CHAPTER 2 REVIEW

KEY TERMS

arc length the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

catenary a curve in the shape of the function $y = a\cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

center of mass the point at which the total mass of the system could be concentrated without changing the moment

centroid the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

cross-section the intersection of a plane and a solid object

density function a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

disk method a special case of the slicing method used with solids of revolution when the slices are disks

doubling time if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

exponential decay systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$

exponential growth systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$

frustum a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

half-life if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

Hooke's law this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx,$ where k is a constant

hydrostatic pressure the pressure exerted by water on a submerged object

lamina a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

method of cylindrical shells a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

moment if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i;$ if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b],$ then the moments of the region with respect to the x- and y-axes are given by $M_x = \rho\int_a^b \frac{[f(x)]^2}{2}dx$ and $M_y = \rho\int_a^b xf(x)dx,$ respectively

slicing method a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

solid of revolution a solid generated by revolving a region in a plane around a line in that plane

surface area the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

symmetry principle the symmetry principle states that if a region R is symmetric about a line l, then the centroid of R lies on l

theorem of Pappus for volume this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

washer method a special case of the slicing method used with solids of revolution when the slices are washers

work the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

KEY EQUATIONS

- **Area between two curves, integrating on the x-axis**

  $$A = \int_a^b [f(x) - g(x)]dx$$

- **Area between two curves, integrating on the y-axis**

  $$A = \int_c^d [u(y) - v(y)]dy$$

- **Disk Method along the x-axis**

  $$V = \int_a^b \pi[f(x)]^2\,dx$$

- **Disk Method along the y-axis**

  $$V = \int_c^d \pi[g(y)]^2\,dy$$

- **Washer Method**

  $$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$$

- **Method of Cylindrical Shells**

  $$V = \int_a^b (2\pi x f(x))dx$$

- **Arc Length of a Function of x**

  $$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

- **Arc Length of a Function of y**

  $$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy$$

- **Surface Area of a Function of x**

  $$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + (f'(x))^2}\right)dx$$

- **Mass of a one-dimensional object**

  $$m = \int_a^b \rho(x)dx$$

- **Mass of a circular object**

  $$m = \int_0^r 2\pi x\rho(x)dx$$

- **Work done on an object**
  $W = \int_a^b F(x)dx$
- **Hydrostatic force on a plate**
  $F = \int_a^b \rho w(x)s(x)dx$
- **Mass of a lamina**
  $m = \rho \int_a^b f(x)dx$
- **Moments of a lamina**
  $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and $M_y = \rho \int_a^b x f(x)dx$
- **Center of mass of a lamina**
  $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$
- **Natural logarithm function**
- $\ln x = \int_1^x \frac{1}{t} dt$ Z
- **Exponential function** $y = e^x$
- $\ln y = \ln(e^x) = x$ Z

KEY CONCEPTS

2.1 Areas between Curves

- Just as definite integrals can be used to find the area under a curve, they can also be used to find the area between two curves.
- To find the area between two curves defined by functions, integrate the difference of the functions.
- If the graphs of the functions cross, or if the region is complex, use the absolute value of the difference of the functions. In this case, it may be necessary to evaluate two or more integrals and add the results to find the area of the region.
- Sometimes it can be easier to integrate with respect to y to find the area. The principles are the same regardless of which variable is used as the variable of integration.

2.2 Determining Volumes by Slicing

- Definite integrals can be used to find the volumes of solids. Using the slicing method, we can find a volume by integrating the cross-sectional area.
- For solids of revolution, the volume slices are often disks and the cross-sections are circles. The method of disks involves applying the method of slicing in the particular case in which the cross-sections are circles, and using the formula for the area of a circle.
- If a solid of revolution has a cavity in the center, the volume slices are washers. With the method of washers, the area of the inner circle is subtracted from the area of the outer circle before integrating.

2.3 Volumes of Revolution: Cylindrical Shells

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.

- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

2.4 Arc Length of a Curve and Surface Area

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of y.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate. It may be necessary to use a computer or calculator to approximate the values of the integrals.

2.5 Physical Applications

- Several physical applications of the definite integral are common in engineering and physics.
- Definite integrals can be used to determine the mass of an object if its density function is known.
- Work can also be calculated from integrating a force function, or when counteracting the force of gravity, as in a pumping problem.
- Definite integrals can also be used to calculate the force exerted on an object submerged in a liquid.

2.6 Moments and Centers of Mass

- Mathematically, the center of mass of a system is the point at which the total mass of the system could be concentrated without changing the moment. Loosely speaking, the center of mass can be thought of as the balancing point of the system.
- For point masses distributed along a number line, the moment of the system with respect to the origin is $M = \sum_{i=1}^{n} m_i x_i$. For point masses distributed in a plane, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \sum_{i=1}^{n} m_i y_i$ and $M_y = \sum_{i=1}^{n} m_i x_i$, respectively.
- For a lamina bounded above by a function $f(x)$, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and $M_y = \rho \int_a^b x f(x) dx$.
- The x- and y-coordinates of the center of mass can be found by dividing the moments around the y-axis and around the x-axis, respectively, by the total mass. The symmetry principle says that if a region is symmetric with respect to a line, then the centroid of the region lies on the line.
- The theorem of Pappus for volume says that if a region is revolved around an external axis, the volume of the resulting solid is equal to the area of the region multiplied by the distance traveled by the centroid of the region.

