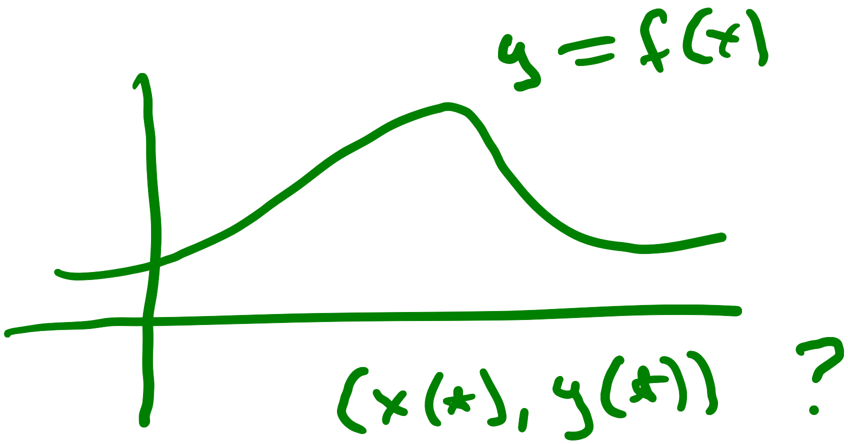
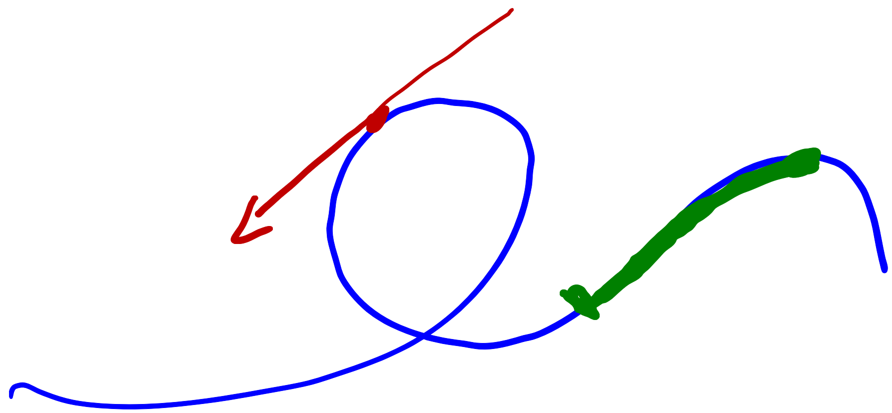
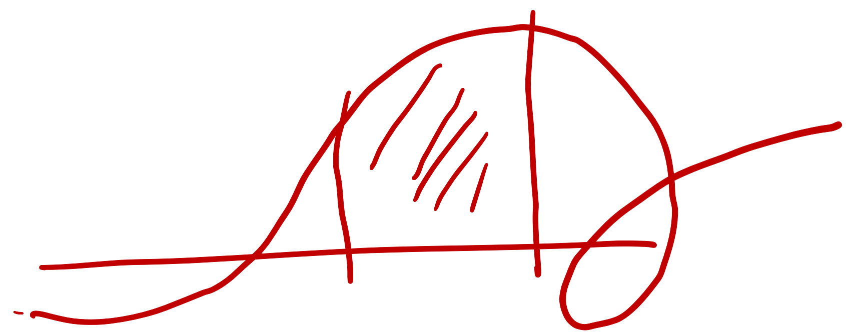
MAT 126.01, Prof. Bishop, Thursday, Nov 19, 2020
Section 7.2 Calculus of Parametric Curves

- Find derivatives and tangents to parametric curves
- Find area under a parametric curve.
- Arclength of parametric curve.
- Area of rotated parametric curve.

