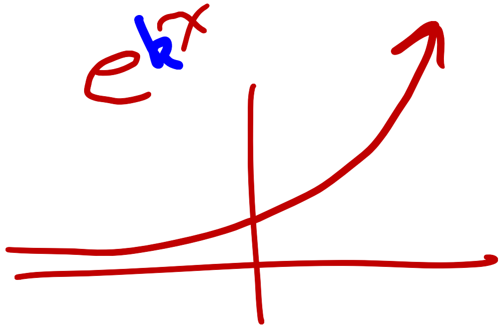
MAT 126.01, Prof. Bishop, Thursday, Oct 22, 2020
Section 2.8: Exponential Growth and Decay
Quiz 7 review

Definition of exponential growth:

$$y = y_0\ e^{kt}, k > 0.$$

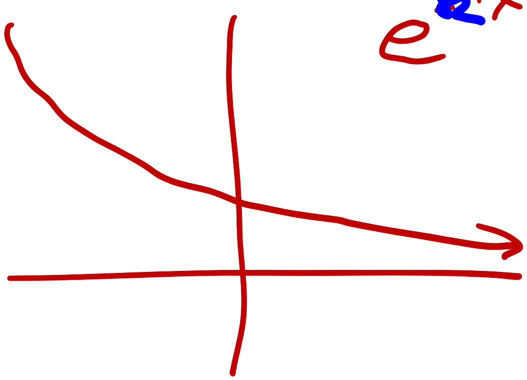
$$y = y_0\ \exp(kt), k > 0.$$

e^{kx} $\quad k > 0$



growth

e^{kx} $\quad k < 0$



decay

$$\boxed{(e^{kt})' = e^{kt} \cdot k}$$

$a = e^k$, $\quad k = \ln a$

$(a^t)' = a^t \cdot \ln a$

A population of bacteria grows according to

$$f(t) = 200e^{.02t}$$

$y_0 e^{kt}$ $y_0 = 200$

$k = .02$

where t is in minutes. How many bacteria are there after 5 hours?

5 hours $= 60 \cdot 5$ minutes $= 300$ minutes

$$f(300) = 200 \cdot e^{(.02)300} = 80686$$

When are there 100,000 bacteria?

$$f(t) = 100000$$

$$2\cancel{0}\cancel{0}\, e^{.02t} = 1000\cancel{0}\cancel{0}$$

$$e^{.02t} = 500$$

$$\ln(e^{.02t}) = \ln(500)$$

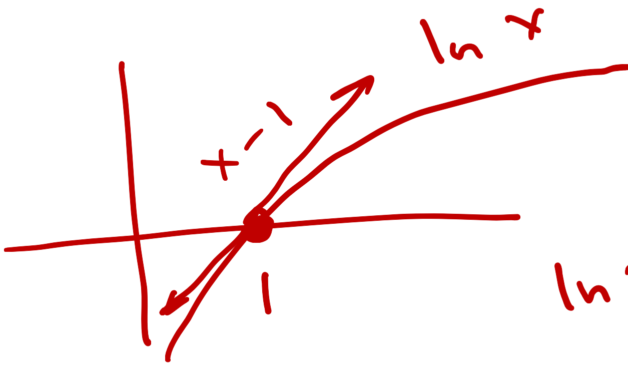
$$.02t = \ln(500)$$

$$t = \frac{\ln(500)}{.02} = 310.73 \text{ minutes}$$

Evaluate

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \quad = e}$$

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n \ln \underbrace{\left(1+\frac{1}{n}\right)}_{x}$$



$$\ln x \approx x-1$$

$$(\ln x)' = \frac{1}{x}$$

$$x = 1 \qquad = 1$$

$$= \lim_{n\to\infty} n\left(1+\frac{1}{n}-1\right)$$

$$= \lim_{n\to\infty} n\cdot\frac{1}{n}$$

$$= 1$$

If a bank account has an annual interest rate of r percent paid once a year, and P is the original amount, then after one year the account is worth

$$P(1+r).$$

After t years it is worth

$$P(1+r)^t.$$

Compounded n times in one year gives

$$P(1+\frac{r}{n})^n.$$

monthly $n=12$

weekly $n=52$

daily $n=365$

\$100 into bank accont: I get 5% interest. paid over 12 times.

$$\underbrace{100\left(1+\frac{.05}{12}\right)}\cdot 1 + \underbrace{100\left(1+\frac{.05}{12}\right)}\cdot\frac{.05}{12}$$

$$100\left(1+\frac{.05}{12}\right)\left(1+\frac{.05}{12}\right) = 100\left(1+\frac{5}{12}\right)^2$$

Continuous compounding is the limit compounding period $\to 0$.

$$\lim_{n\to\infty} P(1 + \frac{r}{n})^n = Pe^r.$$

If we continuously compound for t years the value is

$$Pe^{rt}.$$

Assume the fish population in a pond grows exponentially. Suppose it starts at 500 and after 6 months is a 1000. How long before the population hits 10,000?

