MAT 126.01, Prof. Bishop, Tuesday, Nov 5, 2020
Section 3.5 Other Strategies for Integration
Section 3.6 Numerical Integration
Today's material will not be on the midterm or final.

We have been learning how to intergrate, but in practice, most integrals that are easy enough to have a simple formula have been computed already.

A table of common integral is found in your textbook; more extensive lists can be found in reference books or online.

MAT 331

There are various software packages like `MATLAB`, `MAPLE` and `MATHEMATICA` that can compute integrals symbolically or numerically.

$\int e^{x^2} = ?$

$\int_0^1 e^{x^2} = \ldots\ldots$

Symbolic = gives a formula

Numerical = gives a decimal answer

$\int x^2 = \frac{1}{3}x^3$

$\int_0^1 x^2 = \frac{1}{3}$

I have posted links to some tables of integrals on the class webpage.

Of the various computer packages, I am most familiar with **MATLAB** and **MATHEMATICA**.

You can use these on the SBU virtual SINC site or download them from SBU DoIT (free while you are a student using the university license).

I will demonstrate **MATLAB**.

① Download to machine

② Virtual SINC site.

Go to SBU Virtual SINC Site:

`https://vdi.cloud.stonybrook.edu/appblast/webclient/index.html#/launchitems`

The link is also on the class webpage.

Enter NetID and password

Click "Virtual SINC Site"

You should get a Desktop (you may be asked to install some VM software the first time you do this).

On the Desktop, click the Windows icon in the lower left corner

To download onto your computer:

Go to SBU DoIT Software Catalog:
`https://it.stonybrook.edu/services/software-catalog/browse`

Scroll down to `MATLAB` and click on name.

This leads to a page with three options: create account, donload, activate. You will have to create a Mathworks account (this is the company that makes `MATLAB`), then download the program. When you download, there are many extra add-ons available (machine learning, signal processing,..) but you won't need most of these. The Symbolic Math Package is important for doing symbolic integration. Once downloaded, you will be prompted to activate the sofware (and this has to be done every year).

There are detailed instructions on the DoIT page.

Let's assume you have `MATLAB` running.

Symbolic indefinite integrals:

```
>> syms x
>> int( x^3 * sin(x))
ans =
cos(x)*(- x^3 + 6*x) + sin(x)*(3*x^2 - 6)
```

The line **`syms x`** tells the program to treat x as a symbol, not a number.

Note that `^` is used for powers and `*` for multiplication.

Symbolic definite integrals:

```
>> int(sin(x))
ans =
-cos(x)

>> int(sin(x),0,pi)
ans =
2
```

```
>> int((x^3+x^2+1)/(x^4-8))
ans = log(x - 2^(3/4))*(2^(1/4)/8 + 2^(3/4)/32 + 1/4)
+ log(9 - (9*2^(1/2))/4 - 9*x*(1/4 - (9*2^(1/2))/32)^(1/2))*
  ((1/4 - (9*2^(1/2))/32)^(1/2)/4 + 1/4)
- log(9*x*(1/4 - (9*2^(1/2))/32)^(1/2) - (9*2^(1/2))/4 + 9)*
  ((1/4 - (9*2^(1/2))/32)^(1/2)/4 - 1/4)
- log(x + 2^(3/4))*(2^(1/4)/8 + 2^(3/4)/32 - 1/4)

>> int((x^3+x^2+1)/(x^4-8),0,1)
ans = log(14^(1/4)/2) - (2^(3/4)*(atan(2^(1/4)/2)
+ atanh(2^(1/4)/2)))/16 + (2^(1/4)*(atan(2^(1/4)/2)
- atanh(2^(1/4)/2)))/4

>> double(ans)
ans = -0.2058
```

The command `double` converts symbolic answer to double precision floating point number.

```
>> int((x^3+x^2+1)/(x^4-8),0,1)
ans =
log(14^(1/4)/2) - (2^(3/4)*(atan(2^(1/4)/2)
+ atanh(2^(1/4)/2)))/16 + (2^(1/4)*(atan(2^(1/4)/2)
- atanh(2^(1/4)/2)))/4

>> vpa(ans,200)
ans =
-0.2058400745315316188185966434180970081848654346542
4206471550495397745478526897544019303744919056852
7742802002888184878227219823177489921697222620875686
29938584617042287270507575825114467539956054323524 26
```

`vpa` converts to decimal of given precision.

