## Lecture 11: March 11

**Global good filtrations.** Let us return for the moment to the topic of good filtrations. I said last time that, by the same argument as in the case of $A_n$, every coherent $\mathscr{D}_X$-module locally admits a good filtration. But in fact, good filtrations also exist globally, because of the finiteness inherent in the definitions.

**Lemma 11.1.** *Let $\mathcal{M}$ be an algebraic $\mathscr{D}_X$-module. If $\mathcal{M}$ is coherent, then there exists a good filtration $F_\bullet \mathcal{M}$ by coherent $\mathscr{O}_X$-modules.*

*Proof.* It will be enough to construct an $\mathscr{O}_X$-submodule $\mathscr{F} \subseteq \mathcal{M}$ that is coherent over $\mathscr{O}_X$ and that generates $\mathcal{M}$ as a $\mathscr{D}_X$-module. Once we have that, we can define a filtration by setting

$$F_j \mathcal{M} = F_j \mathscr{D}_X \cdot \mathscr{F} \subseteq \mathcal{M},$$

and for the same reason as in the case of the Weyl algebra, each $F_j \mathcal{M}$ is coherent over $\mathscr{O}_X$, and the filtration $F_\bullet \mathcal{M}$ is good.

Since $X$ is of finite type over $k$, it is quasi-compact, and so we can cover $X$ by finitely many affine open subsets $U_1, \dots, U_m$. Then $\Gamma(U_i, \mathcal{M})$ is finitely generated over $\Gamma(U_i, \mathscr{D}_X)$, and after choosing a finite set of generators and taking the $\Gamma(U_i, \mathscr{O}_X)$-submodule of $\Gamma(U_i, \mathcal{M})$ generated by this set, we certainly obtain a coherent $\mathscr{O}_{U_i}$-module $\mathscr{F}_{U_i} \subseteq \mathcal{M}\big|_{U_i}$ that has the desired properties on $U_i$.

To turn these locally defined subsheaves into global objects, we use the following fact from Hartshorne's book: Suppose that $\mathscr{G}$ is a quasi-coherent sheaf on an algebraic variety $X$. If we have a nonempty open subset $U \subseteq X$, and a coherent subsheaf $\mathscr{F}_U \subseteq \mathscr{G}\big|_U$, then there is a coherent subsheaf $\mathscr{F} \subseteq \mathscr{G}$ such that $\mathscr{F}\big|_U = \mathscr{F}_U$. When applied to our situation, this says that there are coherent $\mathscr{O}_X$-modules $\mathscr{F}_1, \dots, \mathscr{F}_n$ such that $\mathscr{F}_i\big|_{U_i} = \mathscr{F}_{U_i}$. Then the image of

$$\mathscr{F}_1 \oplus \cdots + \oplus \mathscr{F}_m \to \mathcal{M}$$

is an $\mathscr{O}_X$-submodule of $\mathcal{M}$ that is coherent over $\mathscr{O}_X$ (because it is the image of a coherent $\mathscr{O}_X$-module) and generates $\mathcal{M}$ as a $\mathscr{D}_X$-module. □

This result is peculiar to the algebraic setting, and does not hold at all for analytic $\mathscr{D}$-modules.

**Characteristic varieties are involutive.** Recall the definition of the characteristic variety from last time. If $\mathcal{M}$ is a coherent $\mathscr{D}_X$-module, we can choose a global good filtration $F_\bullet \mathcal{M}$, which makes the associated graded module $\operatorname{gr}^F \mathcal{M}$ coherent over $\operatorname{gr}^F \mathscr{D}_X \cong \operatorname{Sym} \mathscr{T}_X$. If $\widetilde{\operatorname{gr}^F \mathcal{M}}$ denotes the corresponding coherent sheaf on the cotangent bundle $T^*X$, then

$$\operatorname{Ch}(\mathcal{M}) = \operatorname{Supp} \widetilde{\operatorname{gr}^F \mathcal{M}}.$$

Equivalently, the characteristic variety is the reduced closed subscheme of the cotangent bundle corresponding to the homegeneous ideal

$$\sqrt{\operatorname{Ann}_{\operatorname{gr}^F \mathscr{D}_X} \operatorname{gr}^F \mathcal{M}} \subseteq \operatorname{gr}^F \mathscr{D}_X.$$

The most important result about the characteristic variety is the following theorem.

**Theorem 11.2.** $\operatorname{Ch}(\mathcal{M})$ *is* involutive *with respect to the natural symplectic structure on $T^*X$. In particular, every irreducible component of* $\operatorname{Ch}(\mathcal{M})$ *has dimension* $\geq n$.

Note that this gives a lot more information about the characteristic variety than Bernstein's inequality. This result was first proved by analytic methods, but Gabber later discovered an algebraic proof. Bernstein's inequality can of course be proved by more elementary means. We are not going to prove Theorem 11.2; instead, I will review some basic facts about symplectic geometry, so that we can understand at least the statement, and where the difficulties lie.

**Symplectic vector spaces.** Let us start with a brief discussion of symplectic vector spaces. Let $V$ be a finite-dimensional vector space over a field $k$. Usually, $k$ will be field of real or complex numbers, but the definition works over any field of characteristic $\neq 2$. A *symplectic form* is a bilinear form

$$\sigma \colon V \otimes_k V \to k$$

that is anti-symmetric and non-degenerate. In other words, one has $\sigma(v,w) = -\sigma(w,v)$ for every $v, w \in V$, and if we denote by $V^* = \operatorname{Hom}_k(V,k)$ the dual vector space, then the induced linear mapping

$$V \to V^*, \quad w \mapsto \sigma(-,w),$$

is an isomorphism (called the "Hamiltonian isomorphism"). For every linear functional $\theta \in V^*$, one therefore has a unique element $H_\theta \in V$ such that $\theta(v) = \sigma(v, H_\theta)$ for all $v \in V$.

