Homework # 1

MAT 566

Due 3/24/22

1. Let n be a positive integer, and let k be an integer with $0 \leq k \leq n$. Let $\mathcal{S} = \{[a_{jk}] \mid a_{jk} = a_{kj} \in \mathbb{R}\}$ be the set of real symmetric $n \times n$ matrices, and and let $\mathbf{H} \in \mathcal{S}$ be the diagonal matrix

$$\mathbf{H} = \begin{bmatrix} \underbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}_{k} & \\ & n{-}k\left\{\begin{matrix} +1 & & \\ & \ddots & \\ & & +1 \end{matrix}\right. \end{bmatrix}.$$

Let $\mathcal{T}$ be the set of real upper-triangular $n \times n$ matrices

$$\begin{bmatrix} * & * & * & * & * & * \\ & * & * & * & * & * \\ & & * & * & * & * \\ & & & * & * & * \\ & & & & * & * \\ & & & & & * \end{bmatrix}$$

and let $\mathbf{I} \in \mathcal{T}$ denote the $n \times n$ identity matrix. Use the inverse function theorem to show that the smooth map $\Psi : \mathcal{T} \to \mathcal{S}$ defined by

$$\Psi(\mathbf{A}) = \mathbf{A}^t \, \mathbf{H} \, \mathbf{A}$$

restricts to some open neighborhood $\mathcal{V}$ of $\mathbf{I} \in \mathcal{T}$ as a diffeomorphism onto some open neighborhood $\mathcal{U}$ of $\mathbf{H} \in \mathcal{S}$.

Hint: If $\mathbf{B} \in \mathcal{T}$, first calculate $\Psi(\mathbf{I} + t\mathbf{B})$ modulo terms of order t^2,

For later use, define $\Phi : \mathcal{U} \to \mathcal{V}$ to be the inverse diffeomorphism $(\Psi|_{\mathcal{V}})^{-1}$.

2. (a) Suppose that a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ has a non-degenerate critical point of index k at $\mathsf{0} \in \mathbb{R}^n$. Prove that there is a linear change of coordinates $\mathbb{R}^n \to \mathbb{R}^n$ so that the Hessian of f at $\mathsf{0}$ becomes $2\mathbf{H}$, where $\mathbf{H}$ is the matrix of problem 1.

(b) Now, after having first made this linear coordinate-change, show that there is then a symmetric-matrix-valued function $[\mathbf{h}] = [h_{jk}] : \mathbb{R}^n \to \mathcal{S}$ such that $[\mathbf{h}(\mathsf{0})] = \mathbf{H}$, and such that

$$f(\mathsf{x}) - f(\mathsf{0}) = \sum_{j,k=1}^{n} h_{jk}(\mathsf{x}) x^j x^k = \mathsf{x}^t \, [\mathbf{h}(\mathsf{x})] \, \mathsf{x}$$

where, for brevity of notation, we have set

$$\mathsf{x} = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}, \qquad \mathsf{x}^t = \begin{bmatrix} x^1 & \cdots & x^n \end{bmatrix}.$$

(c) Let $U \subset \mathbb{R}^n$ be the open neighborhood of $\mathsf{0}$ defined by requiring that $[\mathbf{h}(\mathsf{x})] \in \mathcal{U}$, where $\mathcal{U} \subset \mathcal{S}$ is the domain of Φ, as defined in Problem 1. Define a map $\mathsf{y} : U \to \mathbb{R}^n$ by

$$\mathsf{y}(\mathsf{x}) = \Phi([\mathsf{h}(\mathsf{x})]) \cdot \mathsf{x}$$

where $\cdot$ denotes matrix multiplication. Use the inverse function theorem to show that this map is a diffeomorphism between some neighborhood $U' \subset U$ of $\mathsf{0} \in \mathbb{R}^n$ and a neighborhood $V \subset \mathbb{R}^n$ of $\mathsf{0} \in \mathbb{R}^n$. We may thus regard $\mathsf{x} \mapsto \mathsf{y}$ as a change of coordinates that sends the origin to itself.

