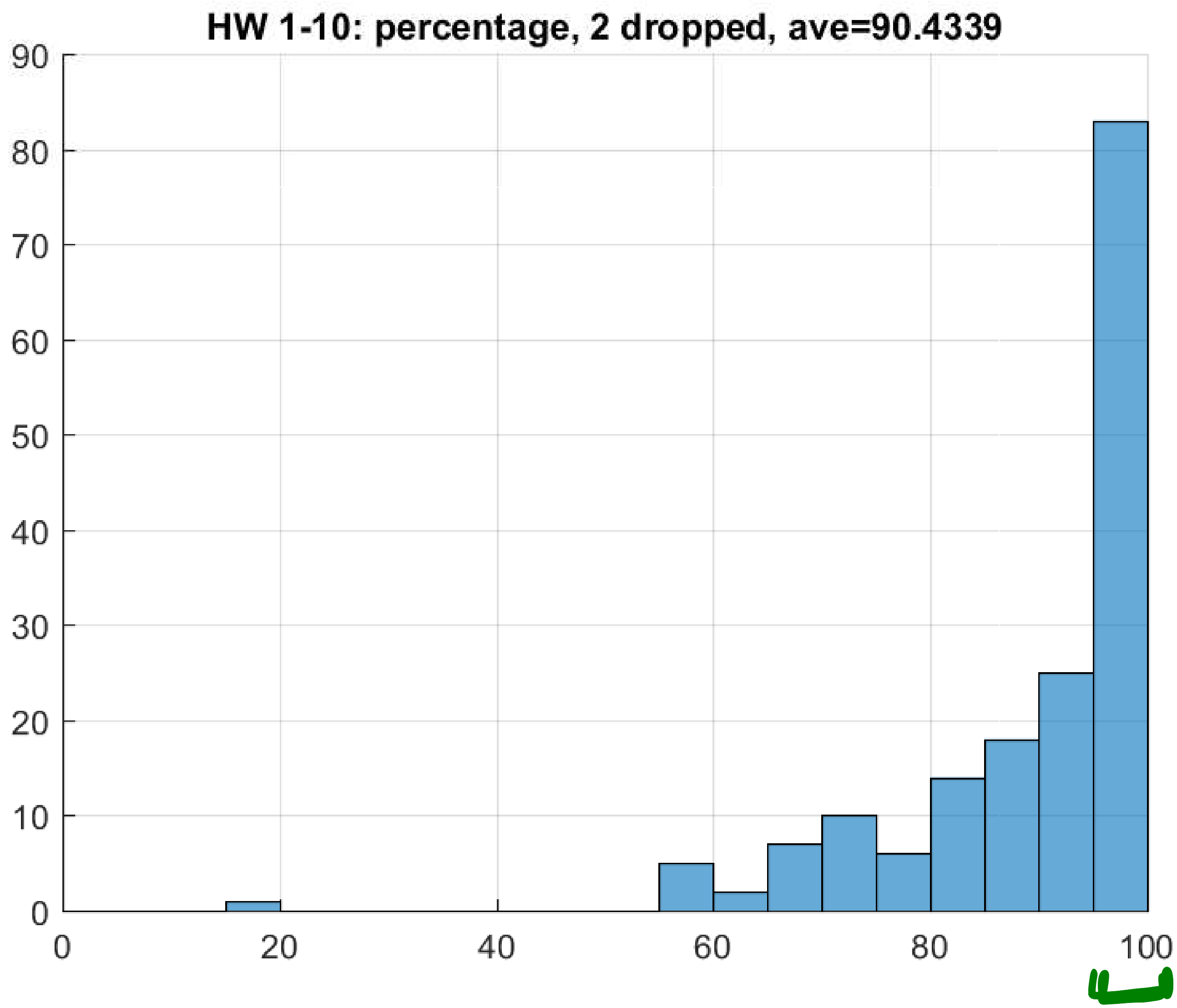
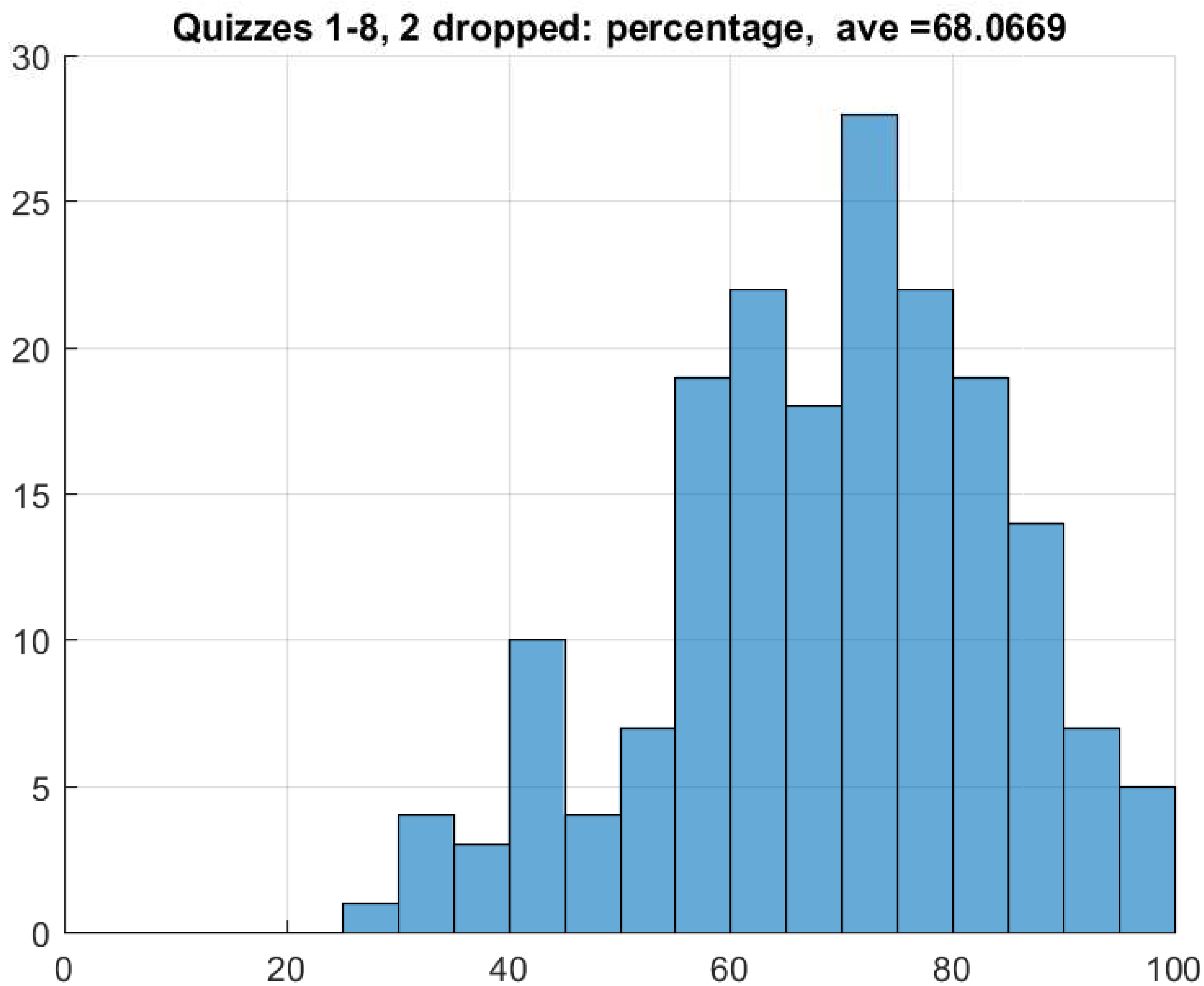
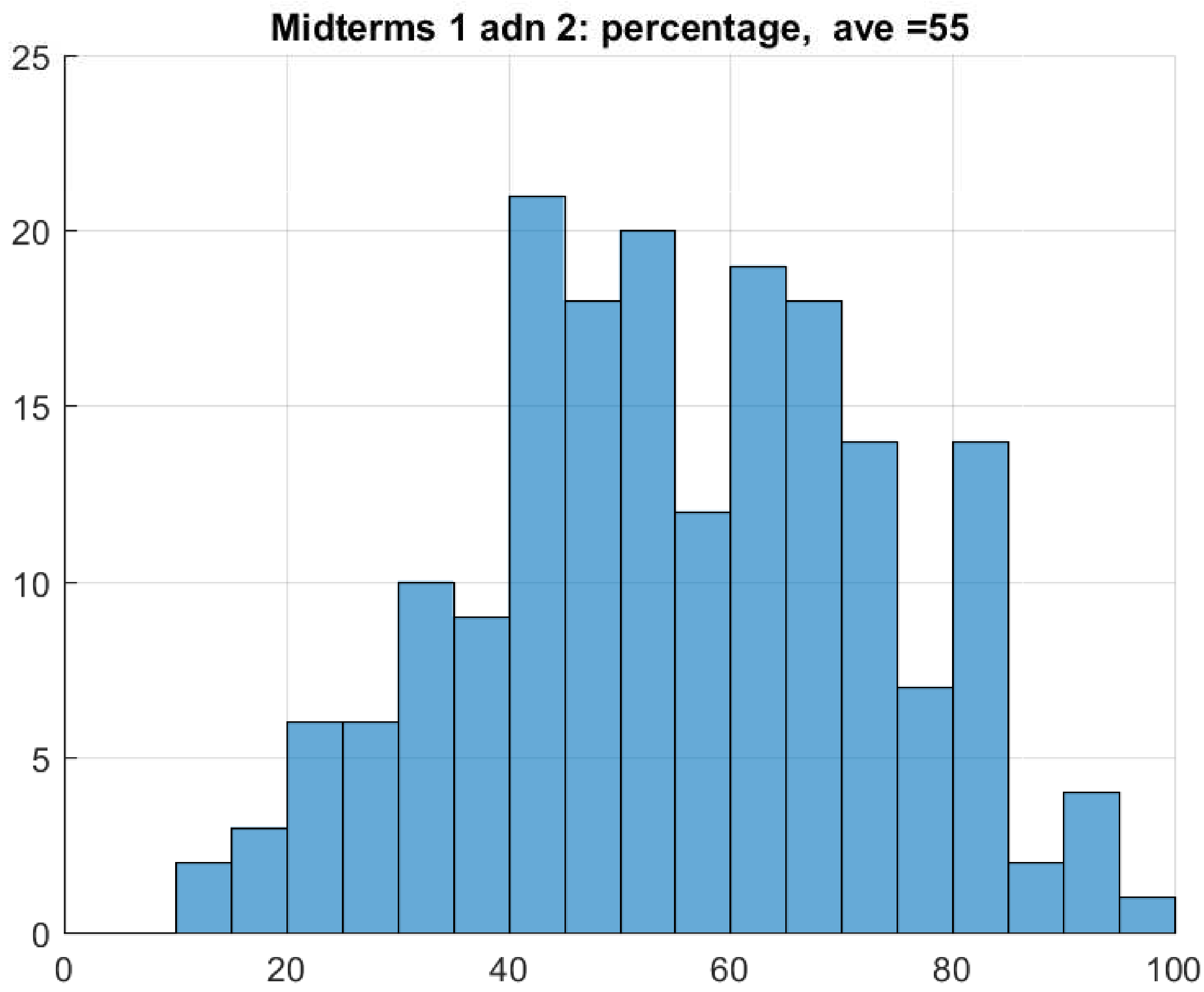
