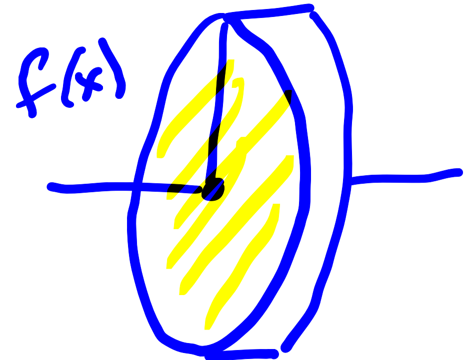
# MAT 126.01, Prof. Bishop, Thursday, Oct 1, 2020
# Section 2.3, Volumes by Shells
# Quiz 5 review

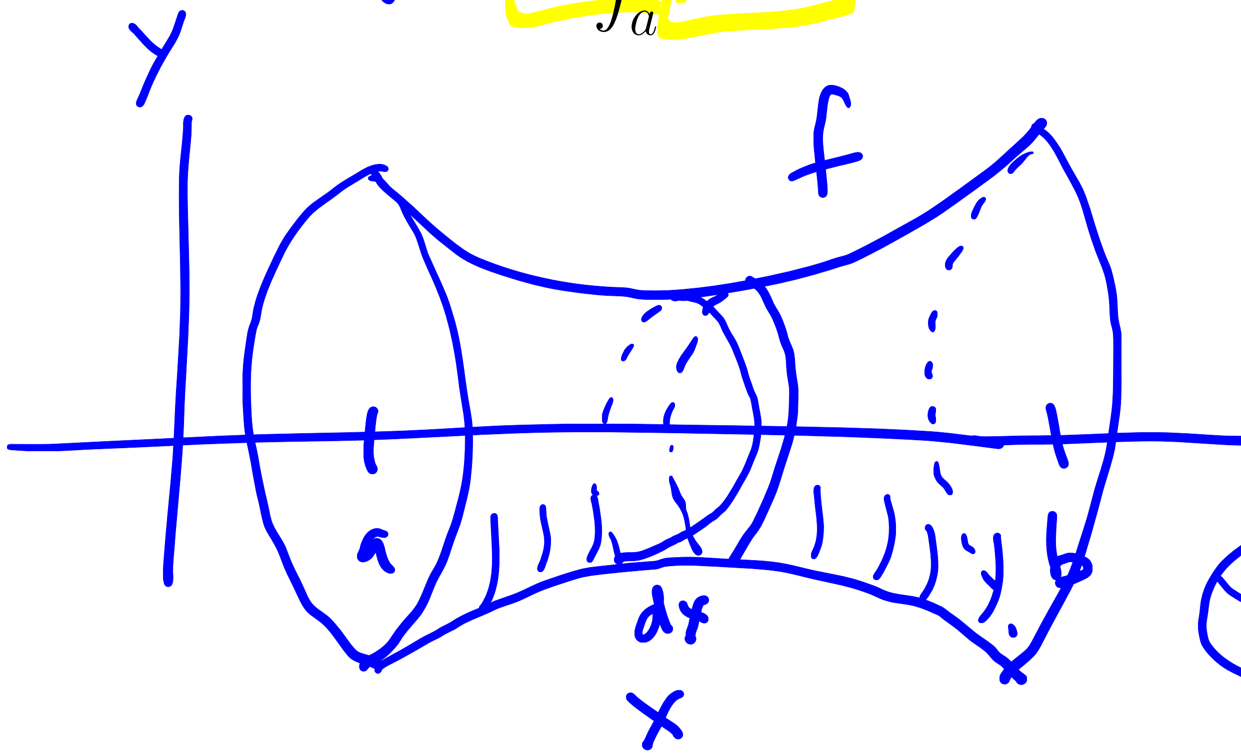
Recall disk method from last time.

If the graph of $f \geq 0$ on $[a, b]$ is revolved around the $x$-axis, the volume of the solid obtained is

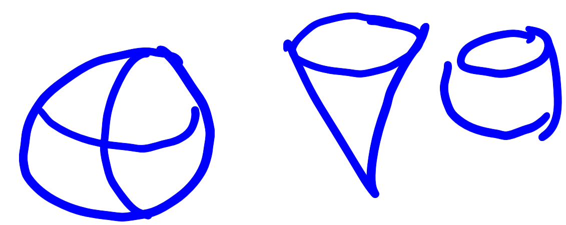
$V = \text{Area} \cdot dx$

$$V = \boxed{\pi} \int_a^b \boxed{f(x)^2}\, dx.$$



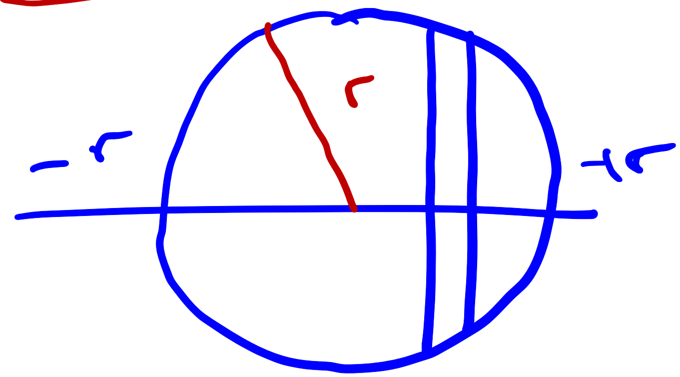



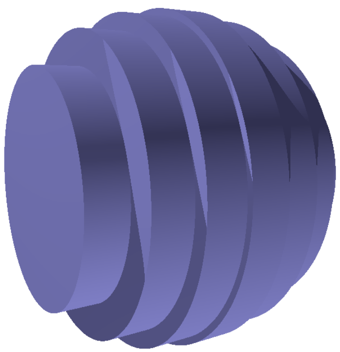

$V = \pi f(x)^2 dx$



Derive volume of sphere $V = \frac{4}{3}\pi r^3$ using disk method.