$$P(t) = P_0 e^{kt}$$

$t =$ is in months

$$P_0 = 500$$

1st step : compute k

$$P(6) = 1000$$

$$P_0 e^{kt} = 1000$$

$$500 e^{k6} = 1000$$

$$e^{k6} = 2$$

$$k6 = \ln 2$$

$$k = \frac{1}{6}\ln 2$$

$$\approx .1155$$

2nd step : Solve for t

$$P(t) = 10000$$

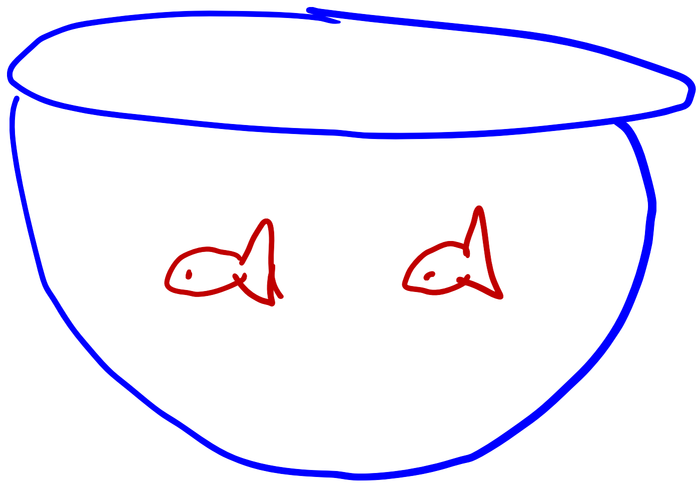
$$500 e^{kt} = 10000$$

$$e^{kt} = 20$$

$$kt = \ln 20$$

$$t = \frac{\ln 20}{k} = \frac{\ln 20}{\frac{1}{6}\ln 2}$$

$$= 25.93$$



N

$$1\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{k}{N}\right)$$

$$\approx 1-\left(\frac{1}{N}+\frac{2}{N}+\cdots\frac{k}{N}\right)+\cdots$$

$$\frac{N-1}{N}$$

$$= 1-\frac{(1+2+\cdots k)}{N}+x$$

$$\frac{N-2}{N}$$

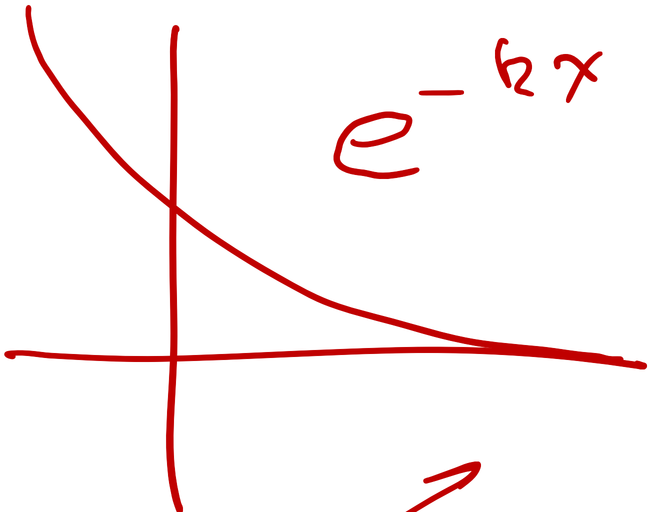
$$= 1-\underbrace{\frac{(k+1)(k)/2}{N}}_{.5}$$

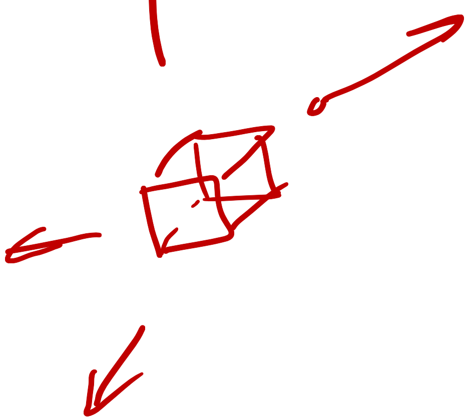
$$.5 \approx$$

$$\frac{(k+1)(k)}{2}\approx\frac{1}{2}N \qquad k\approx\sqrt{N}$$

Exponential decay:

$$y = y_0\, e^{-kx}, k > 0.$$





Carbon 14 is a material that accumulates in living things, and stops accumulating when they die. The amount of Carbon 14 then starts to decay exponentially as it changes to Nitrogen-14.