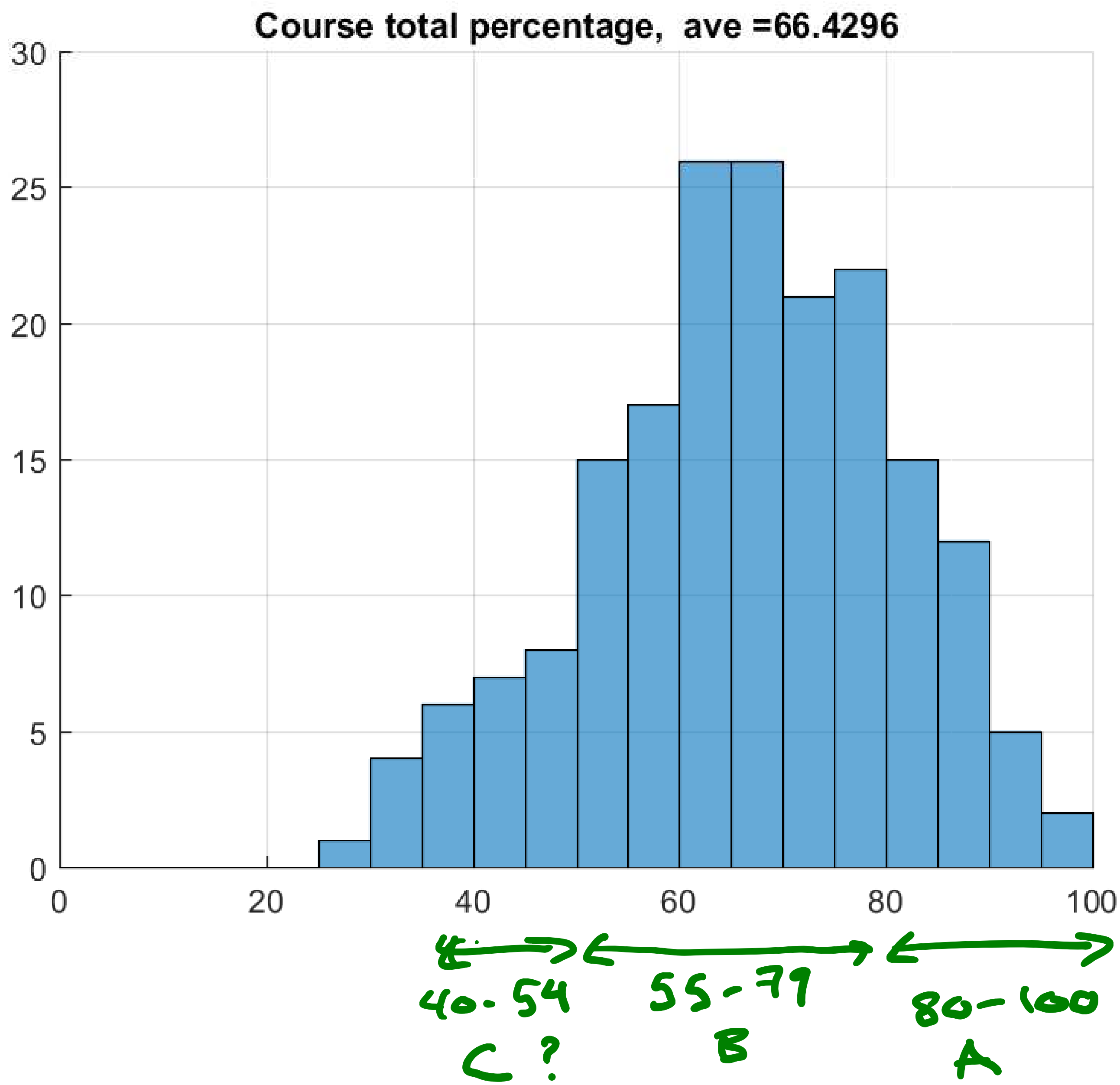
MAT 126.01, Prof. Bishop, Thursday, 12, 2020
Midterm 3 review


