**MAT 126.01, Prof. Bishop, Tuesday Sept. 8, 2020**

**Tuesday, September 8, 2020**
**Section 1.3, The fundamental theorem of caclculus**
**Section 1.4, Integration formulas and the net change theorem**

- The fundamental theorem, parts I and II.
- Sketch of proofs.
- The mean value theorem.
- Using chain rule for functions in limits of integration.
- Definite versus indefinite integral.
- Net change theorem (slight variation of fundamental theorem).
- Position, velocity, acceleration
- Other rates of change

$a$

**The Fundamental Theorem of Calculus, Part I:** If $f$ is continuous on $[a, b]$ and we define

$$F(x) = \int_a^x f(t)dt,$$


then $F' = f$ on $[a, b]$

**The Fundamental Theorem of Calculus, Part II:** If $f$ is continuous on $[a, b]$ and $F$ is any function with $F' = f$, then

$\frac{1}{3}b^3 - \frac{1}{3}a^3$

$$F(b) - F(a) = \int_a^b f(t)dt.$$

$\int x^2$

$F = \frac{1}{3}x^3$

Part I says that any continuous function $f$ has some anti-derivative $F$.

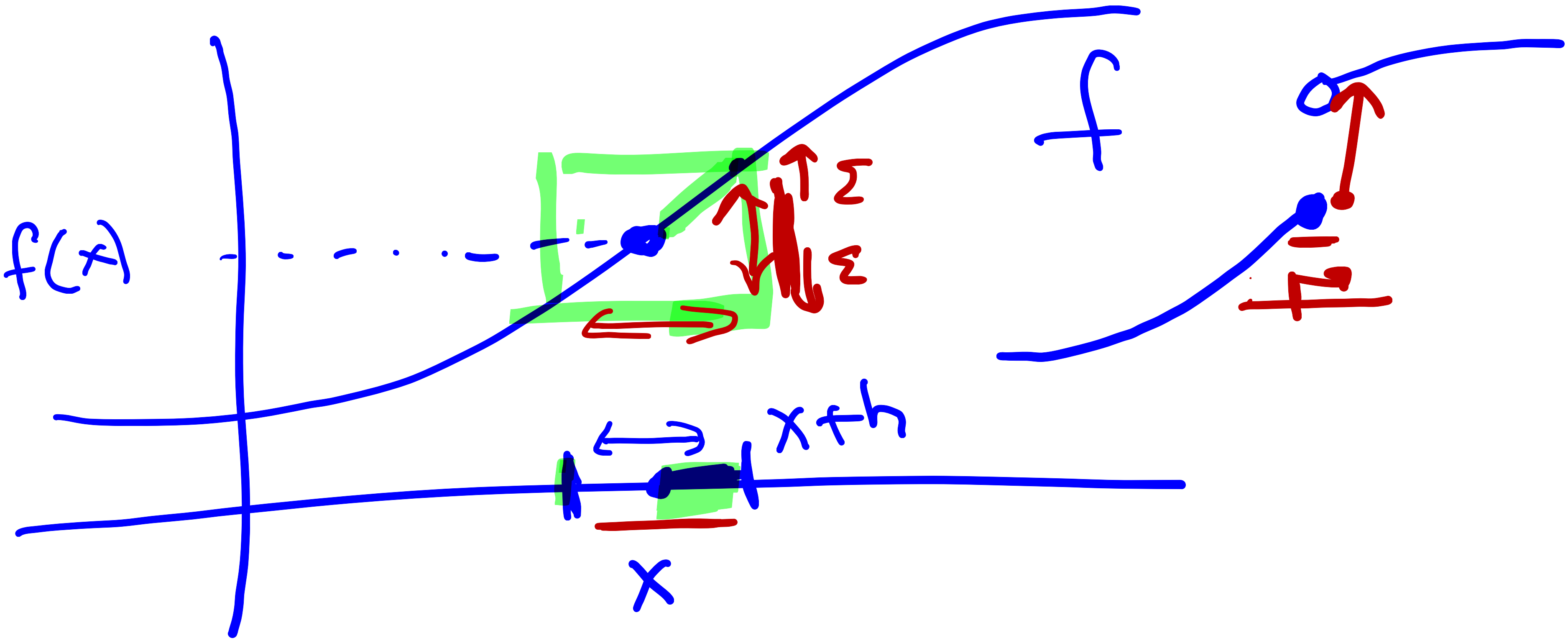
Part II says we can evaluate integral by guessing an anti-derivative!