$\ln xy = \ln x + \ln y$

$0 = \ln x \cdot \frac{1}{x} = \ln x + \ln \frac{1}{x}$

The half-life of Carbon-14 is 5730. Find k in the decay equation

$$y = y_0\, e^{-kt}.$$

$\ln x^p = p \ln x$

$x^{-1} = \frac{1}{x}$

$\ln \frac{1}{x} = -1 \ln x$

$\ln \frac{1}{2} = -\ln 2$

$1 \cdot e^{-kt} = \frac{1}{2}$

$e^{-k\,5730} = \frac{1}{2}$

$-k\,5730 = \ln \frac{1}{2}$

$k = \frac{-1}{5730} \cdot \ln \frac{1}{2} = \frac{\ln 2}{5730}$

$\approx .00012097$

Some pottery contains only 70% of its original Carbon-14. How old is it?

$$1 \cdot e^{-kt} = .7$$

$$-kt = \ln .7$$

$$t = \frac{\ln .7}{-k}$$

$$= \frac{\ln .7}{-\ln 2 / 5730}$$

$$\approx 2948.5$$

Newton's law of cooling: If an object with initial temperature T_0 is placed in an environment with constant temperature T_a (a for ambient) its temperature at time t

$$T = (T_0 - T_a)e^{-kt} + T_a.$$

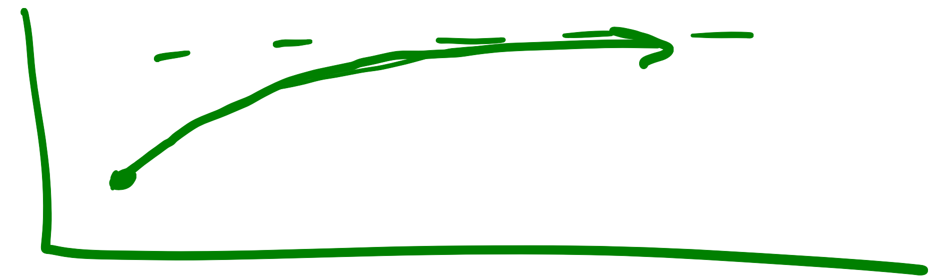
$t = 0 \quad T = T_0$

$t \to \infty \quad T \to T_a$

This says the object goes to the ambient temperature exponentially fast.

$T_0 = 400$

$T_a = 80$

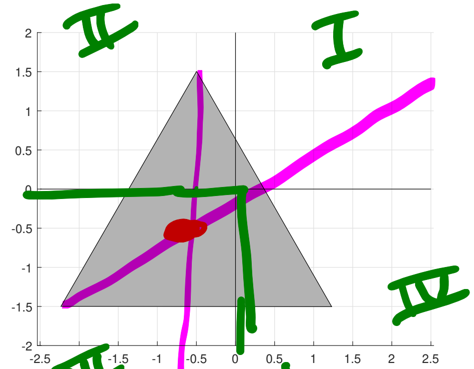


Quiz 7 review:

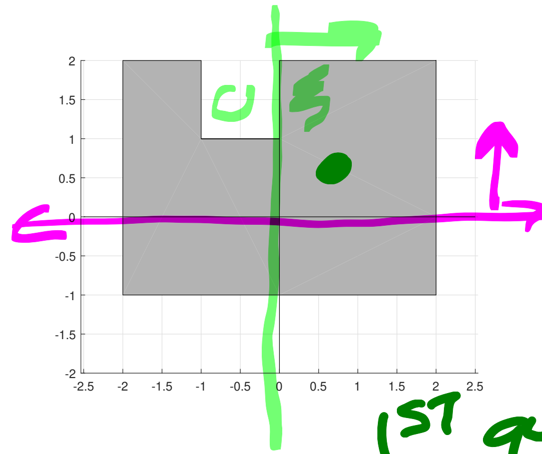
- 2 questions: finding center of mass by eye
- 3 questions: given region find mass, x-moment, y-moment
- 1 question: theorem of Pappus
- 2 questions: derivative of exponentials
- 2 questions: Newton's law of cooling

For each figure say where center of mass is: the origin, or what quadrant, or which axis (e.g., positive x-axis, negative y-axis,...).