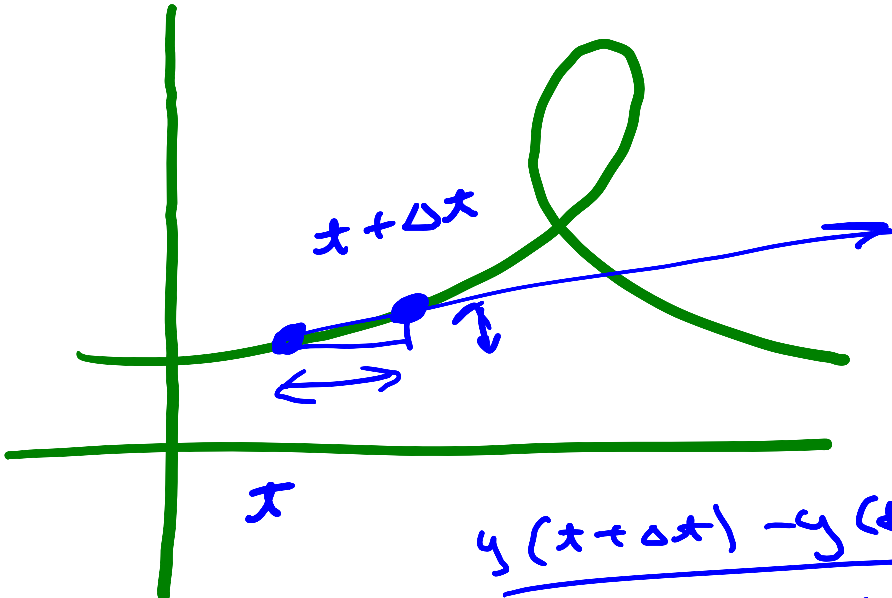






Derivatives of parametric equations:

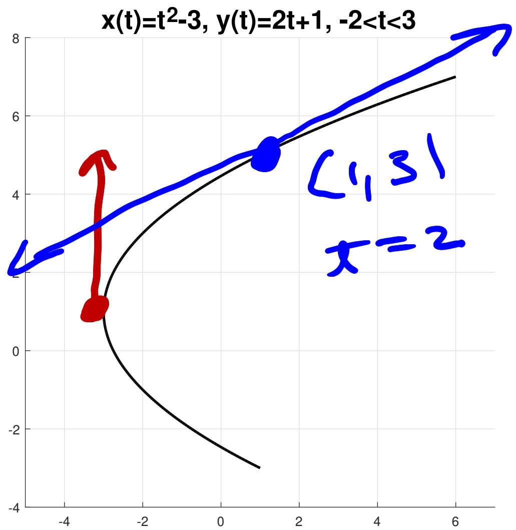
$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$



$$\frac{y(t+\Delta t)-y(t)}{x(t+\Delta t)-x(t)} = \frac{y(t+\Delta t)-y(t)}{\Delta t}\cdot\frac{\Delta t}{x(t+\Delta t)-x(t)}$$

$$\Rightarrow y'(t) \cdot \frac{1}{x'(t)}$$

Example: calcuate derivative at time t of $x(t) = t^2 - 3$, $y(t) = 2t - 1$.



$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2}{2t} = \frac{1}{t}$$

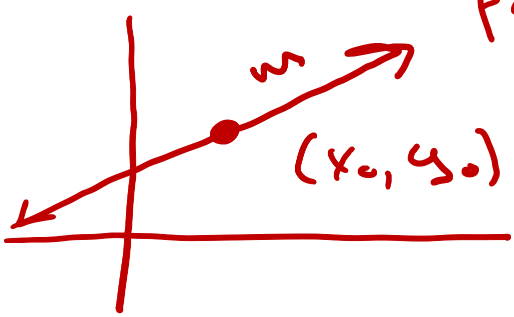
$$y'(t) = (2t-1)' = 2$$

$$x'(t) = (t^2-3)' = 2t$$

What is tangent line at $t = 2$?