2.7 Integrals, Exponential Functions, and Logarithms

- The earlier treatment of logarithms and exponential functions did not define the functions precisely and formally. This section develops the concepts in a mathematically rigorous way.
- The cornerstone of the development is the definition of the natural logarithm in terms of an integral.
- The function e^x is then defined as the inverse of the natural logarithm.
- General exponential functions are defined in terms of e^x, and the corresponding inverse functions are general logarithms.

- Familiar properties of logarithms and exponents still hold in this more rigorous context.

2.8 Exponential Growth and Decay

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

2.9 Calculus of the Hyperbolic Functions

- Hyperbolic functions are defined in terms of exponential functions.
- Term-by-term differentiation yields differentiation formulas for the hyperbolic functions. These differentiation formulas give rise, in turn, to integration formulas.
- With appropriate range restrictions, the hyperbolic functions all have inverses.
- Implicit differentiation yields differentiation formulas for the inverse hyperbolic functions, which in turn give rise to integration formulas.
- The most common physical applications of hyperbolic functions are calculations involving catenaries.

CHAPTER 2 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

435. The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

436. If the force is constant, the amount of work to move an object from $x = a$ to $x = b$ is $F(b - a)$.

437. The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

438. If the half-life of seaborgium-266 is 360 ms, then $k = (\ln(2))/360$.

For the following exercises, use the requested method to determine the volume of the solid.

439. The volume that has a base of the ellipse $x^2/4 + y^2/9 = 1$ and cross-sections of an equilateral triangle perpendicular to the y-axis. Use the method of slicing.

440. $y = x^2 - x$, from $x = 1$ to $x = 4$, rotated around the y-axis using the washer method

441. $x = y^2$ and $x = 3y$ rotated around the y-axis using the washer method

442. $x = 2y^2 - y^3$, $x = 0$, and $y = 0$ rotated around the x-axis using cylindrical shells

For the following exercises, find

a. the area of the region,

b. the volume of the solid when rotated around the x-axis, and

c. the volume of the solid when rotated around the y-axis. Use whichever method seems most appropriate to you.

443. $y = x^3$, $x = 0$, $y = 0$, and $x = 2$

444. $y = x^2 - x$ and $x = 0$

445. **[T]** $y = \ln(x) + 2$ and $y = x$

446. $y = x^2$ and $y = \sqrt{x}$

447. $y = 5 + x$, $y = x^2$, $x = 0$, and $x = 1$

448. Below $x^2 + y^2 = 1$ and above $y = 1 - x$

449. Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius 4.

450. Find the center of mass for $\rho = \tan^2 x$ on $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

451. Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

For the following exercises, find the requested arc lengths.

452. The length of x for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

453. The length of y for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

For the following exercises, find the surface area and volume when the given curves are revolved around the specified axis.

454. The shape created by revolving the region between $y = 4 + x$, $y = 3 - x$, $x = 0$, and $x = 2$ rotated around the y-axis.

455. The loudspeaker created by revolving $y = 1/x$ from $x = 1$ to $x = 4$ around the x-axis.

For the following exercises, consider the Karun-3 dam in Iran. Its shape can be approximated as an isosceles triangle with height 205 m and width 388 m. Assume the current depth of the water is 180 m. The density of water is 1000 kg/m^3.

456. Find the total force on the wall of the dam.

457. You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and $45°\text{F}$ outside and the temperature of the body is $78°\text{F}$. You know the cooling constant is $k = 0.00824°\text{F/min}$. When did the victim die, assuming that a human's temperature is $98°\text{F}$?

For the following exercise, consider the stock market crash in 1929 in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.

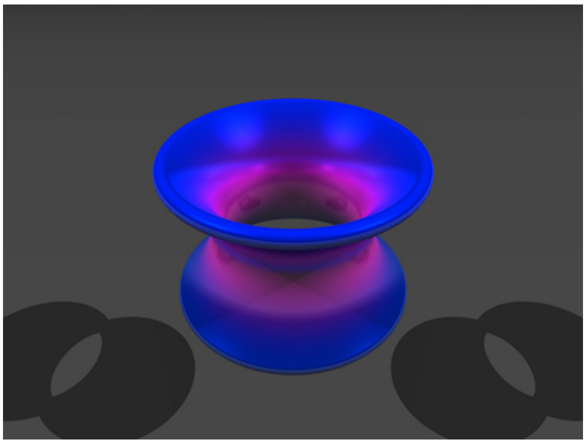
Years after 1920	Value ($)
1	63.90
3	100
5	110
7	160
9	381.17

Source: http://stockcharts.com/freecharts/historical/djia19201940.html

458. **[T]** The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in 2014 ?

For the following exercises, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.

459. Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-axis, as shown here.



460. Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-axis.