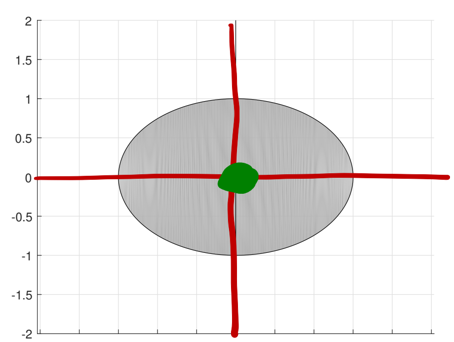
$x<0$
$y<0$



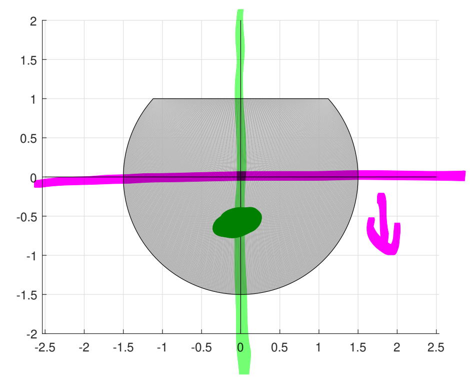
3rd quad



1st quad

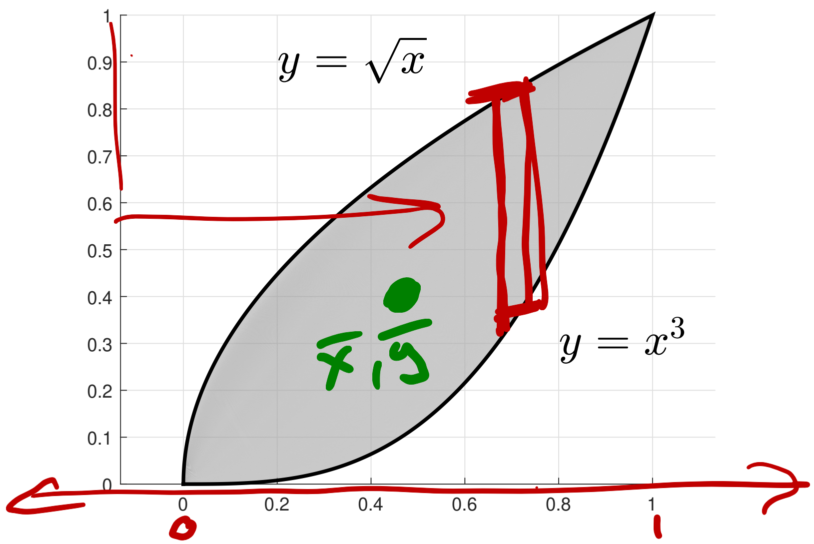


origin



neg y-axis

Compute the mass, x-moment and y-moment for the following region.



density $\rho = 1$.

$\bar{x} = \frac{M_y}{m}$ $\bar{y} = \frac{M_x}{m}$

③ mass = area $= \int_0^1 \sqrt{x} - x^3 \, dx = \frac{2}{3} x^{3/2} - \frac{1}{4} x^4 \Big|_0^1$

$= \left(\frac{2}{3} - \frac{1}{4}\right) - (0 - 0)$

$= \frac{8-3}{12} = 5/12$

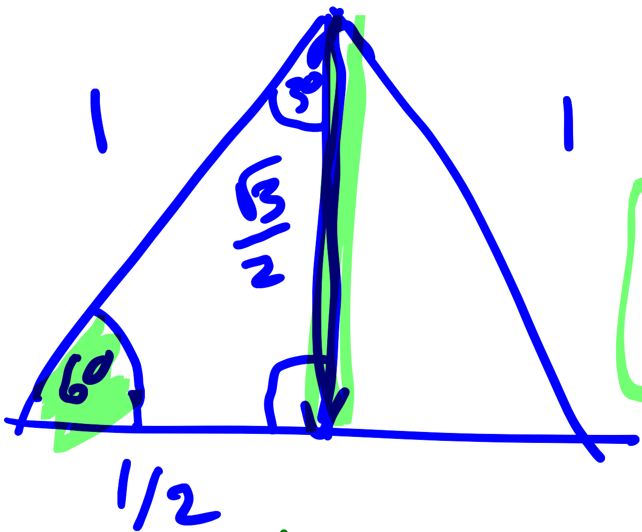
④ x-moment $= \int_0^1 \frac{1}{2} f(x)^2 - g(x)^2 \, dx$

$= \frac{1}{2} \int_0^1 x - x^6 \, dx$

⑤ y-moment $= \int_0^1 x\left(f(x) - g(x)\right) dx$

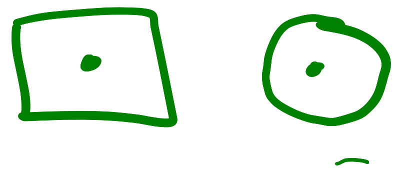
$= \int_0^1 x\left(\sqrt{x} - x^3\right) dx$

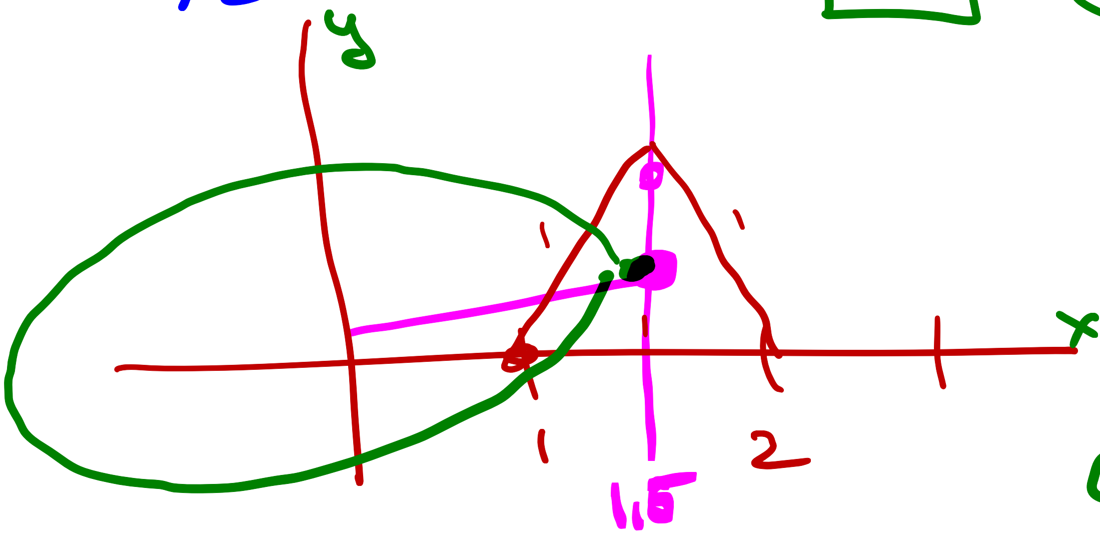
An equilateral triangle with base $[1, 2]$ on the x-axis is rotated around the y-axis. Use the Theorem of Pappus to compute the volume of this region.