point-slope formula

slope = m



$$x_0 = x(2) = 1$$

$$y_0 = y(2) = 3$$

$$(y - y_0) = m(x - x_0)$$

$$m = \frac{dy}{dx}(2) = \frac{1}{2}$$

$$(y-3) = \frac{1}{2}(x-1)$$

Second derivatives:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}$$

Find second derivative of

$$x(t) = t^2 - 3, \qquad y(t) = 2t - 1.$$

$$x'(t) = 2t \qquad y'(t) = 2$$

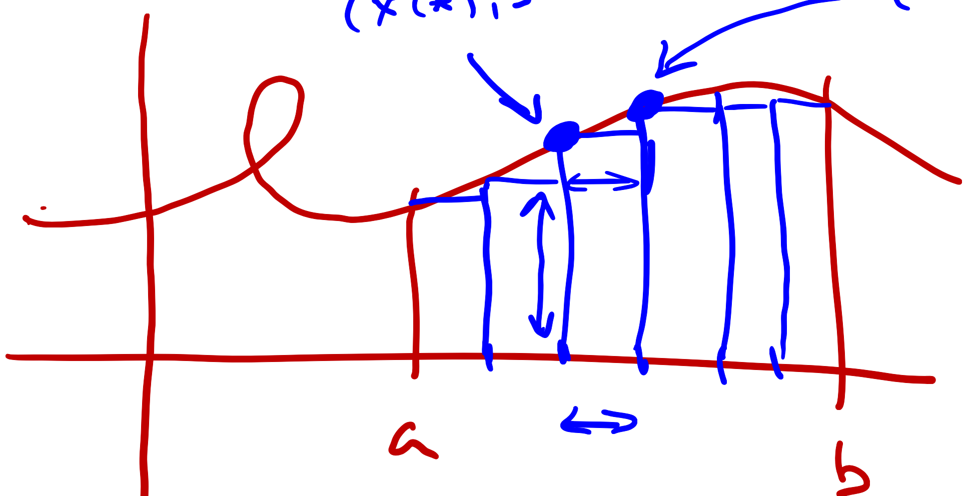
$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{2}{2t} = \frac{1}{t}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{1}{t^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-1/t^2}{2t} = -\frac{1}{2t^3}$$

Area under a parametric curve:

$$A = \int_a^b y(t)x'(t)dt.$$



$$\frac{x(t+\Delta t) - x(t)}{\Delta t} \to x'(t)$$

$$x(t+\Delta t) - x(t) \approx x'(t)\,\Delta t$$

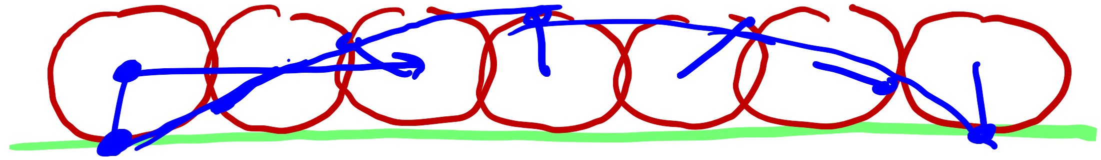
height $= y(t)$

width $= \underline{x(t+\Delta t) - x(t)} \approx x'(t)\,\Delta t$

area $\approx \sum y(t)\cdot x'(t)\,\Delta t$

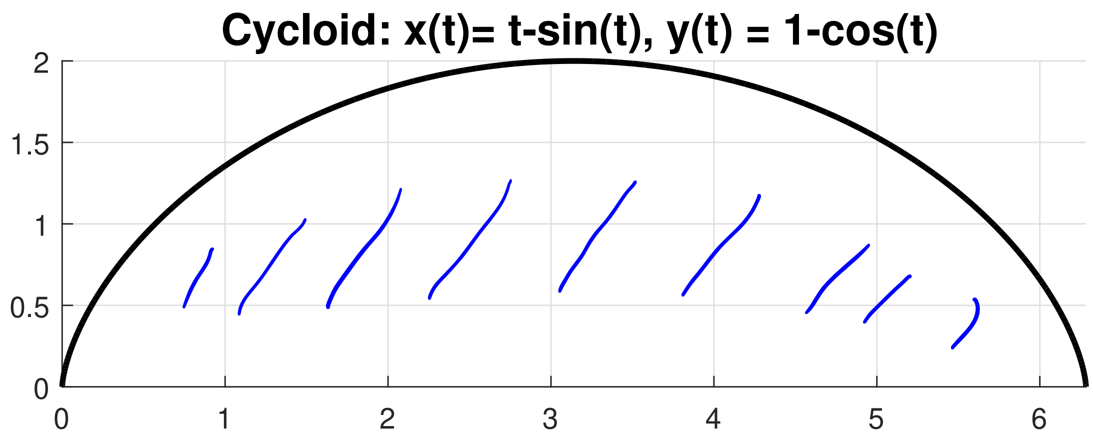
$$\to \int y(t)\,x'(t)\,dt$$

Area under a cycloid.