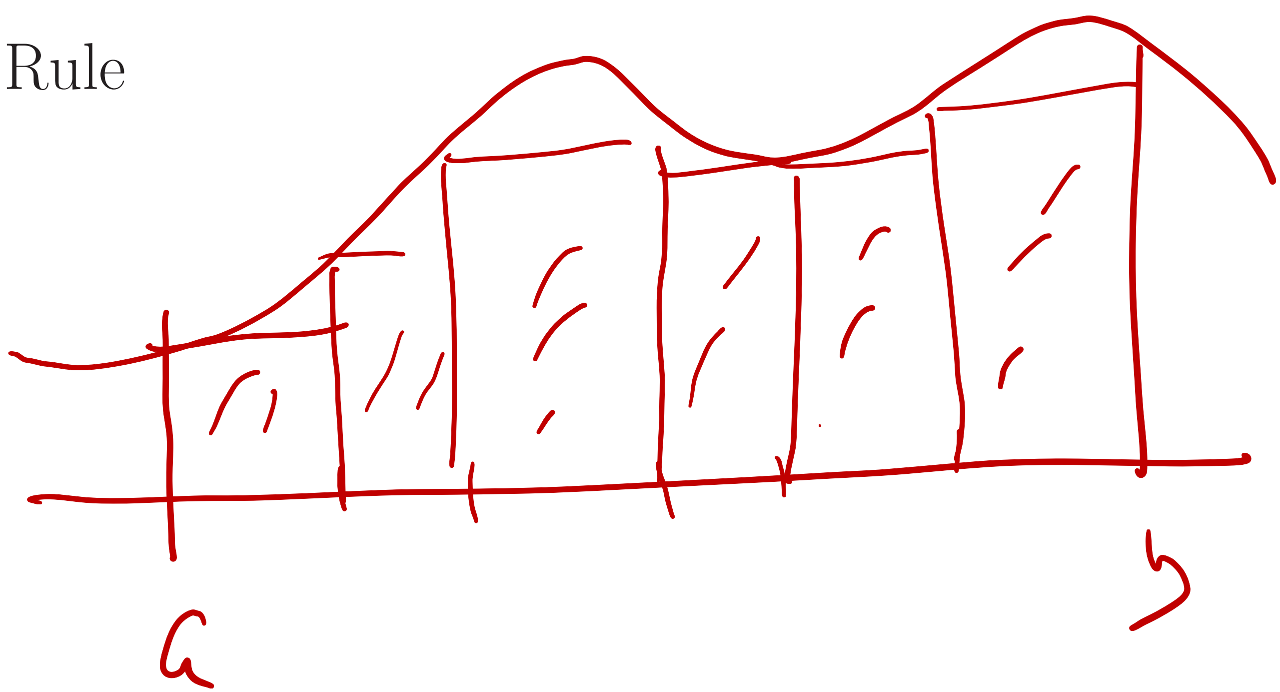
Section 3.6: Numerical Integration

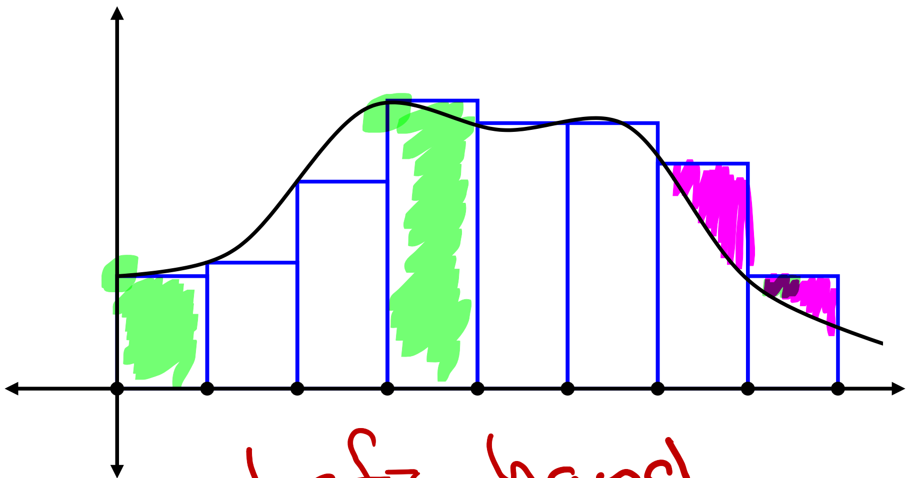
- Left and Right hand rules

- Trapezoid Rule

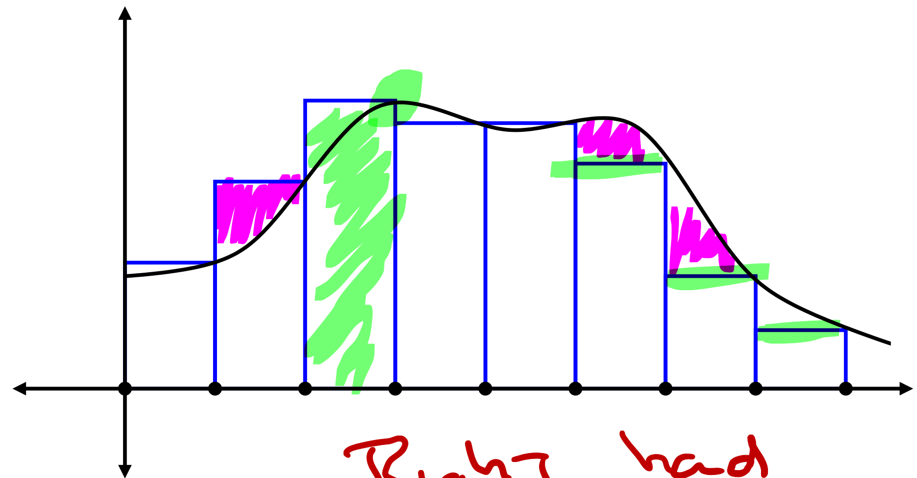
- Midpoint Rule

- Simpson's Rule

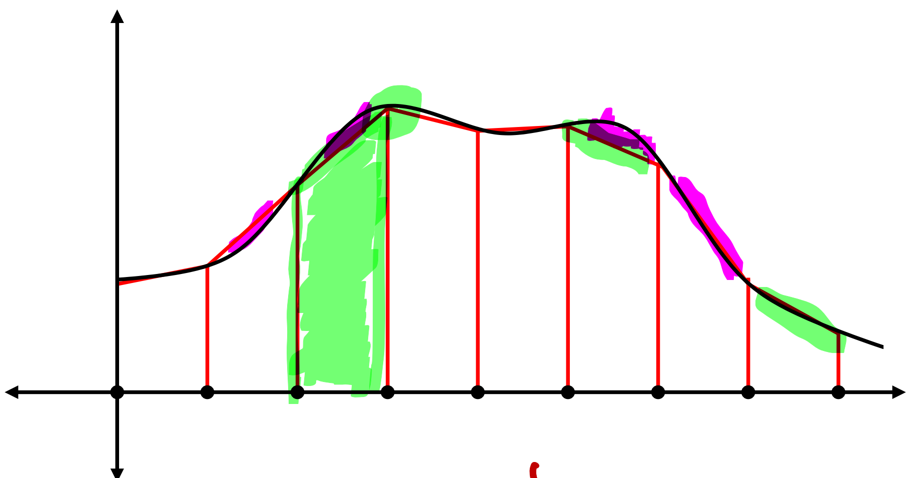




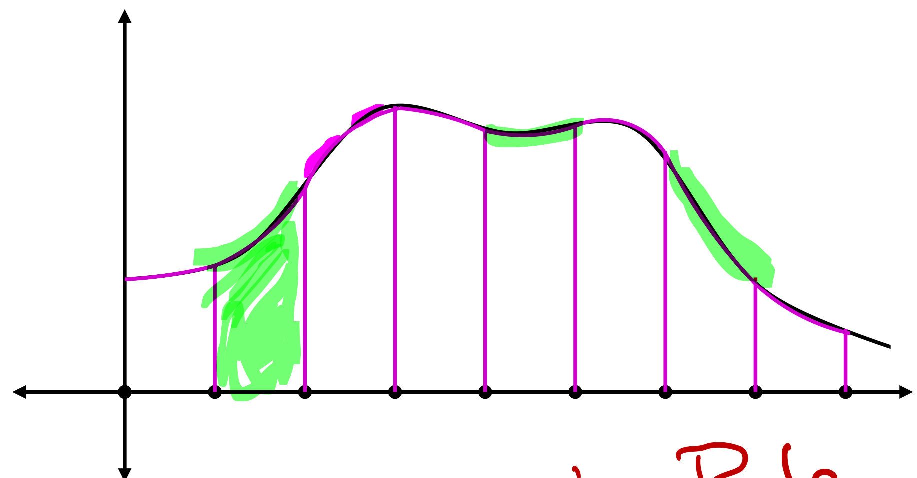
Left hand



Right hand



Trapezoid.



Simpson's Rule

$$\text{Trap} = \frac{LHR + RHR}{2}$$

Error bound for Left and Right hand Rules:

if $M = \max |f'|$ over $[a, b]$.

$$\text{error} \le \frac{M(b-a)^2}{n}.$$

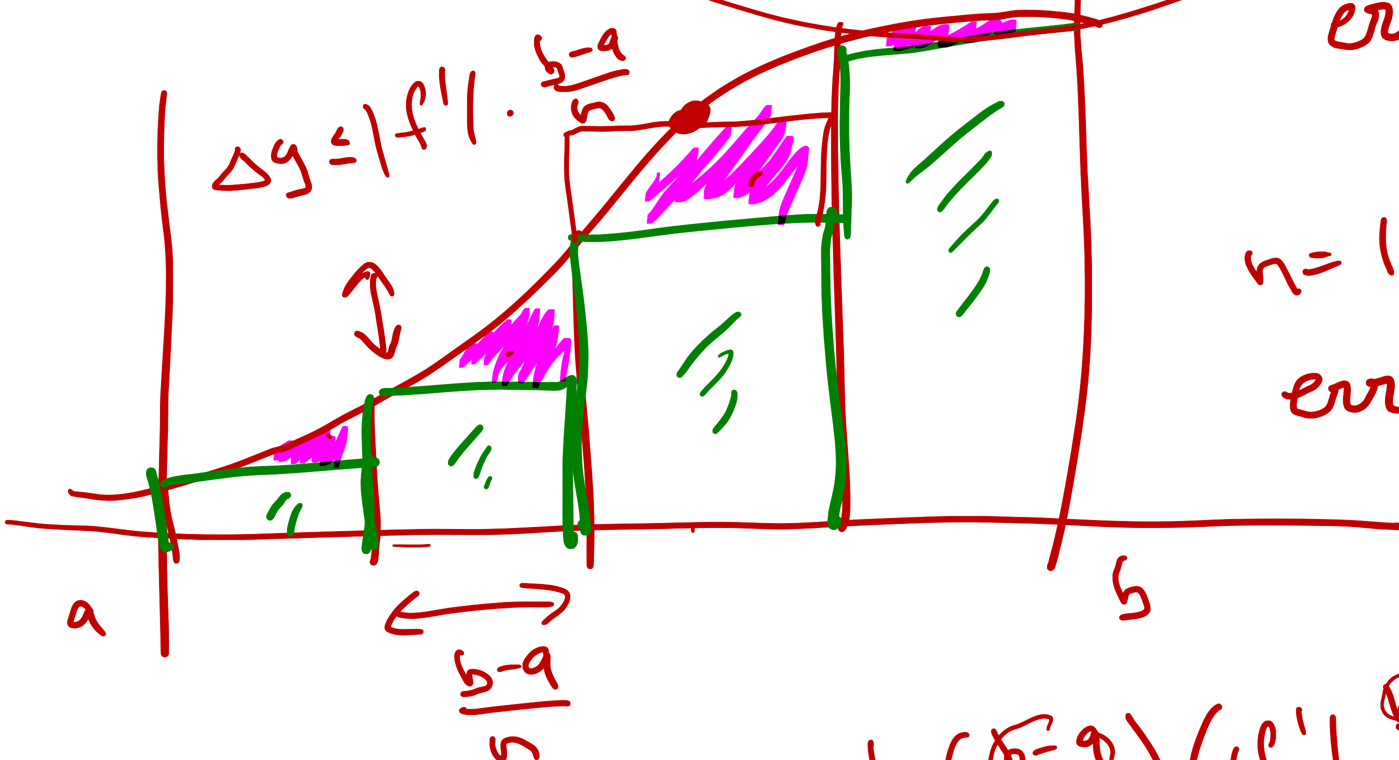
$n = 100$

error $\frac{1}{100} \approx .01$

$n = 10^6$

error $\approx .000001$

$\Delta y \le |f'| \cdot \frac{b-a}{n}$



$$\frac{1}{2}\left(\frac{b-a}{n}\right)\left(|f'| \frac{b-a}{n}\right) \cdot n$$

$$\frac{1}{2} \frac{(b-a)^2}{n}$$

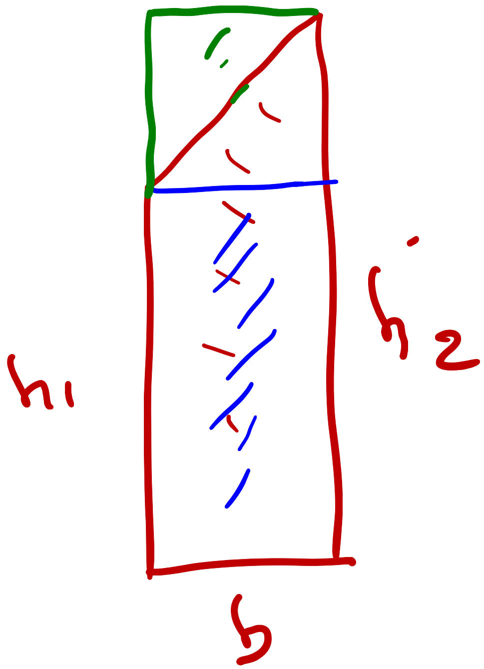
Trapzoid Rule is average of Left and Right hand rules.