HW 1-10: percentage, 2 dropped, ave=90.4339
90
80
70
60
50
40
30
20
10
0
0
20
40
60
80
100


Quizzes 1-8, 2 dropped: percentage, ave =68.0669
30
25
20
15
10
5
0
0
20
40
60
80
100


Midterms 1 adn 2: percentage, ave =55
25
20
15
10
5
0
0
20
40
60
80
100



Course total percentage, ave =66.4296
0
5
10
15
20
25
30
0
20
40
60
80
100
40-54
C ?
55-79
B
80-100
A

Midterm is 25 points, 6 pages.

Several problems have multiple parts worth one point each.

No books or calculators are allowed.

One sheet (2-sided) of formulas and notes are allowed.

A sheet is on the webpage. You may bring it, or make your own.

- **Page 1:** 2 integration by parts, 1 theorem of Pappus
- **Page 2:** 3 problems on Newton's law of cooling.
- **Page 3:** 5 on center of mass. Compute area, integrals for M_y, M_x, give x, y coordinates.
- **Page 4:** 5 on partial fractions: long division, find A and B, integrate. Formula/Graph.
- **Page 5:** 4 problems: two indefinite integrals, then use to evaluate improper integral.
- **Page 6:** 5 on trig integrals: 3 choosing strategy. 2 setting up arclength, evaluate.