$$x(t) = t - \sin(t), \qquad y(t) = 1 - \cos(t).$$

$0 \le t \le 2\pi$



Area = 3π

$$\int_0^{2\pi} y(t)\, x'(t)\, dt = \int_0^{2\pi} (1-\cos t)(1-\cos t)\, dt$$

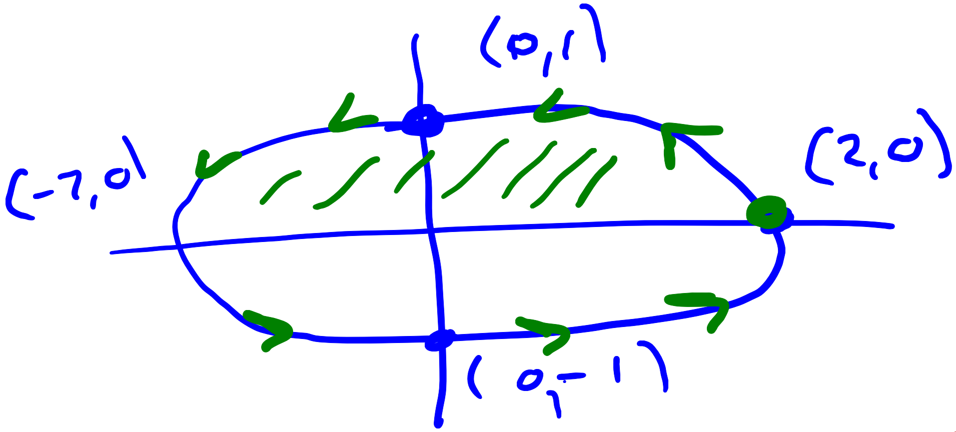
$$= \int 1 - 2\cos t + \cos^2 t \, dt$$

$$\int_0^{2\pi} 1 = 2\pi \qquad \int_0^{2\pi} -2\cos t = -2 \sin t \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \tfrac{1}{2}(1+\cos 2t)\, dt = \int_0^{2\pi} \tfrac{1}{2} + \frac{\cos 2t}{2}\, dt$$

$$= \pi + \frac{\sin 2t}{4}\Big|_0^{2\pi} = \pi + 0$$

What is the area of the ellipse $\frac{1}{4}x^2 + y^2 \leq 1$?



$x(t) = 2\cos(t)$

$y(t) = \sin t$

upper half $= \int_0^{\pi} y(t)\, x'(t)\, dt$

$= \int_0^{\pi} \sin t \cdot (-2\sin t)\, dt$

$= -2 \int_0^{\pi} \sin^2 t\, dt$

$= -2 \int_0^{\pi} \frac{1 - \cos 2t}{2}\, dt$

$\vdots$

$= \pi$

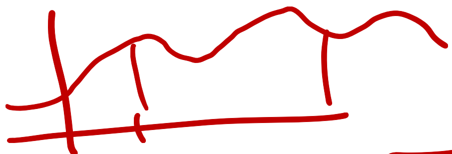
total area $= 2 \times$ upper $= 2\pi$

Arclength of a parametric curve:

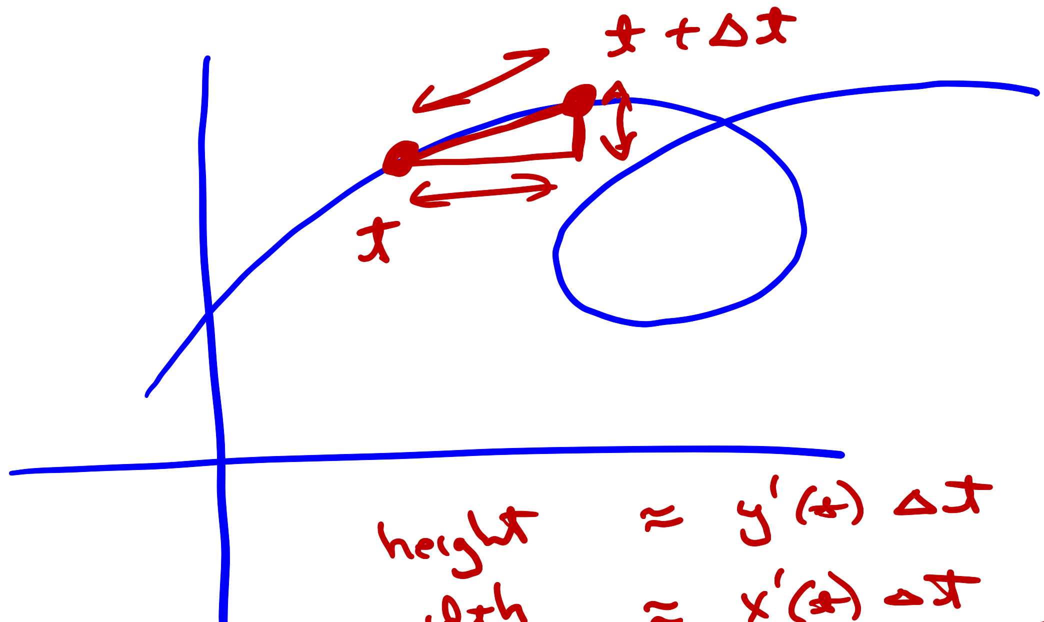
$$\text{length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

