POINTS OF INCREASE FOR RANDOM WALKS

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Abstract: Say that a sequence $S_0, \dots, S_n$ has a (global) *point of increase* at k if S_k is maximal among $S_0, \dots, S_k$ and minimal among $S_k, \dots, S_n$. We give an elementary proof that an n-step symmetric random walk on the line has a (global) point of increase with probability comparable to $1/\log n$. (No moment assumptions are needed). This implies the classical fact, due to Dvoretzky, Erdős and Kakutani (1961), that Brownian motion has no points of increase.

1 Introduction

A real-valued function f has a **global point of increase in the interval** $(\mathbf{a}, \mathbf{b})$ if there is a point t_0 in the interval such that $f(t) \le f(t_0)$ for all $t \in (a, t_0)$ and $f(t_0) \le f(t)$ for all $t \in (t_0, b)$. Dvoretzky, Erdős and Kakutani (1961) proved that Brownian motion almost surely has no global points of increase in any time interval. Knight (1981) and Berman (1983) noted that this follows from properties of the local time of Brownian motion; elegant direct proofs were given by Adelman (1985) and Burdzy (1990). The aim of this note is to show that the nonincrease phenomenon holds for arbitrary symmetric random walks, and can thus be viewed as a combinatorial consequence of fluctuations in random sums.

Definition: Say that a sequence of real numbers $s_0, s_1, \dots, s_n$ has a (global) **point of increase** at k if $s_i \le s_k$ for $i = 0, 1, \dots, k-1$ and $s_k \le s_j$ for $j = k+1, \dots, n$.

Theorem 1.1 *Let $S_0, S_1, \dots, S_n$ be a random walk where the independent identically distributed increments $S_i - S_{i-1}$ have a symmetric distribution,* or *have mean 0 and finite variance. Then*

$$\mathbf{P}[S_0, \dots, S_n \text{ has a point of increase}] \le \frac{C}{\log n}\ ,$$

for $n > 1$, where C does not depend on n.

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As we shall see in Section 4, this estimate is sharp except for the value of C.

Proof of nonincrease of Brownian motion:

To deduce this, it suffices to apply Theorem 1.1 to simple random walk on the integers. Indeed it clearly suffices to show that the Brownian motion $\{B(t)\}_{t\geq 0}$ almost surely has no global points of increase in a fixed rational time interval (a,b). Sampling the Brownian motion when it visits a lattice yields a simple random walk; by refining the lattice, we may make this walk as long as we wish, which will complete the proof. More precisely, for any vertical spacing $h>0$ define τ_0 to be the first $t \geq a$ such that $B(t)$ is an integral multiple of h, and for $i \geq 0$ let τ_{i+1} be the minimal $t \geq \tau_i$ such that $|B(t)-B(\tau_i)| = h$. Then

$$\left\{ \frac{B(\tau_i)-B(\tau_0)}{h} \ : \ i \geq 0 \ \text{ and } \ \tau_i < b \right\}$$

is a finite portion of a simple random walk. If the Brownian motion has a (global) point of increase in (a,b) at the point t_0, then this random walk has a point of increase at the integer k where τ_k is closest to t_0. Thus by Theorem 1.1,

$$\mathbf{P}\,[\text{B.M. has a global point of increase in } \ (a,b)] \leq \frac{C}{\log n} + \mathbf{P}[\tau_n \geq b]. \tag{1}$$

Since the event $[\tau_n \geq b]$ can happen only if the B.M. increment satisfies $|B(b)-B(a)| \leq (n+1)h$, the probability in (1) can be made arbitrarily small by first taking n large and then picking $h>0$ very small. □

2 Proof of the upper bound on the probability of increase

Notation: For the rest of the paper, let $X_1, X_2, \ldots$ be i.i.d. random variables, and let $S_k = \sum_{i=1}^k X_i$ be their partial sums. Denote

$$p_n = \mathbf{P}[S_i \geq 0 \ \text{ for all } \ 1 \leq i \leq n]\,. \tag{2}$$

Observe that the event that $[S_n$ is largest among $S_0, S_1, \ldots S_n]$ is precisely the event that the reversed random walk $X_n + \ldots + X_{n-k+1}$ is nonnegative for all $k = 1, \ldots, n$; thus this event also has probability p_n. To see that this event is positively correlated with the event in (2), we need Harris' inequality.

Proposition 2.1 (Harris 1960) *Let $X_1, \ldots, X_n$ be independent random variables, and let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be nondecreasing functions. (I.e., f and g are nondecreasing in each coordinate.) Then*

$$\mathbf{E}[f(X_1, \ldots, X_n) \cdot g(X_1, \ldots, X_n)] \geq \mathbf{E}[f(X_1, \ldots, X_n)] \cdot \mathbf{E}[g(X_1, \ldots, X_n)] \, .$$

For a proof of this inequality see, e.g., Kesten (1982) pp. 72–73.