Page 1:

1. Integrate by parts: $\int \underbrace{x}_{u}\underbrace{e^x dx}_{dv}$.

$du = 1\,dx$

$v = e^x$

$= uv - \int v\,du$

$= xe^x - \int e^x 1\,dx$

$\boxed{= xe^x - e^x}$

$\downarrow$

$1 \cdot \cancel{e^x} + xe^x - \cancel{e^x}$ ✓

2. Use integration by parts to evaluate $\int x^2 \sin(2x)dx$.

$$\int \underbrace{x^2}_{u} \underbrace{\sin(2x)dx}_{dv}$$

$$du = 2x\,dx \qquad v = \frac{-1}{2}\cos(2x)$$

$$= uv - \int v\,du$$

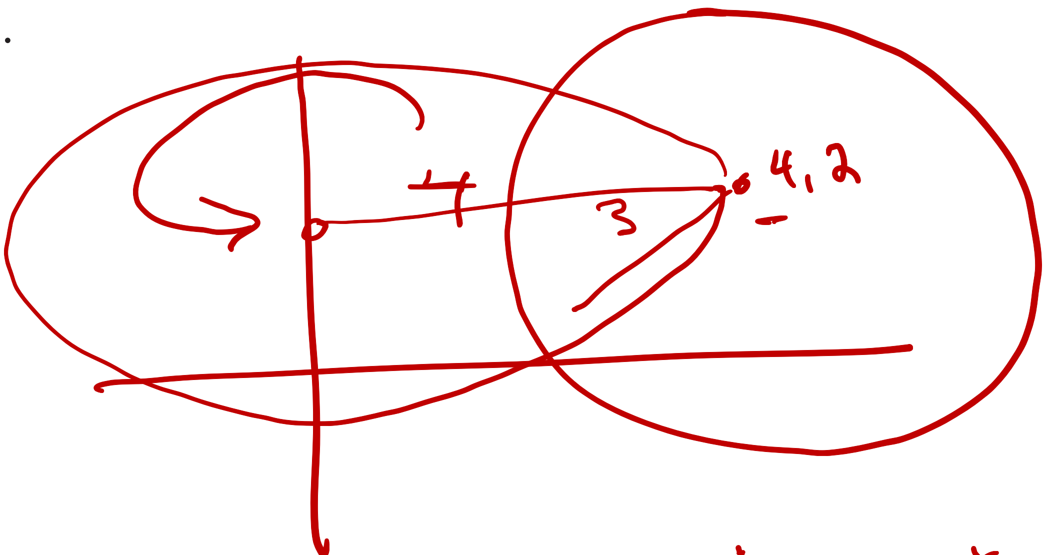
$$= x^2\left(-\frac{1}{2}\cos(2x)\right) - \int \left(-\frac{1}{\cancel{2}}\right)\cos(2x)(\cancel{2}x)\,dx$$

$$= -\frac{x^2}{2}\cos(2x) + \underbrace{\int x\cos(2x)\,dx}_{\substack{u = x \qquad dv = \cos(2x) \\ du = dx \qquad v = \frac{1}{2}\sin 2x}}$$

$$= -\frac{x^2}{2}\cos(2x) + \left[\frac{x}{2}\sin 2x - \underbrace{\int 1 \cdot \frac{1}{2}\sin 2x}_{-\frac{1}{2}\cdot\frac{1}{2}\cos(2x)}\right]$$

$$= -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) - \frac{1}{4}\cos(2x)$$

3. Let S be a disk of radius 3 centered at $(x, y) = (4, 2)$. Use the theorem of Pappus to compute the volume obtained by rotating S around the y-axis.



Vol = Area × dist traveled by center of mass

$= (\pi \cdot 9) \cdot (2\pi \cdot 4)$

$= 9 \cdot \pi \cdot 8\pi$

$\boxed{= 72\pi^2.}$

Page 2:

4. A turkey at 60° is placed in a 400° degree oven. Give the formula for the turkey's temperature at time t according to Newton's law of cooling.

T_0 T_a

$$T = (T_0 - T_a)e^{-kt} + T_a$$

$$T = \boxed{-340e^{-kt} + 400 = 400 - 340e^{-kt}}$$

5. If the turkey in the previous problem is at 150° degrees after one hour, what is the value of $k > 0$?

$$150 = 400 - 340e^{-k\cdot 1}$$

$$340e^{-k} = 250$$

$$e^{-k} = \frac{250}{340}$$

$$-k = \ln\frac{250}{340}$$

$$k = -\ln\frac{250}{340} = \boxed{\ln\frac{340}{250}}$$

6. Using the equation from Problem 4, when does the turkey reach 200°? (leave k as a symbol)

$$200 = 400 - 340e^{-kt}$$

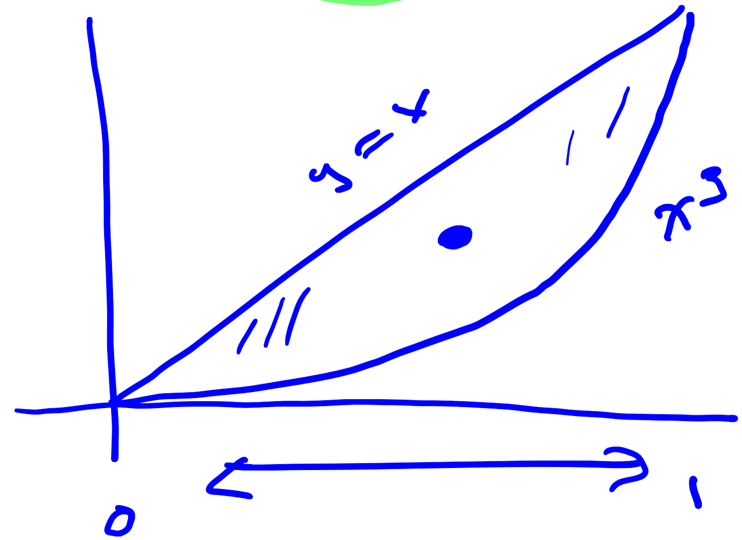
$$\cancel{340}e^{-kt} = \frac{200}{340}$$

$$-kt = \ln 200/340$$

$$\boxed{t = \frac{\ln\frac{340}{200}}{k}}$$

Page 3:

7. What is the area of the region $S = \{(x, y) : 0 \le x \le 1, x^3 \le y \le x\}$?

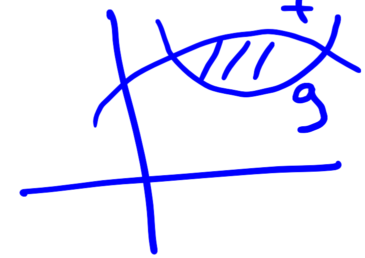


$$\text{Area} = \int_0^1 x - x^3 \, dx$$

$$= \frac{1}{2}x^2 - \frac{1}{4}x^4 \Big|_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) - (0 - 0) = \boxed{\frac{1}{4}}$$

8. For the region above, what is the formula for M_y, the moment around the y-axis?



$$M_y = \int_0^1 x\,(f(x) - g(x))\,dx$$

$$= \boxed{\int_0^1 x\,(x - x^3)\,dx}$$

$$= \int_0^1 x^2 - x^4 = \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 \Big|_0^1$$

9. For the region above, what is the formula for M_x, the moment around the x-axis?

$$M_x = \int_0^1 \tfrac{1}{2}\left(f(x)^2 - g(x)^2\right)dx$$

$$\boxed{= \tfrac{1}{2}\int_0^1 x^2 - x^6\,dx}$$

10. For the region above, what $\overline{x}$, the x-coordinate of the center of mass?

$$\overline{x} = \frac{M_y}{\text{area}} = \frac{2/15}{1/4} = 4 \cdot \frac{2}{15} = \boxed{8/15}$$

$$M_y = \int_0^1 x^2 - x^4 = \frac{1}{3}x^3 - \frac{1}{5}x^5 \Big|_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{5}\right) - (0-0)$$

$$= \frac{5-3}{15} = \frac{2}{15}$$

11. For the region above, what $\overline{y}$, the y-coordinate of the center of mass?

$$\overline{y} = \frac{M_x}{\text{area}} = \frac{2/21}{1/4} = 4\,\frac{2}{21} = \boxed{8/21}$$

$$M_y = \frac{1}{2}\int_0^1 x^2 - x^6 = \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{1}{7}x^7\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{7}\right)$$

$$= \frac{1}{2}\left(\frac{7-3}{21}\right) = \frac{1}{2}\,\frac{4}{21} = \frac{2}{21}$$

Page 4: Problems 12-15 all involve the same rational function and each step depends on the previous ones. Take extra care to check your answers, e.g., put your answers over a common denominator or plug in some values to check them.

12. Use long division of polynomials to write $r(x) = (2x^3+3x^2-4)/(x^2-4)$ in the form $p_1(x) + p_2(x)/q(x)$ where p_2 has lower degree than q.

$$
\begin{array}{r|l}
 & 2x + 3 \\
\hline
x^2-4 & 2x^3 + 3x^2 + 0x - 4 \\
 & \underline{2x^3 \qquad\quad - 8x \qquad} \\
 & 0 \quad 3x^2 + 8x - 4 \\
 & \quad \underline{3x^2 \qquad\quad - 12} \\
 & \quad 0 \quad 8x + 8
\end{array}
$$

$$= \underbrace{2x+3}_{} + \underbrace{\frac{8x+8}{x^2-4}}_{}$$

Write the fractional part of r in its partial fraction expansion:

$$\frac{p_2(x)}{q(x)} = \frac{A}{X-2} + \frac{B}{X+2}.$$

13. Find A.

14. Find B.

$$\cancel{(x-2)}\cancel{(x+2)}\frac{8x+8}{\cancel{(x-2)}\cancel{(x+2)}} = \frac{A\,\cancel{(x-2)}(x+2)}{\cancel{x-2}} + \frac{B\,(x-2)\cancel{(x+2)}}{\cancel{x+2}}$$

$$8x+8 = A(x+2) + B(x-2)$$
$$= (A+B)x + 2A - 2B$$

$x = 2$

$8 \cdot 2 + 8 = A\,4 + B \cdot 0$

$24 = 4A$

$6 = A$

$x = -2$

$-16 + 8 = 0A - 4B$

$-8 = -4B$, $B = 2$

$8 = A + B.$

$\cancel{8 = 2A - 2B}$

$+$

$4 = A - B$

$12 = 2A + 0$

$\boxed{6 = A}$, $\boxed{B = 2}$

$\therefore$?

15. Compute $\int_0^1 \frac{2x^3+3x^2-4}{x^2-4}dx$.

$$\int_0^1 2x+3+\frac{6}{x-2}+\frac{2}{x+2}$$

$$\left[x^2+3x+6\ln|x-2|+2\ln|x+2|\right]_0^1$$

$$=\left[1+3+6\cdot 0+2\ln 3\right]-\left[0+0+6\ln 2+2\ln 2\right]$$

$$\boxed{=4+2\ln 3-8\ln 2}$$

16. $\boxed{d}$ Which a possible partial fraction expansion for the graph below?