For graphs $y = f(x)$



$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx$$



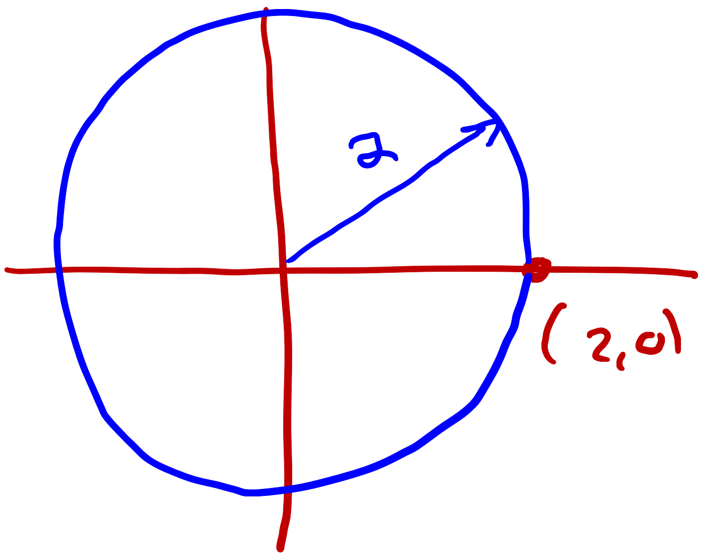
height $\approx y'(t)\,\Delta t$

width $\approx x'(t)\,\Delta t$

$$\text{hyp} = \sqrt{(x'\Delta t)^2 + (y'\Delta t)^2}$$

$$= \sqrt{(x')^2 + (y')^2} \cdot \Delta t$$

What is length of circle $x(t) = 2\cos(t)$, $y(t) = 2\sin(t)$. $0 \le t \le 2\pi$