Lemma 2.2

- **(i)** $\mathbf{P}[0 \leq S_i \leq S_n \ \textit{for all} \ \ 1 \leq i \leq n] \geq p_n^2$.
- **(ii)** *If the increments X_i have a symmetric distribution or have mean 0 and finite variance, then there are positive constants C_1, C_2 such that $C_1 n^{-1/2} \leq p_n \leq C_2 n^{-1/2}$ for all $n \geq 1$.*

PROOF:

- **(i)** Let $f(x_1, \ldots, x_n) := 1$ if all the partial sums $x_1 + \ldots + x_k$ for $k = 1, \ldots, n$ are nonnegative, and $f(x_1, \ldots, x_n) := 0$ otherwise. Also, define $g(x_1, \ldots, x_n) := f(x_n, \ldots, x_1)$. Then f and g are nondecreasing functions, and applying the Harris inequality concludes the proof.
- **(ii)** For simple RW, the estimate follows easily from the reflection principle; for the general argument, see Feller (1966), Section XII.8.

□

We now state an extension of Theorem 1.1.

Theorem 2.3 *For any random walk $\{S_j\}$ on the line,*

$$\mathbf{P}[S_0, \ldots, S_n \ \textit{has a point of increase}] \leq 2 \frac{\sum_{k=0}^{n} p_k p_{n-k}}{\sum_{k=0}^{\lfloor n/2 \rfloor} p_k^2} \, . \qquad (3)$$

Proof of Theorem 2.3. The idea is simple: The expected number of points of increase is the numerator in (3), and given that there is at least one such point, the expected number is bounded below by the denominator; the ratio of these expectations gives the required probability.

To carry this out, denote by $I_n(k)$ the event that k is a point of increase for $S_0, S_1, \ldots, S_n$ and by $F_n(k) := I_n(k) \setminus \cup_{i=0}^{k-1} I_n(i)$ the event that k is the first such point. The events that $[S_k$ is largest among $S_0, S_1, \ldots S_k]$ and that $[S_k$ is smallest among $S_k, S_{k+1}, \ldots S_n]$ are independent, and therefore $\mathbf{P}[I_n(k)] = p_k p_{n-k}$.

Observe that if S_j is minimal among $S_j, \ldots, S_n$, then any point of increase for $S_0, \ldots, S_j$ is automatically a point of increase for $S_0, \ldots, S_n$. Therefore for $j \leq k$ we can write

$$\begin{aligned} &F_n(j) \cap I_n(k) \ = \\ &\quad F_j(j) \cap \{S_j \leq S_i \leq S_k \text{ for all } i \in [j,k]\} \cap \{S_k \text{ is minimal among } S_k, \ldots, S_n\}\,. \end{aligned} \tag{4}$$

The three events on the right-hand side are independent, as they involve disjoint sets of summands; the second of these events is of the type considered in Lemma 2.2(i). Thus

$$\begin{aligned} \mathbf{P}[F_n(j) \cap I_n(k)] &\geq \mathbf{P}[F_j(j)]\, p_{k-j}^2\, p_{n-k} \\ &\geq p_{k-j}^2\, \mathbf{P}[F_j(j)]\, \mathbf{P}\left[S_j \text{ is minimal among } S_j, \ldots, S_n\right], \end{aligned}$$

since $p_{n-k} \geq p_{n-j}$. Here the two events on the right are independent, and their intersection is precisely $F_n(j)$. Consequently $\mathbf{P}[F_n(j) \cap I_n(k)] \geq p_{k-j}^2 \mathbf{P}[F_n(j)]$.

Decomposing the event $I_n(k)$ according to the first point of increase gives

$$\begin{aligned} \sum_{k=0}^{n} p_k p_{n-k} &= \sum_{k=0}^{n} \mathbf{P}[I_n(k)] \geq \sum_{k=0}^{n} \sum_{j=0}^{k} \mathbf{P}[F_n(j) \cap I_n(k)] \\ &\geq \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=j}^{j+\lfloor n/2 \rfloor} p_{k-j}^2 \mathbf{P}[F_n(j)] \geq \sum_{j=0}^{\lfloor n/2 \rfloor} \mathbf{P}[F_n(j)] \sum_{i=0}^{\lfloor n/2 \rfloor} p_i^2\,. \end{aligned}$$

This yields an upper bound on the probability that $\{S_j\}_{j=0}^n$ has a point of increase by time $n/2$; but this RW has a point of increase at time k if and only if the "reversed" RW $\{S_n - S_{n-i}\}_{i=0}^n$ has a point of increase at time $n-k$. Doubling this upper bound proves the theorem.