(a) $\dfrac{A}{x-4} + \dfrac{Bx+C}{(x+4)^2} + \dfrac{D}{1+(x+1)^2}$

(b) $\dfrac{A}{x+3} + \dfrac{Bx+C}{x^2} + \dfrac{D}{1+(x-3)^2}$

(c) $\dfrac{A}{x-2} + \dfrac{Bx+C}{(x+4)^2} + \dfrac{D}{1+(x+1)^2}$

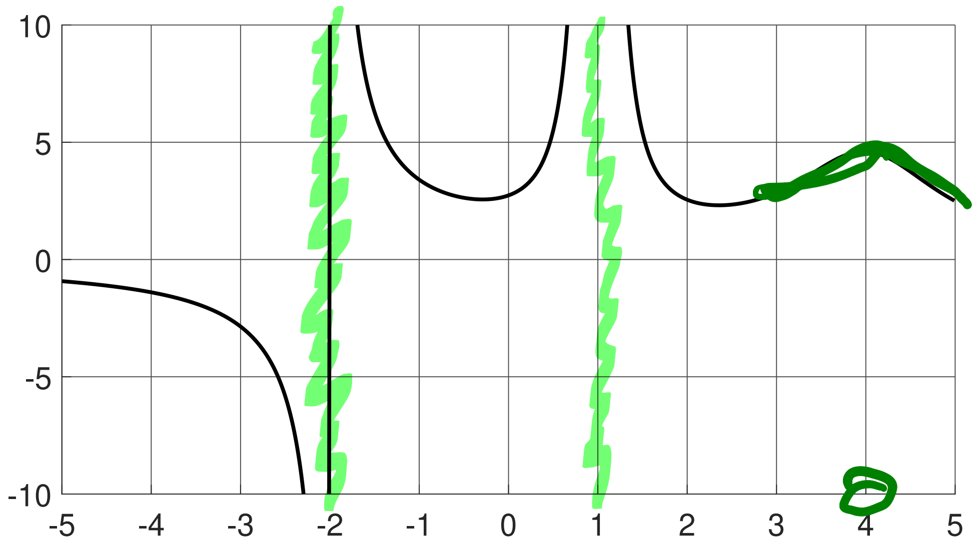
(d) $\dfrac{A}{x+2} + \underbrace{\dfrac{Bx+C}{(x-1)^2}}_{\checkmark} + \underbrace{\dfrac{D}{1+(x-4)^2}}_{\checkmark}$

(e) $\dfrac{A}{x+4} + \dfrac{Bx+C}{(x-4)^2} + \dfrac{D}{1+(x+1)^2}$

(f) $\dfrac{A}{x+4} + \dfrac{Bx+C}{(x-3)^2} + \dfrac{D}{1+x^2}$

(g) $\dfrac{A}{x+5} + \dfrac{Bx+C}{(x+2)^2} + \dfrac{D}{1+(x-4)^2}$

(h) none of these



$\dfrac{1}{x+2}$ $\qquad$ $\dfrac{1}{(x-1)^2}$

Page 5:

17. Evaluate $\int x^3 e^{-x^2} dx$.

$$\int \underbrace{x^2}_{u} \cdot \underbrace{xe^{-x^2}}_{dv}$$

$du = 2x\,dx$

$v = -\frac{1}{2} e^{-x^2}$

$$= x^2\left(-\frac{1}{2} e^{-x^2}\right) + \int \left(\cancel{+}\cancel{\tfrac{1}{2}} e^{-x^2}\right) \cancel{2}x\,dx$$

$$= -\frac{x^2}{2} e^{-x^2} + \int x e^{-x^2} dx$$

$$\boxed{= -\frac{x^2}{2} e^{-x^2} - \frac{1}{2} e^{-x^2}}$$

18. Evaluate the improper integral $\int_1^\infty x^3 e^{-x^2} dx$.

$$= \lim_{t \to \infty} \int_1^t \underline{\qquad} = e^{-x^2}\left(-\frac{x^2}{2} - \frac{1}{2}\right)\Big|_1^t$$

$$= \underbrace{e^{-t^2}\left(-\frac{t^2}{2} - \frac{1}{2}\right)}_{\to 0} - \underbrace{e^{-1}\left(-\frac{1}{2} - \frac{1}{2}\right)}$$

$$= \boxed{e^{-1} = 1/e}$$

19. Evaluate $\int \frac{dx}{x\sqrt{\ln x}}$.

$$= \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

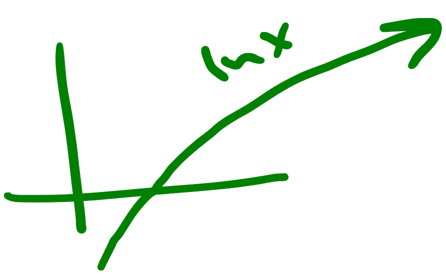
$$= 2u^{1/2}$$

$$= \boxed{2\sqrt{\ln x}}$$

$$\approx \int_2^\infty \frac{1}{x} = \infty$$

20. Evaluate the improper integral $\int_2^\infty \frac{dx}{x\sqrt{\ln x}}$.

$$= \lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \underbrace{2\sqrt{\ln t}} - \underbrace{2\sqrt{\ln 2}}$$



$$= \infty$$

Diverges

Page 6: For each integral, select the appropriate strategy from the list.

21. a $\int \cos^6 x \sin^9 x dx$ 22. i $\int \tan^5 x dx$ 23. f $\int \cos(10x)\cos(7x)dx$

Trigonometric integration strategies:

(a) Replace $\sin^2 x$ by $1 - \cos^2 x$ and the use substitution $u = \cos x$.

(b) Replace $\cos^2 x$ by $1 - \sin^2 x$ and the use substitution $u = \sin x$.

(c) Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ or $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, then (a) or (b).

(d) Use $\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$.

(e) Use $\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$.

(f) Use $\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$.

(g) Rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$. Then let $u = \tan x$.

(h) Rewrite $\tan^k x \sec^j x = \tan^{k-1} \sec^{j-1} x \tan x \sec x$ and use $\tan^2 = \sec^2 - 1$.
Then use $u = \sec x$

(i) Use $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x(\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$.
Repeat if necessary.

(j) Use $\tan^2 x = \sec^2 x - 1$. Then integrate by parts the powers of $\sec x$.

