Homework # 2

MAT 566

Due 5/5/22*

1. (a) Let $\wp : E \to N$ be a smooth complex vector bundle of rank r, and let M be any smooth manifold. Show that that

$$\mathrm{id}_M \times \wp : (M \times E) \to (M \times N)$$

can be canonically be given the structure of a smooth complex vector bundle of rank r. If $\varpi : M \times N \to N$ is the second-factor projection, this new bundle is often denoted $\varpi^* E$, and called the *pull-back* of E via ϖ. Show that this has a natural analogue for real vector bundles.

(b) Let $\wp : E \to N$ be a smooth complex vector bundle of rank r, and suppose that $M \subset N$ is a smooth, embedded submanifold. Show that

$$\wp|_{\wp^{-1}(M)} : \wp^{-1}(M) \to M$$

can be canonically be given the structure of a smooth complex vector bundle of rank r. This new bundle is called the *restriction* of E to the submanifold M, and is often denoted by $j^* E$, where $j : M \hookrightarrow N$ denotes the embedding. Show that the same construction can also be applied to real vector bundles.

(c) Let $\wp : E \to N$ be a smooth complex vector bundle of rank r, and let $\Phi : M \to N$ be any smooth map. By combining parts (a) and (b), show that one can define a smooth complex vector bundle $\Phi^* E \to M$, called the *pull-back* of E via Φ, by first pulling E back to $M \times N$, and then restricting to the graph of Φ. If $p \in M$ is any point, show that the fiber $(\Phi^* E)_p$ is canonically isomorphic to $E_{\Phi(p)} := \wp^{-1}(\Phi(p))$. In particular, $\Phi^* E$ also has rank r. Now describe the analogue for real vector bundles.

*But accepted through 5/9/22.

2. (a) Let $\wp : E \to N$ be a smooth complex vector bundle, and $\Phi : M \to N$ a smooth map. If σ is a local section of E with domain an open subset $U \subset N$, we can then define a local section $\Phi^*\sigma$ of Φ^*E over $\Phi^{-1}(U) \subset M$ by setting $(\Phi^*\sigma)(p) = \sigma(\Phi(p))$ for each $p \in M$, using the canonical isomorphism $(\Phi^*E)_p = E_{\Phi(p)}$. If σ is *smooth*, show that $\Phi^*\sigma$ is also smooth, by using the constructions described in Problem 1.

(b) Use this to show that a smooth local trivialization of E over $U \subset N$ induces a corresponding smooth local trivialization of Φ^*E over $\Phi^{-1}(U) \subset M$.

(c) If E is equipped with a smooth connection ∇, use this to show that there is a unique smooth connection $\widetilde{\nabla}$ on Φ^*E such that

$$\widetilde{\nabla}(\Phi^*\sigma) = \Phi^*(\nabla\sigma)$$

for any smooth section σ of E.

Hint: Explain the precise meaning of the pull-back on the right-hand side.

(d) Show that the curvature of $\widetilde{\nabla}$ is related to the curvature of ∇ by

$$F^{\widetilde{\nabla}} = \Phi^*(F^{\nabla}).$$

Hint: Again, explain the meaning of the pull-back on the right-hand side.

(e) Use this to show that the Chern classes of E are and Φ^*E related by

$$c_j(\Phi^*E) = \Phi^*[c_j(E)]$$

for every integer j, where, on the right-hand side, $\Phi^* : H^{2j}(N) \to H^{2j}(M)$ denotes the usual pull-back in deRam cohomology.

3. (a) Let $\wp : \mathsf{E} \to N$ be a smooth real vector bundle, and let $\Phi : M \to N$ be a smooth map. By imitating the arguments of Problem 2, show that any smooth connection $\triangledown$ on E induces a smooth connection $\widetilde{\triangledown}$ on $\Phi^*\mathsf{E}$. How are the curvatures of these two connections related?

(b) Show that the Pontryagin classes of E and $\Phi^*\mathsf{E}$ are related by

$$p_k(\Phi^*\mathsf{E}) = \Phi^*[p_k(\mathsf{E})]$$

for every integer k, where, on the right-hand side, $\Phi^* : H^{4k}(N) \to H^{4k}(M)$ denotes the usual pull-back in deRam cohomology.

4. (a) Let $E \to M$ be a smooth complex vector bundle, and let $E^* \to M$ be its *dual* vector bundle. Given any connection ∇ on E, show that there is a unique connection $\widehat{\nabla}$ on E^* such that, for any vector field X on M, any smooth section ξ of E and any smooth section φ of E^*,

$$X[\varphi(\xi)] = [\widehat{\nabla}_X \varphi](\xi) + \varphi(\nabla_X \xi).$$

(b) Show that the curvatures of ∇ and $\widehat{\nabla}$ are related by

$$F^{\widehat{\nabla}} = -(F^{\nabla})^{\top},$$

where $^{\top}$ denotes the "transpose" isomorphism $\Omega^2(\mathrm{End}(E)) \to \Omega^2(\mathrm{End}(E^*))$ induced by the *adjoint* isomorphism $\mathrm{End}(E) = E^* \otimes E \cong E \otimes E^* = \mathrm{End}(E^*)$. In other words, given a local frame for E, and then using the associated dual frame for E^*, the curvature matrix of $\widehat{\nabla}$ is minus the transpose of the curvature matrix of ∇.