$$\text{Vol} = (\text{area}) \cdot \underbrace{(\text{distance COM travels})}_{2\pi \,\cdot\, x \text{ coord of center of mass}}$$

$$\boxed{\text{Vol} = \frac{\sqrt{3}}{4} \cdot 3\pi}$$





① Area $= \frac{1}{2} \cdot b \cdot h$

$= \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$

② x-coor $= 1.5$

$2\pi \times 1.5 = 3\pi$

$$\bar{x} = \frac{M_y}{m}$$

Compute the derivative of $f(x) = x^{\sqrt{x}}$.

$x = e^{\ln x}$

$x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}} = e^{\sqrt{x}\ln x}$

$(e^{\sqrt{x}\ln x})' = e^{\sqrt{x}\ln x}\left[\frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\,\frac{1}{x}\right]$

$= x^{\sqrt{x}}\left[\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right]$

$y = x^{\sqrt{x}}$

$\ln y = \sqrt{x}\ln x$

$\frac{1}{y}\frac{dy}{dx} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}$

What is minimal value of f on $(0, 1)$?

Solve $f' = 0$

$x^{\sqrt{x}} \neq 0$

$\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} = 0$

$\frac{1}{2}\ln x + 1 = 0$

$\ln x = -2$

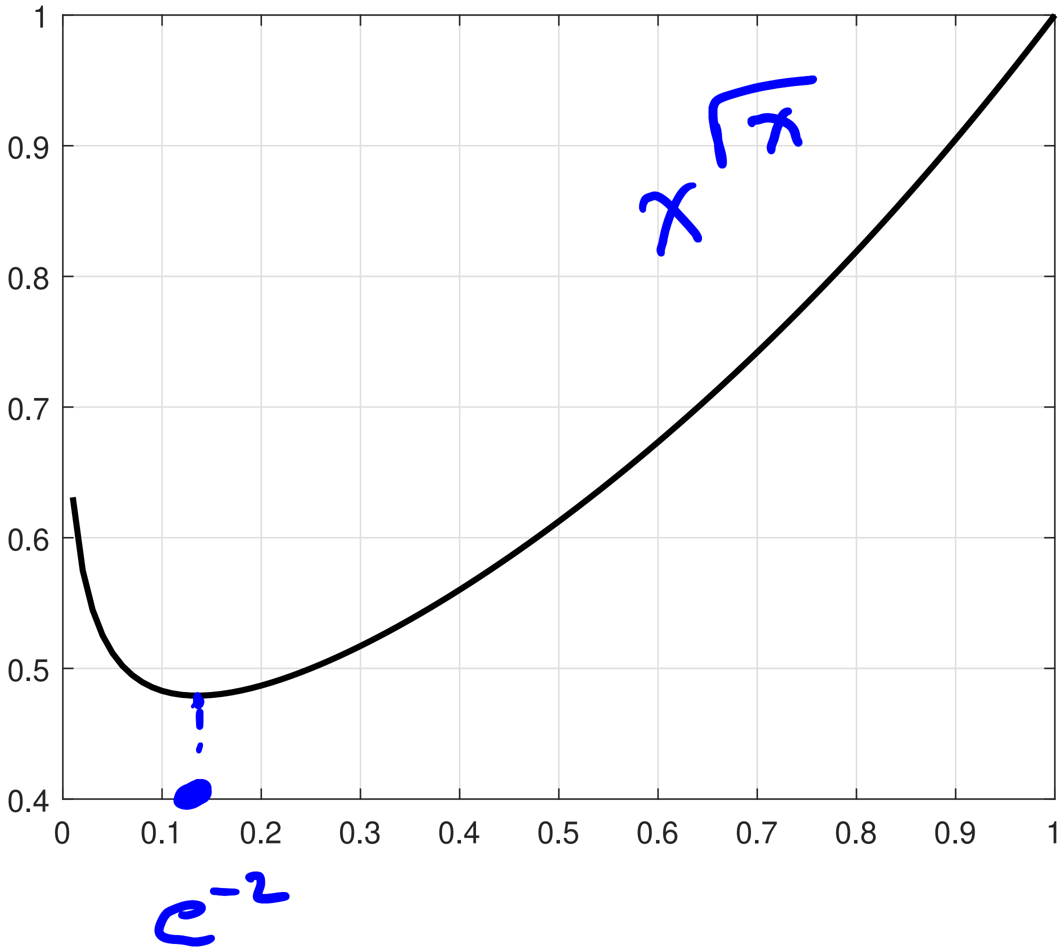
$x = e^{-2}$

$f(e^{-2}) = (e^{-2})^{\sqrt{e^{-2}}}$

$= (e^{-2})^{e^{-1}}$

$= (e^{-2})^{1/e}$

$= e^{-2/e}$



1
0.9
0.8
0.7
0.6
0.5
0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$x^{\sqrt{x}}$
e^{-2}

T_a

A 40° degree turkey is put into a 350° degree oven. What is the temperature as a function of time, according to Newton's law of cooling? Leave k as a parameter.