$f = \sqrt{r^2 - x^2}$



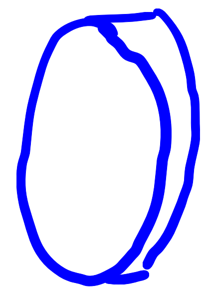



$$V = \pi \int_{-r}^{r} f(x)^2 \, dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2) \, dx$$

$$\vdots$$

$$= \frac{4}{3} \pi r^3$$



"disk"

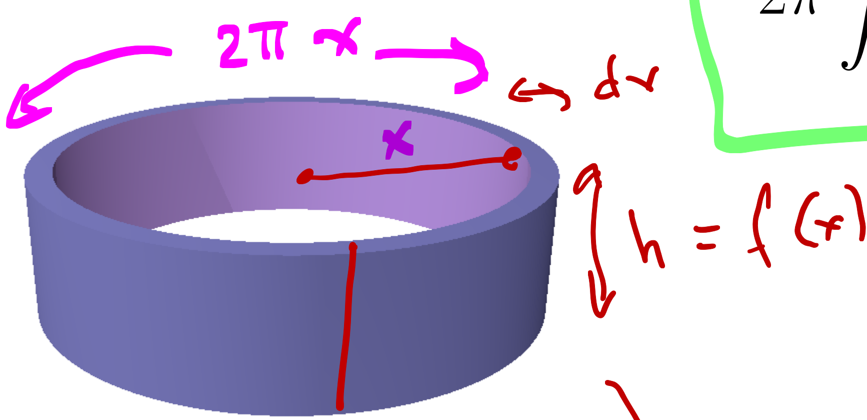
$\cancel{f(x)}$ $x f(x)$

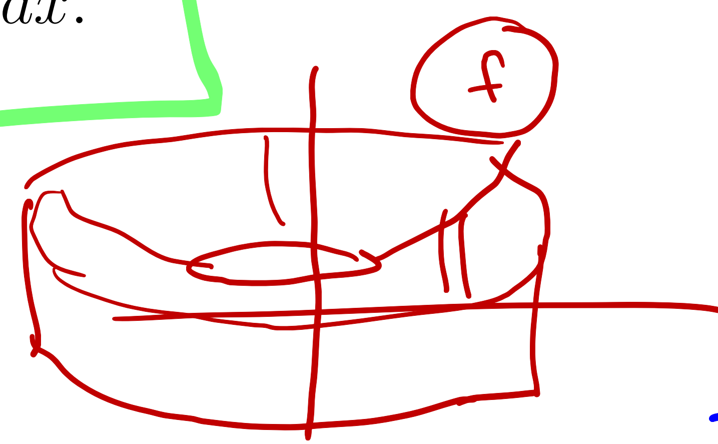
Volumes of revolution: shell method.

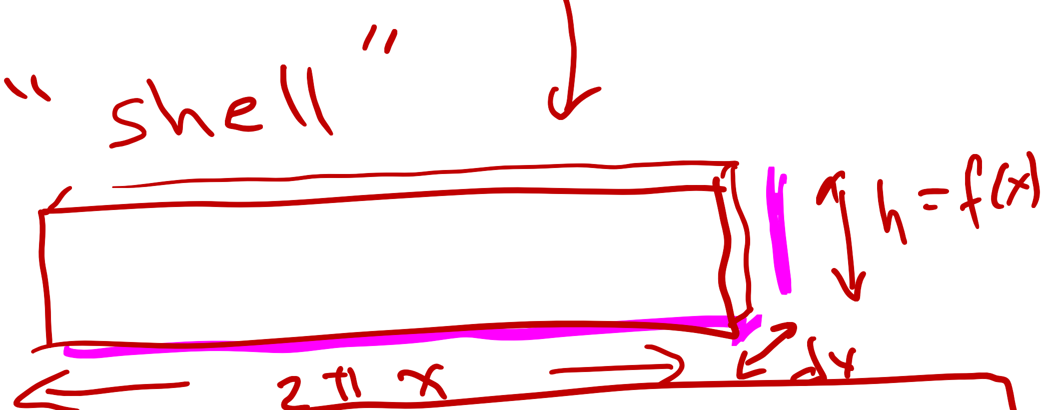
If the graph of $f \geq 0$ on $[a, b]$ is revolved around the $y$-axis, the volume of the solid obtained is

