathcal{M}}_{g_v,k_v+\ell(\mathbf{s}_v)}\times V\,. \tag{3.10}$$

In particular, if

$$\dim^{\mathrm{vir}} \overline{\mathfrak{M}}^{\mathbb{P}_{X,\infty}V}_{g_v,k_v;\mathbf{s}_v}(\mathbb{P}_X V, d_vF) > \dim\big(\overline{\mathcal{M}}_{g_v,k_v+\ell(\mathbf{s}_v)}\times V\big), \tag{3.11}$$

then the relative invariant corresponding to the vertex $v\in\Gamma_V$ vanishes and such bipartite graph Γ does not contribute to the left-hand side of (1.9).

By the second equation in (1.2) and the condition $|\mathbf{s}_v|=d_v$, (3.11) is equivalent to

$$d_v + (n-3)(1-g_v) + k_v + \ell(\mathbf{s}_v) > n-1 + \begin{cases} 0, & \text{if } g_v=0,\ k_v+\ell(\mathbf{s}_v)\le 2;\\ 3g_v-3+k_v+\ell(\mathbf{s}_v), & \text{otherwise.}\end{cases}$$

If $g_v=0$, either $d_v\in\mathbb{Z}^+$ (and thus $\ell(\mathbf{s}_v)\in\mathbb{Z}^+$) or $k_v\ge 3$ for stability reason. Thus, the relative invariant corresponding to a vertex $v\in\Gamma_V$ with $g_v=0$ is zero unless $d_v=1$, $k_v=0$, and $\mathbf{s}_v=(1)$. In this remaining case, the only nonzero relative invariant is

$$\mathrm{GW}^{\mathbb{P}_X V,\mathbb{P}_{X,\infty}V}_{0,F;(1)}\big(1,1;\mathrm{PD}_V([\mathrm{pt}])\big) = 1.$$

In particular, the contribution to the left-hand side of (1.9) from the simplest graph, i.e. as in the first diagram in Figure 4, is

$$\frac{1}{(A\cdot V)!}\mathrm{GW}^{X,V}_{g,A;\mathbf{1}_{A\cdot V}}\big(\kappa;\alpha;1^{A\cdot V}\big) \equiv \frac{1}{(A\cdot V)!}\mathrm{GW}^{X,V}_{g,A;\mathbf{1}_{A\cdot V}}\big(\kappa;\alpha\big)\,. \tag{3.12}$$

All other nonzero contributions to the left-hand side of (1.9) can come only from graphs Γ such that $(d_v,k_v,\mathbf{s}_v)=(1,0,(1))$ for all $v\in\Gamma_V$ with $g_v=0$ and $g_v\in\mathbb{Z}^+$ for some $v\in\Gamma_V$. Since there are no such graphs if $g=0$, this concludes the proof of the $g=0$ case of (1.9).

We next show that the relative invariants corresponding to $v\in\Gamma_V$ with $g_v\in\mathbb{Z}^+$ also vanish under the assumptions of (1.8). If $g_v\ge 2$ and $\nu\in\Gamma_{g_v,0}(V,J_V)$, the composition with the projection $\pi_{X,V}$ induces a continuous map

$$\pi_{X,V} : \overline{\mathfrak{M}}^{\mathbb{P}_{X,\infty}V}_{g_v,k_v;\mathbf{s}_v}\big(\mathbb{P}_X V, d_vF; J, \pi^*_{X,V}\nu\big) \longrightarrow \overline{\mathfrak{M}}_{g_v,0}\big(V,0;J_V,\nu\big). \tag{3.13}$$

Since

$$\dim\overline{\mathfrak{M}}_{g_v,0}\big(V,0;J_V,\nu\big) = \dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{g_v,0}\big(V,0\big) = (n-4)(1-g_v) \qquad \forall\ g_v\ge 2$$

for a generic $\nu\in\Gamma_{g_v,0}(V,J_V)$, the moduli spaces in (3.13) are empty if $g_v\ge 2$ and $n\ge 5$. In particular, the relative invariants vanish in these cases.

If $g_v=1$, then $d_v,\ell(\mathbf{s}_v)\in\mathbb{Z}^+$ by the first assumption in (1.8). For a generic $\nu\in\Gamma_{1,1}(V,J_V)$,

$$\pi_{X,V} : \overline{\mathfrak{M}}^{\mathbb{P}_{X,\infty}V}_{1,k_v;\mathbf{s}_v}\big(\mathbb{P}_X V, d_vF; J, \pi^*_{X,V}\nu\big) \longrightarrow \overline{\mathfrak{M}}_{1,1}\big(V,0;J_V,\nu\big)$$

is then a fibration with typical fiber $\overline{\mathfrak{M}}^{\mathrm{pt}}_{1,k_v;\mathbf{s}_v}(\mathbb{P}^1,d)_{\mathfrak{j}}$, where the subscript $\mathfrak{j}$ denotes the moduli space with $\mathfrak{j}$ fixed on $\overline{\mathcal{M}}_{1,1}$. Since the obstruction bundle for $\overline{\mathfrak{M}}_{1,1}(V,0)$ is given by (3.3),

$$\big[\overline{\mathfrak{M}}_{1,1}\big(V,0;J_V,\nu\big)\big]=e\big(\pi_1^*\mathbb{E}^*\otimes\pi_2^*TV\big)\cap\big[\overline{\mathcal{M}}_{1,1}\times V\big]=\{\mathfrak{j}\}\times V_1+\overline{\mathcal{M}}_{1,1}\times V_0, \tag{3.14}$$

where $V_0,V_1\subset V$ are some cycles of $\mathbb{R}$-dimensions 0 and 2, respectively, and $\mathfrak{j}$ is a fixed element of $\mathcal{M}_{1,1}$. Since

$$\dim^{\mathrm{vir}}\overline{\mathfrak{M}}^{\mathrm{pt}}_{1,k_v;\mathbf{s}_v}(\mathbb{P}^1,d_v)=d_v+k_v+\ell(\mathbf{s}_v)>k_v+\ell(\mathbf{s}_v)=\dim\overline{\mathcal{M}}_{1,k_v+\ell(\mathbf{s}_v)},$$

the integral of the pull-back of any class by (3.10) vanishes on the last term in (3.14). Since

$$\dim^{\mathrm{vir}}\overline{\mathfrak{M}}^{\mathrm{pt}}_{1,k_v;\mathbf{s}_v}(\mathbb{P}^1,d_v)_{\mathfrak{j}}=d_v-1+k_v+\ell(\mathbf{s}_v)>k_v+\ell(\mathbf{s}_v)-1=\dim\overline{\mathcal{M}}_{1,k_v+\ell(\mathbf{s}_v);\mathfrak{j}}\,,$$

the integral of the pull-back of any class by (3.10) vanishes on the first term on the RHS of (3.14) as well.

In summary, the only graph Γ that contributes to the left-hand side of (1.9) via the symplectic sum formula applied to the decomposition (3.7) under the assumptions (1.8) is the graph with

$$|\Gamma_X|=1 \qquad \text{and} \qquad (g_v,d_v,k_v,\mathbf{s}_v)=(0,1,0,(1)) \quad \forall\, v\in\Gamma_V;$$

see the first diagram in Figure 4. Since its contribution is given by (3.12), we have established the first claim of Theorem 1.

Suppose next that $\kappa=1$ and $g\ge 2$ in (1.9), i.e. only the primary insertions are considered. The relative invariants of (X,V) that enter into the symplectic sum formula then count curves that meet generic Poincare duals of all the constraints α_i. If $n=1,2$ and $g_0<g$,

$$\dim^{\mathrm{vir}}\overline{\mathfrak{M}}^{V}_{g_0,k;\mathbf{s}_0}(X,A)<\dim^{\mathrm{vir}}\overline{\mathfrak{M}}_{g,k}(X,A) \tag{3.15}$$

by (1.2). Thus, these relative invariants vanish if $n=1,2$ and the total genus of the vertices in Γ_X is less than g. This happens in particular if $g_v>0$ for any $v\in\Gamma_V$. Along with the paragraph containing (3.12), this establishes the $n=1,2$ cases of the last claim of Theorem 1.

Suppose $g_v\ge 2$ for some $v\in\Gamma_V$ and $n=3$. The dimensions of the two moduli spaces in (3.15) are then the same. The relative invariants of (X,V) that enter into the symplectic sum formula thus count curves that meet V at finitely many distinct points $\{p_j\}$. Since the obstruction bundle for $\overline{\mathfrak{M}}_{g_v,0}(V,0)$ is given by (3.3), the homology class of the subspace of elements of $\overline{\mathfrak{M}}_{g_v,0}(V,0;J_V,\nu)$ that pass through p_j

$$\big[\overline{\mathfrak{M}}_{g_v,0}(V,0;J_V,\nu)|_{p_j}\big]=e\big(\mathbb{E}^*\otimes T_{p_j}V\big)\cap\big[\overline{\mathcal{M}}_{g_v,0}\times\{p_j\}\big]=\lambda_{g_v}^2\cap\big[\overline{\mathcal{M}}_{g_v,0}\big]=0;$$

see (3.4). Thus, by (3.13), the genus g_v relative invariants of $(\mathbb{P}_XV,\mathbb{P}_{X,\infty}V)$ with a relative point insertion vanish in this case as well.

The remaining case of Theorem 1 is $\kappa=1$, $n=4$, and $g=2$. Since $A\neq 0$ in this case, $d_v,\ell(\mathbf{s}_v)\in\mathbb{Z}^+$. For a generic $\nu\in\Gamma_{2,0}(V,J_V)$, the target in (3.13) is a finite set of points, while the dimension of the fiber is

$$d_v+1-g_v+k_v+\ell(\mathbf{s}_v)\ge 1-1+0+1=1.$$

Thus, the genus 2 relative invariants of $(\mathbb{P}_XV,\mathbb{P}_{X,\infty}V)$ with only primary insertions from V vanish. This concludes the proof of the last claim of Theorem 1.

3.3 Extension to virtual cycles

In the process of establishing the first claim of Theorem 1 above, we showed that the relative invariants in the fiber classes of $\mathbb{P}^1$-bundles often vanish. This, more technical, conclusion is summarized, in Lemma 3.2 below. It leads to a version of Theorem 1 for virtual moduli cycles; see Corollary 3.3.

Let (V,ω) be a compact symplectic manifold, $\pi_L\colon L\longrightarrow V$ be a complex line bundle, and

$$\pi_{L,V}\colon \mathbb{P}_L\equiv\mathbb{P}(L\oplus V\times\mathbb{C})\longrightarrow V$$

be the bundle projection map. Given a Hermitian metric ρ (square of the norm) and a ρ-compatible connection ∇ in L, let α denote the connection 1-form on the ρ-circle bundle in L and its extension to $L-V$ via the retraction given by $v\longrightarrow v/|v|$. The closed 2-form

$$\widetilde{\omega}\equiv\pi_{X,V}^*\omega-\epsilon\,\mathrm{d}\bigg(\frac{\alpha}{1+\rho^2}\bigg)$$

on $L-V$ extends to a closed 2-form on $\mathbb{P}_L$, which is symplectic if $\epsilon>0$ is sufficiently small; we will take the symplectic deformation equivalence class of this form to be the default one. Let

$$\mathbb{P}_{L,\infty}=\mathbb{P}(L\oplus 0)\subset\mathbb{P}_L\,.$$

The projection map

$$\varphi\equiv \mathrm{st}\times\pi_{L,V}\colon \overline{\mathfrak{M}}_{g,k;\mathbf{s}}^{\mathbb{P}_{L,\infty}}(\mathbb{P}_L,dF)\longrightarrow\overline{\mathcal{M}}_{g,k+\ell(\mathbf{s})}\times V,$$

where $F\in H_2(\mathbb{P}_L;\mathbb{Z})$ is the fiber class, induces a push-forward on the virtual class:

$$\varphi_*\big[\overline{\mathfrak{M}}_{g,k;\mathbf{s}}^{\mathbb{P}_{L,\infty}}(\mathbb{P}_L,dF)\big]^{\mathrm{vir}}\in H_*\big(\overline{\mathcal{M}}_{g,k+\ell(\mathbf{s})}\times V\big).$$

By the Poincare duality applied on $\overline{\mathcal{M}}_{g,k+\ell(\mathbf{s})}\times V$, this push-forward is determined by the evaluation of cohomology classes pulled back from $\overline{\mathcal{M}}_{g,k+\ell(\mathbf{s})}\times V$ by φ on the virtual class of $\overline{\mathfrak{M}}_{g,k;\mathbf{s}}^{\mathbb{P}_{L,\infty}}(\mathbb{P}_L,dF)$. Thus, Section 3.2 establishes the following statement.

Lemma 3.2. *Let (V,ω) be a compact symplectic manifold of real dimension $2(n-1)$ and $L\longrightarrow V$ be a complex line bundle. If $g,d,k\in\mathbb{Z}^{\ge0}$ and $\mathbf{s}\in(\mathbb{Z}^+)^\ell$ are such that*

$$(g,d)\neq(1,0)\qquad\textit{and}\qquad(n-5)g(g-1)\ge0\,,\tag{3.16}$$

then

$$\varphi_*\big[\overline{\mathfrak{M}}_{g,k;\mathbf{s}}^{\mathbb{P}_{L,\infty}}(\mathbb{P}_L,dF)\big]^{\mathrm{vir}}=\begin{cases}[V], & \textit{if } (g,d,k,\mathbf{s})=(0,1,0,(1));\\ 0, & \textit{otherwise.}\end{cases}$$

Corollary 3.3 (D. Maulik)**.** *Suppose (X,ω) is a projective manifold of real dimension $2n$, $g,k\in\mathbb{Z}^{\ge0}$, $A\in H_2(X;\mathbb{Z})$, and $V\subset X$ is a (g,A)-hollow projective hypersurface such that $A\cdot V\ge0$. If*

$$(g,A)\neq(1,0)\qquad\textit{and}\qquad(n-5)g(g-1)\ge0\,,\tag{3.17}$$

then

$$\big[\overline{\mathfrak{M}}_{g,k}(X,A)\big]^{\mathrm{vir}}=\frac{1}{(A\cdot V)!}\,f_*\big[\overline{\mathfrak{M}}_{g,k;\mathbf{1}_{A\cdot V}}(X,A)\big]^{\mathrm{vir}}\,,\tag{3.18}$$

where f is the morphism between the moduli spaces dropping the relative marked points.

Proof. Let $\Delta\subset\mathbb{C}$ denote a small disk around the origin, $\mathcal{Z}$ be the blowup of $\Delta\times X$ along $0\times V$, and $\pi\colon\mathcal{Z}\longrightarrow\Delta$ be the projection map. Thus,

$$\mathcal{Z}_\lambda = X \quad \forall\,\lambda\in\Delta^*\equiv\Delta-0 \qquad\text{and}\qquad \mathcal{Z}_0\equiv\pi^{-1}(0)=X_1^V\,,$$

with notation as in (3.8).