① $(T_0 - T_a)e^{-kt} + T_a$

$$\boxed{(\cancel{40-350})e^{-kt} + 350}$$

-310

After an hour, the turkey is at 100°. What is the value of k.

②

$$-310e^{-k\cdot 1} + 350 = 100$$

$$-310e^{-k} = -250$$

$$e^{-k} = \frac{250}{310}$$

$$-k = \ln\frac{250}{310}$$

$$k = -\ln\frac{250}{310}$$

$$\boxed{k = \ln\frac{310}{250}}$$

When does the turkey reach 330°?

↑ not on quiz

?

$$-310e^{-kt} + 350 = 330$$

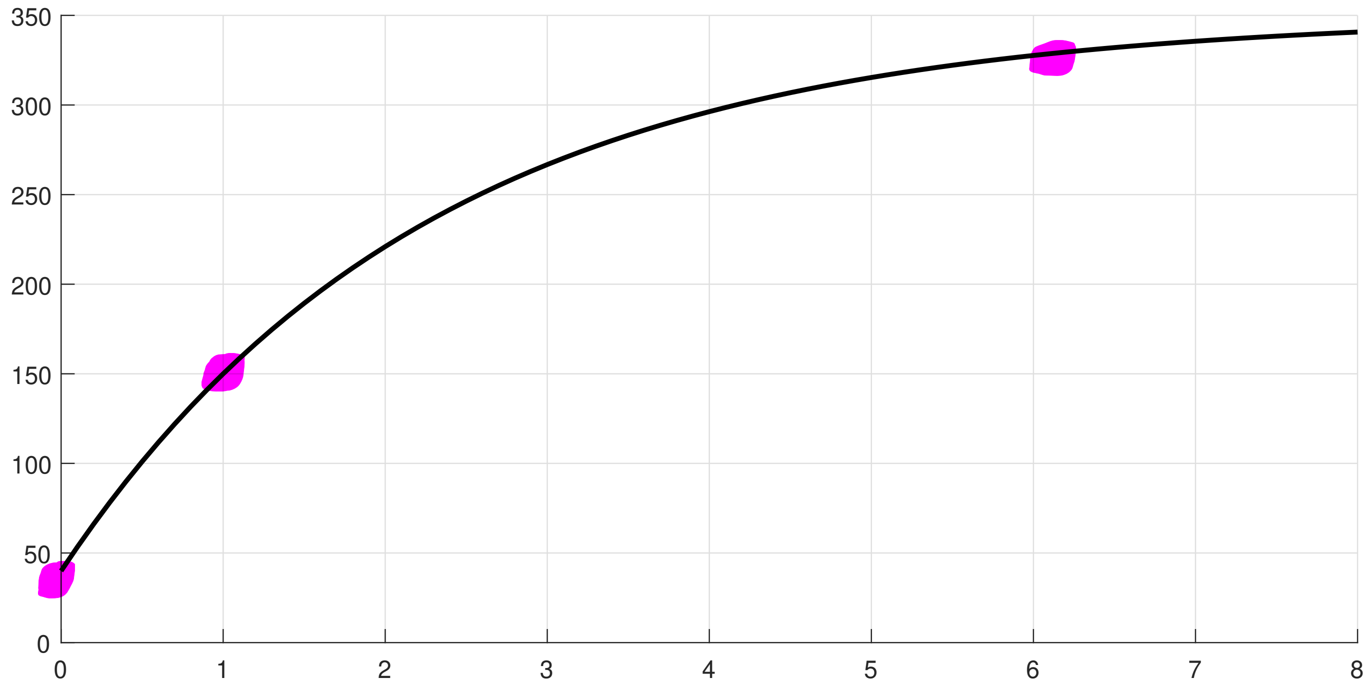
$$-310e^{-kt} = -20$$

$$e^{-kt} = \frac{20}{310} \longrightarrow$$

$$-kt = \ln 20/310$$

$$t = \frac{1}{k}\cdot\ln\frac{310}{20}$$

$$= \left(\ln 310/20\right)/\ln\left(\frac{310}{250}\right)$$



350
300
250
200
150
100
50
0
0
1
2
3
4
5
6
7
8

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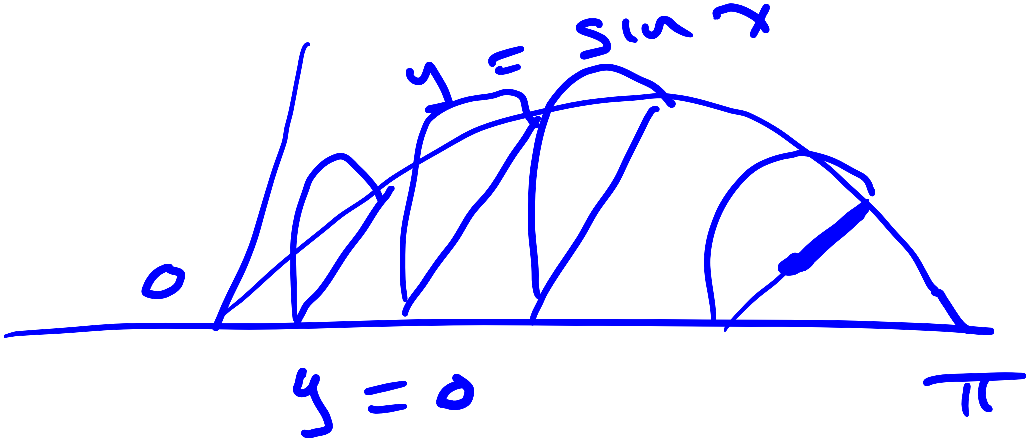
Office Hours

