Study Problems

MAT 360

1. True or False?

_T_ Any two points of $\mathbb{E}^2$ are joined by a unique line.

_F_ Any two points of S^2 are joined by a unique line.

_T_ Any two points of $\mathbb{P}^2$ are joined by a unique line.

_T_ Any two points of $\mathcal{H}^2$ are joined by a unique line.

_F_ Any two lines in $\mathbb{E}^2$ meet in a unique point.

_F_ Any two lines in S^2 meet in a unique point.

_T_ Any two lines in $\mathbb{P}^2$ meet in a unique point.

_F_ Any two lines in $\mathcal{H}^2$ meet in a unique point.

_T_ Any isometry of $\mathbb{P}^2$ is a rotation about some point.

_F_ Any isometry of S^2 is a rotation about some point.

_F_ The isometry group of $\mathbb{E}^2$ is Abelian.

_F_ The isometry group of S^2 is Abelian.

_F_ The isometry group of $\mathbb{P}^2$ is Abelian.

_F_ The isometry group of $\mathcal{H}^2$ is Abelian.

2. Consider the points

$$A = (\sin\beta, 0, \cos\beta), \quad B = (0, \sin\alpha, \cos\alpha), \quad C = (0, 0, 1)$$

in $S^2 \subset \mathbb{R}^3$.

Assume for simplicity that $0 < \alpha, \beta < \pi$!!!

(a) Compute the (spherical) distances $a = BC$, $b = AC$, and $c = AB$.

$$\begin{aligned} a &= \cos^{-1}\langle(0,\sin\alpha,\cos\alpha),(0,0,1)\rangle = \cos^{-1}(\cos\alpha) = \alpha \\ b &= \cos^{-1}\langle(\sin\beta,0,\cos\beta),(0,0,1)\rangle = \cos^{-1}(\cos\beta) = \beta \\ c &= \cos^{-1}\langle(0,\sin\alpha,\cos\alpha),(\sin\beta,0,\cos\beta)\rangle = \cos^{-1}(\cos\alpha\cos\beta) \end{aligned}$$

(b) Show that $\angle ACB$ is a right angle (in the spherical sense).

$$\begin{aligned} \angle ACB &= \cos^{-1}\left\langle \frac{(0,0,1)\times(\sin\beta,0,\cos\beta)}{|(0,0,1)\times(\sin\beta,0,\cos\beta)|}, \frac{(0,0,1)\times(0,\sin\alpha,\cos\alpha)}{|(0,0,1)\times(0,\sin\alpha,\cos\alpha)|} \right\rangle \\ &= \cos^{-1}\left\langle \frac{(0,\sin\beta,0)}{|\sin\beta|}, \frac{(-\sin\alpha,0,0)}{|-\sin\alpha|} \right\rangle \\ &= \cos^{-1}\langle(0,1,0),(-1,0,0)\rangle \\ &= \cos^{-1} 0 \\ &= \frac{\pi}{2}. \end{aligned}$$

(c) Show that $(\cos a)(\cos b) = \cos c$.

By part (a), $\cos c = \cos\alpha\cos\beta = (\cos a)(\cos b)$.

(d) Does one have $a^2 + b^2 = c^2$ when, for example, $\alpha = \beta = \pi/4$?

No, because when $\alpha = \beta = \frac{\pi}{4}$, $a = b = \frac{\pi}{4}$, while

$$c = \cos^{-1}(\cos^2\frac{\pi}{4}) = \cos^{-1}((\frac{1}{\sqrt{2}})^2) = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3},$$

and obviously $(\frac{\pi}{4})^2 + (\frac{\pi}{4})^2 \neq (\frac{\pi}{3})^2$.

3. Consider the points

$$A = (\sinh\beta, 0, \cosh\beta), \quad B = (0, \sinh\alpha, \cosh\alpha), \quad C = (0,0,1)$$

in $\mathcal{H}^2 \subset \mathbb{R}^3$.