As summarized in [19, Section 0], there are moduli stacks $\overline{\mathfrak{M}}_{g,k}(X_1^V,A)$ and $\overline{\mathfrak{M}}_{g,k}(\mathcal{Z},A)$. The former carries a virtual class so that

$$\big[\overline{\mathfrak{M}}_{g,k}(X_1^V,A)\big]^{\text{vir}} = \big[\overline{\mathfrak{M}}_{g,k}(\mathcal{Z}_\lambda,A)\big]^{\text{vir}} = \big[\overline{\mathfrak{M}}_{g,k}(X,A)\big]^{\text{vir}} \tag{3.19}$$

under the inclusion into $\overline{\mathfrak{M}}_{g,k}(\mathcal{Z},A)$. In the case of the given family $\mathcal{Z}\longrightarrow\Delta$, (3.19) can be written as

$$q_*\big[\overline{\mathfrak{M}}_{g,k}(X_1^V,A)\big]^{\text{vir}} = \big[\overline{\mathfrak{M}}_{g,k}(X,A)\big]^{\text{vir}}\,, \tag{3.20}$$

with q as in (3.9). By the last formula on page 201 in [19],

$$\big[\overline{\mathfrak{M}}_{g,k}(X_1^V,A)\big]^{\text{vir}} = \sum_{\Gamma}\frac{\mathbf{m}(\Gamma)}{\ell(\Gamma)!}\Phi_{\Gamma*}\Delta^!\big(\big[\overline{\mathfrak{M}}(X,\Gamma_X)\big]^{\text{vir}}\times\big[\overline{\mathfrak{M}}(\mathbb{P}_XV,\Gamma_V)^{\text{vir}}\big]\big). \tag{3.21}$$

This sum is taken over the same bipartite graphs Γ as in Section 3.2. For such a graph Γ, $\mathbf{m}(\Gamma)$ is the product of the edge labels (of contacts with the common divisor V) and $\ell(\Gamma)$ is the number of edges (of contacts with V). In the notation of Section 3.2, the two moduli spaces appearing on the right-hand side of (3.21) are

$$\prod_{v\in\Gamma_X}\overline{\mathfrak{M}}^V_{g_v,k_v;\mathbf{s}_v}(X,A_v) \qquad\text{and}\qquad \prod_{v\in\Gamma_V}\overline{\mathfrak{M}}^{\mathbb{P}_{X,\infty}V}_{g_v,k_v;\mathbf{s}_v}\big(\mathbb{P}_XV,A_v(\mathbf{s}_v)\big)\,,$$

respectively. The symbol $\Delta^!$ indicates the cap product with the product over the edges of Γ of the pull-back of the diagonal $\Delta_V\subset V^2$ by the evaluation maps at the relative marked points corresponding to the same edge, while Φ_Γ is the morphism given by identifying these marked points.

Since V is (g,A)-hollow, the only possible nonzero summands in (3.21) correspond to Γ with $A_v=0$ for all $v\in\Gamma_V$. For such Γ, the relative evaluation maps are given by the composition with the projection map $\pi_{X,V}\colon\mathbb{P}_XV\longrightarrow V$, while $q\circ\Phi_\Gamma$ factors through $\text{id}\times\varphi$. Combining (3.20) and (3.21), we thus obtain

$$\big[\overline{\mathfrak{M}}_{g,k}(X,A)\big]^{\text{vir}} = \sum_{\Gamma}\frac{\mathbf{m}(\Gamma)}{\ell(\Gamma)!}\Phi_{\Gamma*}\Delta^!\big(\big[\overline{\mathfrak{M}}(X,\Gamma_X)\big]^{\text{vir}}\times\varphi_*\big[\overline{\mathfrak{M}}(\mathbb{P}_XV,\Gamma_V)^{\text{vir}}\big]\big), \tag{3.22}$$

with the sum now taken over bipartite graphs Γ as in Section 3.2 with $A_v=0$ for all $v\in\Gamma_V$. For such graphs Γ, the restrictions (3.17) imply the restrictions (3.16) for all $(g,d)=(g_v,d_v)$ with $v\in\Gamma_V$. By Lemma 3.2, the last term in (3.22) thus vanishes except for the basic graph Γ with $|\Gamma_X|=1$, $(g_v,A_v,k_v)=(0,0,0)$ for all $v\in\Gamma_V$, and all edge labels equal to 1, i.e. as in the first diagram in Figure 4. The summand in (3.22) corresponding to this basic graph gives (3.18). $\square$

Remark 3.4. The equality (3.19) is established in [19] for a general flat degeneration $\pi\colon\mathcal{Z}\longrightarrow\Delta$, with $\mathcal{Z}_0$ consisting of two smooth varieties joined along a smooth hypersurface, only after summing over all classes A of the same degree with respect to an ample line bundle over $\mathcal{Z}$. However, in the given case, the relevant summands on the two sides of (3.19) lie in different spaces and thus must be equal pairwise.

Remark 3.5. The conclusion of Corollary 3.3 should apply to any compact symplectic manifold (X,ω) and (g,A)-hollow symplectic hypersurface V. Unfortunately, the above proof of Corollary 3.3 makes use of the symplectic sum (degeneration) formula for virtual fundamental cycles (not just numbers) in GW-theory, which is not even claimed in the symplectic category in any work we are aware of. In particular, [14] is concerned only with equating GW-invariants (viewed as numbers), contrary to the claim just above [15, (11.4)].

Remark 3.6. The crucial step in our proof of Theorem 1 and Lemma 3.2 is that we start by taking a generic regularization for the maps to V, i.e. a horizontal deformation of the parameters (J,ν) along V, before deforming the parameter ν in the normal direction to V. The order of the deformations is reversed in [15, Section 12], which makes the horizontal directions not even defined and crucially misses out the opportunity to quickly settle most cases of Theorem 1. The argument in [15, Section 11] instead misinterprets [6, (9)] to arrive at the conclusion of Lemma 3.2 in Section 3.3 without the restrictions in (3.16) and the conclusion of Corollary 3.3 without the restrictions in (3.17) or the projective assumptions on X and A.

4 Details on the counter-examples

In Sections 4.1-4.3, we establish the claims made in Examples 1-3, respectively; see Section 1. In the case of Example 1, we give two computations of the relative invariants. In the cases of Examples 2 and 3, we include localization computations of the $\delta=0,1$ numbers as consistency checks; the localization computations for Example 3 are separated off into Section 4.4.

In Sections 4.2-4.4, we use some degree 1 relative GW-invariants of $(\mathbb{P}^1,\infty)$ and rubber relative invariants of $(\mathbb{P}^1,\infty,0)$ with respect to the standard $\mathbb{C}^*$-action. In principle, all such invariants are computed in [6, 28]. As it is not completely trivial to extract actual numbers from the generating series in [6, 28], we include alternative computations for the few numbers relevant to our purposes.

4.1 Genus 1 degree 0 invariants

Let (X,ω,V) be as in Example 1. Fix an ω-tame almost complex structure J on X so that $J(TV)=TV$ and the Nijenhuis condition (2.11) holds.

For a complex structure $\mathfrak{j}$ on a smooth 1-marked genus 1 Riemann surface (Σ,x_1), the space of degree 0 holomorphic maps $u:\Sigma\longrightarrow X$ consists of the constant maps and so is canonically isomorphic to X. The obstruction bundle (i.e. the bundle of the cokernels of the linearizations $D_{J,u}$ of the $\bar\partial_{J,\mathfrak{j}}$-operator at u) is isomorphic to $\mathcal{H}^{0,1}_{\mathfrak{j}}\otimes_{\mathbb{C}}TX$, where $\mathcal{H}^{0,1}_{\mathfrak{j}}$ is the complex one-dimensional space of anti-holomorphic one-forms on Σ. Thus,

$$TX\approx \mathrm{Obs}\longrightarrow \mathrm{Hol}_{\mathfrak{j}}(X,0)\approx X. \tag{4.1}$$

By definition, the 1-marked genus 1 degree 0 fixed $\mathfrak{j}$ absolute GW-invariant with primary insertion $1\in H^*(X)$ is the (signed) number of solutions $u:\Sigma\longrightarrow X$ of

$$\bar\partial_{J,\mathfrak{j}}u\big|_z=\nu\big(z,u(z)\big)\quad\forall\, z\in\Sigma,\qquad u_*[\Sigma]=0\in H_2(X;\mathbb{Z}), \tag{4.2}$$

for a generic element

$$\nu\in\Gamma_{\mathfrak{j}}(X,J)\equiv\Gamma\big(\Sigma\times X,(T^*\Sigma)^{0,1}\otimes_{\mathbb{C}}TX\big).$$

The projection $\bar{\nu}$ of this element to the cokernel of $D_{J,u}$ for each $u\in\mathrm{Hol}_{\mathfrak{j}}(X,0)$ induces a transverse section of the obstruction bundle (4.1). The solutions of (4.2) correspond to the zeros of $\bar{\nu}$, as the obstruction to solving (4.2) vanishes at these points. Thus, the number of solutions of (4.2) is

$$\big\langle e(\mathrm{Obs}),\mathrm{Hol}_{\mathfrak{j}}(X,0)\big\rangle = \big\langle c_n(TX),X\big\rangle = \chi(X).$$

If $\mathfrak{j}\in H^2(\overline{\mathcal{M}}_{1,1})$ is the Poincare dual of a generic element, the absolute GW-invariant $\mathrm{GW}^X_{1,0}(\mathfrak{j};1)$ is half this number, because the group of automorphisms of a generic element of $\overline{\mathcal{M}}_{1,1}$ is $\mathbb{Z}_2$. This establishes the first equality in (1.11).

For maps of degree $A=0$, $A\cdot V=0$ and so the only compatible contact vector is the length 0 vector, which we denote by (). By definition, the 1-marked genus 1 degree 0 fixed $\mathfrak{j}$ GW-invariant of (X,V) with contact vector () and primary insertion $1\in H^*(X)$ is the number of solutions $u\colon\Sigma\longrightarrow X$ of

$$\bar\partial_{J,\mathfrak{j}}u\big|_z = \nu\big(z,u(z)\big) \quad \forall\ z\in\Sigma, \qquad u_*[\Sigma]=0\in H_2(X;\mathbb{Z}), \qquad u(\Sigma)\not\subset V, \tag{4.3}$$

for a generic $\nu\in\Gamma^V_{\mathfrak{j}}(X,J)$, where

$$\Gamma^V_{\mathfrak{j}}(X,J)\subset\Gamma_{\mathfrak{j}}(X,J)$$

is the subspace of elements ν such that

$$\nu|_{\Sigma\times V}\in\Gamma_{\mathfrak{j}}(V,J|_V), \quad \widetilde\nabla_w\nu+J\widetilde\nabla_{Jw}\nu\in(T^*\Sigma)^{0,1}\otimes_{\mathbb{C}}T_xV \quad \forall\ w\in T_xX,\ x\in V.$$

By the first assumption above, the number of maps $u:\Sigma\longrightarrow V\subset X$ that satisfy the first two conditions in (4.3) and fail the third is $\chi(V)$. The total number of maps $u:\Sigma\longrightarrow X$ that satisfy the first two conditions in (4.3) is $\chi(X)$, as in the previous paragraph. Thus, the number of solutions of (4.3) is $\chi(X)-\chi(V)$. Similarly to the previous paragraph, the relative GW-invariant $\mathrm{GW}^{X,V}_{1,0;()}(\mathfrak{j};1)$ is half this number. This establishes the second equality in (1.11).

With α as in (1.12), let $Y\subset X$ be a generic representative of the Poincare dual of α. Since every degree 0 J-holomorphic map is constant,

$$\overline{\mathfrak{M}}_{1,1}(X,0)=\overline{\mathcal{M}}_{1,1}\times X.$$

Similarly to the previously case, the obstruction bundle in this case is isomorphic to

$$\mathrm{Obs}=\pi_1^*\mathbb{E}^*\otimes\pi_2^*TX\longrightarrow\overline{\mathcal{M}}_{1,1}\times X, \tag{4.4}$$

where $\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{1,1}$ is the Hodge line bundle of holomorphic differentials. Its first chern class, $\lambda\equiv c_1(\mathbb{E})$, satisfies

$$\big\langle\lambda,\overline{\mathcal{M}}_{1,1}\big\rangle=\frac{1}{24}\,. \tag{4.5}$$

By definition, the 1-marked genus 1 degree 0 absolute GW-invariant with primary insertion α is the (signed) number of solutions $u\colon(\Sigma,\mathfrak{j})\longrightarrow X$ of

$$\bar\partial_{J,\mathfrak{j}}u\big|_z = \nu\big(z,u(z)\big) \quad \forall\ z\in\Sigma, \qquad u_*[\Sigma]=0\in H_2(X;\mathbb{Z}), \qquad u(x_1)\in Y, \tag{4.6}$$

where Σ is a smooth torus, $x_1\in\Sigma$ is the marked point, ν is a generic element of

$$\Gamma_{1,1}(X,J)\equiv\Gamma\big(\mathcal{U}_{1,1}\times X,(T^*\mathcal{U}_{1,1})^{0,1}\otimes_{\mathbb{C}}TX\big),$$

and $\mathcal{U}_{1,1} \longrightarrow \overline{\mathcal{M}}_{1,1}$ is the universal curve. Similarly to the case considered above, ν induces a transverse section $\bar{\nu}$ of the obstruction bundle (4.4). The solutions of (4.6) correspond to the zeros of $\bar{\nu}$ with $u(x_1)\in Y$. Thus,

$$\begin{aligned}\mathrm{GW}_{1,0}^X(\alpha) &= \big\langle e(\mathrm{Obs}), \overline{\mathcal{M}}_{1,1}\times Y\big\rangle = -\big\langle \lambda, \overline{\mathcal{M}}_{1,1}\big\rangle\big\langle c_{n-1}(X), Y\big\rangle \\ &= -\frac{1}{24}\big\langle \alpha\, c_{n-1}(X), X\big\rangle\,.\end{aligned} \tag{4.7}$$

This establishes the first equality in (1.12).

By definition, the 1-marked genus 1 degree 0 GW-invariant relative to V with contact vector () and primary insertion α is the number of solutions $u\colon(\Sigma,\mathfrak{j})\longrightarrow X$ of

$$\bar{\partial}_{J,\mathfrak{j}}u\big|_z = \nu\big(z,u(z)\big) \quad \forall\, z\in\Sigma, \quad u_*[\Sigma]=0\in H_2(X;\mathbb{Z}), \quad u(x_1)\in Y, \quad u(\Sigma)\not\subset V, \tag{4.8}$$

for a generic $\nu\in\Gamma_{1,1}^V(X,J)$, where $\Gamma_{1,1}^V(X,J)\subset\Gamma_{1,1}(X,J)$ is the subspace of elements ν satisfying (2.14). By the first assumption in (2.14) and previous paragraph, the number of maps $u\colon\Sigma\longrightarrow V\subset X$ that satisfy the first three conditions in (4.8) and fail the fourth is

$$\mathrm{GW}_{1,0}^V(\mathrm{PD}_V(V\cap Y)) = -\frac{1}{24}\big\langle \mathrm{PD}_V(V\cap Y)\, c_{n-2}(Y), Y\big\rangle = -\frac{1}{24}\big\langle \alpha|_V\, c_{n-2}(Y), Y\big\rangle\,. \tag{4.9}$$

Since the total number of maps $u:\Sigma\longrightarrow X$ that satisfy the first three conditions in (4.8) is $\mathrm{GW}_{1,0}^X(\alpha)$, $\mathrm{GW}_{1,0;()}^{X,V}(\alpha)$ is the difference of (4.7) and (4.9), as claimed in the second equality in (1.12).

Remark 4.1. Strictly speaking, the arguments in the last two paragraphs should be applied to the universal curve $\breve{\mathcal{U}}_{1,1}$ over the moduli space $\widetilde{\mathcal{M}}_{1,1}$ of 1-marked genus 1 curves with Prym structures in place of $\mathcal{U}_{1,1}$ and the resulting numbers should then be divided by the order of the covering (2.1) with $(g,k)=(1,1)$. This nuance is taken into account by $\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{1,1}$ being a line orbi-bundle over an orbifold with the chern class given by (4.5).