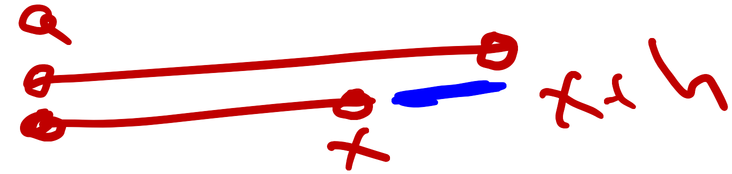
**Proof of Part I:**

By the definition of continuity of $f$ we have

$$f(x) - \epsilon < f(x+h) < f(x) + \epsilon,$$

whenever $h$ is small engough (less than some $\delta > 0$).

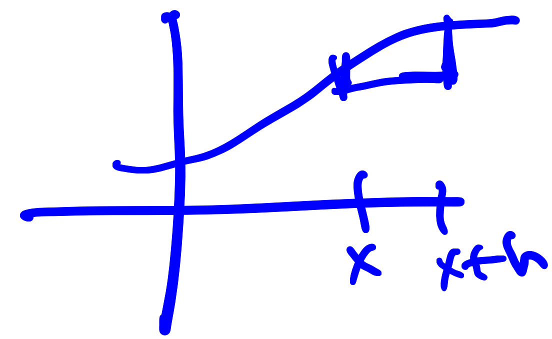





Note that $F(x+h) - F(x)$

$$F'(x) \approx \frac{\cancel{F(x) - F(x+h)}}{h} = \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^{x\cancel{+h}} f(t)dt\right]$$

$$= \frac{1}{h}\int_x^{x+h} f(t)dt$$




Therefore,

$$\frac{1}{h}\int_x^{x+h} \underline{(f(x) - \epsilon)}dt \le \frac{F(x) - F(x+h)}{h} \le \frac{1}{h}\int_x^{x+h} \underline{f(x) + \epsilon}dt$$

$$\frac{1}{\cancel{h}}(f(x) - \epsilon)\cancel{h} \le \frac{F(x) - F(x+h)}{h} \le \frac{1}{\cancel{h}}(f(x) + \epsilon)\cancel{h}$$

$$(f(x) - \epsilon) \le \frac{F(x) - F(x+h)}{h} \le \underline{(f(x) + \epsilon)}$$

$\downarrow f(x)$ $\boxed{|h| < \delta}$

Thus $\underline{F'(x) = f(x)}$.

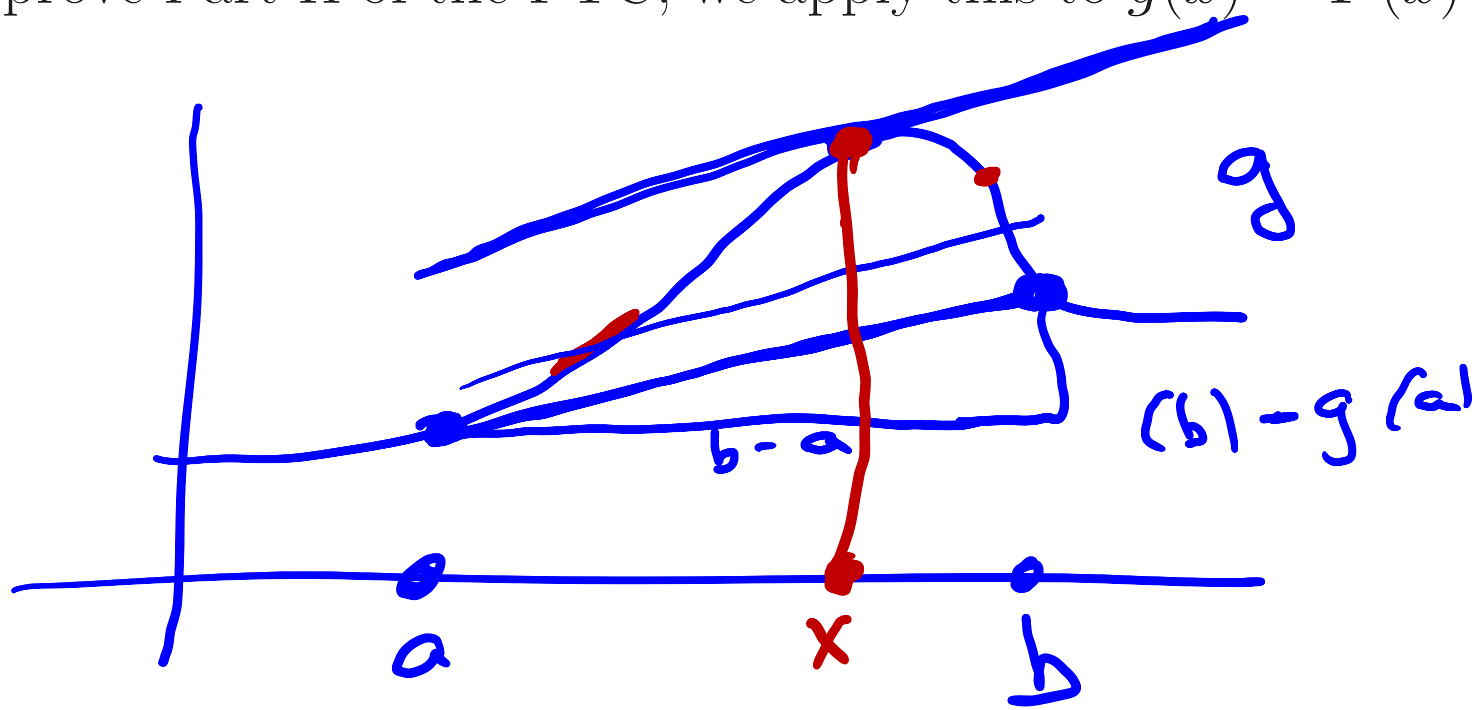
## Proof of Part II:

This uses the mean value theorem from MAT 125:

**Mean Value Theorem:** If $g$ has a continuous derivative $g'$ on $[a, b]$ then there is an $a \leq x \leq b$ so that

$$g'(x) = \frac{g(\overset{b}{\cancel{a}}) - g(\overset{a}{\cancel{b}})}{b - a}$$

To prove Part II of the FTC, we apply this to $g(x) = F(x) = \int_a^x f(t)dt$.

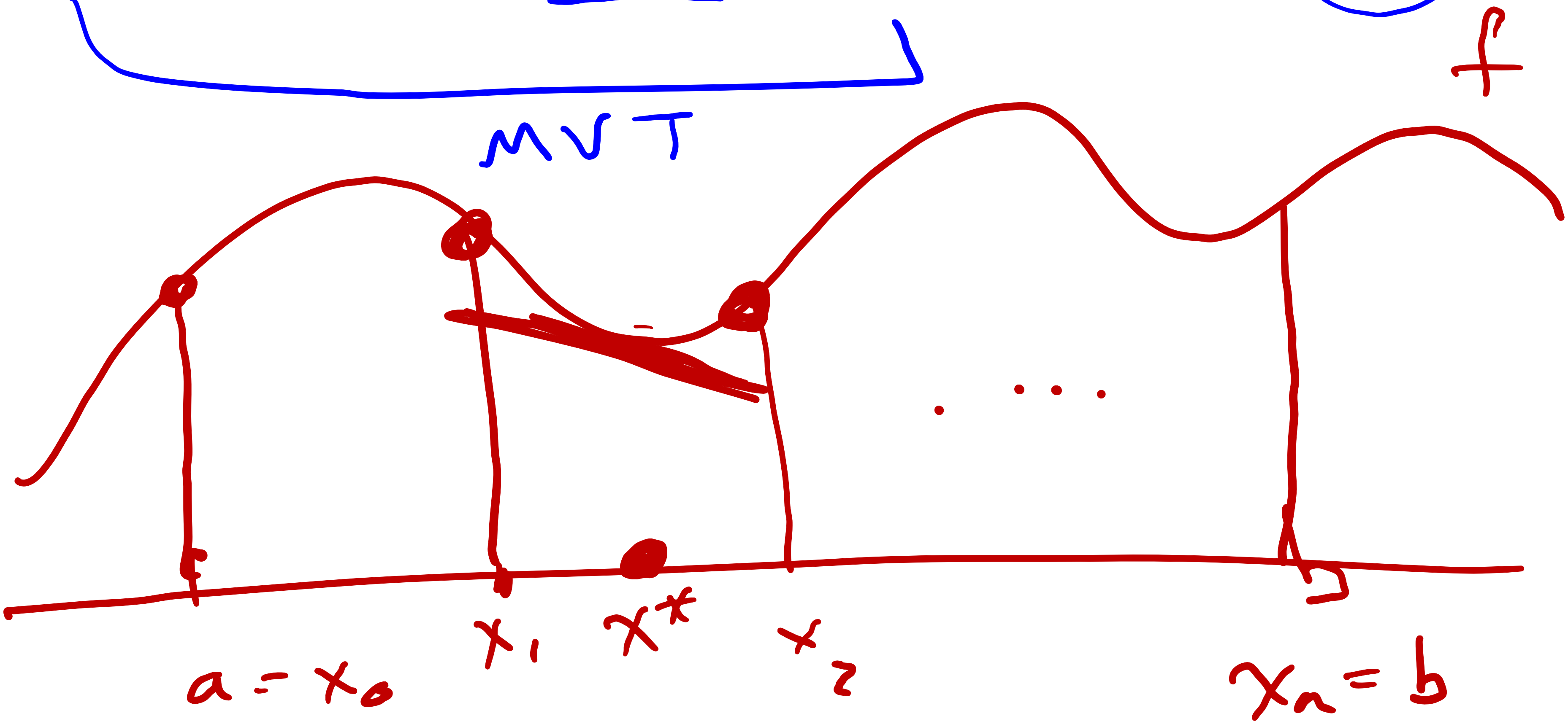


Cut $[a, b]$ into subintervals using $n$ points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

The mean value theorem says there is a point $x_k^*$ in $[x_k, x_{k+1}]$. Hence

$$\underbrace{F(x_{k+1}) - F(x_k) = \underline{(x_{k+1} - x_k)}F'(x_k^*)}_{\text{MVT}} = (x_{k+1} - x_k)f(x_k^*).$$



$x_n = b \qquad x_0 = a$

So

$$
\begin{aligned}
& F(b) - F(a) \\
= \quad & (F(x_n) - \cancel{F(x_{n-1})}) + (\cancel{F(x_{n-1})} - F(x_{n-2})) + \ldots \\
& \quad (F(x_3) - \cancel{F(x_2)}) + (\cancel{F(x_2)} - F(x_1)) + (F(x_1) - F(x_0)) \\
= \quad & \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) \qquad \text{(using Sigma notation)} \\
= \quad & \sum_{k=1}^{n} (x_k - x_{k-1}) f(x_k^*) \qquad \text{(by MVT)} \\
& \to \int_a^b f(x)\,dx = F(b) - F(a)
\end{aligned}
$$

**Definite versus indefinite integrals:**

Definite integral is a number, $\int_a^b f(t)dt$.

Indefinite integral is a function, $F(x) = \int_a^x f(t)dt$.

A function has more that one anti-derivative.

Any two differ by a constant term.

$$x^2 \longrightarrow \frac{1}{3}x^3 + C,$$

$$\left(\frac{1}{3}x^3 + 1\right)' = 3\frac{1}{3}x^2 + 0$$

$$= x^2$$

**Chain Rule of derivatives of integrals:**

If $G(x) = \int_a^{g(x)} f(t)dt$ then

$$G'(x) = F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x),$$

where $F(x) = \int_a^x f(t)dt.$

What is $G'(x)$ if $G(x) = \int_0^{\sin(x)} x^2 dx$?

