Lecture 1: January 28

Overview. The purpose of this course is to give an introduction to the theory of algebraic $\mathscr{D}$-modules. I plan to cover roughly the following topics:

- modules over the Weyl algebra A_n
- $\mathscr{D}$-modules on smooth algebraic varieties
- functors on $\mathscr{D}$-modules (and how they relate to PDE)
- holonomic $\mathscr{D}$-modules, regularity (with a focus on what it means)
- b-functions, localization along a hypersurface
- $\mathscr{D}$-modules of normal crossing type (as a class of examples)
- Riemann-Hilbert correspondence (with proofs in the normal crossing case)
- some applications, either to representation theory or to algebraic geometry

The website for the course,

`http://www.math.stonybrook.edu/~cschnell/mat615`,

contains a list of useful references.

Introduction. Very briefly, $\mathscr{D}$-modules were invented in Japan (by Mikio Sato, Masaki Kashiwara, and others) and France (by Alexander Grothendieck, Zogman Mebkhout, and others). It has its origins in the field of "algebraic analysis", which means the study of partial differential equations with algebraic tools. The theory of algebraic $\mathscr{D}$-modules was further developed by Joseph Bernstein.

Systems of linear equations. $\mathscr{D}$-modules arise naturally from systems of linear partial differential equations. To get a better understanding of how this works, let us first look at the example of a system of linear equations

$$\sum_{j=1}^{q} a_{i,j} x_j = 0, \quad i = 1, \dots, p, \tag{1.1}$$

with coefficients $a_{i,j}$ in a field K (such as $\mathbb{R}$ or $\mathbb{C}$). In linear algebra, one usually transforms such a system in various ways, for example by making a substitution in the unknowns $x_1, \dots, x_q$, or by taking linear combinations of the equations. One can associate to the system in (1.1) a single K-vector space that is invariant under such transformations. Consider the linear mapping

$$\varphi \colon K^p \to K^q, \quad \varphi(y_1, \dots, y_p) = \left(\sum_{i=1}^{p} y_i a_{i,1}, \dots, \sum_{i=1}^{p} y_i a_{i,q} \right),$$

built from the coefficient matrix of the system in (1.1), and define the K-vector space $V = \ker \varphi = K^q / \varphi(K^p)$. It sits in the exact sequence

$$K^p \xrightarrow{\varphi} K^q \xrightarrow{\pi} V \longrightarrow 0,$$

and the solution space to (1.1) can be recovered from V as

$$\operatorname{Hom}_K(V, K) = \{\, f \colon K^q \to K \mid f \circ \varphi = 0 \,\}.$$

Indeed, a linear mapping from V to K is the same thing as a linear mapping $f \colon K^q \to K$ whose composition with φ is equal to zero.

$$\begin{tikzcd} K^p \arrow[r, "\varphi"] \arrow[dr, "0"'] & K^q \arrow[r, "\pi"] \arrow[d, "f"] & V \arrow[dl] \\ & K & \end{tikzcd}$$

Now f is uniquely determined by the q scalars $x_j = f(e_j) \in K$, where e_j denotes the j-th coordinate vector in K^q. Since $f \circ \varphi = 0$, we get

$$\sum_{i,j} y_i a_{i,j} x_j = 0$$

for every $(y_1, \dots, y_p) \in K^p$. This means exactly that $(x_1, \dots, x_q) \in K^q$ is a solution to the system of linear equations in (1.1).

The same construction can be applied to systems of linear equations with coefficients in other rings. For example, let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables, and consider the system of linear equations

$$\sum_{j=1}^{q} f_{i,j} u_j = 0, \quad i = 1, \dots, p, \tag{1.2}$$

with polynomial coefficients $f_{i,j} \in R$. As before, we can associate to the system an R-module $M = \operatorname{coker} \varphi$, defined as the cokernel of the morphism of R-modules

$$\varphi \colon R^p \to R^q, \quad \varphi(v_1, \dots, v_p) = \left(\sum_{i=1}^{p} v_i f_{i,1}, \dots, \sum_{i=1}^{p} v_i f_{i,q} \right),$$

and the space of solutions $(u_1, \dots, u_q) \in R^q$ to the system in (1.2) can be recovered from M as $\operatorname{Hom}_R(M, R)$. This formulation has the advantage that we can describe solutions over other R-algebras S, such as the ring of formal power series $K[[x_1, \dots, x_n]]$, in the same way, by taking $\operatorname{Hom}_R(M, S)$.

Note. The polynomial ring R is noetherian, meaning that every ideal of R is finitely generated. This implies that every submodule of a finitely generated R-module is again finitely generated. In particular, every finitely generated R-module is isomorphic to the cokernel of $\varphi \colon R^p \to R^q$ for some $p, q \in \mathbb{N}$. Studying systems of linear equations with coefficients in R is therefore the same thing as studying finitely generated R-modules.