The absolute invariant $\mathrm{GW}_{1,0}^X(\mathfrak{j};1)$ can also be computed using the same framework as $\mathrm{GW}_{1,0}^X(\alpha)$. If $\sigma\in\overline{\mathcal{M}}_{1,1}$ represents the Poincare dual of $\mathfrak{j}$, (4.7) becomes

$$\mathrm{GW}_{1,0}^X(\mathfrak{j};1) = \big\langle e(\mathrm{Obs}), [\sigma]\times X\big\rangle = \big\langle 1,[\sigma]\big\rangle\big\langle c_n(X), X\big\rangle = \frac{1}{2}\chi(X).$$

Below we recall a similar framework for computing the relative invariants in the algebraic category, based on [20, Section 8], and note that it applies equally well in the symplectic category.

If X is a complex manifold and $V\subset X$ is a complex hypersurface, the sheaf $\mathcal{O}_X(TX)$ of holomorphic vector fields contains the subsheaf $\mathcal{O}_X(TX(-\log V))$ of vector fields with values in TV along V. If $(z_1,\ldots,z_n)$ is a coordinate chart on $U\subset X$ such that $U\cap V$ is the slice $z_n=0$, $\mathcal{O}_U(TX(-\log V))$ is freely generated by the vector fields

$$\frac{\partial}{\partial z_1},\ \ldots,\ \frac{\partial}{\partial z_{n-1}},\ z_n\frac{\partial}{\partial z_n}\,.$$

Thus, $\mathcal{O}_X(TX(-\log V))$ is a locally free sheaf of rank n, i.e. the sheaf of a holomorphic sections of a holomorphic vector bundle $TX(-\log V)$ of rank n. In the symplectic category, such a vector bundle can be constructed using the Symplectic Neighborhood Theorem [25, Theorem 3.30]; the resulting complex vector bundle is well-defined up to equivalence by the deformation equivalence class of ω as a symplectic form on $X\supset V$.

Lemma 4.2. *Suppose (X,ω) is a compact symplectic manifold of real dimension $2n$ and $V\subset X$ is a compact symplectic hypersurface. If $\alpha\in H^{2(n-k)}(X)$, then*

$$\big\langle \alpha\, c_k(TX(-\log V)),X\big\rangle = \big\langle \alpha\, c_k(X),X\big\rangle - \big\langle \alpha\, c_{k-1}(V),V\big\rangle\,.$$

Proof. By definition, there is a short exact sequence of sheaves

$$0\longrightarrow \mathcal{O}_X(TX(-\log V))\longrightarrow \mathcal{O}_X(TX)\longrightarrow \mathcal{O}_V(V)\longrightarrow 0, \tag{4.10}$$

where the second non-trivial homomorphism is the restriction to V followed by the projection to the normal bundle $\mathcal{N}_XV$, which equals to the restriction of the line bundle $\mathcal{O}_X(V)$ to V. Combining (4.10) with the short exact sequence

$$0\longrightarrow \mathcal{O}_X\longrightarrow \mathcal{O}_X(V)\longrightarrow \mathcal{O}_V(V)\longrightarrow 0,$$

we find that

$$\begin{aligned} c\big(\mathcal{O}_X(TX(-\log V))\big) &= c\big(\mathcal{O}_X(TX)\big)\, c\big(\mathcal{O}_V(V)\big)^{-1} = c\big(\mathcal{O}_X(TX)\big)\,\big(c(\mathcal{O}_X(V))c(\mathcal{O}_X)^{-1}\big)^{-1}\\ &= c(X)\,\big(1+\mathrm{PD}_XV\big)^{-1}\,. \end{aligned}$$

Thus,

$$\begin{aligned} \big\langle \alpha\, c_k(TX(-\log V)),X\big\rangle &= \big\langle \alpha\, c_k(X),X\big\rangle - \sum_{i=0}^{k-1}(-1)^i\big\langle \alpha\, c_{k-1-i}(X)\,(\mathrm{PD}_XV)^{1+i},X\big\rangle\\ &= \big\langle \alpha\, c_k(X),X\big\rangle - \sum_{i=0}^{k-1}(-1)^i\big\langle \alpha\, c_{k-1-i}(X)\,c_1(\mathcal{N}_XV)^{i},V\big\rangle\,. \end{aligned} \tag{4.11}$$

Since $c(V)=c(TX)|_V c(\mathcal{N}_XV)^{-1}$, the claim follows from (4.11). □

In the projective setting, the analogue of the obstruction bundle (4.4) for the relative moduli space is

$$\mathrm{Obs}^V = \pi_1^*\mathbb{E}^*\otimes\pi_2^*TX(-\log V)\longrightarrow \overline{\mathcal{M}}_{1,1}\times X\,;$$

see [20, Section 8]. In the symplectic setting, the substance of the first restriction in (2.14) is that ν induces a section of Obs^V. If ν is generic, subject to the conditions in (2.14), this section is transverse to the zero set everywhere and when restricted to $\overline{\mathcal{M}}_{1,1}\times V$. Thus, it has no zeros along V and the two relative invariants in Example 1 are given by

$$\begin{aligned} \mathrm{GW}^{X,V}_{1,0;()}(\mathrm{j};1) &= \big\langle e(\mathrm{Obs}^V),[\sigma]\times X\big\rangle = \big\langle 1,[\sigma]\big\rangle\big\langle c_n(TX(-\log V)),X\big\rangle\,,\\ \mathrm{GW}^{X,V}_{1,0;()}(\alpha) &= \big\langle e(\mathrm{Obs}^V),\overline{\mathcal{M}}_{1,1}\times Y\big\rangle = -\big\langle \lambda,\overline{\mathcal{M}}_{1,1}\big\rangle\big\langle c_{n-1}(TX(-\log V)),Y\big\rangle. \end{aligned}$$

The second equalities in (1.11) and (1.12) now follow from Lemma 4.2 and (4.5).

Finally, we note that (1.11) and (1.12) are consistent with the symplectic sum formula as stated in the second-to-last equation on page 201 in [19] and applied to the decomposition (3.7). Since the degree $A=0$ in this case, there are only two types of bipartite graphs Γ as in Section 3.2 to

Figure 5: The two bipartite graphs Γ contributing to the genus 1 degree 0 GW-invariants of X via the symplectic sum formula applied to (3.7).

sum over: the two possible one-vertex graphs; see Figure 5. The symplectic sum formula in these cases gives

$$\mathrm{GW}_{1,0}^{X}(\mathrm{j};1) = \mathrm{GW}_{1,0;()}^{X,V}(\mathrm{j};1) + \mathrm{GW}_{1,0;()}^{\mathbb{P}_X V,\mathbb{P}_{X,\infty}V}(\mathrm{j};1), \tag{4.12}$$

$$\mathrm{GW}_{1,0}^{X}(\alpha) = \mathrm{GW}_{1,0;()}^{X,V}(\alpha) + \mathrm{GW}_{1,0;()}^{\mathbb{P}_X V,\mathbb{P}_{X,\infty}V}\big(\pi_{X,V}^*(\alpha|_V)\big), \tag{4.13}$$

with $\mathbb{P}_{X,\infty}V\subset\mathbb{P}_X V$ and $\pi_{X,V}:\mathbb{P}_{X,\infty}V\longrightarrow V$ as in (2.5) and (2.6). According to the second equality in (1.11),

$$\begin{aligned}\mathrm{GW}_{1,0;()}^{X,V}(\mathrm{j};1) &= \frac{\chi(X)-\chi(V)}{2},\\ \mathrm{GW}_{1,0;()}^{\mathbb{P}_X V,\mathbb{P}_{X,\infty}V}(\mathrm{j};1) &= \frac{\chi(\mathbb{P}_X V)-\chi(\mathbb{P}_{X,\infty}V)}{2} = \frac{2\chi(V)-\chi(V)}{2}\,.\end{aligned}$$

Thus, (4.12) is consistent with the first equality in (1.11). According to the second equality in (1.12),

$$\begin{aligned}\mathrm{GW}_{1,0;()}^{X,V}(\alpha) &= -\frac{\langle\alpha\, c_{n-1}(X),X\rangle-\langle\alpha|_V\, c_{n-2}(V),V\rangle}{24},\\ \mathrm{GW}_{1,0;()}^{\mathbb{P}_X V,\mathbb{P}_{X,\infty}V}\big(\pi_{X,V}^*(\alpha|_V)\big) &= -\frac{\langle\pi_{X,V}^*(\alpha|_V)\, c_{n-1}(\mathbb{P}_X V),\mathbb{P}_X V\rangle-\langle\alpha|_V\, c_{n-2}(V),V\rangle}{24}\\ &= -\frac{2\,\langle\alpha|_V\, c_{n-2}(V),V\rangle-\langle\alpha|_V\, c_{n-2}(V),V\rangle}{24}\,.\end{aligned}$$

Thus, (4.13) is consistent with the first equality in (1.12).

4.2 Genus 2 degree 1 invariants of $\mathbb{P}^1$

We establish the second equality in (1.13) by applying the symplectic sum formula, as stated in the second-to-last equation on page 201 in [19], to the absolute GW-invariant in (1.13) via the decomposition (3.7) with

$$X=\mathbb{P}^1, \qquad V=V_\delta\equiv\{p_1,\ldots,p_\delta\}, \qquad \mathbb{P}_X V=\{1,\ldots,\delta\}\times\mathbb{P}^1\,.$$

We will make use of some top intersection numbers on the Deligne-Mumford spaces $\overline{\mathcal{M}}_2$ and $\overline{\mathcal{M}}_{2,1}$, as summarized in Tables 1 and 2. The numbers for $\overline{\mathcal{M}}_3$ and $\overline{\mathcal{M}}_{3,1}$, appearing in Tables 1 and 3, will be used in Sections 4.3 and 4.4. These numbers can be obtained from C. Faber's computer program, which is described in [5].

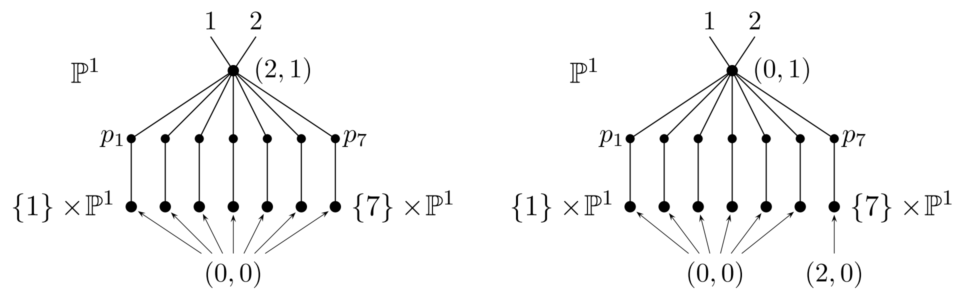

Figure 6: Bipartite graphs Γ that contribute to the absolute GW-invariant in (1.13) via the symplectic sum decomposition with respect to $V_7=\{p_1,\ldots,p_7\}$.

Since the Poincare duals of the two primary insertions in (1.13) vanish on the divisor V_δ (the constraining points can be chosen to be distinct from the δ points in V_δ), $k_v=0$ for all $v\in\Gamma_V$ (the marked points stay on the X-side) if Γ is a bipartite graph as in Section 3.2 contributing to the absolute GW-invariant in (1.13). Furthermore, $A_v=1$ for the unique vertex $v\in\Gamma_X$ and all edge labels are 1 in this case (because the curve on the X-side is of degree 1 and so meets each point in the divisor with order 1). By Section 3.2 or Lemma 3.2 (separately), there are thus only two types of graphs Γ contributing to the absolute GW-invariant in (1.13):

(1) $(g_v,A_v,k_v)=(0,0,0)$ for all $v\in\Gamma_V$ and

(2) the δ graphs with $(g_v,A_v,k_v)=(2,0,0)$ for one element $v\in\Gamma_V$ and $(g_v,A_v,k_v)=(0,0,0)$ for the remaining $\delta-1$ elements $v\in\Gamma_V$;

see Figure 6. There are other bipartite graphs Γ, but they all contain a vertex $v\in\Gamma_V$ with $g_v=1$; by Section 3.2 or Lemma 3.2, such a graph does not contribute to an absolute GW-invariant via the symplectic sum formula. In the setup of Section 3.1, such graphs correspond to configurations with genus 1 components sinking into the divisor; since there are no higher-genus components sinking into the divisor in this case, the argument in Section 3.1 also implies that such a configuration does not contribute.

Thus, by the symplectic sum formula,

$$\mathrm{GW}_{2,1}^{\mathbb{P}^1}(\kappa^4;\mathrm{pt},\mathrm{pt})=\frac{1}{\delta!}\mathrm{GW}_{2,1;\mathbf{1}_\delta}^{\mathbb{P}^1,V_\delta}(\kappa^4;\mathrm{pt},\mathrm{pt})+\frac{\delta}{\delta!}\sum_i\mathrm{GW}_{0,1;\mathbf{1}_\delta}^{\mathbb{P}^1,V_\delta}(\kappa_i;\mathrm{pt},\mathrm{pt})\mathrm{GW}_{2,1;(1)}^{\mathbb{P}^1,\mathrm{pt}}(\kappa_i';1)\,, \tag{4.14}$$

with $\kappa_i\in H^*(\overline{\mathcal{M}}_{0,2+\delta})$ and $\kappa_i'\in H^*(\overline{\mathcal{M}}_{2,1})$ given by

$$\mathrm{gl}^*\kappa^4=\sum_i\kappa_i\otimes\kappa_i'\in H^*(\overline{\mathcal{M}}_{0,2+\delta})\otimes H^*(\overline{\mathcal{M}}_{2,1})=H^*(\overline{\mathcal{M}}_{0,2+\delta}\times\overline{\mathcal{M}}_{2,1}),$$

where

$$\mathrm{gl}\colon\overline{\mathcal{M}}_{0,2+\delta}\times\overline{\mathcal{M}}_{2,1}\longrightarrow\overline{\mathcal{M}}_{2,2}\,,$$

is the morphism obtained by forgetting the last $\delta-1$ points on the genus 0 curve and identifying the marked point of the genus 2 curve with the third marked point on the genus 0 curve. Since κ is the Poincare dual of the divisor represented by the bottom right diagram in Figure 1, it follows that

$$\sum_i\kappa_i\otimes\kappa_i'\equiv\mathrm{gl}^*\kappa^4=1\otimes\psi_1^4, \tag{4.15}$$

λ_1^3	$\lambda_1\lambda_2$
$\frac{1}{2880}$	$\frac{1}{5760}$

λ_1^6	$\lambda_1^4\lambda_2$	$\lambda_1^3\lambda_3$	$\lambda_1^2\lambda_2^2$	$\lambda_1\lambda_2\lambda_3$	λ_2^3	λ_3^2
$\frac{1}{90720}$	$\frac{1}{181440}$	$\frac{1}{725760}$	$\frac{1}{362880}$	$\frac{1}{1451520}$	$\frac{1}{725760}$	0

Table 1: The top intersections of λ-classes on $\overline{\mathcal{M}}_2$ and $\overline{\mathcal{M}}_3$.

where $\psi_1 \in H^*(\overline{\mathcal{M}}_{2,1})$ is the chern class of the universal cotangent line bundle. By Theorem 1,

$$\frac{1}{\delta!}\mathrm{GW}^{\mathbb{P}^1,V_\delta}_{0,1;\mathbf{1}_\delta}(1;\mathrm{pt},\mathrm{pt}) \equiv \frac{1}{\delta!}\mathrm{GW}^{\mathbb{P}^1,V_\delta}_{0,1;\mathbf{1}_\delta}(\mathrm{pt},\mathrm{pt}) = \mathrm{GW}^{\mathbb{P}^1}_{0,1}(\mathrm{pt},\mathrm{pt}) = 1. \tag{4.16}$$

Combining (4.14) with (4.15), (4.16), and Lemma 4.3 below, we conclude that

$$\mathrm{GW}^{\mathbb{P}^1}_{2,1}(\kappa^4;\mathrm{pt},\mathrm{pt}) = \frac{1}{\delta!}\mathrm{GW}^{\mathbb{P}^1,V_\delta}_{2,1;\mathbf{1}_\delta}(\kappa^4;\mathrm{pt},\mathrm{pt}) + \delta\big\langle \psi_1^4, \overline{\mathcal{M}}_{2,1}\big\rangle\,.$$

The second equality in (1.13) now follows from the first column in Table 2.