$$F(x) = \int_0^x t^2 dt$$

$$F'(x) = x^2 \quad \text{FTC}$$

$$G(x) = F(\sin x)$$

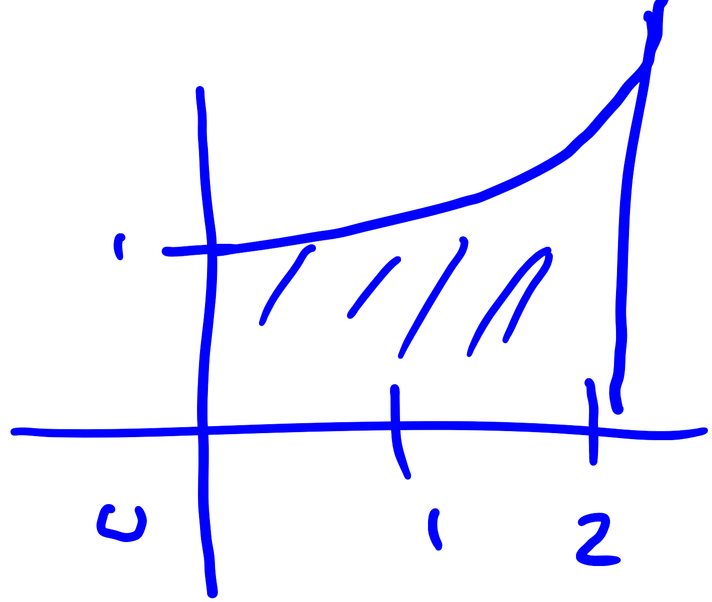
$$G'(x) = F'(\sin x) \cdot \cos x$$

$$= \sin^2 x \cdot \cos x$$

Find $\int_0^2 (x^2+1)dx$

$\boxed{\text{Find } F \text{ so } F' = \underline{x}^2 + 1}$

$(x^n)' = n x^{n-1}$



$F(x) = \frac{1}{3}x^3 + x + \cancel{C}$

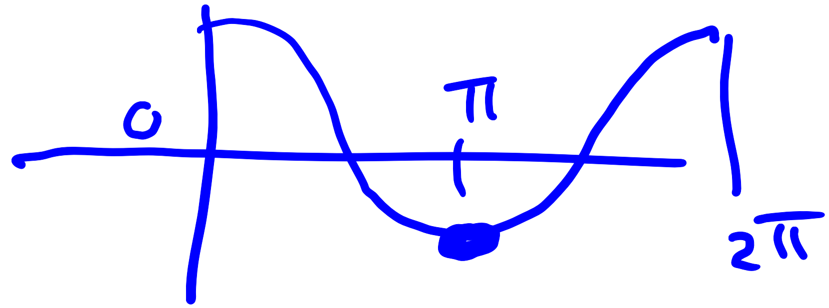
$\int_0^2 (x^2+1)\,dx = F(2) - F(0)$

$= \left(\frac{1}{3}x^3 + x\right)\Big|_0^2$

$= \left(\frac{1}{3}2^3 + 2\right) - \left(\frac{1}{3}0^3 + 0\right)$

$= \frac{8}{3} + 2$

Find $\int_0^\pi \sin(x)dx > 0$ ?

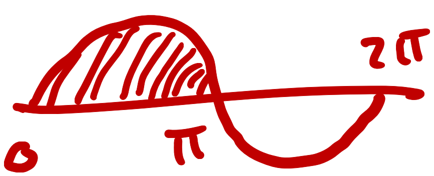



ANY $F' = \sin x$

$F(x) = -\cos x$

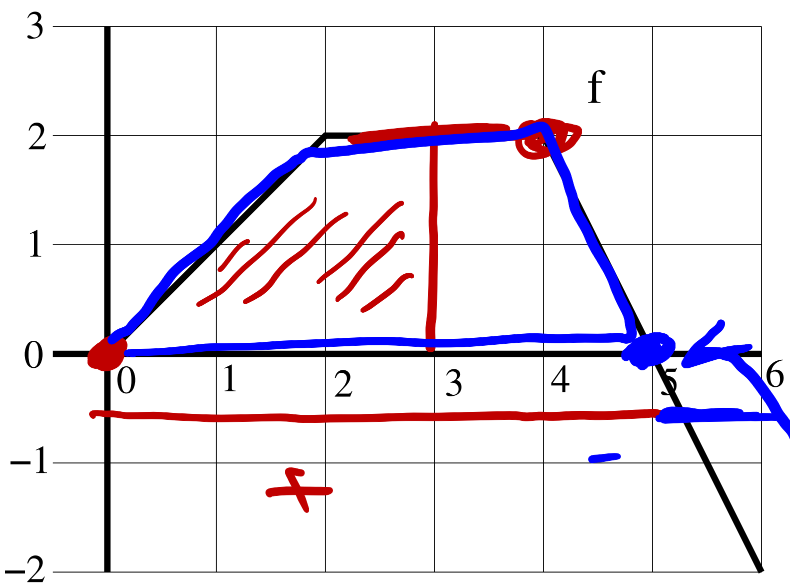
$\int_0^\pi \sin x \, dx = (-\cos x)\Big|_0^\pi$

$= (-\cos(\pi)) - (-\cos(0))$




$= 1 + 1 = 2$

Let $F(x) = \int_0^x f(t)dt$ for $f$ pictured below.