Systems of linear partial differential equations. We now apply the same construction to systems of linear partial differential equations with coefficients in the polynomial ring. The role of the polynomial ring $R = K[x_1, \dots, x_n]$ is played by the *Weyl algebra* $A_n = A_n(K)$. The elements of A_n are linear partial differential operators of the form

$$P = \sum_{i_1, \dots, i_n} f_{i_1, \dots, i_n}(x_1, \dots, x_n) \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}},$$

where $f_{i_1, \dots, i_n} \in R$, and the sum is finite. To simplify the notation, we put $\partial_j = \partial / \partial x_j$, and write the above sum in multi-index notation as

$$P = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha \partial^\beta,$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. We can multiply two differential operators in the obvious way, using the relations

$$[x_i, x_j] = 0, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{i,j}, \tag{1.3}$$

where $\delta_{i,j} = 1$ if $i = j$, and $\delta_{i,j} = 0$ otherwise. The relation $[\partial_i, \partial_j] = 0$ expresses the equality of mixed partial derivatives; the relation $[\partial_i, x_j] = \delta_{i,j}$ is a consequence of the product rule:

$$\frac{\partial}{\partial x_i}(x_j f) = \frac{\partial x_j}{\partial x_i} f + x_j \frac{\partial f}{\partial x_j} = \delta_{i,j} f + x_j \frac{\partial}{\partial x_i} f$$

Multiplication of differential operators turns A_n into a non-commutative ring. Differential operators naturally act on polynomials, by the usual (algebraic) rules for computing derivatives of polynomials; if we denote the action of a differential operator P on a polynomial f by the symbol Pf, we obtain a linear mapping

$$A_n \times R \to R, \quad (P, f) \mapsto Pf.$$

This makes the polynomial ring R into a left module over the Weyl algebra A_n.

The action on polynomials leads to the following more intrinsic description of the Weyl algebra: A_n is the smallest subring of the ring of K-linear endomorphisms

$$\operatorname{Hom}_K\bigl(K[x_1, \dots, x_n], K[x_1, \dots, x_n]\bigr)$$

that contains $K[x_1, \dots, x_n]$ and the partial derivative operators $\partial_1, \dots, \partial_n$. Algebraically, one can also describe the Weyl algebra by generators and relations: A_n is the non-commutative K-algebra generated by the $2n$ symbols $x_1, \dots, x_n, \partial_1, \dots, \partial_n$, subject to the relations in (1.3).

Now suppose that we have a system of linear partial differential equations

$$\sum_{j=1}^{q} P_{i,j} u_j = 0, \quad i = 1, \dots, p, \tag{1.4}$$

with differential operators $P_{i,j} \in A_n$. As before, we consider the morphism of left A_n-modules

$$\varphi\colon A_n^p \to A_n^q, \quad \varphi(Q_1, \dots, Q_p) = \left(\sum_{i=1}^{p} Q_i P_{i,1}, \dots, \sum_{i=1}^{p} Q_i P_{i,q}\right),$$

and associate to the system in (1.4) the left A_n-module

$$M = \operatorname{coker} \varphi = A_n^q / \varphi(A_n^p).$$

Note that it becomes necessary to distinguish between left and right A_n-modules, because A_n is non-commutative. We can again recover the solutions to the system in (1.4) directly from M, as follows. Let S be any commutative K-algebra with an action by differential operators, meaning that S is a left A_n-module. Examples are the polynomial ring $R = K[x_1, \dots, x_n]$, the ring of formal power series $K[[x_1, \dots, x_n]]$, etc. For $K = \mathbb{R}$ or $K = \mathbb{C}$, we might also be interested in the ring of convergent power series, the ring of C^∞-functions, etc. In any of these examples, the solutions in S are given by the formula

$$\operatorname{Hom}_{A_n}(M, S) = \bigl\{\, f\colon A_n^q \to S \bigm| f \circ \varphi = 0 \,\bigr\}$$

Indeed, a morphism of left A_n-modules from M to S is the same thing as a morphism of left A_n-modules $f\colon A_n^q \to A_n$ whose composition with φ is equal to zero.

$$\begin{tikzcd}
A_n^p \arrow[r, "\varphi"] \arrow[dr, "0"'] & A_n^q \arrow[r, "\pi"] \arrow[d, "f"] & M \arrow[dl] \\
& S &
\end{tikzcd}$$

Once again, f is uniquely determined by the q functions $u_j = f(e_j) \in S$, where e_j denotes the j-th coordinate vector in A_n^q. Since $f \circ \varphi = 0$, we get

$$\sum_{i,j} Q_i P_{i,j} u_j = 0$$

for every $(Q_1, \dots, Q_p) \in A_n^p$, and so $(u_1, \dots, u_q) \in S^q$ solves the system of linear partial differential equations in (1.4).

Note. The Weyl algebra A_n is again left noetherian, meaning that every left ideal of A_n is finitely generated. (We will prove this next time.) This implies that every submodule of a finitely generated left A_n-module is again finitely generated. Studying systems of linear partial differential equations with coefficients in R is therefore the same thing as studying finitely generated left A_n-modules.