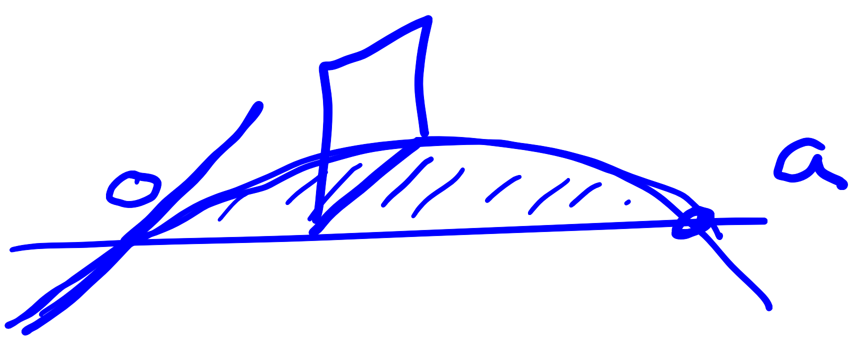
begin $\approx$ 11:20



$A = \frac{1}{2}\pi r^2$

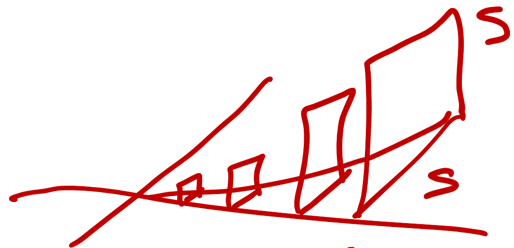
cross section = semi-circle with base on vertical segment.

$$\int_0^{\pi} \frac{1}{2}\pi\left(\frac{\sin x}{2}\right)^2 dx = \frac{\pi}{8}\int_0^{\pi} \sin^2 x \, dx$$

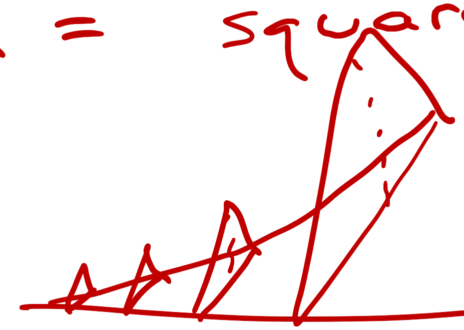


. . . .

cross section = square



$A = s^2$



$A = \frac{1}{2}bh$

$$\int_0^1 (x^2)^2\,dx = \int_0^1 x^4\,dx$$

$$= \frac{1}{5}x^5\Big|_0^1 = \frac{1}{5}$$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x + x\ln^2 x} = \int \frac{\frac{1}{x}\,dx}{1+\ln^2 x} \quad du$$

$$\frac{1}{x} \qquad \frac{1}{1+\ln^2 x}$$

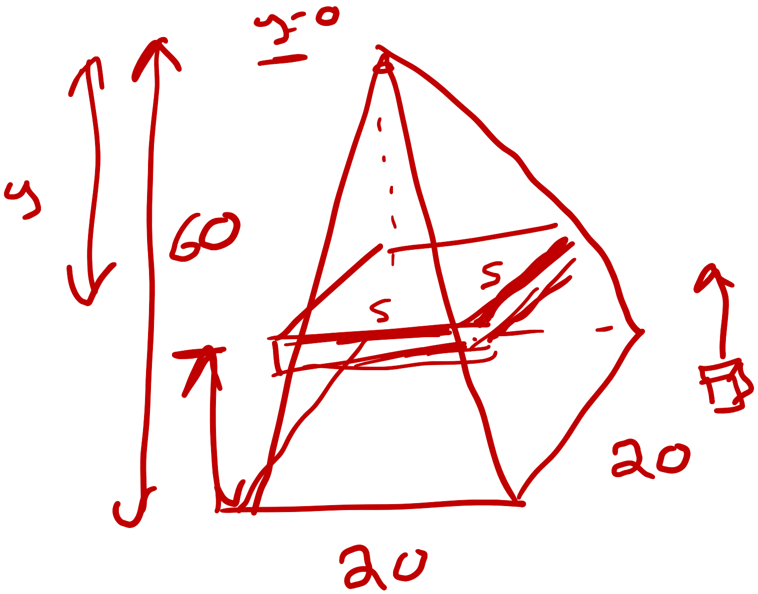
$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \int \frac{du}{1+u^2} \qquad a = 1$$