Error bound for Trapezoid Rule: if $M = \max |f''|$ over $[a, b]$.

$$\text{error} \leq \frac{M(b-a)^3}{12n^2}.$$

$n = 100$

$\frac{1}{100^2} = .0001$



$\text{Area} = b \cdot \frac{h_1 + h_2}{2}$

small box $b \cdot h_1$

bigger box $b \cdot h_2$

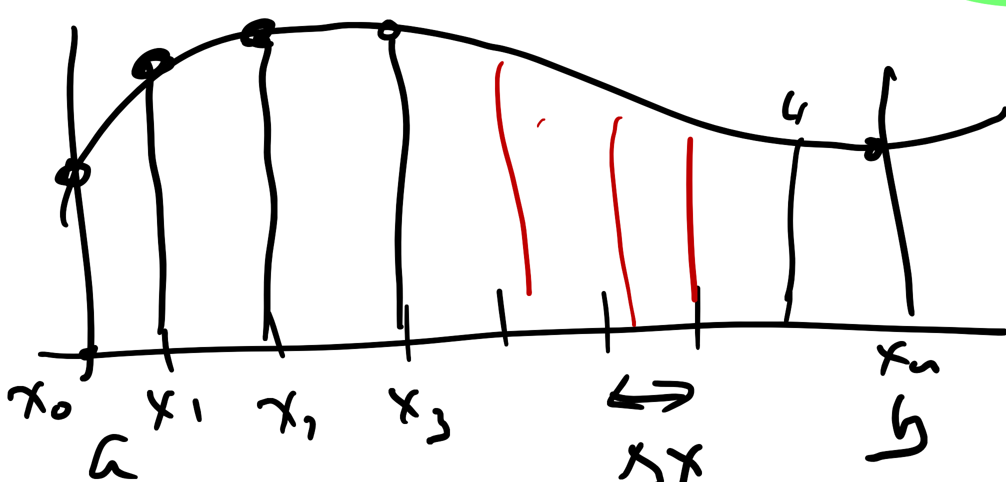
Simpson's rule approximates graph by parabolas. For n even,

$\Delta x = \frac{b-a}{n}$

$$\text{Simpson} = \frac{\Delta x}{3}(f(x_0) + \underline{4f(x_1)} + \underline{2f(x_2)} + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(n)).$$

Error bound for Simpson's Rule: If $M = \max|f''''|$ on $[a, b]$.