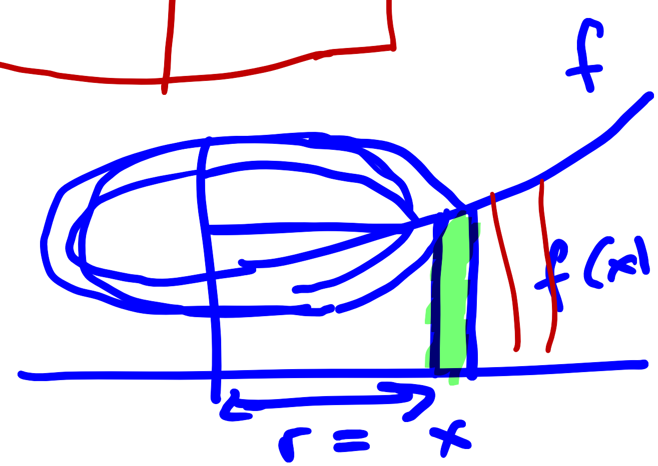
$$\boxed{2\pi \int_a^b x f(x) dx.}$$

✓

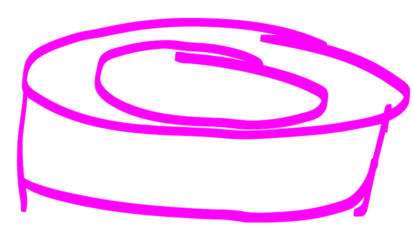


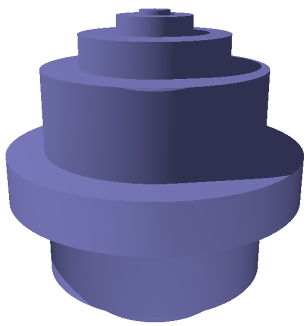



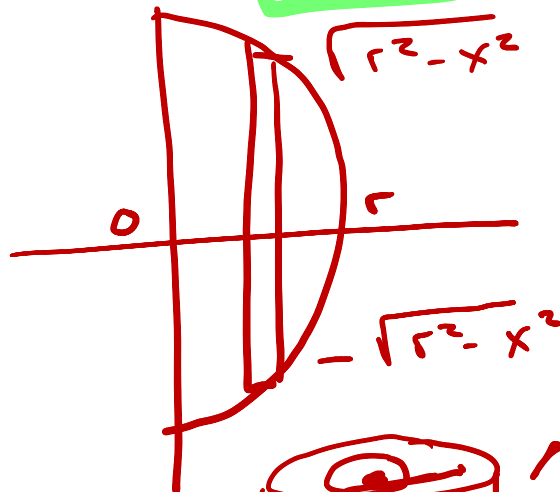



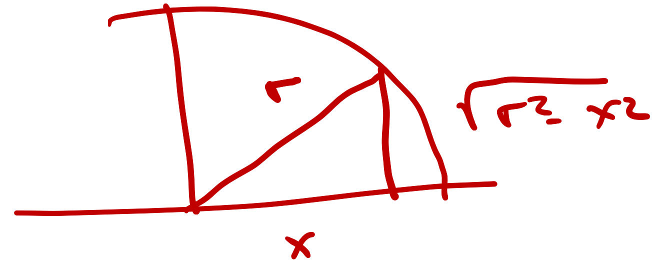

$$\boxed{V \approx 2\pi x\, f(x) \cdot dx}$$

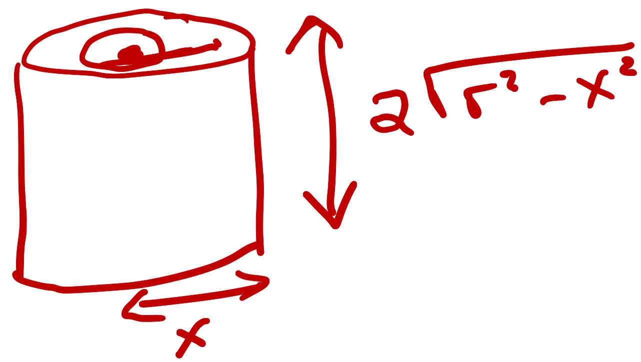


Derive volume of sphere $\boxed{V = \frac{4}{3}\pi r^3}$ using shell method.