$$= \tan^{-1}\left(\frac{u}{a}\right) + C$$

$$= \tan^{-1}(\ln x)$$

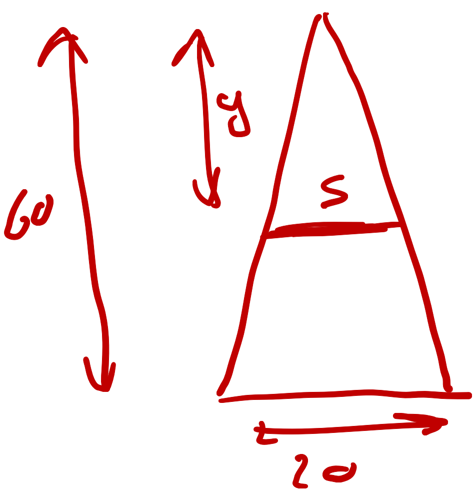


build this from stone

weighs 150 lbs / ft^3.

How much work to lift stones from ground level?

$w = F \cdot d$



$$\frac{60}{20} = \frac{y}{s}$$

$$s = \frac{20}{60} y = \frac{1}{3} y$$

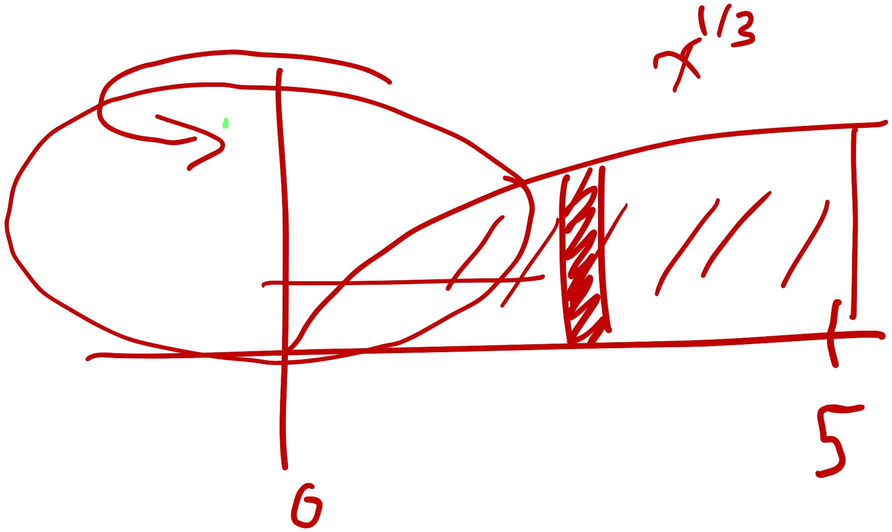
$$150 \cdot \int_0^{60} s^2 \, dy$$

$$= 150 \int_0^{60} \left(\frac{y}{3}\right)^2 (60-y) \, dy$$

$$= \frac{150}{9} \int_0^{60} y^2 \cdot 60 - y^3 \, dy$$

$$= \frac{150}{9} \left(20y^3 - \frac{1}{4} y^4\right)\Big|_0^{60}$$

$$= \frac{150}{9} \left(20 \cdot 60^3 - \frac{1}{4} 60^4\right)$$

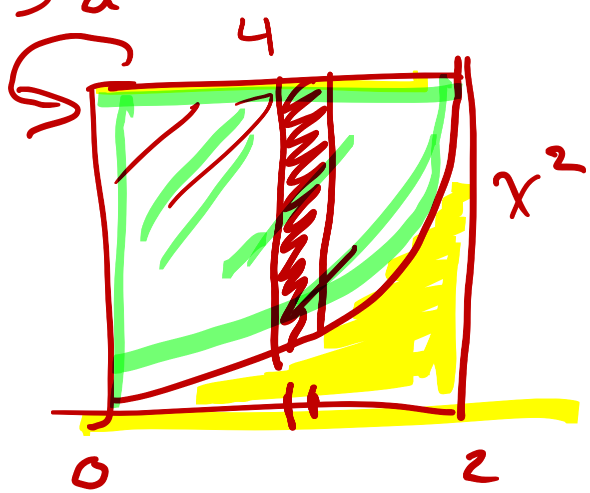


$$\int_a^b 2\pi x\, f(x)\,dx = \int_0^5 2\pi x\, x^{1/3}\,dx$$

$$= 2\pi \int_0^5 x^{4/3}\,dx$$

$$= 2\pi \cdot \frac{3}{7} x^{7/3} \Big|_0^5$$

$$= 2\pi \frac{3}{7} (5)^{7/3} - 0$$



$$\int_0^2 2\pi x(4 - x^2)$$