circum $= 2\pi \cdot r$
$= 4\pi$

$$\text{length} = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}$$

$$x' = -2\sin t$$

$$y' = 2\cos t$$

$$= \int_0^{2\pi} \sqrt{(-2\sin)^2 + (2\cos)^2}\, dt$$

$$= 2 \int_0^{2\pi} \sqrt{\sin^2 + \cos^2}\, dt$$

$$= 2 \int_0^{2\pi} \sqrt{1}\, dt$$

$$= 2 \int_0^{2\pi} 1\, dt$$

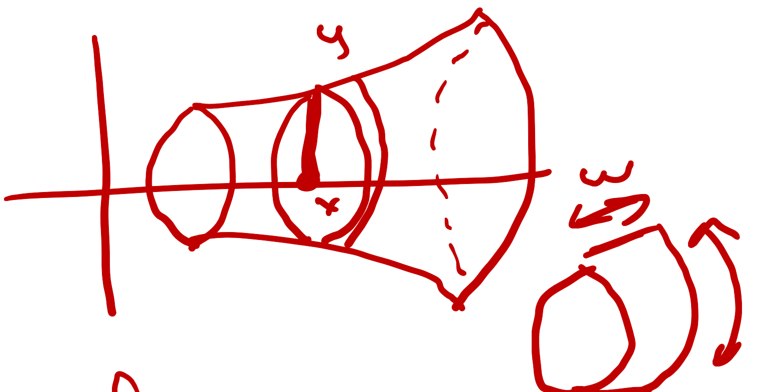
$$= 2 \cdot 2\pi = 4\pi$$

Surface area of revolution

$$\text{area} = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}$$

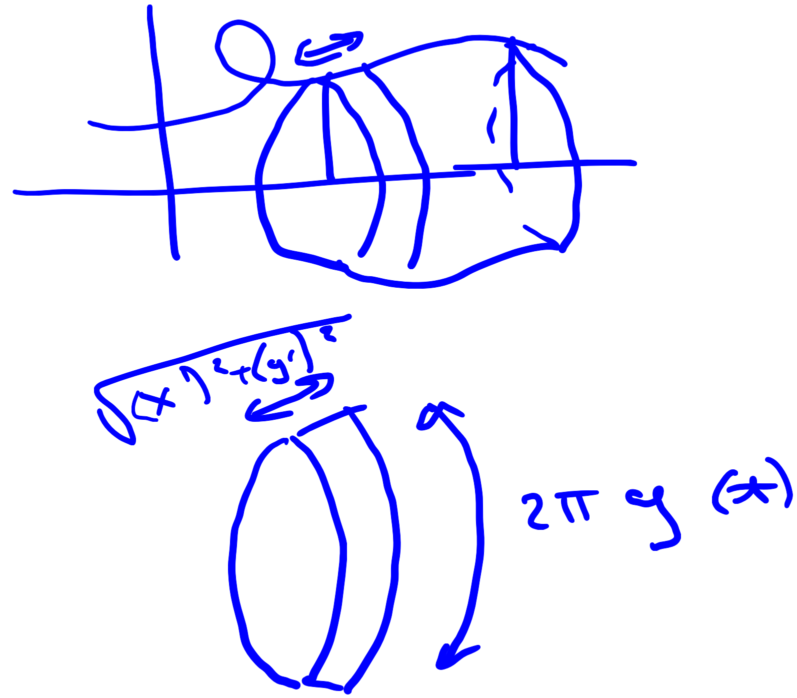
Graph case

$y = f(x)$

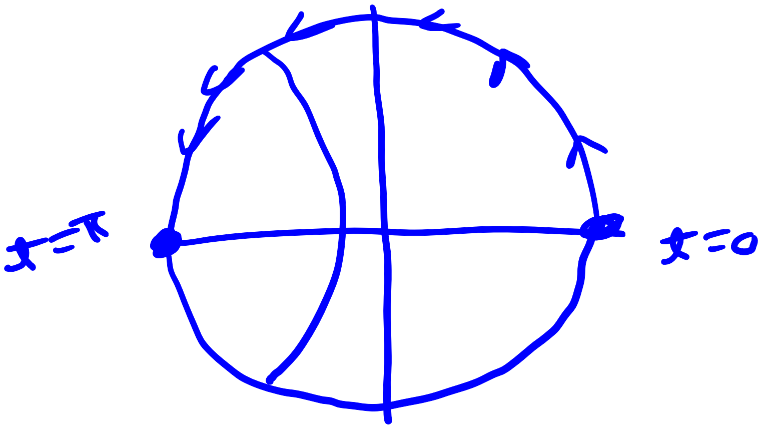


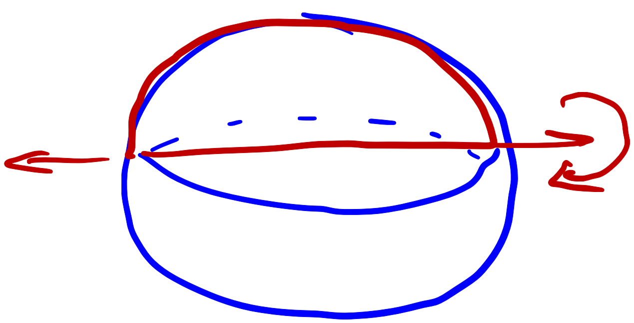
surface area

$$\int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx$$



Find surface area of a sphere of radius r.





$$x(t) = r\cos t \qquad x' = -r\sin t$$

$$y(t) = r\sin t \qquad y' = r\cos t$$

$$\text{Area} = \int_0^{\pi} 2\pi\, y(t)\sqrt{(x')^2+(y')^2}\,dt$$

$$= 2\pi \int_0^{\pi} (r\sin t)\sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt$$

$$= 2\pi r \int_0^{\pi} \sin t \; r\sqrt{\sin^2 t + \cos^2 t}$$

$$= 2\pi r^2 \int_0^{\pi} \sin t \sqrt{1}\,dt$$

$$= 2\pi r^2 \int_0^{\pi} \sin t$$

$$= 2\pi r^2 \left[-\cos t\right]_0^{\pi}$$

$$= 2\pi r^2 \left[-1 - -1\right] = \boxed{4\pi r^2}$$

After break:

1. no quiz
2. recitations <u>online</u>
3. 1 more HW on chap 7
4. Final online Dec 10, 2:15-5:00
   more details later
5. tentative letter grades?
6. Course Evaluations