$$\text{error} \leq \frac{M(b-a)^5}{180n^4}.$$

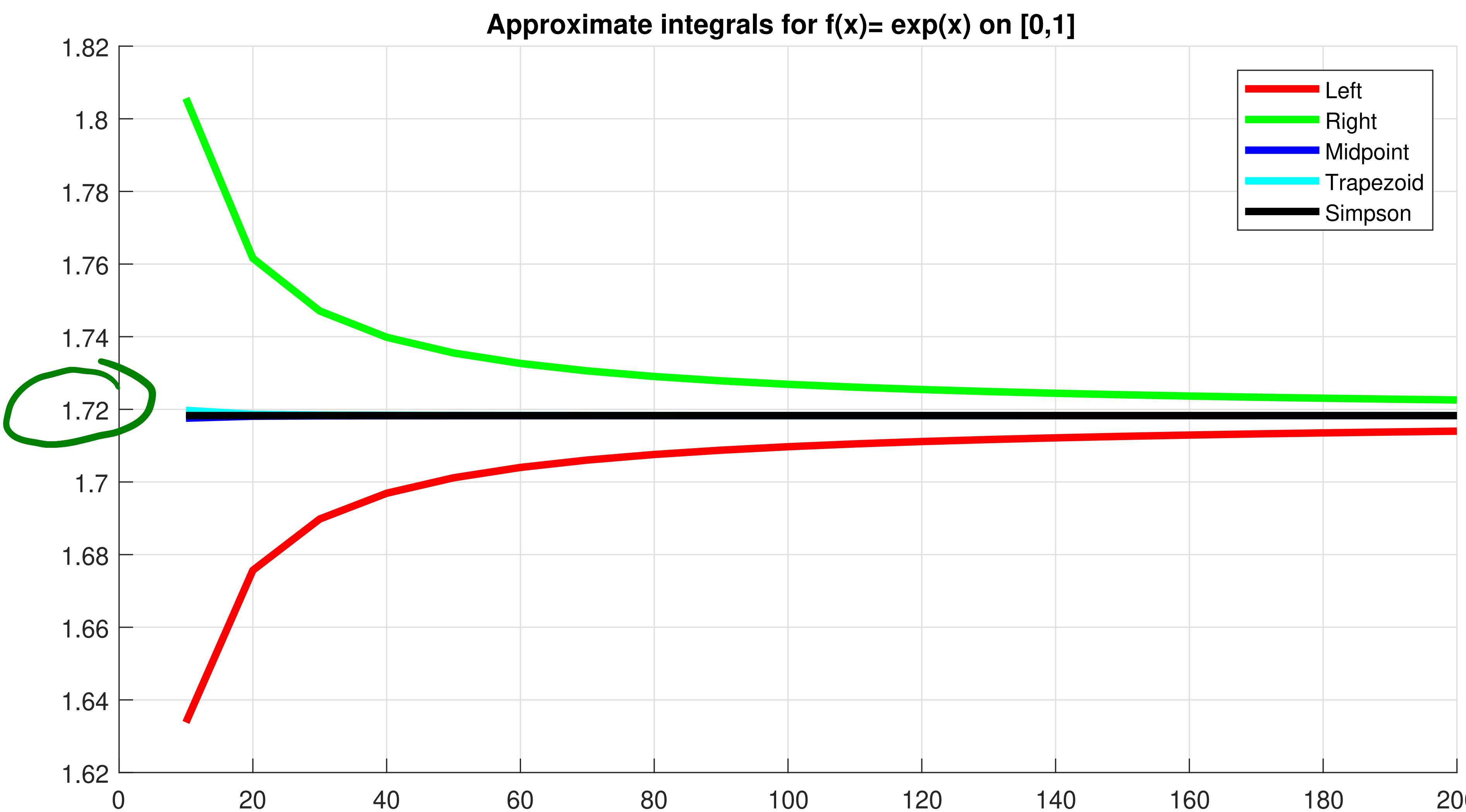


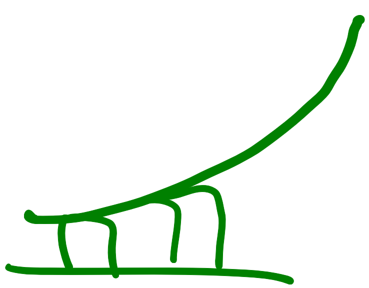
$n = 100$

$\text{error} \approx \frac{1}{100^4}$

$= \frac{1}{10^8}$

$= .00000001$

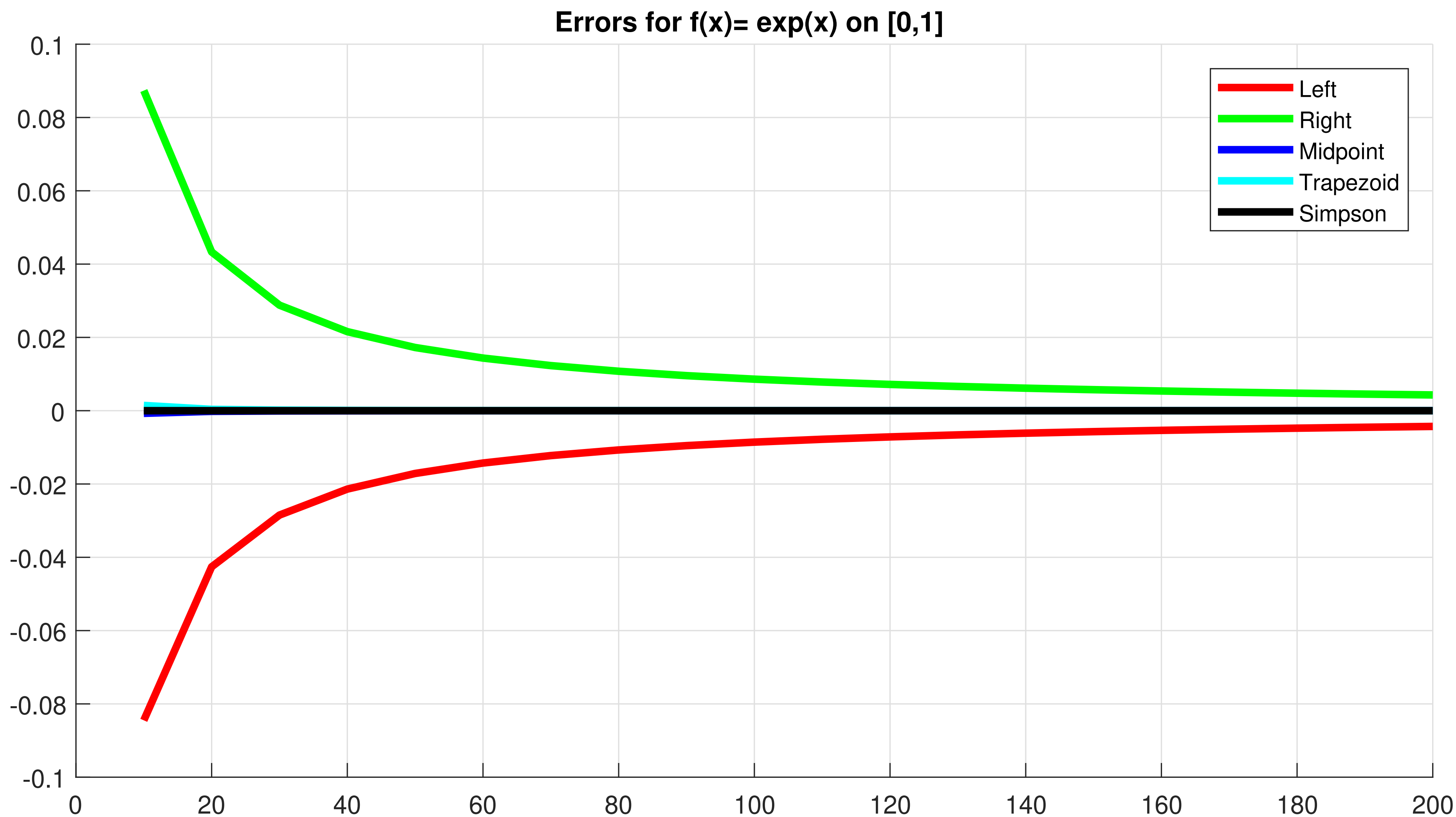


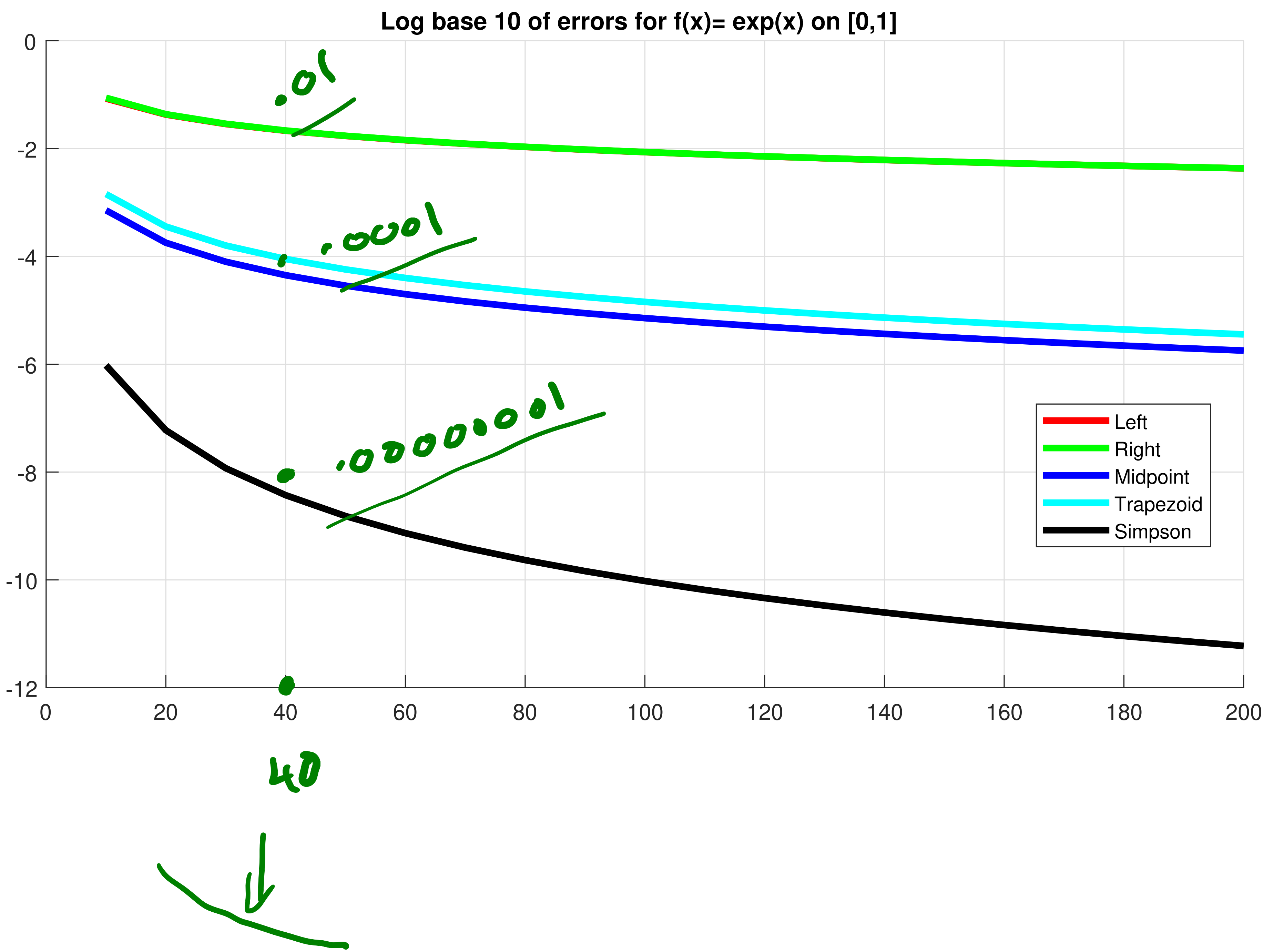


$$\int_0^1 e^x \, dx$$

$n = 10, 20, 30 \ldots$ 200

$e^1 - 1 \approx 1.718.$


Errors for f(x)= exp(x) on [0,1]
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
20
40
60
80
100
120
140
160
180
200
Left
Right
Midpoint
Trapezoid
Simpson



Log base 10 of errors for f(x)= exp(x) on [0,1]
0
-2
-4
-6
-8
-10
-12
0
20
40
60
80
100
120
140
160
180
200
Left
Right
Midpoint
Trapezoid
Simpson
.01
.0001
.00000001
40

Are there methods better than Simpons's Rule?

Are there methods better than Simpons's Rule?

Defnitely yes. But these are a bit too complex to describe here. I sometimes cover these in MAT 331.

Gauss quadrature will estimate the integral of a "nice" function like e^x with

$$\text{error} \approx \frac{1}{n!},$$

$\frac{1}{n^4}$

$n! = n \cdot (n-1)(n-2) \cdots$
$\cdots 4 \cdot 3 \cdot 2 \cdot 1$

using n specially chosen points

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} w_k f(x_k).$$

Figuring out the correct points $\{x_k\}$ and weights $\{w_k\}$ requires more advanced math, like linear algebra. Most serious integration programs use methods like this.

MAT 331

Tue Nov 10, 2020

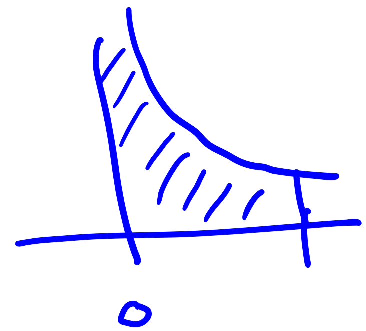
Office Hours

$\approx$ 11:15 — 12:00

(a little shorter today)

Does each improper integral converge or diverge?