One advantage of this point of view is that we can describe the solutions to the system in a uniform way, by applying the *solution functor* $\mathrm{Hom}_{A_n}(M, -)$. We shall see later on that the solution functor is not exact (in the sense of homological algebra), and that it is natural to consider its derived functors. We shall also see that for so-called "regular holonomic" systems, one can recover the system up to isomorphism from its solutions (in the derived sense); this is the content of the famous Riemann-Hilbert correspondence.

Example 1.5. The exponential function $u = e^x$ solves the ordinary differential equation $u' = u$, which we can write in the form $(\partial - 1)u = 0$. The corresponding left A_1-module is $A_1/A_1(\partial - 1)$. The function $v = e^{1/x}$ solves the ordinary differential equation $-x^2v' = v$, whose associated A_1-module is $A_1/A_1(x^2\partial + 1)$. Later on, when we discuss regularity, we shall see how the essential singularity of v at the point $x = 0$ affects the properties of the A_1-module $A_1/A_1(x^2\partial + 1)$.

Another advantage is that we can transform the system in (1.4) without changing the isomorphism class of the A_n-module M.

Example 1.6. Consider the second-order equation $a(x)u'' + b(x)u' + c(x)u = 0$, where $a, b, c \in K[x]$. A standard trick is to transform this into a system of two first-order equations $u_1' - u_2 = 0$ and $au_2' + bu_2 + cu_1 = 0$, by setting $u_1 = u$ and $u_2 = u'$. The first-order system leads to the left A_1-module

$$M_1 = \mathrm{coker}\left(A_1^2 \xrightarrow{\left(\begin{smallmatrix} \partial & -1 \\ c & a\partial + b \end{smallmatrix}\right)} A_1^2 \right)$$

and the second-order system to the left A_1-module

$$M_2 = A_1/A_1(a\partial^2 + b\partial + c)$$

Can you find an isomorphism between M_1 and M_2 as left A_1-modules?

Left and right A_n-modules. I already mentioned that it is necessary to distinguish between left A_n-modules and right A_n-modules, due to the non-commutativity of the Weyl algebra. Left A_n-modules naturally arise from functions, whereas right A_n-modules arise naturally from distributions. Let us look at the example of distributions in more detail. The $\mathbb{R}$-algebra $C_0^\infty(\mathbb{R}^n)$ of all compactly supported C^∞-functions on $\mathbb{R}^n$ is naturally a left $A_n(\mathbb{R})$-module; as before, we denote the action of a differential operator $P \in A_n$ on a test function $\varphi \in C_0^\infty(\mathbb{R}^n)$ by the symbol $P\varphi$. With the topology of uniform convergence of all derivatives on compact subsets, $C_0^\infty(\mathbb{R}^n)$ becomes a topological $\mathbb{R}$-vector space, and we denote by $\mathrm{Db}(\mathbb{R}^n)$ its topological dual. In other words, a distribution $D \in \mathrm{Db}(\mathbb{R}^n)$ is a continuous linear functional from $C_0^\infty(\mathbb{R}^n)$ to $\mathbb{R}$. We write the natural pairing between distributions and test functions as

$$\mathrm{Db}(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \to \mathbb{R}, \quad (D, \varphi) \mapsto \langle D, \varphi \rangle.$$

In analysis, it is also common to use the more suggestive notation

$$\langle D, \varphi \rangle = \int_{\mathbb{R}^n} D\varphi \, d\mu,$$

where $d\mu$ is Lebesgue measure. Using formal integration by parts, $\mathrm{Db}(\mathbb{R}^n)$ naturally becomes a right A_n-module, by defining

$$\langle DP, \varphi\rangle = \langle D, P\varphi\rangle$$

for $D \in \mathrm{Db}(\mathbb{R}^n)$, $P \in A_n$, and $\varphi \in C_0^\infty(\mathbb{R}^n)$. For example, $D\partial_j$ is the distribution obtained by applying D to the test function $\partial\varphi/\partial x_j$. If we take any distribution, and act on it by differential operators, we obtain a right A_n-module inside $\mathrm{Db}(\mathbb{R}^n)$.

Example 1.7. Consider the delta function $\delta_0 \in \mathrm{Db}(\mathbb{R}^n)$, defined by $\langle \delta_0, \varphi\rangle = \varphi(0)$. Clearly, $\delta_0 x_1 = \cdots = \delta_0 x_n = 0$, and in fact, one can show that the right A_n-module generated by δ_0 is isomorphic to

$$A_n/(x_1, \ldots, x_n)A_n.$$

As an $\mathbb{R}$-vector space, this is just $\mathbb{R}[\partial_1, \ldots, \partial_n]$, but the A_n-action is nontrivial.

Exercises.

Exercise 1.1. Construct an isomorphism between the two left A_1-modules M_1 and M_2 in Example 1.6.

Exercise 1.2. Show that if $P \in A_n(\mathbb{R})$ satisfies $(P\varphi)(0) = 0$ for every test function $\varphi \in C_0^\infty(\mathbb{R}^n)$, then $P \in (x_1, \ldots, x_n)A_n$.