(d) Now prove the Morse Lemma by showing that, in these coordinates,

$$f(\mathsf{y}) - f(\mathsf{0}) = \mathsf{y}^t \, \mathbf{H} \, \mathsf{y} = -\sum_{j=1}^{k} (y^j)^2 + \sum_{j=k+1}^{n} (y^j)^2.$$

3. Let $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ be distinct real numbers, and define

$$f : \mathbb{RP}^n \to \mathbb{R}$$

by

$$f([x^0, x^1, \ldots, x^n]) = \frac{\sum_{j=0}^n \lambda_j |x^j|^2}{\sum_{j=0}^n |x^j|^2},$$

where, for any $(x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} - \{\mathsf{0}\}$, we use $[x^0, x^1, \ldots, x^n]$ to denote its equivalence class in $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{\mathsf{0}\})/\mathbb{R}^\times$. (Here the multiplicative group $\mathbb{R}^\times$ of non-zero real numbers acts on $\mathbb{R}^{n+1}$ by scalar multiplication.)

(a) Show that f is a Morse function by explicitly finding all its critical points, and then showing that each one is non-degenerate. Also compute the index of each critical point.

(b) Use (a) to calculate the Euler characteristic $\chi(\mathbb{RP}^n)$.

(c) Use the pull-back of f to the double cover $S^n \to \mathbb{RP}^n$ to compute $\chi(S^n)$.

4. Let $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ again be distinct real numbers, and define

$$f : \mathbb{CP}_n \to \mathbb{R}$$

by

$$f([z^0, z^1, \ldots, z^n]) = \frac{\sum_{j=0}^n \lambda_j |z^j|^2}{\sum_{j=0}^n |z^j|^2},$$

where, for any $(z^0, z^1, \ldots, z^n) \in \mathbb{C}^{n+1} - \{\mathsf{0}\}$, we use $[z^0, z^1, \ldots, z^n]$ to denote its equivalence class in $\mathbb{CP}_n = (\mathbb{C}^{n+1} - \{\mathsf{0}\})/\mathbb{C}^\times$. (Here the multiplicative group $\mathbb{C}^\times$ of non-zero real numbers acts on $\mathbb{C}^{n+1}$ by scalar multiplication.)

(a) Show that f is a Morse function by explicitly finding all its critical points, and then showing that each one is non-degenerate. Also compute the index of each critical point.

(b) Use (a) to calculate the Euler characteristic $\chi(\mathbb{CP}_n)$.

5. Quaternionic projective n-space $\mathbb{HP}_n$ is defined as $(\mathbb{H}^{n+1} - \{\mathsf{0}\})/\mathbb{H}^\times$, where the non-zero quaternions $\mathbb{H}^\times$ act on $\mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$ by scalar multiplication on the right. Show that $\mathbb{HP}_n$ is a smooth compact $4n$-manifold. Then, by analogy with Problem 4, construct a Morse function on $\mathbb{HP}_n$, and use it to calculate $\chi(\mathbb{HP}_n)$.

6. Let $f : M \to \mathbb{R}$ is a Morse function on an n-manifold M, and let ν_k be the number of index-k critical points of f. Recall that we have proved the *weak Morse inequality*

$$b_k(M) \leq \nu_k, \quad k = 0, \ldots, n,$$

and the Euler-characteristic formula

$$\chi(M) = \sum_{j=0}^{n} (-1)^j \nu_j.$$

(a) Show that these facts imply the following result:

Theorem. If every critical point has even index, then $b_k(M) = \nu_k$ for every k.

(b) Use this result to calculate the Betti numbers of $\mathbb{CP}_n$ and $\mathbb{HP}_n$.

Hint. Use the Morse functions of Problems 4 and 5.

(c) Use this same method to calculate the Betti numbers of S^{2n}.

Hint. Do not use the Morse function of Problem 3!

7. Let $f_1 : M_1 \to \mathbb{R}$ and $f_2 : M_2 \to \mathbb{R}$ be Morse functions on two smooth manifolds. Letting $\varpi_1 : M_1 \times M_2 \to M_1$ and $\varpi_2 : M_1 \times M_2 \to M_2$ denote the factor projections, now define $f : M_1 \times M_2 \to \mathbb{R}$ by

$$f = f_1 \circ \varpi_1 + f_2 \circ \varpi_2.$$

(a) Prove that f is a Morse function. What are its critical points? What is the Morse index of each critical point?

(b) Use this to prove that $\chi(M_1 \times M_2) = \chi(M_1)\,\chi(M_2)$.

(c) If f_1 and f_2 only have critical points of even index, find a formula for the Betti numbers $b_j(M_1 \times M_2)$ in terms of the Betti numbers of M_1 and M_2.