Lemma 4.3 (C.-C. Liu). *If* $\mathrm{st}: \overline{\mathfrak{M}}^{\mathrm{pt}}_{2,0;(1)}(\mathbb{P}^1,1) \longrightarrow \overline{\mathcal{M}}_{2,1}$ *is the forgetful morphism dropping the map to* $\mathbb{P}^1$*, then*

$$\mathrm{st}_*\big[\overline{\mathfrak{M}}^{\mathrm{pt}}_{2,0;(1)}(\mathbb{P}^1,1)\big]^{\mathrm{vir}} = \big[\overline{\mathcal{M}}_{2,1}\big]. \tag{4.17}$$

Since $\overline{\mathcal{M}}_{2,1}$ is smooth (as an orbifold) and irreducible, (4.17) is equivalent to

$$\big\langle \mathrm{st}^*\sigma, [\overline{\mathfrak{M}}^{\mathrm{pt}}_{2,0;(1)}(\mathbb{P}^1,1)]^{\mathrm{vir}}\big\rangle = 1, \tag{4.18}$$

where $\sigma \in H^8(\overline{\mathcal{M}}_{2,1})$ is the Poincare dual of a generic element (Σ, x_1) of $\overline{\mathcal{M}}_{2,1}$. We give two proofs of (4.18) below. The first argument applies the virtual localization theorems of [9, 10] as in [11, Chapter 27]. The second proof applies the obstruction analysis of [31] as in [32, Section 4].

Proof 1 of (4.18). We use the standard $(\mathbb{C}^*)$-action on $\mathbb{P}^1$. It has two fixed points,

$$p_1 = [1,0] \qquad \text{and} \qquad p_2 = [0,1],$$

and lifts linearly to an action on $\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \mathbb{P}^1$. As in [11, Chapter 27], we let

$$\alpha_i = c_1\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)\big|_{p_i} \in H^*_{(\mathbb{C}^*)^2} \equiv H^*\big(B((\mathbb{C}^*)^2)\big) = H^*(\mathbb{P}^\infty \times \mathbb{P}^\infty) = \mathbb{C}[\alpha_1,\alpha_2]\,.$$

The fixed loci of the induced action on $\overline{\mathfrak{M}}^{p_2}_{2,0;(1)}(\mathbb{P}^1,1)$ consist of maps sending components of positive genus to either the fixed point p_1 or the rubber $\mathbb{P}^1$ attached to the fixed point p_2. The three

ψ_1^4	$\psi_1^3\lambda_1$	$\psi_1^2\lambda_1^2$	$\psi_1^2\lambda_2$	$\psi_1\lambda_1^3$	$\psi_1\lambda_1\lambda_2$
$\frac{1}{1152}$	$\frac{1}{480}$	$\frac{7}{2880}$	$\frac{7}{5760}$	$\frac{1}{1440}$	$\frac{1}{2880}$

Table 2: The top intersections of λ-classes and ψ_1 on $\overline{\mathcal{M}}_{2,1}$.

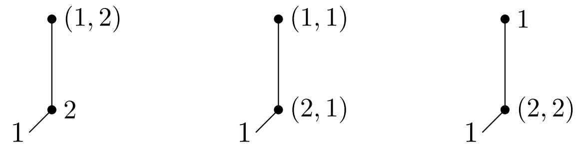

Figure 7: The three graphs describing the $(\mathbb{C}^*)^2$-fixed loci of $\overline{\mathfrak{M}}_{2,0;(1)}^{p_2}(\mathbb{P}^1, 1)$.

graphs describing these fixed loci in the notation of [11, Chapter 27] are shown in Figure 7. In these diagrams, the first vertex label indicates the corresponding fixed point of $\mathbb{P}^1$, while the second indicates the genus of the component taken there, if any. The edge degree is 1 in all cases, corresponding to the degree 1 cover from $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$.

The morphism st takes the fixed locus represented by the middle diagram in Figure 7 to the closure in $\overline{\mathcal{M}}_{2,1}$ of the locus consisting of two-component maps. Thus, $\mathrm{st}^*\sigma$ vanishes on this locus and the middle diagram does not contribute to (4.18) via the virtual localization theorem of [9].

The locus represented by the first diagram in Figure 7 is isomorphic to $\overline{\mathcal{M}}_{2,1}$ and is cut down by $\mathrm{st}^*\sigma$ to a single point. The space of deformations of this locus consists of moving the node and of smoothing the node; after restricting to the cut-down space, the equivariant chern class of both of these line bundles equal to the equivariant chern class of $T\mathbb{P}^1$ at p_1, which is $\alpha_1 - \alpha_2$ in this case; see [11, Exercise 27.1.3]. The obstruction bundle after cutting down by $\mathrm{st}^*\sigma$ is

$$H^1(\Sigma; T_{p_1}\mathbb{P}^1) = \big(H^0(\Sigma; T^*\Sigma \otimes T^*_{p_1}\mathbb{P}^1)\big)^* \approx T_{p_1}\mathbb{P}^1 \oplus T_{p_1}\mathbb{P}^1\,;$$

its equivariant euler class is $(\alpha_1 - \alpha_2)^2$. Thus, the contribution of the first diagram in Figure 7 to (4.18) is

$$\int_{\overline{\mathcal{M}}_{2,1}} \mathrm{st}^*\sigma \, \frac{(\alpha_1 - \alpha_2)^2}{(\alpha_1 - \alpha_2) \cdot (\alpha_1 - \alpha_2)} = 1\,;$$

see [9, (7)] or [10, Theorem 3.6].

The locus represented by the last diagram in Figure 7 is isomorphic to the (rubber) moduli space $\overline{\mathfrak{M}}_{2,0;(1),(1)}^{0,\infty}(\mathbb{P}^1, 1)_\sim$ of relative morphisms to the non-rigid target $(\mathbb{P}^1, 0, \infty)$ with the standard $\mathbb{C}^*$-action; see [10, Section 2.4]. Since the virtual dimension of this moduli space is 3, the restriction of $\mathrm{st}^*\sigma$ to this fixed locus vanishes. By [10, Theorem 3.6], the last diagram in Figure 7 thus does not contribute to (4.18). Combining this with the conclusion of the two previous paragraphs, we obtain (4.18). $\square$

Proof 2 of (4.18). Let $(\Sigma, \mathfrak{j}, x_1)$ be a generic element of $\overline{\mathcal{M}}_{2,1}$ as before. The number (4.18) is the number of solutions $u\colon \Sigma \longrightarrow \mathbb{P}^1$ of

$$\bar{\partial}_{J,\mathfrak{j}} u\big|_z = \nu\big(z, u(z)\big) \quad \forall\, z \in \Sigma, \qquad u_*[\Sigma] = [\mathbb{P}^1] \in H_2(\mathbb{P}^1; \mathbb{Z}), \tag{4.19}$$

for a generic $\nu \in \Gamma_{\mathfrak{j}}^{\mathrm{pt}}(\mathbb{P}^1, J)$, where

$$\Gamma_{\mathfrak{j}}^{\mathrm{pt}}(\mathbb{P}^1, J) \subset \Gamma\big(\Sigma \times \mathbb{P}^1, (T^*\Sigma)^{0,1} \otimes_{\mathbb{C}} T\mathbb{P}^1\big)$$

is the subspace of elements ν such that

$$\nu|_{\Sigma\times\mathrm{pt}}=0, \qquad \nabla_w\nu+J\nabla_{Jw}\nu=0 \quad \forall\, w\in T_{\mathrm{pt}}\mathbb{P}^1. \tag{4.20}$$

The moduli space of degree 1 holomorphic maps $(\Sigma,\mathfrak{j})\longrightarrow\mathbb{P}^1$ and its obstruction bundle are given by

$$\mathcal{H}^{0,1}_\Sigma\otimes T\mathbb{P}^1\approx\mathrm{Obs}\longrightarrow\mathrm{Hol}_{\mathfrak{j}}(\mathbb{P}^1,1)\approx\mathbb{P}^1\,, \tag{4.21}$$

where $\mathcal{H}^{0,1}_\Sigma$ is the space of harmonic $(0,1)$-forms on Σ.

The space of deformations of the domain of the elements in $\mathrm{Hol}_{\mathfrak{j}}(\mathbb{P}^1,1)$ is the product of the two tangent bundles at the node, i.e.

$$T_{x_1}\Sigma\otimes T\mathbb{P}^1\approx T\mathbb{P}^1\longrightarrow\mathbb{P}^1\,. \tag{4.22}$$

Each smoothing parameter υ in this line bundle determines an approximately $(J,\mathfrak{j})$-holomorphic map $u_\upsilon\colon\Sigma\longrightarrow\mathbb{P}^1$; see [31, Section 3.3]. The first-order term of the projection $\pi^{0,1}_{\upsilon,-}\bar\partial_{J,\mathfrak{j}}u_\upsilon$ of $\bar\partial_{J,\mathfrak{j}}u_\upsilon$ to Obs is given by

$$\big\{L(\upsilon_1\otimes\upsilon_2)\big\}(\psi)=\psi_{x_1}(\upsilon_1)\big\{\mathrm{d}_{x_2}u\big\}(\upsilon_2)\in T_{u(x_2)}\mathbb{P}^1 \qquad \forall\;\psi\in\mathcal{H}^{1,0}_\Sigma\,,$$

where $x_2\in\mathbb{P}^1$ is the node of the rational component of the domain of the map; see [32, Lemma 4.5]. Since L is injective in this case, the solutions of (4.19) correspond to the zeros of the section of

$$\mathrm{Obs}/\mathrm{Im}\,L\longrightarrow\mathrm{Hol}_{\mathfrak{j}}(\mathbb{P}^1,1) \tag{4.23}$$

induced by a generic ν, excluding the one with $u(x_2)=\mathrm{pt}$; see the proof of [32, Corollary 4.7]. Thus, the number of solutions of (4.19) is

$$\big\langle e(\mathrm{Obs}/\mathrm{Im}\,L),\mathrm{Hol}_{\mathfrak{j}}(\mathbb{P}^1,1)\big\rangle-1=\big\langle c_1(\mathcal{H}^{0,1}_\Sigma\otimes T\mathbb{P}^1)-c_1(T_{x_1}\Sigma\otimes T\mathbb{P}^1),\mathbb{P}^1\big\rangle-1=1.$$

This establishes (4.18). □

We next use the virtual localization theorem of [9] to compute the absolute invariant and the $\delta=1$ case of the relative invariant in Example 2. We continue with the localization setup of the first proof of (4.18) and compute

$$\begin{aligned}&\int_{[\overline{\mathfrak{M}}_{2,2}(\mathbb{P}^1,1)]^{\mathrm{vir}}}\mathrm{st}^*\kappa^4\,\mathrm{ev}_1^*\mathcal{O}_{\mathbb{P}^1}(1-\alpha_2)\,\mathrm{ev}_2^*\mathcal{O}_{\mathbb{P}^1}(1-\alpha_2) \qquad \text{and}\\ &\int_{[\overline{\mathfrak{M}}^{p_2}_{2,2;(1)}(\mathbb{P}^1,1)]^{\mathrm{vir}}}\mathrm{st}^*\kappa^4\,\mathrm{ev}_1^*\mathcal{O}_{\mathbb{P}^1}(1-\alpha_2)\,\mathrm{ev}_2^*\mathcal{O}_{\mathbb{P}^1}(1-\alpha_2)\,.\end{aligned} \tag{4.24}$$

The $(\mathbb{C}^*)^2$-fixed loci consist of maps sending the positive-genus components and the absolute marked points to the fixed points p_1 and p_2. Since the equivariant chern class of $\mathcal{O}_{\mathbb{P}^1}(1-\alpha_2)$ vanishes at p_2, the only graphs possibly contributing to the integrals in (4.24) must have both absolute marked points sent to p_1. Since the morphism st takes fixed loci with a positive-genus component at both fixed points to the closure in $\overline{\mathcal{M}}_{2,2}$ of the locus consisting of two genus 1 curves, $\mathrm{st}^*\kappa^4$ vanishes on such fixed loci as well. The two remaining graphs possibly contributing to each of the integrals in (4.24) are shown in Figure 8.

Figure 8: The two pairs of graphs possibly contributing to the two integrals in (4.24).

The locus represented by the first diagram in Figure 8 is isomorphic to $\overline{\mathcal{M}}_{2,3}$. The space of deformations of this locus consists of moving the node and of smoothing the node; the equivariant chern classes of these line bundles are $\alpha_1-\alpha_2$ and $\alpha_1-\alpha_2-\psi_3$, respectively. The euler class of the obstruction bundle is given by

$$e\big(\mathbb{E}^*\otimes T_{p_1}\mathbb{P}^1\big) = \lambda_2 - (\alpha_1-\alpha_2)\lambda_1 + (\alpha_1-\alpha_2)^2\,.$$

By [9, (7)], the contribution of the first diagram in Figure 8 to the first integral in (4.24) is thus

$$\begin{split}\int_{\overline{\mathcal{M}}_{2,3}} \mathrm{st}^*\kappa^4\,(\alpha_1-\alpha_2)^2\,\frac{\lambda_2-(\alpha_1-\alpha_2)\lambda_1+(\alpha_1-\alpha_2)^2}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_2-\psi_3)} &= \int_{\overline{\mathcal{M}}_{2,3}} \mathrm{st}^*\kappa^4\big(\lambda_2-\lambda_1\psi_3+\psi_3^2\big)\\ = 0+4\langle\psi_1^3\lambda_1,\overline{\mathcal{M}}_{2,1}\rangle - \langle (f^*\psi_1)^3\psi_2^2,\overline{\mathcal{M}}_{2,2}\rangle &= 4\langle\psi_1^3\lambda_1,\overline{\mathcal{M}}_{2,1}\rangle - \langle\psi_1^3\psi_2^2,\overline{\mathcal{M}}_{2,2}\rangle\,,\end{split} \tag{4.25}$$

where $f:\overline{\mathcal{M}}_{2,2}\longrightarrow\overline{\mathcal{M}}_{2,1}$ is the forgetful morphism. The second equality above applies the dilaton equation [11, Exercise 25.2.7] to the middle term, while the last equality follows from [11, Lemma 25.2.3].

The locus represented by the second diagram in Figure 8 is isomorphic to $\overline{\mathcal{M}}_{2,1}$. The space of deformations of this locus consists of moving and smoothing the two nodes; the total equivariant euler class of the deformations is

$$(\alpha_1-\alpha_2)(\alpha_2-\alpha_1)(\alpha_1-\alpha_2)(\alpha_2-\alpha_1-\psi_1).$$

The euler class of the obstruction bundle is given by

$$e\big(\mathbb{E}^*\otimes T_{p_2}\mathbb{P}^1\big) = \lambda_2 - (\alpha_2-\alpha_1)\lambda_1 + (\alpha_2-\alpha_1)^2\,.$$

By [9, (7)], the contribution of the second diagram in Figure 8 to the first integral in (4.24) is thus

$$\int_{\overline{\mathcal{M}}_{2,1}} \mathrm{st}^*\kappa^4\,(\alpha_1-\alpha_2)^2\,\frac{\lambda_2-(\alpha_2-\alpha_1)\lambda_1+(\alpha_1-\alpha_2)^2}{(\alpha_1-\alpha_2)^3(\alpha_1-\alpha_2+\psi_1)} = \int_{\overline{\mathcal{M}}_{2,1}}\psi_1^4 = \frac{1}{1152}\,. \tag{4.26}$$

Combining (4.25) and (4.26) with

$$\langle\psi_1^3\psi_2^2,\overline{\mathcal{M}}_{2,2}\rangle = \frac{29}{5760}$$

from C. Faber's program, we obtain the first equality in (1.13).