(c) Use this to show that the Chern classes of E and E^* are related by

$$c_j(E^*) = (-1)^j c_j(E).$$

(d) If a complex vector bundle E is isomorphic to its complex-conjugate bundle $\overline{E}$, show that the Chern class $c_j(E)$ vanishes whenever j is *odd*.

(e) If $\mathsf{E} \to M$ is a real vector bundle, we defined its Pontryagin classes to be

$$p_k(\mathsf{E}) = (-1)^k c_{2k}(\mathsf{E} \otimes \mathbb{C}) \in H^{4k}(M).$$

Notice that this definition only uses the Chern classes c_j of the complexified bundle $\mathsf{E} \otimes \mathbb{C} = \mathsf{E} \oplus i\mathsf{E}$ when j is *even*. Why can we safely ignore the corresponding Chern classes c_j for *odd* values of j?

5. If $\mathsf{E} \to M$ is a real vector bundle, its *total Pontryagin class* is defined by

$$p(\mathsf{E}) := 1 + p_1(\mathsf{E}) + p_2(\mathsf{E}) + p_3(\mathsf{E}) + \cdots \quad \in \bigoplus_k H^{4k}(M).$$

Prove that

$$p(\mathsf{E}_1 \oplus \mathsf{E}_2) = p(\mathsf{E}_1)\, p(\mathsf{E}_2)$$

for any Whitney sum of $\mathsf{E}_1 \oplus \mathsf{E}_2$ of two smooth real vector bundles.

6. Let $M^n \subset \mathbb{R}^{n+2}$ be a compact oriented codimension-2 submanifold of Euclidean space. The normal bundle $\nu = (TM)^\perp$ of M is thus an oriented rank-2 vector bundle over M, and carries a canonical positive-definite inner product induced by the Euclidean inner product on $\mathbb{R}^{n+2}$. Let $J : \nu \to \nu$ denote rotation through $+90°$ with respect to this inner product and the natural orientation. Since $J^2 = -I$, observe that this allows us to view ν as the underlying oriented real bundle of a complex line bundle $L \to M$.

(a) Show that $M \subset \mathbb{R}^{n+2}$ has a finite cover $\{U_j \mid j = 1, \ldots, m\}$ by open sets $U_j \subset \mathbb{R}^{n+2}$ on which there exist smooth functions $f_j : U_j \to \mathbb{C}$ such that

- $f_j^{-1}(0) = U_j \cap M$;
- $df_j \neq 0$ at every point of $U_j \cap M$; and
- $df_j \circ J = \mathsf{i}\, df_j$ at every point of of $U_j \cap M$, where $\mathsf{i} = \sqrt{-1}$.

Now extend this to a finite cover $\{U_j \mid j = 0, \ldots, m\}$ of $\mathbb{R}^{n+2}$ by setting $U_0 = \mathbb{R}^{n+2} - M$, and equip U_0 with the constant function $f_0 \equiv 1$.

(b) Whenever $U_j \cap U_k \neq \varnothing$, show that there is a unique smooth function

$$\phi_{jk} : U_j \cap U_k \to \mathbb{C}^\times := \mathbb{C} - \mathsf{0}$$

such that $f_k = f_j \phi_{jk}$ on $U_j \cap U_k$.

(c) Show that the ϕ_{jk} satisfy

$$\phi_{jk}\phi_{kj} = 1 \text{ on } U_j \cap U_k, \quad \text{and} \quad \phi_{jk}\phi_{k\ell}\phi_{\ell j} = 1 \text{ on } U_j \cap U_k \cap U_\ell,$$

and so are the transition functions for a complex line bundle $\mathscr{L} \to \mathbb{R}^{n+2}$.

(d) Show that the restriction of $\mathscr{L}$ to M is isomorphic to $L \to M$. Then use this to prove that the normal bundle $\nu \to M$ is trivial.

(e) Use this to show that $p_k(TM) = 0$ for every $k > 0$.

7.(a) Show that $p_1(T\mathbb{CP}_n) \neq 0$ if $n \geq 2$.

(b) Prove that $\mathbb{CP}_3$ cannot be smoothly embedded in $\mathbb{R}^8$.

(c) Prove that $\mathbb{CP}_2 \times S^m$ cannot be smoothly embedded in $\mathbb{R}^{m+6}$.