$F(0) = 0$

$F' = f$

chain rule

$G(x) = F(2x)$

$G'(x) = F'(2x) \cdot 2$

$= F'(4) \cdot 2$

FTC

$= f(4) \cdot 2$

$= 4$

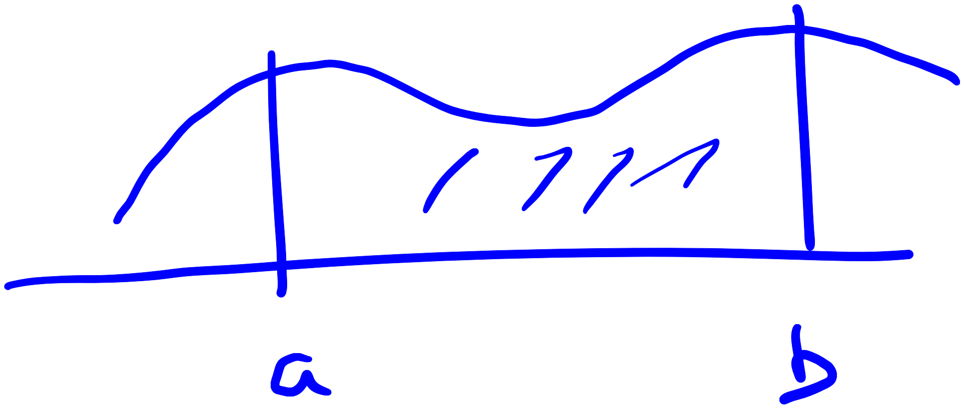
(1) What is $F(3)$? $= 4$

(2) What is $F'(3)$? $= f(3) = 2$

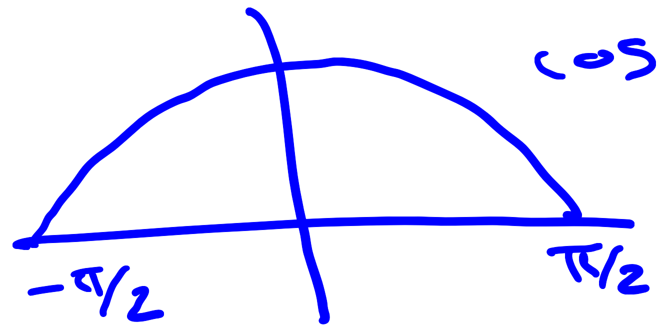
(3) At what $x$ does $F$ takes its maximum on $[0, 6]$? 5!

(4) If $G(x) = \int_0^{2x} f(t)dt$, what is $G'(2)$?

What is the average value of $\cos(x)$ on the interval $[-\pi/2, \pi/2]$?




$$\frac{1}{b-a}\int_a^b f(x)\,dx$$




$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos x\,dx$$

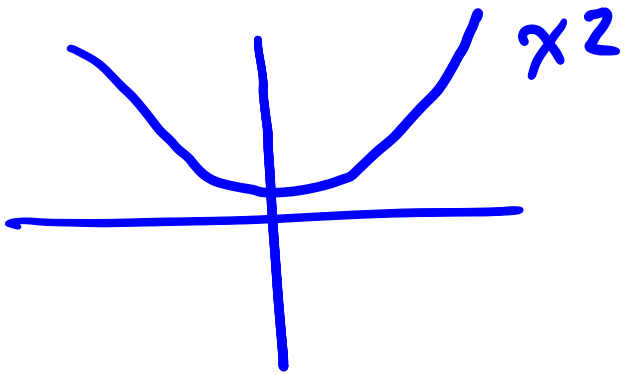
$$= \frac{1}{\pi} \sin x \Big|_{-\pi/2}^{\pi/2}$$

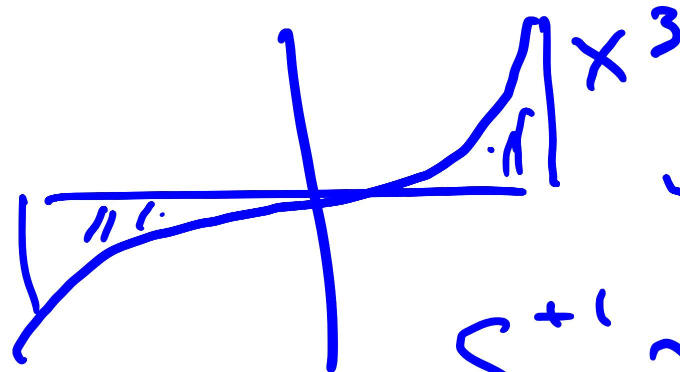
$$= \frac{1}{\pi}\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right]$$

$$1 - (-1)$$

$$= 2/\pi$$

What is the average value of $x^n$, $n \geq 1$ on the interval $[-1, 1]$?