The contribution of the fixed locus of $\overline{\mathfrak{M}}^{p_2}_{2,2;(1)}(\mathbb{P}^1,1)]^{\mathrm{vir}}$ represented by the third diagram in Figure 8 to the second integral in (4.24) is the same as of the first diagram to the the first integral in (4.24). The locus represented by the fourth diagram in Figure 8 is isomorphic to $\overline{\mathfrak{M}}^{0,\infty}_{2,0;(1),(1)}(\mathbb{P}^1,1)_\sim$. Since the virtual dimension of this moduli space is 3, the restriction of $\mathrm{st}^*\kappa^4$ to this fixed locus vanishes and so the latter does not contribute to the second integral in (4.24). Along with the two previous paragraphs, this provides a direct check of the $d\!=\!1$ case of the second equality in (1.13).

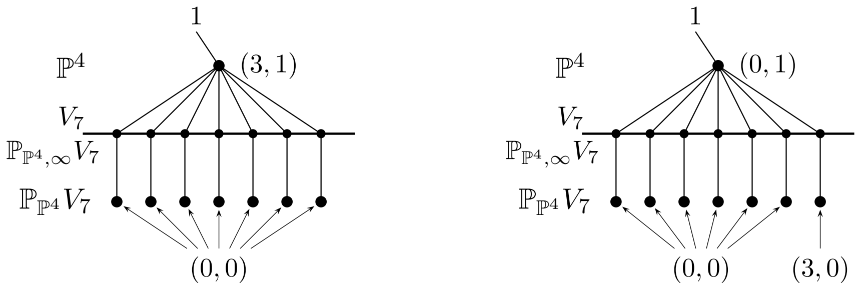

Figure 9: Bipartite graphs Γ that contribute to the absolute GW-invariant in (1.14) via the symplectic sum decomposition with respect to V_7.

4.3 Genus 3 degree 1 primary invariants of $\mathbb{P}^4$

We establish the second equality in (1.14) by applying the symplectic sum formula, as stated in the second-to-last equation on page 201 in [19], to the absolute GW-invariant in (1.14) via the decomposition (3.7) with $X=\mathbb{P}^4$ and $V=V_\delta$, where $V_\delta\subset\mathbb{P}^4$ is a smooth degree δ hypersurface.

Since the Poincare dual of the primary insertion in (1.14) vanishes on the hypersurface V_δ (the constraining point can be chosen outside of V_δ), $k_v=0$ for all $v\in\Gamma_V$ (the marked point stays on the X-side) if Γ is a bipartite graph as in Section 3.2 contributing to the absolute GW-invariant in (1.14). Furthermore, $A_v=1$ for the unique vertex $v\in\Gamma_X$. By Section 3.2 or Lemma 3.2 (separately), the only graphs Γ that may contribute to the absolute GW-invariant in (1.14) satisfy

(1) $(g_v,A_v,k_v)=(0,0,0)$ for all $v\in\Gamma_V$ or

(2) $(g_v,A_v,k_v)=(3,0,0)$ for one element $v\in\Gamma_V$ and $(g_v,A_v,k_v)=(0,0,0)$ for the remaining elements $v\in\Gamma_V$.

There are other bipartite graphs Γ, but they all contain a vertex $v\in\Gamma_V$ with $g_v\in\{1,2\}$; by Section 3.2, such a graph does not contribute to an absolute GW-invariant with primary insertions via the symplectic sum formula. By Section 3.2 or Lemma 3.2, the label of the edge leaving a vertex $v\in\Gamma_V$ with $g_v=0$ in a contributing graph Γ is 1 (and thus omitted in our diagrams). By the proof of Lemma 3.2 in Section 3.2, the same is the case if $g_v=3$; otherwise, the fiber of the projection in (3.13) would have positive dimension, while too many conditions would be imposed on the curve on the X-side. The same conclusions can be drawn from Section 3.1.

In summary, there are only two graphs that may contribute to the absolute GW-invariant in (1.14) via the symplectic sum formula; they are shown in Figure 9. Thus,

$$\mathrm{GW}^{\mathbb{P}^4}_{3,1}(\mathrm{pt})=\frac{1}{\delta!}\mathrm{GW}^{\mathbb{P}^4,V_\delta}_{3,1;\mathbf{1}_\delta}(\mathrm{pt})+\frac{\delta}{\delta!}\,\mathrm{GW}^{\mathbb{P}^4,V_\delta}_{0,1;\mathbf{1}_\delta}(1;\mathrm{pt};1^{\delta-1},\mathrm{pt})\,\mathrm{GW}^{\mathbb{P}_{\mathbb{P}^4}V_\delta,\mathbb{P}_{\mathbb{P}^4,\infty}V_\delta}_{3,F;(1)}(1;1;1)\,, \qquad (4.27)$$

where $F\in H_2(\mathbb{P}_{\mathbb{P}^4}V_\delta;\mathbb{Z})$ is the fiber class. The first insertion 1 in the last two relative invariants in (4.27) indicates that no constraint is imposed on the domain of the maps by pulling back a class κ from a Deligne-Mumford space of curves. The relative insertions $(1^{\delta-1},\mathrm{pt})$ and 1 in these invariants (shown after the second semi-column in each case) arise from the Kunneth decomposition of the diagonal Δ_V in V^2; the point insertion on the first of these invariants corresponds to the pairing with the second invariant, which arises from a zero-dimensional relative moduli space. It is immediate from the $g=0$ part of the argument in Section 3.1 that

$$\frac{\delta}{\delta!}\,\mathrm{GW}^{\mathbb{P}^4,V_\delta}_{0,1;\mathbf{1}_\delta}(1;\mathrm{pt};1^{\delta-1},\mathrm{pt}) = \mathrm{GW}^{\mathbb{P}^4}_{0,1}(\mathrm{pt},\mathrm{pt}) = 1. \tag{4.28}$$

Combining (4.27) with (4.28) and Lemma 4.4 below, we conclude that

$$\mathrm{GW}^{\mathbb{P}^4}_{3,1}(\mathrm{pt}) = \frac{1}{\delta!}\mathrm{GW}^{\mathbb{P}^4,V_\delta}_{3,1;\mathbf{1}_\delta}(\mathrm{pt}) + \frac{\langle c_1(V_\delta)c_2(V_\delta)-c_3(V_\delta),V_\delta\rangle}{362880}\,.$$

The second equality in (1.14) now follows from

$$c(V_\delta) = \big((1+x)^5(1+\delta x)^{-1}\big)\big|_{V_\delta} \in H^*(V_\delta;\mathbb{Z}),$$

where $x=c_1(\mathcal{O}_{\mathbb{P}^4}(1))\in H^2(\mathbb{P}^4;\mathbb{Z})$ is the standard generator.

Lemma 4.4. *Let (V,ω) be a compact symplectic manifold of real dimension 6 and $L\longrightarrow V$ be a complex line bundle. With notation as at the beginning of Section 3.3, the virtual dimension of the genus 3 relative moduli space $\overline{\mathfrak{M}}^{\mathbb{P}_L,\infty}_{3,0;(1)}(\mathbb{P}_L,F)$ is 0 and*

$$\deg\big[\overline{\mathfrak{M}}^{\mathbb{P}_L,\infty}_{3,0;(1)}(\mathbb{P}_L,F)\big]^{\mathrm{vir}} = \frac{\langle c_1(V)c_2(V)-c_3(V),V\rangle}{362880}\,.$$

Proof. The first claim is immediate from the second equation in (1.2). In order to establish the second claim, we proceed as in Section 3.2 by first choosing a generic deformation $\nu\in\Gamma_{3,0}(V,J_V)$. Lifting J_V and ν to $\mathbb{P}_L\longrightarrow V$ as in Section 3.2, we obtain a fibration

$$\pi_{L,V}\colon \overline{\mathfrak{M}}^{\mathbb{P}_L,\infty}_{3,0;(1)}\big(\mathbb{P}_L,F;J,\pi^*_{X,V}\nu\big) \longrightarrow \overline{\mathfrak{M}}_{3,0}\big(V,0;J_V,\nu\big) \tag{4.29}$$

as in (3.13). In this case, the base is zero-dimensional. Since the obstruction bundle for $\overline{\mathfrak{M}}_{3,0}(V,0)$ is given by (3.3) with $(g_v,k_v+\ell_v)=(3,0)$, the degree of this base is

$$\begin{aligned}\big\langle e(\pi_1^*\mathbb{E}^*\otimes\pi_2^*TV),\overline{\mathcal{M}}_3\times V\big\rangle &= \langle\lambda_1\lambda_2\lambda_3,\overline{\mathcal{M}}_3\rangle\big\langle c_1(V)c_2(V)-3c_3(V),V\big\rangle\\ &+\langle\lambda_2^3,\overline{\mathcal{M}}_3\rangle\big\langle c_3(V),V\big\rangle+\langle\lambda_3^2,\overline{\mathcal{M}}_3\rangle\big\langle c_1(V)^3-3c_1(V)c_2(V)+3c_3(V),V\big\rangle\,.\end{aligned}$$

The three intersection numbers on $\overline{\mathcal{M}}_3$ above are provided by Table 1 and (3.4). The second claim of Lemma 4.4 now follows from Lemma 4.5 below. □

Lemma 4.5. *If $\mathrm{st}\colon\overline{\mathfrak{M}}^{\mathrm{pt}}_{3,0;(1)}(\mathbb{P}^1,1)\longrightarrow\overline{\mathcal{M}}_3$ is the forgetful morphism dropping the map to $\mathbb{P}^1$ and the marked point, then*

$$\mathrm{st}_*\big[\overline{\mathfrak{M}}^{\mathrm{pt}}_{3,0;(1)}(\mathbb{P}^1,1)\big]^{\mathrm{vir}} = 4\big[\overline{\mathcal{M}}_3\big]. \tag{4.30}$$

Since $\overline{\mathcal{M}}_3$ is smooth (as an orbifold) and irreducible, (4.30) is equivalent to

$$\big\langle\mathrm{st}^*\sigma,[\overline{\mathfrak{M}}^{\mathrm{pt}}_{3,0;(1)}(\mathbb{P}^1,1)]^{\mathrm{vir}}\big\rangle = 4, \tag{4.31}$$

where $\sigma\in H^{12}(\overline{\mathcal{M}}_3)$ is the Poincare dual of a generic element Σ of $\overline{\mathcal{M}}_3$. We give two proofs of (4.31) below, which are similar to the two proofs of (4.18).

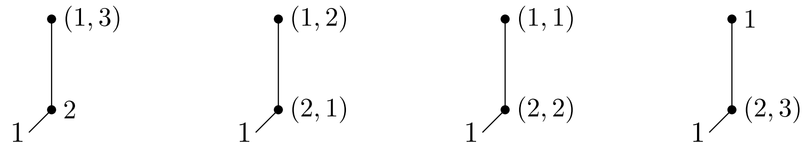

Figure 10: The four graphs describing the $(\mathbb{C}^*)^2$-fixed loci of $\overline{\mathfrak{M}}^{p_2}_{3,0;(1)}(\mathbb{P}^1,1)$

Proof 1 of (4.31). We continue with the localization setup in the first proof of (4.18). The fixed loci of the induced action on $\overline{\mathfrak{M}}^{p_2}_{3,0;(1)}(\mathbb{P}^1,1)$ again consist of maps sending components of positive genus to either the fixed point p_1 or the rubber $\mathbb{P}^1$ attached to the fixed point p_2. The four graphs describing these fixed loci, in the notation of [11, Chapter 27] and Figure 7, are shown in Figure 10.

The morphism st takes the fixed loci represented by the two middle diagrams in Figure 10 to the closure in $\overline{\mathcal{M}}_3$ of the locus consisting of two-component maps. Thus, $\mathrm{st}^*\sigma$ vanishes on these loci and the two middle diagrams do not contribute to (4.31) via the virtual localization theorem of [9].

The locus represented by the first diagram in Figure 10 is isomorphic to $\overline{\mathcal{M}}_{3,1}$ and is cut down by $\mathrm{st}^*\sigma$ to the curve Σ (which encodes the position of the node). The space of deformations of this locus consists of moving the node and of smoothing the node; its euler class equals

$$(\alpha_1-\alpha_2)(\alpha_1-\alpha_2+c_1(T\Sigma))$$

after restricting to the cut-down space. The obstruction bundle after cutting down by $\mathrm{st}^*\sigma$ is

$$H^1(\Sigma;T_{p_1}\mathbb{P}^1)=\big(H^0(\Sigma;T^*\Sigma\otimes T^*_{p_1}\mathbb{P}^1)\big)^*\approx T_{p_1}\mathbb{P}^1\oplus T_{p_1}\mathbb{P}^1\oplus T_{p_1}\mathbb{P}^1\,;$$

its equivariant euler class is $(\alpha_1-\alpha_2)^3$. Thus, the contribution of the first diagram in Figure 10 to (4.31) is

$$\int_{\overline{\mathcal{M}}_{3,1}}\mathrm{st}^*\sigma\,\frac{(\alpha_1-\alpha_2)^3}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_2+c_1(T\Sigma))}=-\int_\Sigma c_1(T\Sigma)=4\,;$$

see [9, (7)] or [10, Theorem 3.6].

The locus represented by the last diagram in Figure 10 is isomorphic to $\overline{\mathfrak{M}}^{0,\infty}_{3,0;(1),(1)}(\mathbb{P}^1,1)_\sim$. Since the virtual dimension of this moduli space is 5, the restriction of $\mathrm{st}^*\sigma$ to this fixed locus vanishes. By [10, Theorem 3.6], the last diagram in Figure 10 thus does not contribute to (4.31). Combining this with the conclusion of the two previous paragraphs, we obtain (4.31). $\square$

Proof 2 of (4.31). Let (Σ,j) be a generic element of $\overline{\mathcal{M}}_3$ as before. The first paragraph of the second proof of (4.18) applies to the present situation; the only change is that the base in (4.21) is replaced by

$$\Sigma\times\mathrm{Hol}_{\mathrm{j}}(\mathbb{P}^1,1)\approx\Sigma\times\mathbb{P}^1\,.$$

The line bundle of smoothing parameters (4.22) now becomes

$$T\Sigma\otimes T\mathbb{P}^1\longrightarrow\Sigma\times\mathbb{P}^1\,.$$

Analogously to the sentence containing (4.23), the solutions of the analogue of (4.19) in this situation correspond to the zeros of the section of

$$\mathrm{Obs}/\mathrm{Im}\,L\longrightarrow\Sigma\times\mathrm{Hol}_{\mathrm{j}}(\mathbb{P}^1,1),$$

with L as before, induced by a generic admissible ν, excluding the ones with $u(x_2)=\mathrm{pt}$. Without the first restriction on ν in (4.20), the number of such zeros would have been

$$\big\langle e(\mathrm{Obs}/\mathrm{Im}\,L), \Sigma\times\mathrm{Hol_j}(\mathbb{P}^1,1)\big\rangle = \big\langle c_1(\mathbb{C}^3/T\Sigma),\Sigma\big\rangle\big\langle c_1(T\mathbb{P}^1),\mathbb{P}^1\big\rangle = 8.$$

The contribution to this number from the vanishing of $\bar{\nu}$ along $\Sigma\times\mathrm{pt}$ is the number of zeros of an affine bundle map

$$T_{\mathrm{pt}}\mathbb{P}^1 \oplus T\Sigma\otimes T_{\mathrm{pt}}\mathbb{P}^1 \longrightarrow \mathcal{H}^{0,1}_{\Sigma}\otimes T_{\mathrm{pt}}\mathbb{P}^1$$

with an injective linear part. Thus, the latter number is

$$\big\langle e(\mathbb{C}^3/(\mathbb{C}\oplus T\Sigma)),\Sigma\big\rangle = 4\,.$$

The number in (4.31) is the difference of the two numbers above. □

4.4 The $\delta=0,1$ numbers in Example 3

We now use the virtual localization theorem of [9] to compute the absolute invariant and the virtual localization theorem of [10] to compute the $\delta=1$ case of the relative invariant in Example 3.