(a) Compute the hyperbolic distances $a = BC$, $b = AC$, and $c = AB$.

$$\begin{aligned}
a &= \cosh^{-1}\left|\mathbb{B}\Big((0,\sinh\alpha,\cosh\alpha),(0,0,1)\Big)\right| = \cosh^{-1}(\cosh\alpha) = \alpha \\
b &= \cosh^{-1}\left|\mathbb{B}\Big(\sinh\beta,0,\cosh\beta),(0,0,1)\Big)\right| = \cosh^{-1}(\cosh\beta) = \beta \\
c &= \cosh^{-1}\left|\mathbb{B}\Big(0,\sinh\alpha,\cosh\alpha),(\sinh\beta,0,\cosh\beta)\Big)\right| = \cosh^{-1}(\cosh\alpha\cosh\beta)
\end{aligned}$$

(b) Show that $\angle ACB$ is a right angle (in the hyperbolic sense).

$$\begin{aligned}
\angle ACB &= \cos^{-1}\mathbb{B}\Big(\frac{(0,0,1)\times_{\mathbb{B}}(\sinh\beta,0,\cosh\beta)}{|(0,0,1)\times_{\mathbb{B}}(\sinh\beta,0,\cosh\beta)|_{\mathbb{B}}}, \frac{(0,0,1)\times_{\mathbb{B}}(0,\sinh\alpha,\cosh\alpha)}{|(0,0,1)\times_{\mathbb{B}}(0,\sinh\alpha,\cosh\alpha)|_{\mathbb{B}}}\Big) \\
&= \cos^{-1}\mathbb{B}\Big(\frac{(0,\sinh\beta,0)}{|\sinh\beta|}, \frac{(-\sinh\alpha,0,0)}{|-\sinh\alpha|}\Big) \\
&= \cos^{-1}\mathbb{B}\left((0,1,0),(-1,0,0)\right) \\
&= \cos^{-1} 0 \\
&= \frac{\pi}{2}.
\end{aligned}$$

(c) Show that $(\cosh a)(\cosh b) = \cosh c$.

By part (a), $\cosh c = \cosh\alpha\cosh\beta = (\cosh a)(\cosh b)$.

(d) Does one have $a^2 + b^2 = c^2$ when, for example, $\alpha = \beta = \cosh^{-1}(\sqrt{2})$?

No. When $\alpha = \beta = \cosh^{-1}(\sqrt{2})$, $a = b = \ln(1+\sqrt{2})$, whereas $c = \cosh^{-1}(2) = \ln(2+\sqrt{3})$, so that $c - \sqrt{a^2+b^2} = \ln\frac{2+\sqrt{3}}{(1+\sqrt{2})^{\sqrt{2}}} \neq 0$.

4. (a) Suppose that $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map such that

$$\mathbb{B}\Big(L(\vec{v}), L(\vec{w})\Big) = \mathbb{B}\Big(\vec{v}, \vec{w}\Big)$$

for every $\vec{v}, \vec{w} \in \mathbb{R}^3$, and such that

$$\mathbb{B}\Big(L\left((0,0,1)\right)\ ,\ (0,0,1)\Big) < 0,$$

where $\mathbb{B}$ denotes the Minkowski inner product

$$\mathbb{B}(\vec{v}, \vec{w}) = v_1 w_1 + v_2 w_2 - v_3 w_3.$$

Prove that L maps $\mathcal{H}^2 \subset \mathbb{R}^3$ to itself, and that

$$L|_{\mathcal{H}^2} : \mathcal{H}^2 \to \mathcal{H}^2$$

is an isometry.