□

Proof of Theorem 1.1. To bound the numerator in (3), we can use symmetry to deduce from Lemma 2.2(ii) that

$$\sum_{k=0}^{n} p_k p_{n-k} \;\leq\; 2 + 2 \sum_{k=1}^{\lfloor n/2 \rfloor} p_k p_{n-k}$$

$$\leq\; 2 + 2C_2 \sum_{k=1}^{\lfloor n/2 \rfloor} k^{-1/2}(n-k)^{-1/2} \;\leq\; 2 + 4C_2 n^{-1/2} \sum_{k=1}^{\lfloor n/2 \rfloor} k^{-1/2} \,,$$

which is bounded above because the last sum is $O(n^{1/2})$. Since Lemma 2.2(ii) implies that the denominator in (3) is at least $C_1^2 \log\lfloor n/2 \rfloor$, this completes the proof. □

Remark: For *Symmetric* random walks, there is an alternative way to bound the numerator in (3) via comparison to strict maxima: Denoting $\alpha = \mathbf{P}[X_1 > 0]$ and using the symmetry of the step distribution, we see that the probability that the walk has a *strict* maximum at time k is at least $p_{k-1} \cdot \mathbf{P}[X_k > 0] \cdot \mathbf{P}[X_{k+1} < 0] \cdot p_{n-k-1} \geq \alpha^2 p_k p_{n-k}$. Hence the expected number of points of increase satisfies

$$\sum_{k=0}^{n} p_k p_{n-k} \leq \alpha^{-2}\, \mathbf{E}[\,\text{number of strict maxima among } S_0, \ldots S_n] \leq \alpha^{-2} \,.$$

Thus the probability that $S_0, \ldots, S_n$ has a point of increase is at most $2(\alpha C_1)^{-2}/\log\lfloor n/2 \rfloor$.

3 A lower bound for the probability of increase

Proposition 3.1 *For any random walk on the line*

$$\mathbf{P}[S_0, \ldots, S_n \text{ has a point of increase}] \geq \frac{\sum_{k=0}^{n} p_k p_{2n-k}}{2 \sum_{k=0}^{\lfloor n/2 \rfloor} p_k^2} \,. \qquad (5)$$

In particular if the increments have a symmetric distribution, or *have mean 0 and finite variance, then*[2] $\mathbf{P}[S_0, \ldots, S_n$ *has a point of increase*$] \asymp 1/\log n$ *for* $n > 1$.

[2]the symbol $\asymp$ means that the ratio of the two sides is bounded above and below by positive constants which do not depend on n.

PROOF: First we record an easy converse to Lemma 2.2(i):

$$\mathbf{P}[0 \le S_i \le S_k \ \text{ for all } \ 1 \le i \le k] \ \le$$
$$\mathbf{P}\left[\{0 \le S_i \ \text{ for all } \ i \in (0, \lfloor k/2 \rfloor]\} \cap \{S_i \le S_k \ \text{ for all } \ i \in [\lceil k/2 \rceil, k)\}\right] \ = \ p^2_{\lfloor k/2 \rfloor} \, .$$

Now the decomposition (4) in the proof of Theorem 2.3, combined with the last inequality, show that

$$\sum_{k=0}^{n} p_k p_{2n-k} = \sum_{k=0}^{n} \mathbf{P}[I_{2n}(k)] = \sum_{k=0}^{n} \sum_{j=0}^{k} \mathbf{P}[I_{2n}(k) \cap F_{2n}(j)] \ \le \sum_{j=0}^{n} \mathbf{P}[F_n(j)] \sum_{i=0}^{n} p^2_{\lfloor i/2 \rfloor} \, .$$

This implies (5). The assertion concerning symmetric or mean 0, finite variance walks follows from Lemma 2.2(ii) and the proof of Theorem 1.1. □

In conclusion, we note that some conditions for nonincrease of Lévy processes have been given by Bertoin (1991) and Doney (1994); it would be interesting to compare these conditions to the estimates in Theorem 2.3 and Proposition 3.1. It is natural to ask whether the assumption of independent increments in Theorem 1.1 can be relaxed; rather than attempt a general statement in this direction, we mention a concrete example.

Conjecture: *Denote* $S_k(\theta) = \sum_{j=1}^{k} \cos(2^j \theta)$, *and let* λ *be Lebesgue measure on* $[0, 2\pi]$. *Then we conjecture that for* $n > 1$,

$$\lambda\{\theta \, : \, S_0(\theta), S_1(\theta), \ldots, S_n(\theta) \ \textit{has a point of increase}\} \asymp \frac{1}{\log n} \, .$$

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