We apply [9, (7)] with the $\mathbb{C}^*$-action on $\mathbb{P}^4$ given by

$$c\cdot[Z_1,Z_2,Z_3,Z_4,Z_5] = [Z_1,cZ_2,cZ_3,c^{-1}Z_4,c^{-1}Z_5]$$

and its linear lift to $\mathcal{O}_{\mathbb{P}^4}(1)$ defined in the same way. The fixed locus of this action consists of

$$p_1\equiv[1,0,0,0,0], \qquad \mathbb{P}^1_{23}\equiv\big\{[0,Z_2,Z_3,0,0]\in\mathbb{P}^4\big\}, \qquad \mathbb{P}^1_{45}\equiv\big\{[0,0,0,Z_4,Z_5]\in\mathbb{P}^4\big\}.$$

Let

$$\alpha = c_1\big(\mathcal{O}_{\mathbb{P}^4}(1)\big)\big|_{p_2}\in H^*_{\mathbb{C}^*}.$$

We denote by

$$\overline{\mathfrak{M}}_{3,1}(\mathbb{P}^4,1)_{p_1}\subset\overline{\mathfrak{M}}_{3,1}(\mathbb{P}^4,1)$$

the preimage of p_1 under the evaluation morphism ev_1. We will compute

$$\int_{[\overline{\mathfrak{M}}_{3,1}(\mathbb{P}^4,1)_{p_1}]^{\mathrm{vir}}} 1. \tag{4.32}$$

The $\mathbb{C}^*$-fixed loci of this moduli space consist of maps sending the positive-genus components to p_1 or a point on $\mathbb{P}^1_{23}$ or $\mathbb{P}^1_{45}$ with the image of a degree 1 rational component running between p_1 and a point on either $\mathbb{P}^1_{23}$ or $\mathbb{P}^1_{45}$. The four types of graphs possibly contributing the integral in (4.32) are shown in the left half of Figure 11, where $\pm$ on the bottom vertex indicates whether it lies on $\mathbb{P}^1_{23}$ or $\mathbb{P}^1_{45}$, respectively. In the computations below, we first assume that $i=+$.

The locus represented by the first diagram in Figure 11 is isomorphic to $\overline{\mathcal{M}}_{3,2}\times\mathbb{P}^1$. The space of deformations of this locus consists of smoothing the node and turning the line around it away from $\mathbb{P}^1_{23}$; the equivariant euler class of the space of deformations is thus

$$(-\alpha-x-\psi_2)(2\alpha+x)^2 = -(\alpha+x+\psi_2)(2\alpha+x)^2\,,$$

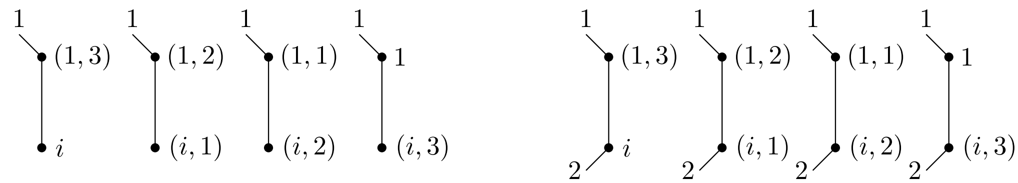

Figure 11: The two sets of graphs possibly contributing to the integrals (4.32) and (4.37), with $i\in\{+,-\}$ in the first case and $i\in\{2,3,4,5\}$ in the second case.

where $x=c_1(\mathcal{O}_{\mathbb{P}^1}(1))\in H^1(\mathbb{P}^1;\mathbb{Z})$ is the standard generator. The euler class of the obstruction bundle is given by

$$e\big(\mathbb{E}_3^*\otimes T_{p_1}\mathbb{P}^4\big)=\big(\lambda_3-\alpha\lambda_2+\alpha^2\lambda_1-\alpha^3\big)^2\big(\lambda_3+\alpha\lambda_2+\alpha^2\lambda_1+\alpha^3\big)^2.$$

By [27, (5.3)],

$$\big(\lambda_3-\alpha\lambda_2+\alpha^2\lambda_1-\alpha^3\big)\big(\lambda_3+\alpha\lambda_2+\alpha^2\lambda_1+\alpha^3\big)=-\alpha^6.$$

The contribution of the first diagram in Figure 11 to (4.32) is thus

$$-\int_{\overline{\mathcal{M}}_{3,2}\times\mathbb{P}^1}\frac{\alpha^{12}}{(\alpha+x+\psi_2)(2\alpha+x)^2}=\frac{5}{2}\langle x,\mathbb{P}^1\rangle\big\langle\psi_2^8,\overline{\mathcal{M}}_{3,2}\big\rangle=\frac{5}{2}\big\langle\psi_1^7,\overline{\mathcal{M}}_{3,1}\big\rangle=\frac{5}{165888}. \tag{4.33}$$

The second equality above applies the dilaton equation [11, Exercise 25.2.7]; the last follows from the first column in Table 3.

The locus represented by the second diagram in Figure 11 is isomorphic to $\overline{\mathcal{M}}_{2,2}\times\overline{\mathcal{M}}_{1,1}\times\mathbb{P}^1$. The space of deformations of this locus consists of smoothing the two nodes and moving the bottom node away from $\mathbb{P}^1_{23}$; the equivariant euler class of the space of deformations is thus

$$(-\alpha-x-\psi_{\rm t})(\alpha+x-\psi_{\rm b})(\alpha+x)(2\alpha+x)^2=-(\alpha+x+\psi_{\rm t})(\alpha+x-\psi_{\rm b})(\alpha+x)(2\alpha+x)^2,$$

where $\psi_{\rm t}\in H^*(\overline{\mathcal{M}}_{2,2})$ and $\psi_{\rm b}\in H^*(\overline{\mathcal{M}}_{1,1})$. The euler class of the obstruction bundle is given by

$$\begin{aligned}&e\big(\mathbb{E}_2^*\otimes T_{p_1}\mathbb{P}^4\big)e\big(\mathbb{E}_1^*\otimes T\mathbb{P}^4|_{\mathbb{P}^1_{23}}\big)\\&\qquad=\big(\lambda_2-\alpha\lambda_1+\alpha^2\big)^2\big(\lambda_2+\alpha\lambda_1+\alpha^2\big)^2\big(\lambda_{\rm b}-2x\big)\big(\lambda_{\rm b}-(\alpha+x)\big)\big(\lambda_{\rm b}-(2\alpha+x)\big)^2\\&\qquad=4\alpha\big(\lambda_2-\alpha\lambda_1+\alpha^2\big)^2\big(\lambda_2+\alpha\lambda_1+\alpha^2\big)^2(3\lambda_{\rm b}x-2\alpha x+\alpha\lambda_{\rm b})\big(\lambda_{\rm b}-(\alpha+x)\big),\end{aligned}$$

where $\lambda_{\rm b}\in H^*(\overline{\mathcal{M}}_{1,1})$. By [27, (5.3)],

$$\big(\lambda_2-\alpha\lambda_1+\alpha^2\big)\big(\lambda_2+\alpha\lambda_1+\alpha^2\big)=\alpha^4.$$

Since $\psi_1=\lambda$ on $\overline{\mathcal{M}}_{1,1}$, the contribution of the second diagram in Figure 11 to (4.32) is thus

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{2,2}\times\overline{\mathcal{M}}_{1,1}\times\mathbb{P}^1}\frac{4\alpha^9(3\lambda_{\rm b}x-2\alpha x+\alpha\lambda_{\rm b})}{(\alpha+x+\psi_{\rm t})(\alpha+x)(2\alpha+x)^2}&=5\langle\lambda,\overline{\mathcal{M}}_{1,1}\rangle\langle x,\mathbb{P}^1\rangle\big\langle\psi_2^5,\overline{\mathcal{M}}_{2,2}\big\rangle\\&=\frac{5}{24}\big\langle\psi_1^4,\overline{\mathcal{M}}_{2,1}\big\rangle=\frac{5}{27648}.\end{aligned} \tag{4.34}$$

The second equality above applies the dilaton equation [11, Exercise 25.2.7]; the last follows from the first column in Table 2.

ψ_1^7	$\psi_1^6\lambda_1$	$\psi_5^2\lambda_1^2$	$\psi_1^5\lambda_2$	$\psi_1^4\lambda_1^3$	$\psi_1^4\lambda_1\lambda_2$	$\psi_1^4\lambda_3$
$\frac{1}{82944}$	$\frac{7}{138240}$	$\frac{41}{290304}$	$\frac{41}{580608}$	$\frac{23}{96768}$	$\frac{23}{193536}$	$\frac{31}{967680}$

$\psi_1^3\lambda_1^4$	$\psi_1^3\lambda_1^2\lambda_2$	$\psi_1^3\lambda_1\lambda_3$	$\psi_1^3\lambda_2^2$	$\psi_1^2\lambda_1^5$	$\psi_1^2\lambda_1^3\lambda_2$	$\psi_1^2\lambda_1^2\lambda_3$	$\psi_1^2\lambda_1\lambda_2^2$	$\psi_1^2\lambda_2\lambda_3$
$\frac{41}{181440}$	$\frac{41}{362880}$	$\frac{41}{1451520}$	$\frac{41}{725760}$	$\frac{1}{7560}$	$\frac{1}{15120}$	$\frac{1}{60480}$	$\frac{1}{30240}$	$\frac{1}{120960}$

Table 3: The top intersections of λ-classes and ψ_1^i with $i\ge 2$ on $\overline{\mathcal{M}}_{3,1}$; the intersections with ψ_1^1 are obtained by multiplying the corresponding numbers in Table 1 by 4.

The locus represented by the third diagram in Figure 11 is isomorphic to $\overline{\mathcal{M}}_{1,2}\times\overline{\mathcal{M}}_{2,1}\times\mathbb{P}^1$. The euler class of its deformation space is as in the previous paragraph. The euler class of the obstruction bundle is given by

$$\begin{aligned} &e\big(\mathbb{E}_1^*\otimes T_{p_1}\mathbb{P}^4\big)e\big(\mathbb{E}_2^*\otimes T\mathbb{P}^4|_{\mathbb{P}^1_{23}}\big)\\ &\qquad=\big(\lambda_{\mathrm{t}}-\alpha\big)^2\big(\lambda_{\mathrm{t}}+\alpha\big)^2\big(\lambda_2-2x\lambda_1\big)\big(\lambda_2-(\alpha+x)\lambda_1+(\alpha+x)^2\big)\big(\lambda_2-(2\alpha+x)\lambda_1+(2\alpha+x)^2\big)^2, \end{aligned}$$

where $\lambda_{\mathrm{t}}\in H^*(\overline{\mathcal{M}}_{1,2})$. Since $\lambda^2=0$ on $\overline{\mathcal{M}}_{1,2}$ and $\lambda_2^2, 2\lambda_2-\lambda_1^2=0$ on $\overline{\mathcal{M}}_2$, the contribution of the third diagram in Figure 11 to (4.32) is thus

$$\begin{aligned} &-\int_{\overline{\mathcal{M}}_{1,2}\times\overline{\mathcal{M}}_{2,1}\times\mathbb{P}^1}\frac{4\alpha^8\lambda_1(2\alpha^2\lambda_1-4\alpha\lambda_1^2-(8\alpha^2-24\alpha\lambda_1+29\lambda_1^2)x)}{(\alpha+x+\psi_1)(\alpha+x-\psi_2)(\alpha+x)(2\alpha+x)^2}\\ &\qquad=\big\langle\psi_1^2,\overline{\mathcal{M}}_{1,2}\big\rangle\langle x,\mathbb{P}^1\rangle\big\langle\lambda_1^3\psi_1-8\lambda_1^2\psi_1^2+8\lambda_1\psi_1^3,\overline{\mathcal{M}}_{2,1}\big\rangle=-\frac{1}{11520}\,. \end{aligned}\tag{4.35}$$

The second equality above applies the dilaton equation [11, Exercise 25.2.7] and uses the first column in Table 1 and the third in Table 2.

The locus represented by the fourth diagram in Figure 11 is isomorphic to $\overline{\mathcal{M}}_{3,1}\times\mathbb{P}^1$. The space of deformations of this locus consists of smoothing the (bottom) node and moving it from $\mathbb{P}^1_{23}$; the equivariant euler class of the space of deformations is thus

$$(\alpha+x-\psi_1)(\alpha+x)(2\alpha+x)^2\,.$$

The euler class of the obstruction bundle is given by

$$\begin{aligned} e\big(\mathbb{E}_3^*\otimes T\mathbb{P}^4|_{\mathbb{P}^1_{23}}\big)&=\big(\lambda_3-2x\lambda_2\big)\big(\lambda_3-(\alpha+x)\lambda_2+(\alpha+x)^2\lambda_1-(\alpha+x)^3\big)\\ &\qquad\times\big(\lambda_3-(2\alpha+x)\lambda_2+(2\alpha+x)^2\lambda_1-(2\alpha+x)^3\big)^2. \end{aligned}$$

Since $\lambda_1^2=2\lambda_2$, $\lambda_2^2=2\lambda_1\lambda_3$, and $\lambda_3^2=0$ on $\overline{\mathcal{M}}_{3,1}$, the contribution of the fourth diagram in Figure 11 to (4.32) is thus

$$\begin{aligned} &\int_{\overline{\mathcal{M}}_{3,1}\times\mathbb{P}^1}\frac{\alpha^6(16\alpha^2\lambda_1^4-16\alpha\lambda_1^5+9\lambda_1^6-64\alpha^3\lambda_3)}{(\alpha+x-\psi_1)(\alpha+x)(2\alpha+x)^2}\\ &\qquad+\int_{\overline{\mathcal{M}}_{3,1}}\frac{\alpha^2(128\alpha^4\lambda_1^2-256\alpha^3\lambda_1^3+424\alpha^2\lambda_1^4-308\alpha\lambda_1^5+141\lambda_1^6-768\alpha^3\lambda_3)}{8(\alpha-\psi_1)}\\ &=\frac{1}{8}\langle x,\mathbb{P}^1\rangle\big\langle 69\lambda_1^6\psi_1-148\lambda_1^5\psi_1^2+232\lambda_1^4\psi_1^3-256\lambda_1^3\psi_1^4+128\lambda_3\psi_1^4+128\lambda_1^2\psi_1^5,\overline{\mathcal{M}}_{3,1}\big\rangle=-\frac{1}{2880}\,. \end{aligned}\tag{4.36}$$

Combining the numbers in (4.33)-(4.36) and multiplying the result by 2 (to account for $i=\pm$), we obtain the first equality in (1.14). This conclusion agrees with A. Gathmann's *growi* program.