$$2\pi \int_0^r x \, 2\sqrt{r^2 - x^2} \, dx$$

$u = r^2 - x^2$

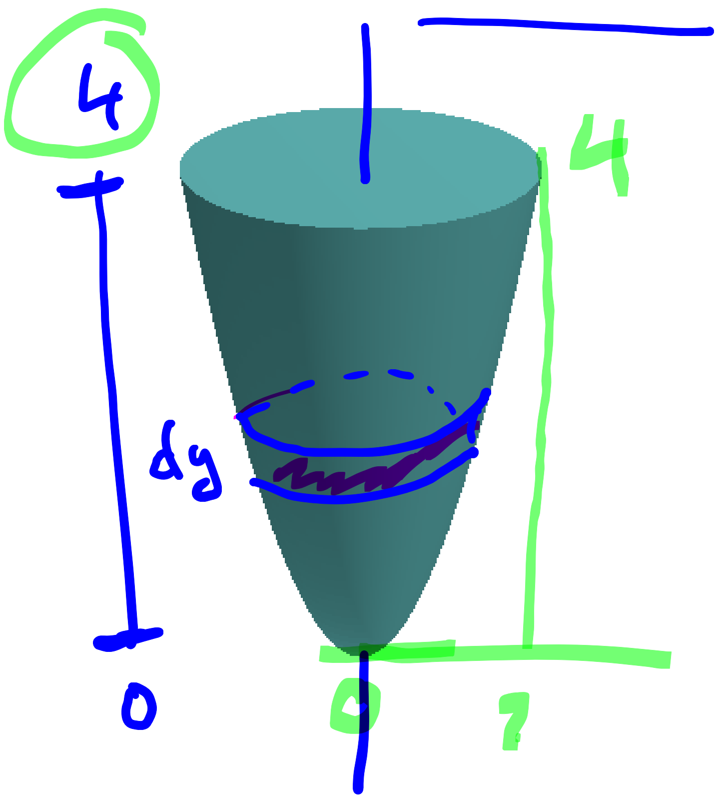
$du = -2x \, dx$

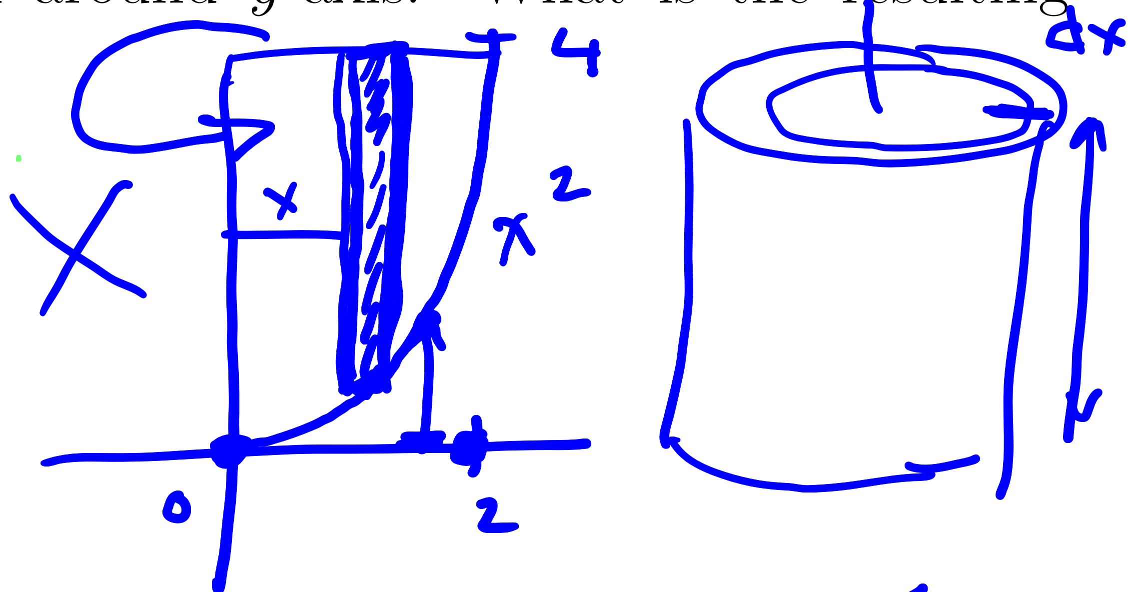
$$-2\pi \int \sqrt{u} \, du = -2\pi \left(\frac{2}{3}\right) u^{3/2} = -\frac{2\pi}{3} \left(r^2 - x^2\right)^{3/2} \Big|_0^r$$

$$= -\frac{4\pi}{3} \left[ \left(r^2 - r^2\right)^{3/2} - \left(r^2 - 0\right)^{3/2} \right]$$

$$= -\frac{4\pi}{3} \left[ 0 - r^3 \right] = \boxed{\frac{4\pi}{3} r^3}$$

Suppose $x^2$ on $[0, 2]$ is revolved around $y$-axis. What is the resulting volume? Use shell method.




$$V = 2\pi \int_0^2 x\, f(x)\, dx$$

$$= 2\pi \int_0^2 x\,(4 - x^2)\, dx$$

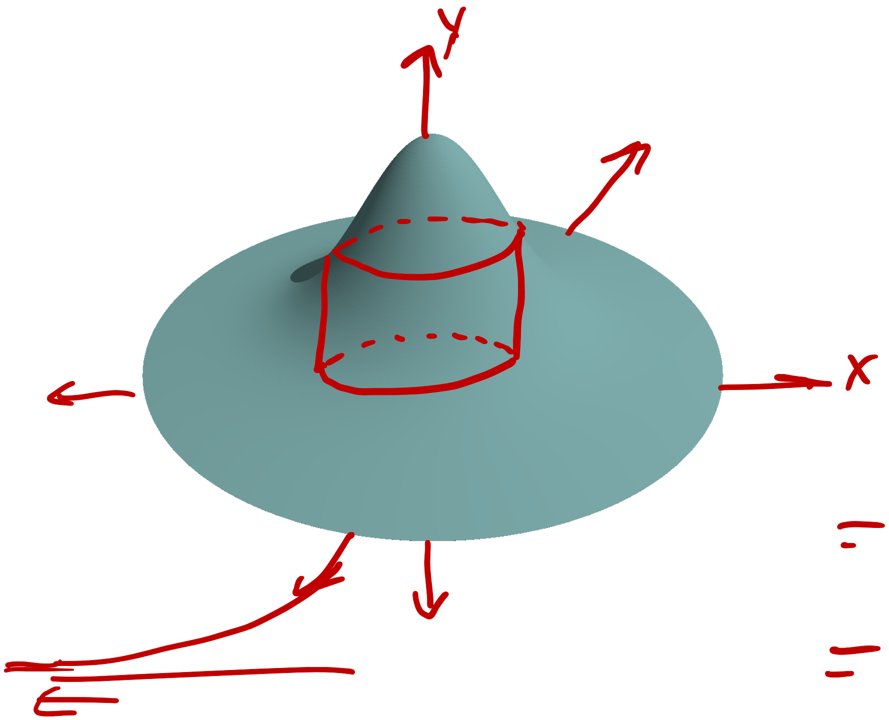
$$= 2\pi \int_0^2 4x - x^3\, dx$$

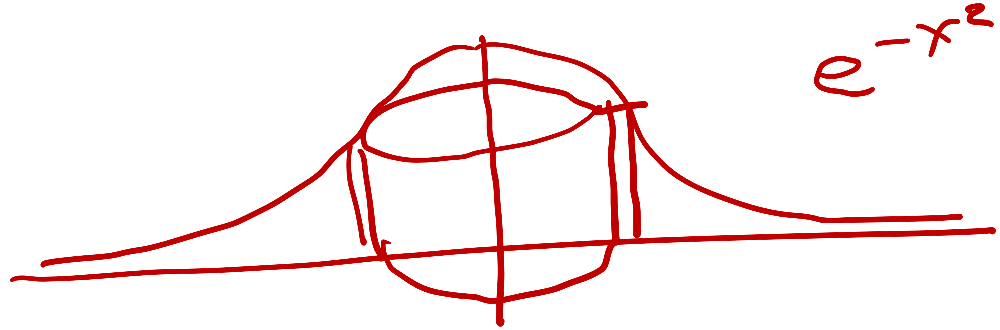
$$= 2\pi \left(2x^2 - \frac{1}{4}x^4\right)\Big|_0^2$$