24. What integral gives the arclength of $y = x^2$ over $[0, 1/2]$?

$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx \qquad \boxed{\int_0^{1/2} \sqrt{1+4x^2}\,dx}$$

$$f(x) = x^2 \qquad f'(x) = 2x$$

25. Use a trigonometric substitution to compute this arclength.

$$\sqrt{1+a^2x^2}$$

$$x = \frac{1}{a}\tan\theta$$

$$= \frac{1}{2}\tan\theta$$

$$dx = \frac{1}{2}\sec^2\theta$$

$$x = 0 \qquad \theta = 0$$

$$x = 1/2$$

$$\frac{1}{2} = \frac{1}{2}\tan\theta$$

$$1 = \tan\theta$$

$$\pi/4 = \theta$$

$$= \int \sqrt{1+4\left(\tfrac{1}{2}\tan\right)^2}\,\frac{1}{2}\sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sec\cdot\sec^2\,d\theta$$

see below

$$= \frac{1}{2}\int_0^{\pi/4} \sec^3\theta\,d\theta$$

$$= \frac{1}{2}\Big[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4}$$

$$= \frac{1}{2}\Big[\frac{1}{2}\sqrt{2}\cdot 1 + \frac{1}{2}\ln(\sqrt{2}+1)\Big]$$

$$-\frac{1}{2}\left[0+\frac{1}{2}\ln|1+0|\right]$$

$$\ln 1$$

$$0$$

$$=\boxed{\frac{1}{4}\sqrt{2}+\frac{1}{4}\ln\left(\sqrt{2}+1\right)}$$

$$\int\sqrt{1+4x^2}$$

$$1+a^2x^2$$

$$x=\frac{1}{a}\tan\theta$$

$$a=2$$

$$\frac{\sin^2}{\cos^2}+\frac{\cos^2}{\cos^2}=\frac{1}{\cos^2}$$

$$\tan^2+1=\sec^2$$

$$\sqrt{1+4\left(\frac{1}{2}\tan\theta\right)^2}$$

$$\sqrt{1+4\frac{1}{4}\tan^2\theta}$$

$$\sqrt{1+\tan^2\theta}$$

$$\sqrt{\sec^2\theta}$$

$$\sec\theta$$

MAT 126 Nov 12

Office Hours

start $\approx$ 11:20

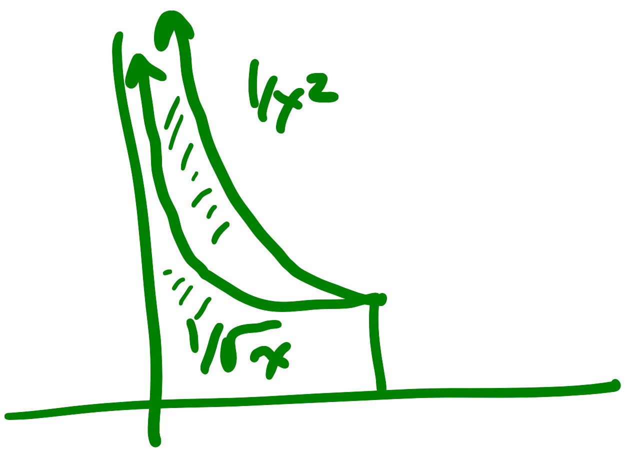
Say whether each improper integral converges or diverges.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx$$

Converges

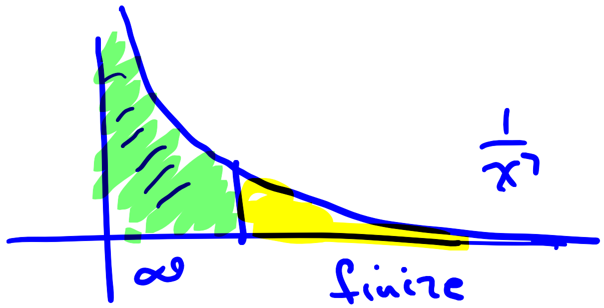
$$\int_0^1 x^p\,dx = \begin{cases} \infty & p \le -1 \\ \cancel{\infty}\ \text{converges} & p > -1 \end{cases}$$

$$\int_0^1 \frac{1}{x^2}\,dx = \int_0^1 x^{-2}\,dx = \text{Divergent.}$$



$$\int_0^1 \frac{1}{x} = \infty$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \int_1^\infty x^{-2}\,dx = \boxed{\text{Converges}}$$



$$\int_1^\infty x^{p} = \begin{cases} \infty & p \geq -1 \\ \text{finite} & p < -1 \end{cases}$$

$$\int_1^\infty \frac{x^6}{x^8+1}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx < \infty$$

$$\leq \frac{x^6}{x^8} = \frac{1}{x^2}$$

Converges.

$$\int_1^\infty \frac{x^5+3x^4+2x+5}{3x^6+x^2+10x+2}\,dx = \text{Diverges}$$

$$\approx \frac{x^5}{3x^6} = \frac{1}{3x} = \frac{1}{3}\cdot\frac{1}{x}$$

$$\int_1^\infty \underbrace{x^{100}}\,\underbrace{e^{-x}}\,dx \le \int_1^\infty x^{100}x^{-102}$$