We next apply [10, Theorem 3.6] with the action of $\mathbb{T}\equiv(\mathbb{C}^*)^2$ on $\mathbb{P}^4$ given by

$$(c_1,c_2)\cdot[Z_1,Z_2,Z_3,Z_4,Z_5] = [Z_1,c_1Z_2,c_1^{-1}Z_3,c_2Z_4,c_2^{-1}Z_5]$$

and its linear lift to $\mathcal{O}_{\mathbb{P}^4}(1)$ defined in the same way. The fixed locus of this action consists of the five points

$$p_1\equiv[1,0,0,0,0], \qquad \ldots, \qquad p_5\equiv[0,0,0,0,1].$$

Let

$$\alpha_1=c_1\big(\mathcal{O}_{\mathbb{P}^4}(1)\big)\big|_{p_2}\in H^*_{\mathbb{T}}, \quad \alpha_2=c_1\big(\mathcal{O}_{\mathbb{P}^4}(1)\big)\big|_{p_4}\in H^*_{\mathbb{T}}, \quad V=\big\{[0,Z_2,Z_3,Z_4,Z_5]\in\mathbb{P}^4\big\}.$$

We denote by

$$\overline{\mathfrak{M}}^V_{3,1;(1)}(\mathbb{P}^4,1)_{p_1}\subset\overline{\mathfrak{M}}^V_{3,1;(1)}(\mathbb{P}^4,1)$$

the preimage of p_1 under the evaluation morphism ev_1. We will compute

$$\int_{[\overline{\mathfrak{M}}^V_{3,1;(1)}(\mathbb{P}^4,1)_{p_1}]^{\mathrm{vir}}}1\,. \tag{4.37}$$

The $\mathbb{C}^*$-fixed loci of this moduli space consist of maps sending the positive-genus components to p_1 and at most one of the fixed points p_i with $i=2,3,4,5$; the image of the non-contracted degree 1 rational tail runs between p_1 and one of the fixed points p_i with $i=2,3,4,5$. The four types of graphs possibly contributing to the integrals in (4.37) are shown in the right half of Figure 11. In the computations below, we first assume that $i=2$.

The locus represented by the first diagram in the right half of Figure 11 is isomorphic to $\overline{\mathcal{M}}_{3,2}$. Its deformations consist of smoothing the node and turning the line around it away from p_2; the equivariant euler class of the space of deformations is thus

$$(-\alpha_1-\psi_2)\big(\alpha_1-(-\alpha_1)\big)(\alpha_1-\alpha_2)\big(\alpha_1-(-\alpha_2)\big)=-2\alpha_1(\alpha_1^2-\alpha_2^2)(\alpha_1+\psi_2)\,.$$

The obstruction bundle is as for the first diagram in Figure 11, but its euler class is now given by

$$\prod_{j=1,2}\big((\alpha_j^3-\alpha_j^2\lambda_1+\alpha_j\lambda_2-\lambda_3)(\alpha_j^3+\alpha_j^2\lambda_1+\alpha_j\lambda_2+\lambda_3)\big)=\alpha_1^6\alpha_2^6\,;$$

the equality holds by [27, (5.3)]. The contribution of the fifth diagram in Figure 11 to (4.37) is thus

$$\begin{aligned}-\int_{\overline{\mathcal{M}}_{3,2}}\frac{\alpha_1^6\alpha_2^6}{2\alpha_1(\alpha_1^2-\alpha_2^2)(\alpha_1+\psi_2)}&=-\frac{1}{2}\cdot\frac{\alpha_2^6}{\alpha_1^4(\alpha_1^2-\alpha_2^2)}\big\langle\psi_2^8,\overline{\mathcal{M}}_{3,2}\big\rangle\\&=-\frac{1}{2}\cdot\frac{\alpha_2^6}{\alpha_1^4(\alpha_1^2-\alpha_2^2)}\big\langle\psi_1^7,\overline{\mathcal{M}}_{3,1}\big\rangle=-\frac{1}{2}\cdot\frac{1}{82944}\cdot\frac{\alpha_2^6}{\alpha_1^4(\alpha_1^2-\alpha_2^2)}\,.\end{aligned}\tag{4.38}$$

The second equality above applies the dilaton equation [11, Exercise 25.2.7]; the last follows from the first column in Table 3.

The locus represented by the second diagram in the right half of Figure 11 is isomorphic to

$$F_2 \equiv \overline{\mathcal{M}}_{2,2} \times \overline{\mathfrak{M}}^{0,\infty}_{1,0;(1),(1)}(\mathbb{P}^1,1)_{\sim}\,.$$

The equivariant euler class of the space of deformations becomes

$$-2\alpha_1(\alpha_1^2-\alpha_2^2)(\alpha_1+\psi_{\rm t})(\alpha_1-\psi_{\rm b}),$$

where $\psi_{\rm t}\in H^*(\overline{\mathcal{M}}_{2,2})$ and $\psi_{\rm b}=\psi_\infty$ is on the rubber moduli space; see [10, Section 3.3]. The euler class of the obstruction bundle is now given by

$$\begin{aligned} &e\big(\mathbb{E}_2^*\otimes T_{p_1}\mathbb{P}^4\big)e\big(\mathbb{E}_1^*\otimes T_{p_2}V\big)\\ &\qquad= \prod_{j=1,2}\big((\alpha_j^2-\alpha_j\lambda_1+\lambda_2)(\alpha_j^2+\alpha_j\lambda_1+\lambda_2)\big)\cdot(2\alpha_1-\lambda_{\rm b})(\alpha_1-\alpha_2-\lambda_{\rm b})(\alpha_1+\alpha_2-\lambda_{\rm b})\\ &\qquad\cong -\alpha_1^4\alpha_2^4(5\alpha_1^2-\alpha_2^2)\lambda_{\rm b} \mod H_{\mathbb{T}}^*\subset H_{\mathbb{T}}^*(F_2)\,, \end{aligned}$$

where $\lambda_{\rm b}\in H^*(F_2)$ is the pull-back of $\lambda\in H^*(\overline{\mathcal{M}}_{1,1})$ by either forgetful morphism f from the second factor. Since

$$\overline{\mathfrak{M}}^{0,\infty}_{1,0;(1),(1)}(\mathbb{P}^1,1)_{\sim} \approx \overline{\mathcal{M}}_{1,1}\times \overline{\mathfrak{M}}^{0,\infty}_{0,1;(1),(1)}(\mathbb{P}^1,1)_{\sim}$$

as spaces and the last factor above is a point, $\psi_{\rm b}$ vanishes on the virtual class of the second factor in F_2. The second proofs of (4.18) and (4.31) readily show that

$$f_*\big[\overline{\mathfrak{M}}^{0,\infty}_{1,0;(1),(1)}(\mathbb{P}^1,1)_{\sim}\big]^{\rm vir} = \big[\overline{\mathcal{M}}_{1,1}\big]\,.$$

Thus, the contribution of the sixth diagram in Figure 11 to (4.37) is

$$\begin{aligned} \int_{[F_2]^{\rm vir}}\frac{\alpha_1^4\alpha_2^4(5\alpha_1^2-\alpha_2^2)\lambda_{\rm b}}{2\alpha_1(\alpha_1^2-\alpha_2^2)(\alpha_1+\psi_{\rm t})\alpha_1} &= -\frac{1}{2}\cdot\frac{\alpha_2^4(5\alpha_1^2-\alpha_2^2)}{\alpha_1^4(\alpha_1^2-\alpha_2^2)}\cdot\big\langle\psi_2^5,\overline{\mathcal{M}}_{2,2}\big\rangle\langle\lambda,\overline{\mathcal{M}}_{1,1}\rangle\\ &= -\frac{1}{2}\cdot\frac{\alpha_2^4(5\alpha_1^2-\alpha_2^2)}{\alpha_1^4(\alpha_1^2-\alpha_2^2)}\cdot\frac{1}{24}\big\langle\psi_2^4,\overline{\mathcal{M}}_{2,1}\big\rangle = -\frac{1}{2}\cdot\frac{1}{27648}\cdot\frac{\alpha_2^4(5\alpha_1^2-\alpha_2^2)}{\alpha_1^4(\alpha_1^2-\alpha_2^2)}\,. \end{aligned} \tag{4.39}$$

The second equality above applies the dilaton equation [11, Exercise 25.2.7]; the last follows from the first column in Table 2.

The locus represented by the third diagram in the right half of Figure 11 is isomorphic to

$$F_3 \equiv \overline{\mathcal{M}}_{1,2}\times\overline{\mathfrak{M}}^{0,\infty}_{2,0;(1),(1)}(\mathbb{P}^1,1)_{\sim} \equiv F_{3;1}\times F_{3;2}\,.$$

The equivariant euler class of its deformation space is as in the previous paragraph. The euler class of the obstruction bundle is now given by

$$\begin{aligned} e\big(\mathbb{E}_1^*\otimes T_{p_1}\mathbb{P}^4\big)e\big(\mathbb{E}_2^*\otimes T_{p_2}V\big) &= \prod_{j=1,2}\big((\alpha_j-\lambda_{\rm t})(\alpha_j+\lambda_{\rm t})\big)\cdot(4\alpha_1^2-2\alpha_1\lambda_1+\lambda_2)\\ &\qquad\times\big((\alpha_1-\alpha_2)^2-(\alpha_1-\alpha_2)\lambda_1+\lambda_2\big)\big((\alpha_1+\alpha_2)^2-(\alpha_1+\alpha_2)\lambda_1+\lambda_2\big)\\ &\cong -\alpha_1^3\alpha_2^2(9\alpha_1^2-\alpha_2^2)\lambda_1^3+\frac{1}{2}\alpha_1^2\alpha_2^2(25\alpha_1^4-10\alpha_1^2\alpha_2^2+\alpha_2^4)\lambda_1^2 \mod H_{\mathbb{T}}^*\otimes H^{\le 2}(F_{3;2})\subset H_{\mathbb{T}}^*(F_3)\,, \end{aligned}$$

where $\lambda_{\mathrm{t}} \in H^*(\overline{\mathcal{M}}_{1,2})$ and $\lambda_1, \lambda_2 \in H^*(F_{3;2})$ are the pull-backs of the Hodge classes $\lambda_1, \lambda_2 \in H^*(\overline{\mathcal{M}}_2)$ by the forgetful morphism f from the second factor. Since

$$\overline{\mathfrak{M}}^{0,\infty}_{2,0;(1),(1)}(\mathbb{P}^1,1)_\sim \approx \overline{\mathcal{M}}_{2,1} \times \mathfrak{M}^{0,\infty}_{0,1;(1),(1)}(\mathbb{P}^1,1)_\sim \cup \big(\overline{\mathcal{M}}_{1,1}\times\overline{\mathcal{M}}_{1,1}\times\overline{\mathfrak{M}}^{0,\infty}_{0,2;(1),(1)}(\mathbb{P}^1,1)_\sim\big)/\mathbb{Z}_2$$

as spaces and the last factors in the two spaces on the RHS above are zero- and one-dimensional, ψ_{b}^2 vanishes on the virtual class of $F_{3;2}$. Since λ_1^3 vanishes on the divisor in $\overline{\mathcal{M}}_2$ consisting of two-component curves and

$$\big\langle \psi_\infty^{k-1}, \overline{\mathfrak{M}}^{0,\infty}_{0,k;(1),(1)}(\mathbb{P}^1,1)_\sim\big\rangle = 1 \qquad \forall\, k\ge 3, \tag{4.40}$$

the second proofs of (4.18) and (4.31) readily show that

$$\begin{aligned}\big\langle \lambda_1^3, [\overline{\mathfrak{M}}^{0,\infty}_{2,0;(1),(1)}(\mathbb{P}^1,1)_\sim]^{\mathrm{vir}}\big\rangle &= \big\langle e(\mathbb{C}^2/T\Sigma_2),\Sigma_2\big\rangle\big\langle\lambda_1^3,[\overline{\mathcal{M}}_2]\big\rangle = 2\big\langle\lambda_1^3,[\overline{\mathcal{M}}_2]\big\rangle\,;\\ \big\langle \psi_\infty\lambda_1^2, [\overline{\mathfrak{M}}^{0,\infty}_{2,0;(1),(1)}(\mathbb{P}^1,1)_\sim]^{\mathrm{vir}}\big\rangle &= \big\langle\lambda_1^2,\overline{\mathcal{M}}_{1,1}\big\rangle^2\,.\end{aligned}$$

Thus, the contribution of the seventh diagram in Figure 11 to (4.37) is

$$\begin{aligned}&\int_{[F_3]^{\mathrm{vir}}}\frac{\alpha_1^3\alpha_2^2(9\alpha_1^2-\alpha_2^2)\lambda_1^3}{2\alpha_1(\alpha_1^2-\alpha_2^2)(\alpha_1+\psi_{\mathrm{t}})\alpha_1} - \frac{1}{2}\int_{[F_3]^{\mathrm{vir}}}\frac{\alpha_1^2\alpha_2^2(25\alpha_1^4-10\alpha_1^2\alpha_2^2+\alpha_2^4)\lambda_1^2\psi_\infty}{2\alpha_1(\alpha_1^2-\alpha_2^2)(\alpha_1+\psi_{\mathrm{t}})\alpha_1^2}\\ &= \frac{\langle\psi_{\mathrm{t}}^2,\overline{\mathcal{M}}_{1,2}\rangle}{2\,\alpha_1^4(\alpha_1^2-\alpha_2^2)}\Big(\alpha_1^2\alpha_2^2(9\alpha_1^2-\alpha_2^2)\cdot 2\big\langle\lambda_1^3,[\overline{\mathcal{M}}_2]\big\rangle - \alpha_2^2(25\alpha_1^4-10\alpha_1^2\alpha_2^2+\alpha_2^4)\cdot\frac{\langle\lambda,\overline{\mathcal{M}}_{1,1}\rangle^2}{2}\Big)\\ &= -\frac{1}{2}\cdot\frac{1}{138240}\cdot\frac{\alpha_2^2(89\alpha_1^4-46\alpha_1^2\alpha_2^2+5\alpha_2^4)}{\alpha_1^4(\alpha_1^2-\alpha_2^2)}\,;\end{aligned} \tag{4.41}$$

the last equality follows from Table 1.