$\boxed{D}$ $\int_0^1 \frac{1}{x^2}\,dx = \infty$



$\int_0^1 x^{-2}\,dx$

$\int_0^1 \frac{1}{x}\,dx = \infty$

$\int_0^1 x^p\,dx$ diverges if $x < -1$

$\boxed{C}$ $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx$

$p = -1/2$

$$\lim_{t \to 0} \int_t^1 x^{-1/2}\,dx = 2x^{1/2}\Big|_t^1$$

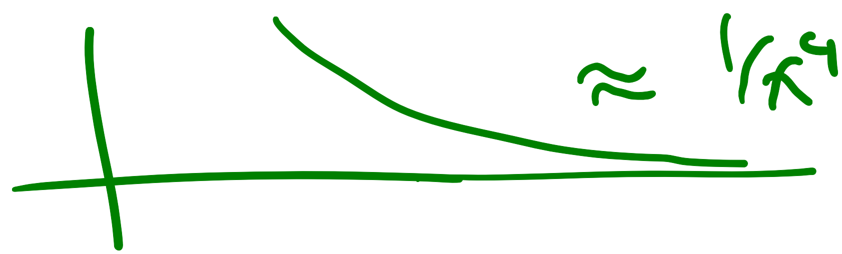
$$= 2 - 2t^{1/2}$$

$$\to 2$$

$\boxed{c}$

$$\int_1^\infty \frac{x^6 + x^4 + 1}{x^{10} + 3x^8 + 5}\,dx$$

$$\approx \frac{x^6}{x^{10}} = \frac{1}{x^4} = x^{-4}$$

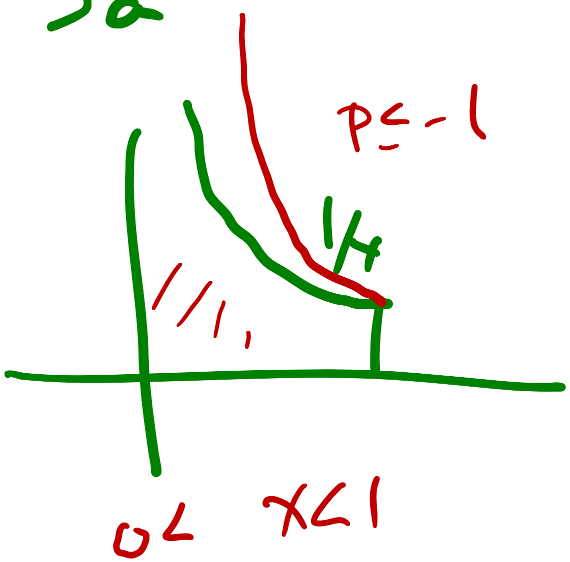


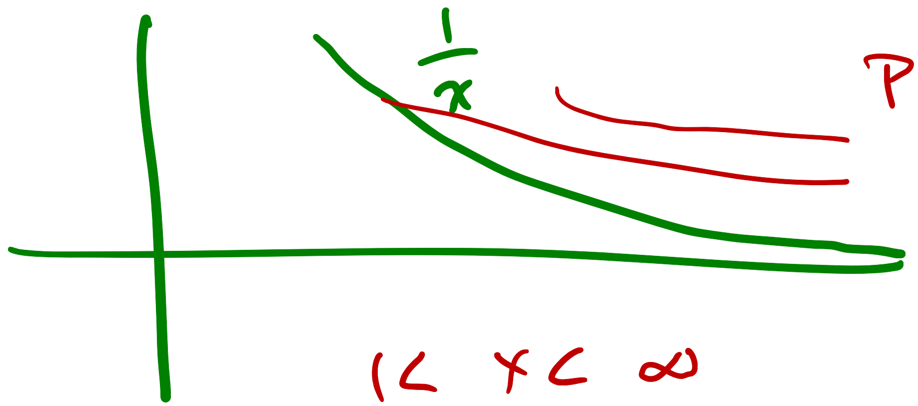
$\int_a^\infty x^p\,dx$ $\int_a^\infty \frac{1}{x} = \infty$