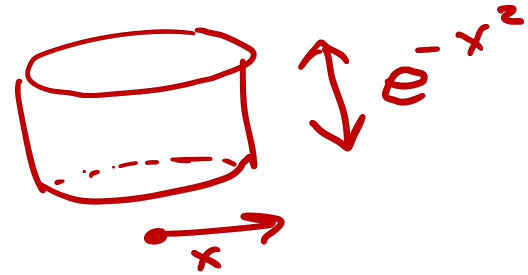
$$= 2\pi \left((8 - 4) - (0 - 0)\right)$$

$$= 8\pi$$

Suppose $f(x) = e^{-x^2}$ on $[0, \infty)$ is revolved aroind the $y$-axis. What is the volume generated?




$$= 2\pi \int_0^\infty x\, f(x)\, dx$$

$$= 2\pi \int_0^\infty x e^{-x^2}\, dx$$

$u = -x^2 \quad du = -2x\, dx$

$$= \frac{2\pi}{-2} \int e^u\, du = -\pi e^u$$

$$= -\pi e^{-x^2} \Big|_0^\infty$$

$$-\pi [\underbrace{e^{-\infty}}_{0} - \underbrace{e^{-0}}_{1}]$$

$$\boxed{= \pi}$$

**Quiz 5:** 10 questions, Multiple Choice

3 problems on recognizing $x$-axis rotations. match formula to picture

Find integral formula for disk method

Evaluate same integral

same integral

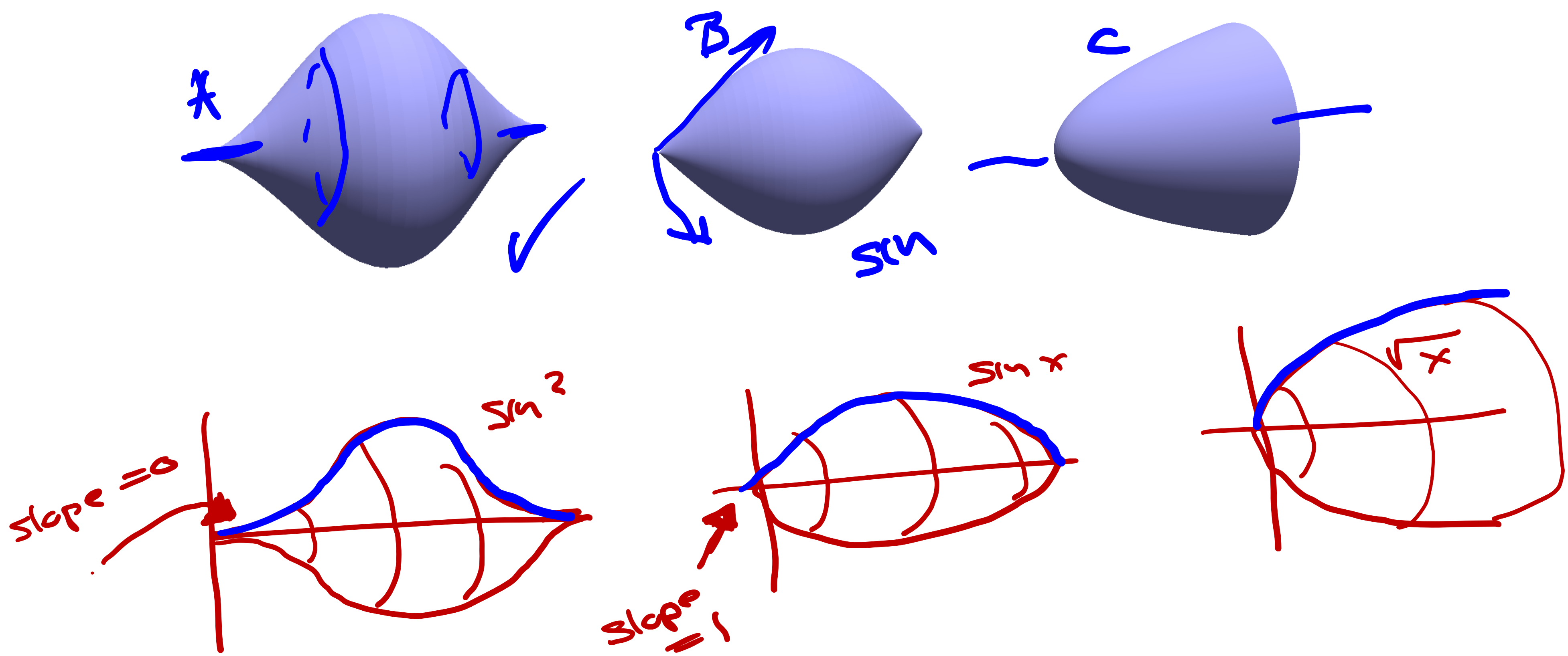
3 problems on recognizing $y$-axis rotations. match formula to picture

Find integral formula using shell method.