The locus represented by the last diagram in Figure 11 is isomorphic to

$$F_4 \equiv \overline{\mathfrak{M}}^{0,\infty}_{3,0;(1),(1)}(\mathbb{P}^1,1)_\sim\,.$$

The equivariant euler class of its deformation space is reduced to $2\alpha_1(\alpha_1^2-\alpha_2^2)(\alpha_1-\psi_{\mathrm{b}})$. The euler class of the obstruction bundle becomes

$$\begin{aligned}e\big(\mathbb{E}_3^*\otimes T_{p_2}V\big) &= (8\alpha_1^3-4\alpha_1^2\lambda_1+2\alpha_1\lambda_2-\lambda_3)\big((\alpha_1-\alpha_2)^3-(\alpha_1-\alpha_2)^2\lambda_1+(\alpha_1-\alpha_2)\lambda_2-\lambda_3\big)\\ &\qquad\big((\alpha_1+\alpha_2)^3-(\alpha_1+\alpha_2)^2\lambda_1+(\alpha_1+\alpha_2)\lambda_2-\lambda_3\big)\\ &\cong -\frac{1}{2}\alpha_1^2(9\alpha_1^2-\alpha_2^2)\lambda_1^5+\frac{1}{4}\alpha_1(45\alpha_1^4-14\alpha_1^2\alpha_2^2+\alpha_2^4)\lambda_1^4\\ &\quad-(18\alpha_1^6-20\alpha_1^4\alpha_2^2+2\alpha_1^2\alpha_2^4)\lambda_1^3-(17\alpha_1^6+45\alpha_1^4\alpha_2^2+3\alpha_1^2\alpha_2^4-\alpha_2^6)\lambda_3 \quad \mathrm{mod}\ H_{\mathbb{T}}^*\otimes H^{\le 4}(F_4),\end{aligned}$$

where $\lambda_i \in H^*(F_4)$ is the pull-back of the Hodge class $\lambda_i \in H^*(\overline{\mathcal{M}}_3)$ by the forgetful morphism f. Since

$$\begin{aligned}\overline{\mathfrak{M}}^{0,\infty}_{3,0;(1),(1)}(\mathbb{P}^1,1)_\sim \approx \overline{\mathcal{M}}_{3,1} \times \mathfrak{M}^{0,\infty}_{0,1;(1),(1)}(\mathbb{P}^1,1)_\sim &\cup \overline{\mathcal{M}}_{2,1}\times\overline{\mathcal{M}}_{1,1}\times\overline{\mathfrak{M}}^{0,\infty}_{0,2;(1),(1)}(\mathbb{P}^1,1)_\sim\\ &\cup\big((\overline{\mathcal{M}}_{1,1})^3\times\overline{\mathfrak{M}}^{0,\infty}_{0,3;(1),(1)}(\mathbb{P}^1,1)_\sim\big)/\mathbb{S}_3\end{aligned}$$

as spaces and the last factors in the three spaces on the RHS above are zero-, one-, and two-dimensional, respectively, ψ_{b}^3 vanishes on the virtual class of F_4. Since λ_1^5 vanishes on the divisor in $\overline{\mathcal{M}}_3$ consisting of two-component curves and λ_1^4 vanishes on the subvariety consisting of four-component curves (three elliptic curves attached to a $\mathbb{P}^1$), (4.40) and the second proofs of (4.18) and (4.31) give

$$\begin{aligned}
\big\langle\lambda_1^5,[\overline{\mathfrak{M}}^{0,\infty}_{3,0;(1),(1)}(\mathbb{P}^1,1)_\sim]^{\mathrm{vir}}\big\rangle &= \big\langle\lambda_1^5 e(\mathbb{E}^*/T\Sigma_3),[\overline{\mathcal{M}}_{3,1}]\big\rangle = \big\langle\lambda_1^5\psi_1^2-\lambda_1^6\psi_1,[\overline{\mathcal{M}}_{3,1}]\big\rangle\,;\\
\big\langle\psi_\infty\lambda_1^4,[\overline{\mathfrak{M}}^{0,\infty}_{3,0;(1),(1)}(\mathbb{P}^1,1)_\sim]^{\mathrm{vir}}\big\rangle &= 8\big\langle\lambda,\overline{\mathcal{M}}_{1,1}\big\rangle\big\langle\lambda_1^3,\overline{\mathcal{M}}_2\big\rangle;\\
\big\langle\psi_\infty^2\lambda_1^3,[\overline{\mathfrak{M}}^{0,\infty}_{3,0;(1),(1)}(\mathbb{P}^1,1)_\sim]^{\mathrm{vir}}\big\rangle &= \big\langle\lambda,\overline{\mathcal{M}}_{1,1}\big\rangle^3,\quad \big\langle\psi_\infty^2\lambda_3,[\overline{\mathfrak{M}}^{0,\infty}_{3,0;(1),(1)}(\mathbb{P}^1,1)_\sim]^{\mathrm{vir}}\big\rangle=\frac{1}{6}\big\langle\lambda,\overline{\mathcal{M}}_{1,1}\big\rangle^3.
\end{aligned}$$

Thus, the contribution of the last diagram in Figure 11 to (4.37) is

$$-\frac{1}{2}\int_{[F_4]^{\mathrm{vir}}}\frac{\alpha_1^2(9\alpha_1^2-\alpha_2^2)\lambda_1^5}{2\alpha_1(\alpha_1^2-\alpha_2^2)\alpha_1}+\frac{1}{4}\int_{[F_4]^{\mathrm{vir}}}\frac{\alpha_1(45\alpha_1^4-14\alpha_1^2\alpha_2^2+\alpha_2^4)\lambda_1^4\psi_\infty}{2\alpha_1(\alpha_1^2-\alpha_2^2)\alpha_1^2}\\
-\frac{1}{6}\int_{[F_4]^{\mathrm{vir}}}\frac{(125\alpha_1^6-75\alpha_1^4\alpha_2^2+15\alpha_1^2\alpha_2^4-\alpha_2^6)\lambda_1^3\psi_\infty^2}{2\alpha_1(\alpha_1^2-\alpha_2^2)\alpha_1^3}.$$

The preceding set of equations reduces this to

$$\begin{aligned}
&\frac{1}{2\,\alpha_1^4(\alpha_1^2-\alpha_2^2)}\Bigg(-\frac{1}{2}\alpha_1^4\big(9\alpha_1^2-\alpha_2^2\big)\big(\big\langle\lambda_1^5\psi_1^2,[\overline{\mathcal{M}}_{3,1}]\big\rangle-4\big\langle\lambda_1^6,[\overline{\mathcal{M}}_3]\big\rangle\big)\\
&\qquad+\alpha_1^2\big(45\alpha_1^4-14\alpha_1^2\alpha_2^2+\alpha_2^4\big)\cdot 2\big\langle\lambda,\overline{\mathcal{M}}_{1,1}\big\rangle\big\langle\lambda_1^3,\overline{\mathcal{M}}_2\big\rangle\\
&\qquad-\big(125\alpha_1^6-75\alpha_1^4\alpha_2^2+15\alpha_1^2\alpha_2^4-\alpha_2^6\big)\frac{\big\langle\lambda,\overline{\mathcal{M}}_{1,1}\big\rangle^3}{6}\Bigg)\\
&=-\frac{1}{2}\cdot\frac{1}{2903040}\cdot\frac{1747\alpha_1^6-1577\alpha_1^4\alpha_2^2+441\alpha_1^2\alpha_2^4-35\alpha_2^6}{\alpha_1^4(\alpha_1^2-\alpha_2^2)}.
\end{aligned}\tag{4.42}$$

The sum of (4.38), (4.39), (4.41), and (4.42) multiplied by 2 (to account for $i=3$) is

$$-\frac{1}{2903040}\cdot\frac{1747\alpha_1^6+292\alpha_1^4\alpha_2^2}{\alpha_1^4(\alpha_1^2-\alpha_2^2)}.$$

Adding in the same expression with α_1 and α_2 interchanged (to account for $i=4,5$), we find that

$$\mathrm{GW}^{\mathbb{P}^4,V_1}_{3,1;(1)}(\mathrm{pt})=-\frac{97}{193536}.$$

Along with the first equality in (1.14), this confirms the $\delta=1$ case of the second equality in (1.14).

5 The Cieliebak-Mohnke approach to GW-invariants

Theorem 1 and Examples 1-3 answer a key question arising in recent attempts to adapt the idea of [2] to constructing positive-genus GW-invariants geometrically. In this section, we review this approach and discuss its connections with Theorem 1 and Examples 1-3.

Suppose (X,ω) is a compact symplectic manifold such that ω represents a rational cohomology class. By [3, Theorem 1], the Poincare dual of every sufficiently large integer multiple $\delta\omega$ of ω can be represented by a symplectic hypersurface V in (X,ω). If $A\in H_2(X;\mathbb{Z})-\{0\}$ can be represented by a J-holomorphic map $u\colon\Sigma\longrightarrow X$ for some ω-tame almost complex structure, then

$$A\cdot V=\delta\,\omega(A)>0.$$

The idea of [2] is to define the primary genus 0 GW-invariants by counting J-holomorphic maps $\mathbb{P}^1\longrightarrow X$ that pass through generic representatives of constraints of the appropriate total dimension and send $A\cdot V$ points of $\mathbb{P}^1$ to V and dividing the resulting number by $(A\cdot V)!$. In order to ensure that the sets of maps being counted are finite, the almost complex structure J on X is allowed to vary with the domain of the map in a coherent way. For δ sufficiently large and a generic coherent family of J's, every non-constant J-holomorphic map $u:\mathbb{P}^1\longrightarrow X$ of ω-energy at most $\omega(A)$ intersects $X–V$ and sends at least three distinct points of the domain to V; see [2, Proposition 8.13].

The almost complex structures J used in [2] are required to be compatible with V, in the sense that $J(TV)\subset TV$. A coherent family of such complex structures can be viewed as a special case of a *single* pair (J,ν), with ν as (2.2) satisfying the first condition in (2.14). By a standard cobordism argument, the resulting count of (J,ν)-maps is independent of a generic pair (J,ν) compatible with (ω,V). As in [2, Section 10], the independence of the counts on V can be shown by defining such counts with respect to two transverse Donaldson's hypersurfaces, V and V', that are compatible with the same ω-tame almost complex structure J on X. In particular, the dimension-counting argument at the beginning of Section 3.1 can be easily modified to show that the number of maps does not change if an additional J-holomorphic hypersurface V' is added.

Remark 5.1. The counts defined in [2] have not been directly shown to be invariant under deformations of ω, which is a central property of GW-invariants in symplectic topology. This could be established by showing that two Donaldson's divisors with respect to deformation equivalent symplectic forms and of the same degree are deformation equivalent through Donaldson's divisors. While it remains unknown whether this is the case, the invariance of the counts of [2] under small deformations of ω is studied directly in [16].

Remark 5.2. Pairs (J,ν) as in Section 2 have been standard on the symplectic side of GW-theory at least since [29, 30]. Using such pairs in [2] would have avoided the need for an elaborate coherency condition on families of almost complex structure and would have simplified the transversality issues. The first part of Section 2 and a slight variation of the genus 0 portion of Section 3.1 in the present paper would have sufficed for the purposes of [2]. This would have also extended the construction to genus 0 invariants with constraints pulled back from the Deligne-Mumford space, as well as to genus 1.

By [17, Section 3.2], any topological component of the preimage of a J-holomorphic hypersurface V in X under the limit $u\colon\Sigma\longrightarrow X$ of a sequence of J-holomorphic maps $u_k:\Sigma_k\longrightarrow X$ from smooth domains meeting $X-V$ comes with a holomorphic section of the pull-back of the normal bundle to V. By [13, Section 6], this conclusion also applies to (J,ν)-holomorphic maps, if $J(TV)=TV$ and ν satisfies the first condition in (2.14). If J and ν also satisfy (2.11) and the second condition in (2.14), spaces of maps from stable domains that satisfy this limiting condition are of dimension at least two less than the space of maps from smooth domains which meet $X-V$. If all relevant domains are stable, invariants of (X,ω,V) can then be defined by counting such maps; see Section 2.

The attempts in [8, 15] to extend the approach of [2] to positive-genus GW-invariants utilize the ideas outlined in the previous paragraph. A crucial claim of [8, 15] is that the resulting counts of relative genus g degree d (J,ν)-maps are independent of the choice of (g,A)-hollow hypersurface V (at least, if it is a Donaldson's hypersurface). As illustrated by Theorem 1 and Examples 1-3, this claim is often, but not always, true. As illustrated by the direct proof of Theorem 1 in Section 3.1, it is true precisely when the ideas outlined in the previous paragraph are *not* needed to define the relative counts. In these cases, the argument in Section 3.1 implies that the counts do not change when a second J-holomorphic hypersurface V_2 is added.

In principle, the approach of [2] could be adapted to constructing positive-genus GW-invariants by subtracting lower-genus contributions with appropriate coefficients if the real dimension of the target X is 8 or less. These coefficients are determined by the chern classes of the divisor V, the top intersections of λ-classes on $\overline{\mathcal{M}}_g$, and the relative GW-theory of $\mathbb{P}^1$. While all of these are computable in some sense, it does not appear that the resulting coefficients would have reasonably simple expressions.

Simons Center for Geometry and Physics, Stony Brook University, NY 11794
mtehrani@scgp.stonybrook.edu

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794
azinger@math.stonybrook.edu

References

[1] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. 128 (1997), no. 1, 45–88.

[2] K. Cieliebak and K. Mohnke, *Symplectic hypersurfaces and transversality in Gromov-Witten theory*, J. Symplectic Geom. 5 (2007), no. 3, 281–356.

[3] S. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. 44 (1996), no. 4, 666–705.

[4] M. Farajzadeh Tehrani and A. Zinger, *On Symplectic Sum Formulas in Gromov-Witten Theory*, arXiv/1404.1898.

[5] C. Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians*, *New Trends in Algebraic Geometry* (Warwick, 1996), 93–109, London Math. Soc. Lecture Note, Ser. 264, Cambridge Univ. Press, 1999.

[6] K. Faber and R. Pandharipande, *Relative maps and tautological classes*, J. EMS 7 (2005), no. 1, 13–49.

[7] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology 38 (1999), no. 5, 933–1048.

[8] A. Gerstenberger, *Geometric transversality in higher genus Gromov-Witten theory*, arXiv/1309.1426.

[9] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. 135 (1999), 487–518.

[10] T. Graber and R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. 130 (2005), no. 1, 1–37.

[11] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror Symmetry*, Clay Math. Inst., AMS, 2003.

[12] J. Hu, T.-J. Li, and Y. Ruan, *Birational cobordism invariance of uniruled symplectic manifolds*, Invent. Math. 172 (2008), no. 2, 231–275.

[13] E. Ionel and T. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. 157 (2003), no. 1, 45–96.

[14] E. Ionel and T. Parker, *The symplectic sum formula for Gromov-Witten invariants*, Ann. of Math. 159 (2004), no. 3, 935–1025.

[15] E. Ionel and T. Parker, *A natural Gromov-Witten virtual fundamental class*, arXiv/1302.3472v2.

[16] A. Krestiachine, *Donaldson hypersurfaces and Gromov-Witten invariants*, Ph.D. Thesis HU Berlin, in preparation.

[17] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. 145 (2001), no. 1, 151–218.

[18] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. 57 (2001), no. 3, 509–578.

[19] J. Li, *A degeneration formula for GW-invariants*, J. Differential Geom. 60 (2002), no. 1, 199–293.

[20] J. Li, *Lecture notes on relative GW-invariants*, *Intersection Theory and Moduli*, ICTP Lect. Notes, XIX, 41–96, ICTP, 2004.

[21] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, in *Topics in Symplectic 4-Manifolds*, 47–83, Internat. Press 1998.

[22] E. Looijenga, *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J. Algebraic Geom. 3 (1994), 283–293.

[23] D. Maulik and R. Pandharipande, *A topological view of Gromov-Witten theory*, Topology 45 (2006), no. 5, 887–918.

[24] D. McDuff and D. Salamon, J-holomorphic curves and quantum cohomology, University Lecture Series 6, AMS 1994.

[25] D. McDuff and D. Salamon, *Symplectic Topology*, 2nd Ed., Oxford University Press, 1998.

[26] D. McDuff and D. Salamon, *J-Holomorphic Curves and Symplectic Topology*, 2nd Ed., AMS Colloquium Publications 52, AMS 2012.

[27] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, *Arithmetic and Geometry*, Vol. II, 271–328, Progr. Math. 36, Birkhäuser, 1983.

[28] A. Okounkov and R. Pandharipande, *Virasoro constraints for target curves*, Invent. Math. 163 (2006), no. 1, 47–108.

[29] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. 42 (1995), no. 2, 259–367.

[30] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. 130 (1997), no. 3, 455–516.

[31] A. Zinger, *Enumerative vs. symplectic invariants and obstruction bundles*, J. Symplectic Geom. 2 (2004), no. 4, 445–543.

[32] A. Zinger, *Enumeration of genus-two curves with a fixed complex structure in* $\mathbb{P}^2$ *and* $\mathbb{P}^3$, J. Differential Geom. 65 (2003), no. 3, 341–467.

[33] A. Zinger, *Basic Riemannian geometry and Sobolev estimates used in symplectic topology*, arXiv/1012.3980.