$n = \text{odd}$

$$\int_{-1}^{+1} x^n = 0$$

$$\frac{1}{2}\int_{-1}^{1} x^n \, dx$$

$$= \frac{1}{2}\frac{1}{n+1} x^{n+1}\Big|_{-1}^{1}$$

$$= \frac{1}{2} \frac{1}{n+1}\left[1^{n+1} - (-1)^{n+1}\right]$$

$$= \begin{cases} \cancel{\frac{1}{2}} \frac{1}{n+1} \cdot \cancel{2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

What is the area trapped between $4-x^2$ and the $x$-axis for $-2 \leq x \leq 2$?

**Net Change Theorem:**

If $F$ is a quantity that changes at rate $F'$, then

$$F(b) = F(a) + \int_a^b F'(t)dt,$$

i.e., the final amount is the initial amount plus the integral of the derivative.

Velocity is the derivative of position.

Acceleration is the derivative of velocity.

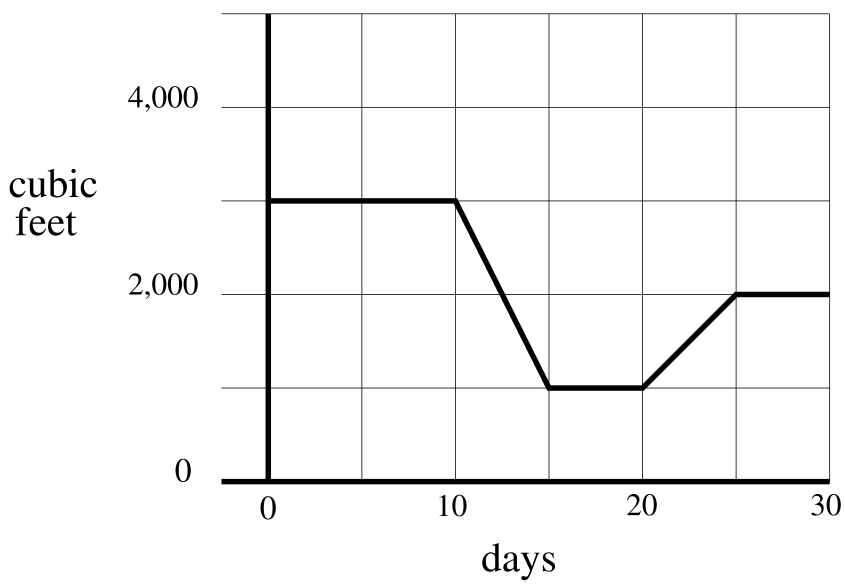
Suppose a particle starts at position 0 with zero velocity.

It's acceleration at time $t$ is $a(t) = \sin(t)$.

What is it's velocity $v(t)$ as a function of $t$?

What is it's postition $p(t)$ as a function of $t$?

The rate a which a factory produces a material (in cubic feet per day) is raphed below. What is the total amount produced in the first 15 days? In the second 15 days? If they sell the all the material for \$10 per cubic feet, what are the sales for the whole month?



A function is **odd** if $f(-x) = -f(x)$. Examples: $x$, $x^2$, $\sin(x)$.
A function is **even** if $f(-x) = f(x)$. Examples: $x^2$, $x^4$, $\cos(x)$.

If $f$ is odd, then $\int_{-a}^{a} f(x)dx = 0$ for any $a$.

If $f$ is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ for any $a > 0$.

**Side Remark:** The fundamental theorem says every continuous function $f$ has an anti-derivative $F$ given by the integral, but $F$ need not have a simple formula, even if $f$ does.

For example $f(x) = e^{-x^2}$ is important in physics and statistics, but there is no simple formula for the anti-derivative $F$.

In MAT 127 you will learn about power series expansions and that

$$f(x) = e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n}x^n + \ldots,$$

and that the anti-deriviative is

$$F(x)x + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{(n+1)}x^{n+1} + \ldots,$$