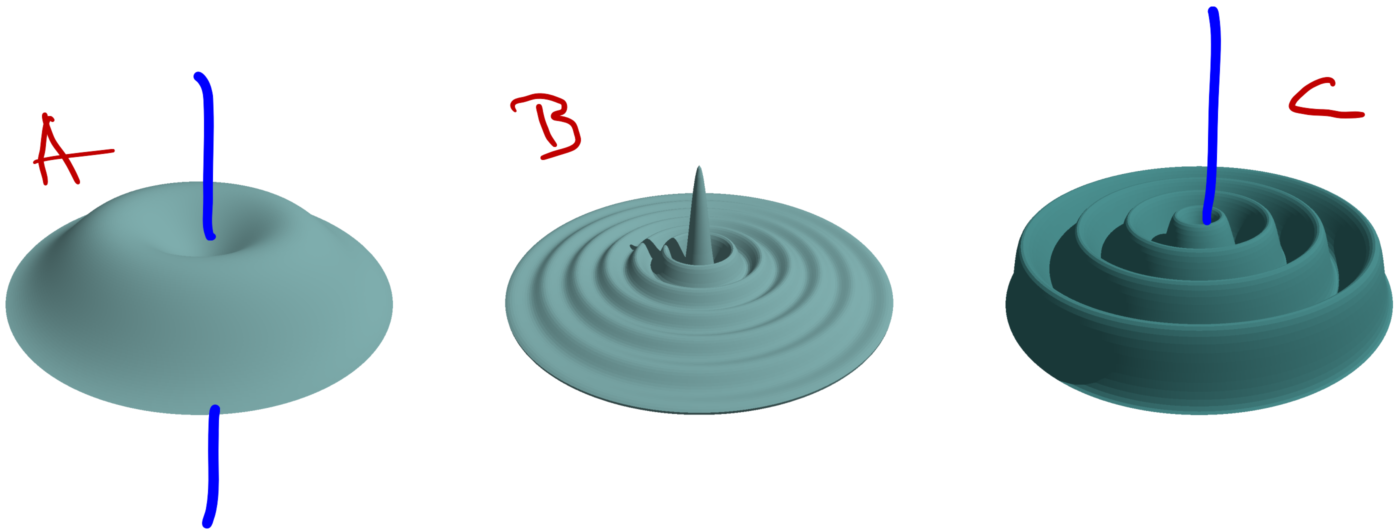
Compute a volume using shell method.

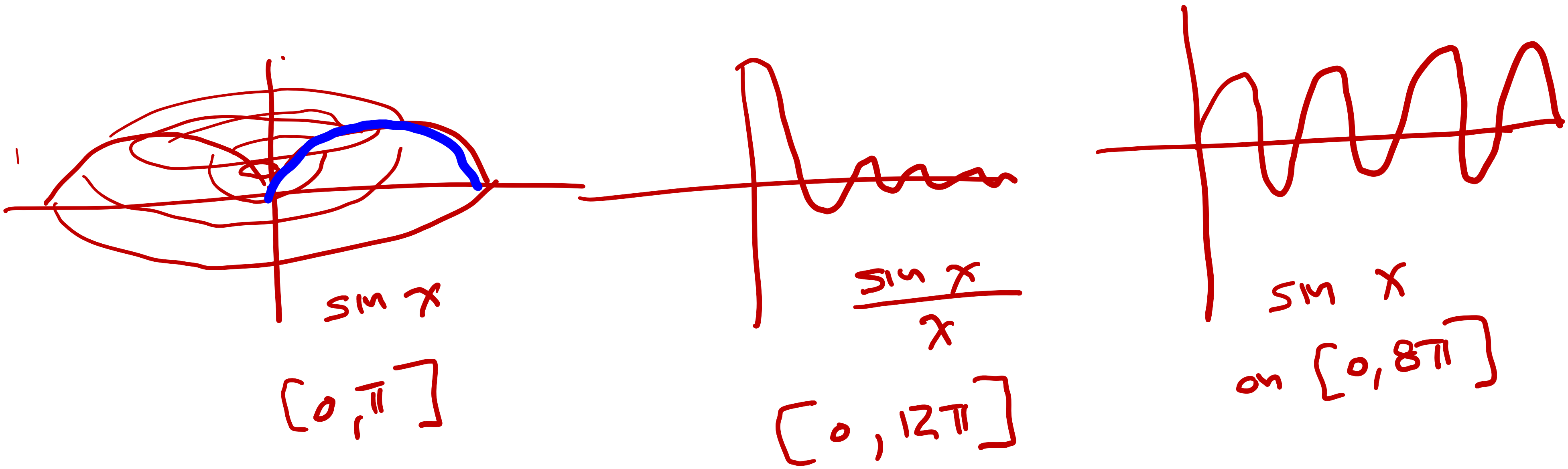
different integrals

Rotate $\sin^2(x)$ on $[0, \pi]$ around the $x$-axis. $\boxed{A}$

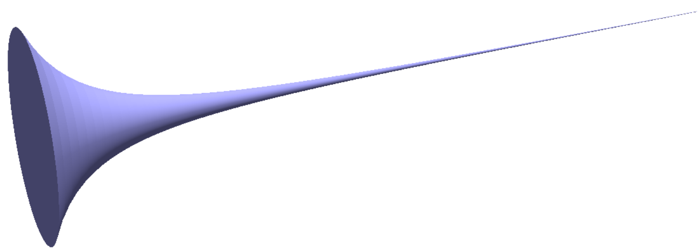


Rotate $\underline{\sin(x)}$ on $[0, \text{[illegible]}\pi]$ around the $y$-axis. $\boxed{A}$



The region below is $\{0 \le y \le e^{-x} : 0 \le x < \infty\}$, rotated around the $x$-axis. What is the integral formula for the volume using the disk method?