Recall that $\mathcal{H}^2 \subset \mathbb{R}^3$ is the set of $\vec{v} = (v_1, v_2, v_3)$ with $\mathbb{B}(\vec{v}, \vec{v}) = -1$ and $v_3 > 0$. For any such $\vec{v}$, we then have $\mathbb{B}(L(\vec{v}), L(\vec{v})) = \mathbb{B}(\vec{v}, \vec{v}) = -1$. In particular, if $\vec{w} = L(\vec{v})$, then $w_3^2 = 1 + w_1^2 + w_2^2 \geq 1$, so that $w_3 = -\mathbb{B}(L(\vec{v}), (0,0,1)) \neq 0$ for all $\vec{v} \in \mathcal{H}^2$. Because $f(\vec{v}) = -\mathbb{B}(L(\vec{v}), (0,0,1))$ is continuous function, and since any two points of $\mathcal{H}^2$ can be joined by a curve in $\mathcal{H}^2 \subset \mathbb{R}^3$, the intermediate value theorem implies that the sign of $f(\vec{v}) = w_3$ is the same for all $\vec{v} \in \mathcal{H}^2$. But since $f((0,0,1)) = -\mathbb{B}\Big(L\left((0,0,1)\right)\ ,\ (0,0,1)\Big) > 0$, we conclude that $f(\vec{v}) > 0$ for all $\vec{v} \in \mathcal{H}^2$. Hence $L(\mathcal{H}^2) \subset \mathcal{H}^2$.

If L is such a linear transformation, and if $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is the standard basis for $\mathbb{R}^3$, then we have

$$\mathbb{B}(L(\vec{e}_j), L(\vec{e}_k)) = g_{jk}$$

where g_{jk} are the entries of the matrix

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The condition that $\mathbb{B}\Big(L(\vec{v}), L(\vec{w})\Big) = \mathbb{B}\Big(\vec{v}, \vec{w}\Big)$ is thus equivalent to the requirement that the matrix A of L satisfy $A^t G A = G$, or in other words that

$$G A^t G A = I,$$

where A^t denotes the transpose matrix of A. In particular, L is always invertible; indeed, L^{-1} is the linear transformation with matrix $A^{-1} = GA^tG$. This inverse L^{-1} automatically satisfies $\mathbb{B}\Big(L^{-1}(\vec{v}), L^{-1}(\vec{w})\Big) = \mathbb{B}\Big(LL^{-1}(\vec{v}), LL^{-1}(\vec{w})\Big) = \mathbb{B}\Big(\vec{v}, \vec{w}\Big)$. The condition that $\mathbb{B}\Big(L\left((0,0,1)\right)\ ,\ (0,0,1)\Big) < 0$ just says that the a_{33} entry of A is positive; and since A and GA^tG have the same entry in this slot, we conclude that L^{-1} also sends $\mathcal{H}^2$ to itself. This proves that $L|_{\mathcal{H}^2}$ is a bijection.

Now if $\vec{v}, \vec{w} \in \mathcal{H}^2$, we have

$$d(L(\vec{v}), L(\vec{w})) = \cosh^{-1}|\mathbb{B}(L(\vec{v}), L(\vec{w}))| = \cosh^{-1}|\mathbb{B}(\vec{v}, \vec{w})| = d(\vec{v}, \vec{w})$$

so $L|_{\mathcal{H}^2}$ is an isometry.

(b) Use this to show that the linear map with matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & \sinh\alpha \\ 0 & \sinh\alpha & \cosh\alpha \end{bmatrix}$$

is an isometry of $\mathcal{H}^2$ for any $\alpha \in \mathbb{R}$.

Let us check that the above matrix A satisfies $A^tGA = G$. Indeed,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & \sinh\alpha \\ 0 & \sinh\alpha & \cosh\alpha \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & \sinh\alpha \\ 0 & \sinh\alpha & \cosh\alpha \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & -\sinh\alpha \\ 0 & \sinh\alpha & -\cosh\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & \sinh\alpha \\ 0 & \sinh\alpha & \cosh\alpha \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh^2\alpha - \sinh^2\alpha & 0 \\ 0 & 0 & \sinh^2\alpha - \cosh^2\alpha \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Since the a_{33} entry in the matrix is also positive, this linear transformation is an isometry of $\mathcal{H}^2$.