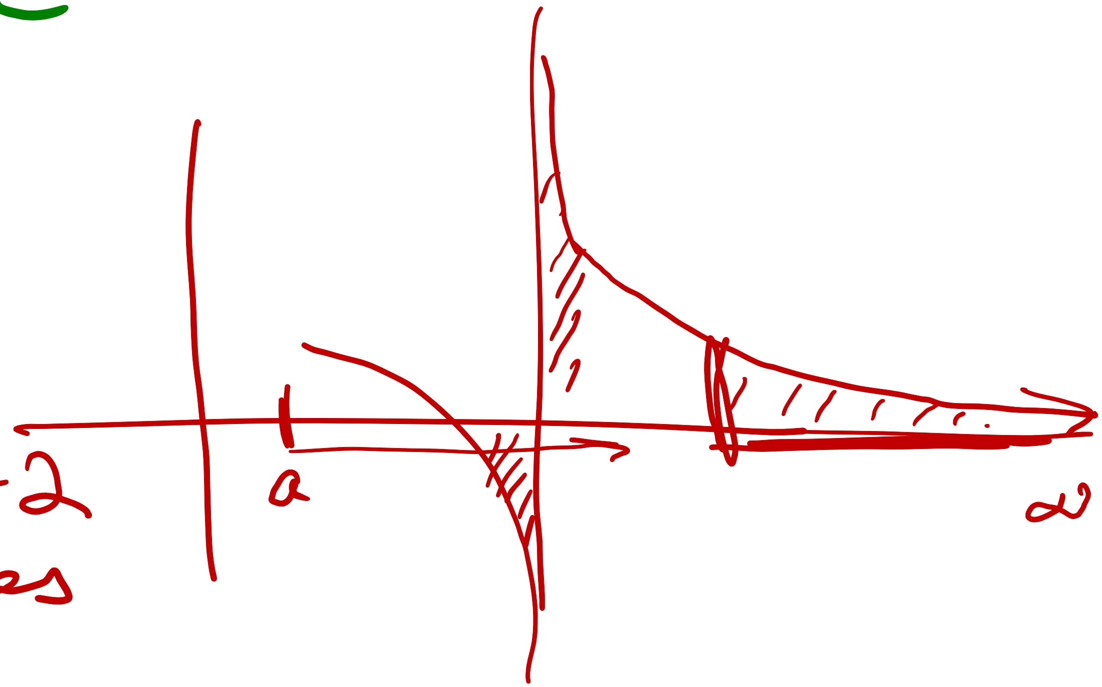
$$= \int_1^\infty \frac{1}{x^2} < \infty$$

if x is big enough $x \ge a$

$$e^{-x} < x^{-102}$$

$\boxed{\text{converges}}$

$$\int_0^\infty \frac{P(x)}{q(x)}$$

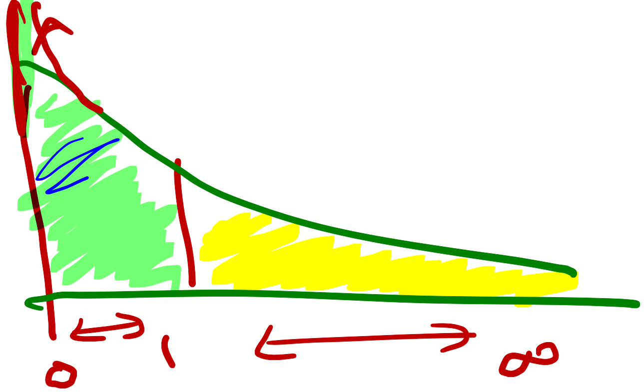


$\deg(P) \le \deg(q) - 2$

converges

$\deg(P) \ge \deg(q) - 1$

diverges



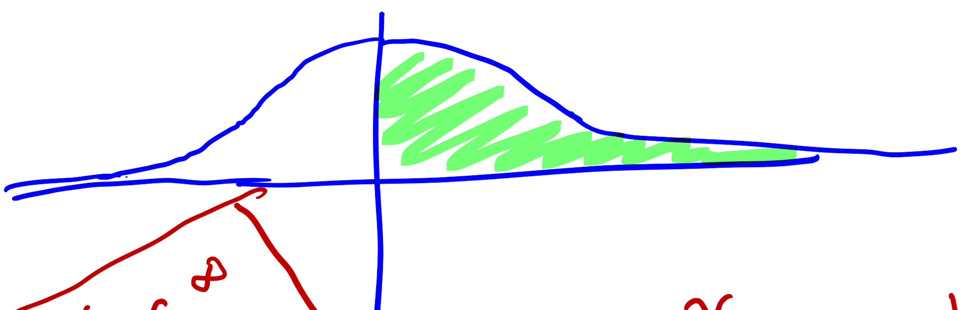
$$\int_0^\infty \frac{1}{x^2}\,dx$$

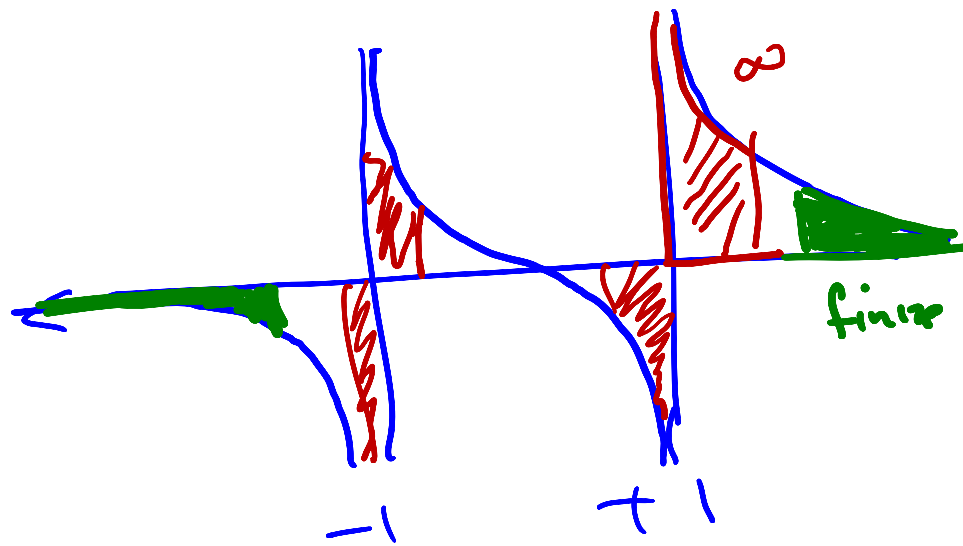
For any p either

$$\int_0^1 x^p\,dx = \infty \quad \text{or} \quad \int_1^\infty x^p\,dx = \infty$$

$$\int_0^\infty \frac{1}{x^2+1}\,dx < \infty$$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$$





$$\boxed{\int_0^1 \quad \int_0^\infty \quad \int_1^\infty}$$

$$\int_a^\infty \frac{x}{x^2-1} \approx \frac{1}{x}$$