$A = \pi r^2$
$= \pi f(x)^2$

Compute the volume of the region above.

A) $\int_a^b \pi f(x)^2\,dx$

$u = -2x \quad du = -2dx$

$$= \int_0^\infty \pi \left(e^{-x}\right)^2 dx$$

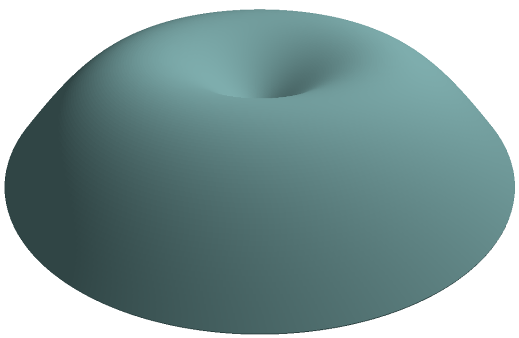
$$= \boxed{\frac{\pi}{-2}\int_0^\infty (-2)\,e^{-2x}\,dx} = \frac{\pi}{-2}\int_0^\infty e^{-2x}(-2dx) = -\frac{\pi}{2}\int e^u du$$

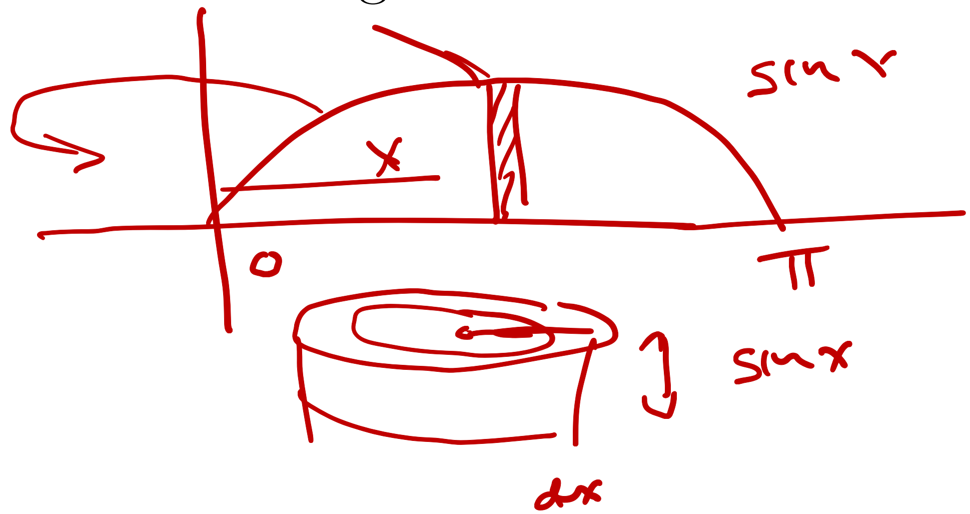
B)

$$= \pi\left(-\frac{1}{2}\right)e^{-2x}\Big|_0^\infty = \frac{-\pi}{2}\left(e^{-\infty} - e^0\right)$$

$$\boxed{= \pi/2}$$

Suppose the region $\{(x, y) : 0 < \sin(x), 0 \leq x \leq \pi\}$ is rotated aroind the $y$-axis. What integral gives the volume using the shell method?

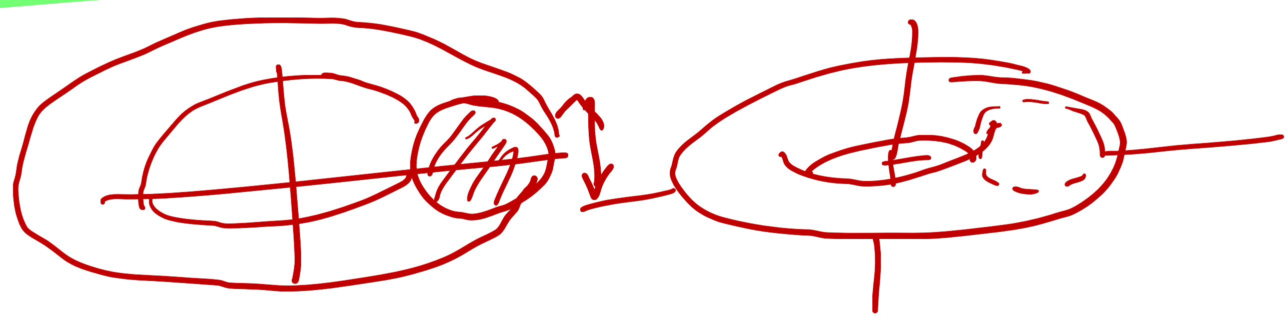




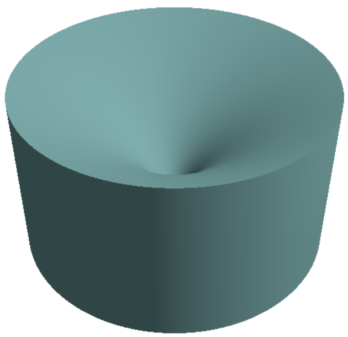

$$V = \int_0^{\pi} (2\pi x) \sin(x)\, dx$$

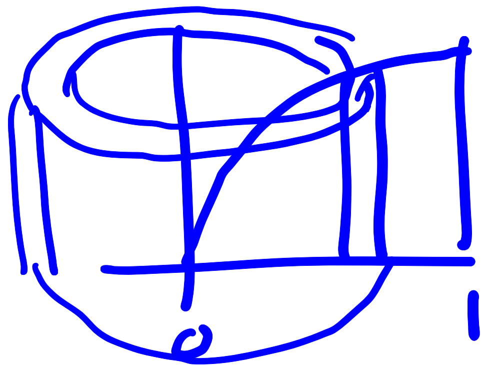
$$\boxed{= 2\pi \int_0^{\pi} x \sin(x)\, dx}$$