(c) Similarly, show that the linear map with matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an isometry of $\mathcal{H}^2$ for any $\theta \in \mathbb{R}$.

This time, the key calculation is

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Since the a_{33} entry in the matrix is again positive, this linear transformation also induces an isometry of $\mathcal{H}^2$.

(d) Using the isometries constructed in (c) and (b), in that order, show that if $x \in \mathcal{H}^2$ is any point, then there is an isometry $\mathcal{H}^2 \to \mathcal{H}^2$ which sends x to the point $(0, 0, 1)$.

The transformations constructed in (c) are ordinary rotations of $\mathbb{R}^3$ around the x_3-axis. By applying such a rotation, we may move x to a point of the x_1x_3-plane with $x_3 > 0$. Moreover, this new point will still lie in $\mathcal{H}^2$, and so will be of the form $x_3 = \cosh u$, $x_1 = \sinh u$ for some u. Now apply the transformation constructed in (b), with $\alpha = -u$, to move our point to $(0, 0, 1)$.

5. (a) Find the equation for the projective line ℓ_1 joining the points $[1,1,1]$ and $[2,3,1]$ in $\mathbb{P}_2$.

Since $(1,1,1)\times(2,3,1)=(-2,1,1)$, the line in question is given by

$$-2x_1+x_2+x_3=0.$$

As a double-check, notice that two solutions of this equation are $(x_1,x_2,x_3)=(1,1,1)$ and $(x_1,x_2,x_3)=(2,3,1)$.

(b) Find the equation for the projective line ℓ_2 which passes through $[0,0,1]$ and is orthogonal (in the elliptic sense) to ℓ_1 at $\ell_1\cap\ell_2$.

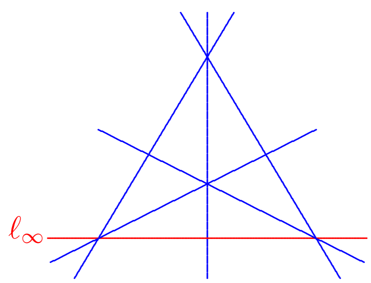
This is equivalent to finding the line which joins the pole of ℓ_1 to $[0,0,1]$. Now the pole of ℓ_1 is a multiple of $(-2,1,1)$ by part (a), and since

$$(-2,1,1)\times(0,0,1)=(1,2,0),$$

the line in question is therefore given by the equation

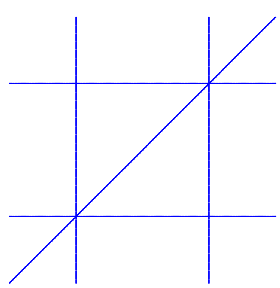
$$x_1+2x_2=0.$$

6. Consider the following configuration of lines in the projective place $\mathbb{P}^2$:



Delete the indicated line ℓ_∞, and identify the complement $\mathbb{P}^2-\ell_\infty$ with the affine plane $\mathbb{R}^2$. Draw the resulting configuration of ordinary lines in the affine plane.

Answer:



7. We have seen that every isometry of $\mathbb{E}^2$ can be represented by a 3×3 matrix with real coefficients. But every isometry of S^2 can *also* be represented by a 3×3 matrix. Describe the set of 3×3 matrices which simultaneously represent isometries of *both* $\mathbb{E}^2$ and S^2. Does this collection of matrices form a group?

The intersection of two subgroups is always a subgroup. In our case, we are taking the intersection of two subgroups of $GL(3)$, so the intersection will certainly be a group. In our case, we are looking at the set of elements of $O(3)$ which send $\vec{e}_3 = (0, 0, 1)$ to itself. The first two columns of such a matrix must be orthogonal to $\vec{e}_3$, and mutually orhtogonal. So the group in question is

$$\left\{ \begin{pmatrix} \cos\theta & \mp\sin\theta & 0 \\ \sin\theta & \pm\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

which is of course isomorphic to $O(2)$.