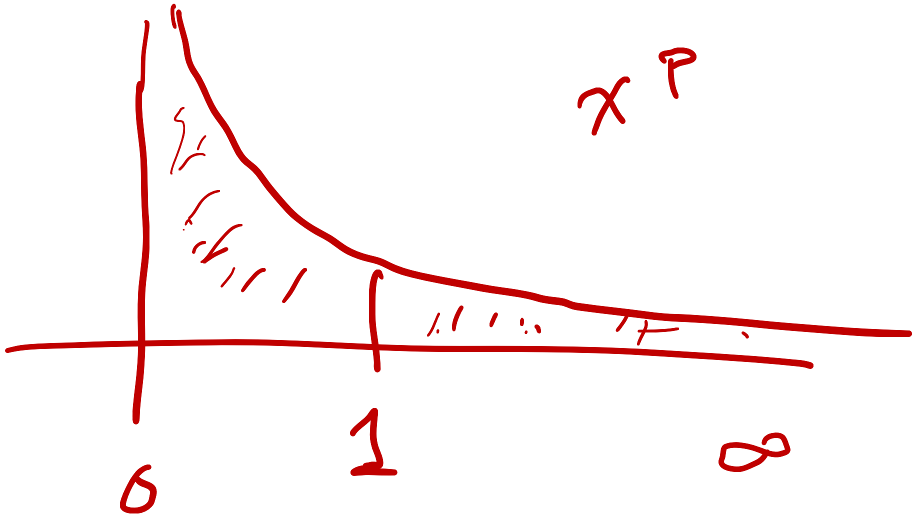
diverges when $p \geq -1$

$\int_0^1$ diverges $p \leq -1$

$\int_a^\infty$ diverges $p \geq -1$







$p \le -1$ $\qquad$ $p \ge -1$

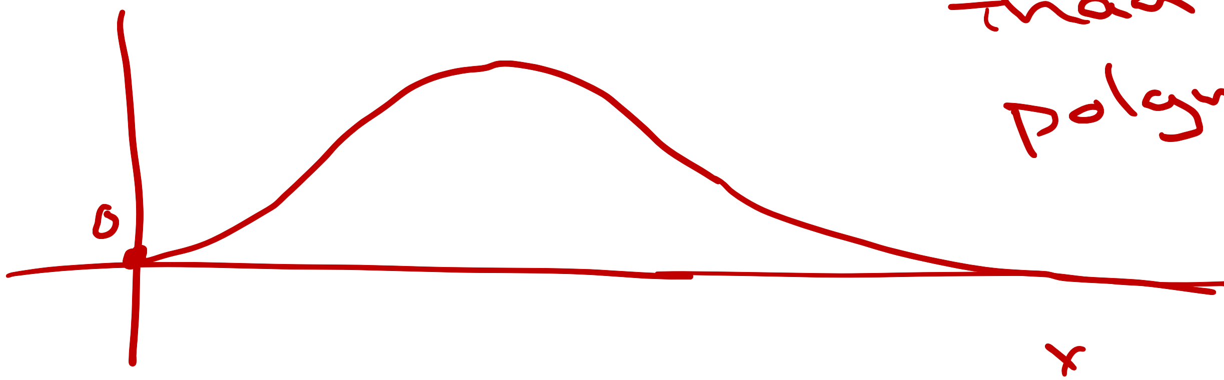
$$\int_0^\infty x^p\,dx = \infty$$

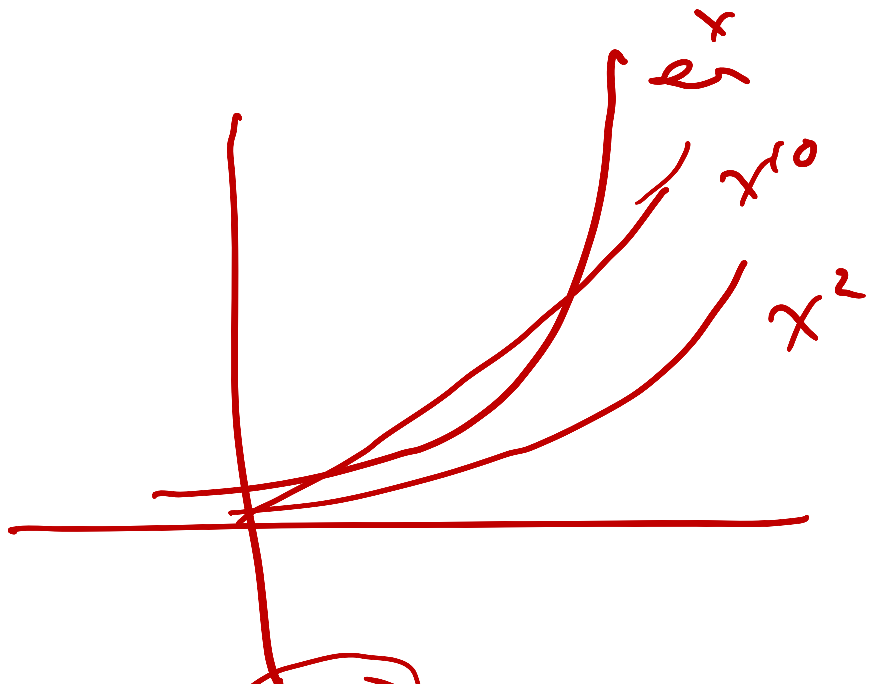
all p.

$$\int_0^\infty x^p\,dx = \underbrace{\int_0^1 x^p\,dx}_{=\infty \text{ if } p \le -1} + \underbrace{\int_1^\infty x^p\,dx}_{=\infty \text{ if } p \ge -1}$$

$$\int_0^\infty x^8 e^{-x}\,dx$$

e^x grows faster than any polynomial.





Fact: $e^{-x} \leq x^{-n}$ for any n if x is large enough.

$$\int_a^\infty \underbrace{x^8 e^{-x}} \le \int_a^\infty x^8 \underbrace{\frac{1}{x^{10}}} = \int_a^\infty \underbrace{\frac{1}{x^2}}$$

$$< \infty$$

$$= \int_a^\infty x^{-2}$$

$$-2 < -1$$

$$\int_0^\infty \sin(x) e^{-x} \, dx$$

$$|\sin(x)| \le 1$$

$$\left| \int_0^\infty \sin(x) e^{-x} \right|$$

$$\leq \int_0^\infty |\sin x \, e^{-x}|$$

$$\leq \int_0^\infty 1 \cdot e^{-x} \, dx$$

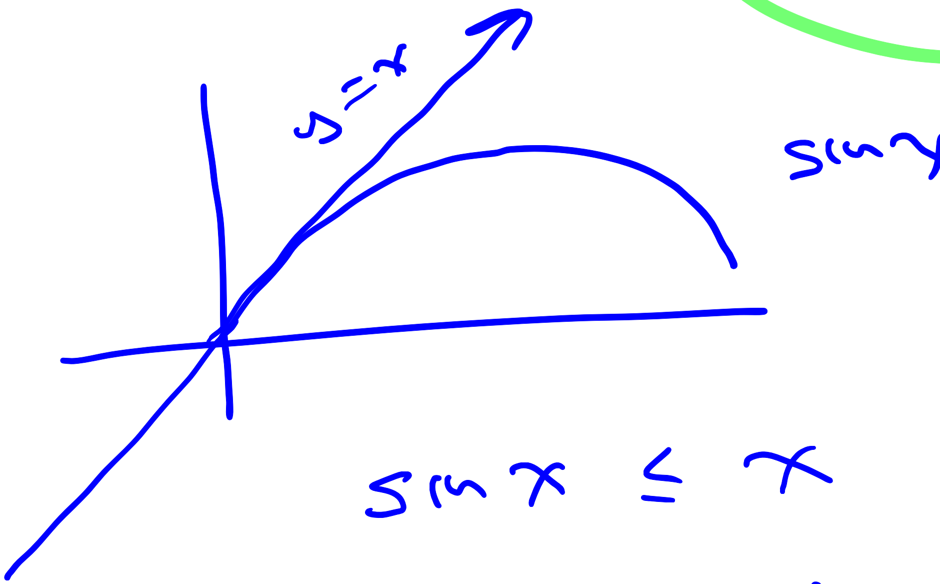
$$\lim_{t \to \infty} -e^{-x} \Big|_0^t$$

$$= 1 - e^{-t}$$

$$\to 1$$

$$\int_0^1 \frac{1}{\sin x}\,dx$$

Converge or diverge?



$\sin x \le x$ on $[0,1]$

$$\frac{1}{\sin x} \ge \frac{1}{x}$$

$$\int_0^1 \frac{1}{\sin x} \ge \int_0^1 \frac{1}{x} = \infty$$

$$(9-x^2)\frac{dy}{dx} = 1 \qquad y(0) = 2$$

$$\frac{dy}{dx} = \frac{1}{9-x^2}$$

$$y = \int \frac{1}{9-x^2} + C$$

$$\frac{1}{9-x^2} = \frac{1}{(3-x)(3+x)} = \frac{A}{3-x} + \frac{B}{3+x}$$

$$\frac{0\cdot x+1}{9-x^2} = \frac{(3+x)A + (3-x)B}{(3-x)(3+x)}$$

$$0 = A - B$$

$$1 = 3A + 3B$$

$$0x + 1 = 3A + Ax + 3B - Bx$$

$$= (A - B)x + 3A + 3B$$

$$0 = A - B$$

$$1 = 3A + 3B$$

$$0 = 3A - 3B$$

$$1 = 6A + 0$$

$$-\frac{1}{6} = A \qquad B = \frac{1}{6}$$

$$y = \int \frac{1}{9 - x^2} + C$$

$$= \int \left(-\frac{1}{6}\right)\frac{1}{3-x} + \left(\frac{1}{6}\right)\frac{1}{3+x}\,dx + C$$

$$= \frac{1}{6}\int \frac{1}{x-3} + \frac{1}{6}\int \frac{1}{x+3} + C$$

$$= \boxed{\frac{1}{6}\ln|x-3| + \frac{1}{6}\ln|x+3|} + C$$

$$y(0) = 2$$

$$2 = y(0) = \frac{1}{6}\ln 3 + \frac{1}{6}\ln 3 + C$$

$$\boxed{2 - \frac{1}{3}\ln 3} = c$$