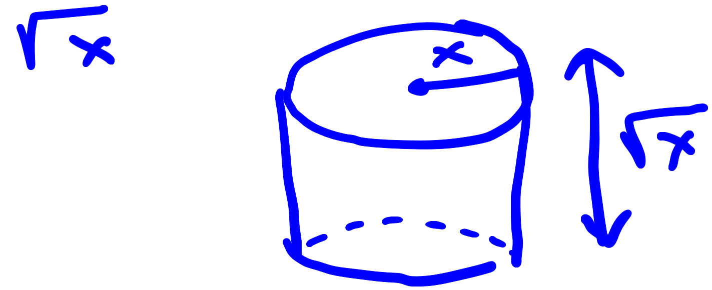
On quiz:



Suppose the region $\{(x, y) : 0 \leq \sqrt{x}, 0 \leq x \leq 1\}$ is rotated around the $y$-axis. Compute the volume of this regions (give a numerical answer).





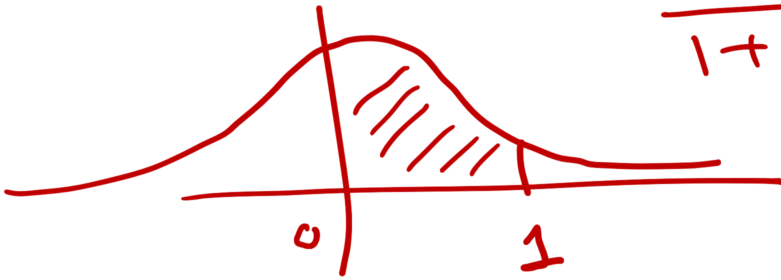

$$V = 2\pi \int_0^1 x\sqrt{x}\, dx = 2\pi \int_0^1 x^{3/2}\, dx$$

$$= 2\pi \left[\frac{2}{5} x^{5/2}\right]_0^1$$

$$= \frac{4\pi}{5}\left[1^{5/2} - 0^{5/2}\right] = \frac{4\pi}{5}$$

Thur Oct 1 2020

office hours




$$\frac{1}{1+x^2}$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a}$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \boxed{\tan^{-1}(x)\Big|_0^1} = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \pi/4 - 0$$

$$\boxed{= \pi/4}$$

$$\frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int e^x = e^x$$

$$\int \frac{1}{x} = \ln x$$

$$\int \ln = x\ln x - x$$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + C$$

$$\frac{1}{\sqrt{a^2-x^2}}$$

$a = 2$

$$\int \frac{1}{\sqrt{16-9x^2}} = \int \frac{1}{3}\frac{1}{\sqrt{(16/9)-x^2}}$$

$a = 4/3$

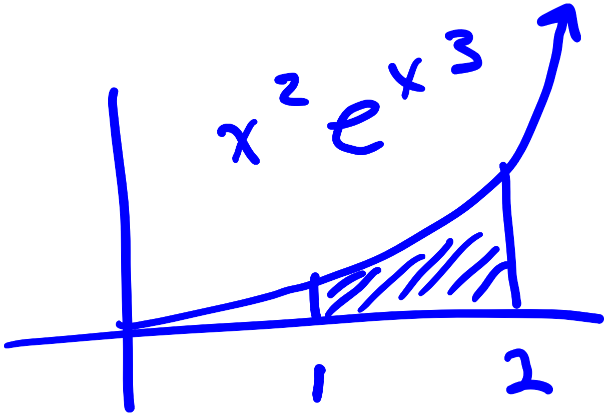
$$\sqrt{16-9x^2} = \sqrt{9\left(\frac{16}{9}-x^2\right)}$$

$$= 3\sqrt{\frac{16}{9}-x^2}$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{a^2-x^2}}$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{x}{4/3}\right)$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right) + C$$




$$\int_1^2 x^2 e^{x^3}\,dx$$

$$u = x^3 \qquad du = 3x^2$$

$$= \frac{1}{3}\int_1^2 (3x^2)\,e^{x^2}\,dx$$

$$= \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u = \frac{1}{3}e^{x^3}\Big|_1^2$$

$$= \frac{1}{3}\left(e^8 - e^1\right)$$

$$\frac{1}{-2}\int \frac{(-2)\ln x}{x\sqrt{1-\ln^2 x}}\,dx = ?$$

$$u = 1-\ln^2 x \qquad du = -2\ln x \cdot \frac{1}{x}\,dx$$

$$-\frac{1}{2}\int \frac{1}{\sqrt{u}} = -\frac{1}{2}\int u^{1/2} = -\left(\frac{1}{2}\right)\frac{2}{3}u^{3/2}$$

$$= -\frac{1}{3}\left(1-\ln^2 x\right)^{3/2} + C$$

$$\int \frac{1}{x\sqrt{1-\ln^2 x}}\,dx$$

$$\boxed{u = \ln x} \qquad du = \frac{1}{x}\,dx$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1}(u) + C = \underline{\sin^{-1}(\ln x) + C}$$

$a = 1$