The dimension of a symplectic vector space is always an even number. One way to see this is as follows. Pick a nonzero vector $w \in V$, and consider the linear subspace $L = k \cdot w \subseteq V$. Since $\sigma(w,w) = 0$, one has $L$ contained in the subspace

$$L^\perp = \big\{ v \in V \bigm| \sigma(v,w) = 0 \big\}.$$

The fact that $\sigma$ is nondegenerate implies that $L^\perp = \dim V - 1$. One easily checks that the quotient space $L^\perp/L$, with the bilinear form induced by $\sigma$, is again a symplectic vector space. Since $\dim V = 2 + \dim L^\perp/L$, the claim now follows by induction.

*Example* 11.3. If $V$ is any finite-dimensional $k$-vector space, then $V \oplus V^*$ is a symplectic vector space, with symplectic form given by

$$\big((v_1,\theta_1),(v_2,\theta_2)\big) \mapsto \theta_1(v_2) - \theta_2(v_1).$$

In fact, every symplectic vector space is isomorphic to this model (after a suitable choice of basis).

Given a subspace $W \subseteq V$, one defines

$$W^\perp = \big\{ v \in V \bigm| \sigma(v,w) = 0 \text{ for every } w \in W \big\}.$$

Under the Hamiltonian isomorphism $V \cong V^*$, the subspace $W^\perp$ corresponds exactly to the kernel of the restriction homomorphism $V^* \to W^*$, and therefore

$$\dim W + \dim W^\perp = \dim V.$$

**Definition 11.4.** Let $W \subseteq V$ be a linear subspace.

(1) $W$ is called *involutive* if $W^\perp \subseteq W$; then $\dim W \geq \frac{1}{2} \dim V$.
(2) $W$ is called *Lagrangian* if $W^\perp = W$; then $\dim W = \frac{1}{2} \dim V$.
(3) $W$ is called *isotropic* if $W^\perp \supseteq W$; then $\dim W \leq \frac{1}{2} \dim V$.

Note that an involutive (or isotropic) subspace is Lagrangian iff $\dim W = \frac{1}{2} \dim V$.

*Example* 11.5. Consider the symplectic vector space $V \oplus V^*$. If $W \subseteq V$ is any linear subspace, then $W \oplus \ker(V^* \to W^*)$ is always a Lagrangian subspace of $V \oplus V^*$. It is clearly isotropic: if $v_1, v_2$ are vectors in $W$, and $\theta_1, \theta_2$ are linear functionals whose restriction to $W$ is trivial, then $\theta_1(v_2) - \theta_2(v_1) = 0$. Since

$$\dim W + \dim \ker(V^* \to W^*) = \dim V$$

is exactly half the dimension of $V \oplus V^*$, it follows that the subspace is Lagragian.

**Symplectic algebraic varieties.** A nonsingular algebraic variety $X$ is called *symplectic* if the tangent space $T_x X$ at every closed point $x \in X$ is a symplectic vector space, and the symplectic forms vary in an algebraic way from point to point. More precisely, there should exist a global algebraic two-form $\sigma \in \Gamma(X, \Omega^2_{X/k})$ whose restriction $\sigma_x \colon T_x X \otimes_k T_x X \to k$ gives a symplectic form on $T_x X$ for every closed point $x \in X$. Of course, this implies that $\dim X$ is even.

*Example* 11.6. The example we care about is the cotangent bundle $T^*X$ of a nonsingular algebraic variety $X$ of dimension $n$. Note that $\dim T^*X = 2n$. If we choose local coordinates $x_1, \dots, x_n$ on $X$, then the differentials $dx_1, \dots, dx_n$ give a local trivialization for $\Omega^1_{X/k}$, and so we obtain local coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ on the cotangent bundle. In these coordinates,

$$\sigma_X = \sum_{i=1}^n d\xi_i \wedge dx_i$$

is a symplectic form. Indeed, at any closed point $(x, \xi) \in T^*X$, we have

$$T_{(x,\xi)}\big(T^*X\big) = T_x X \oplus (T_x X)^*,$$

because the fiber of $p \colon T^*X \to X$ over the point $x$ is the cotangent space $(T_x X)^*$, and because a vector space is isomorphic to its own tangent space. Under this isomorphism, the two-form $\sigma_X$ corresponds exactly to the standard symplectic form in Example 11.3. In more functorial language, one can describe $\sigma_X$ as follows. As with any vector bundle, the pullback $p^*\Omega^1_{X/k}$ has a tautological global section, whose image under $p^*\Omega^1_{X/k} \to \Omega^1_{T^*X/k}$ gives a one-form

$$\alpha_X \in \Gamma\big(T^*X, \Omega^1_{T^*X/k}\big).$$

In local coordinates as above, one has $\alpha_X = \sum_i \xi_i \, dx_i$. Then

$$\sigma_X = d\alpha_X \in \Gamma\big(T^*X, \Omega^2_{T^*X/k}\big)$$

is the symplectic form from above.

Let $X$ be a nonsingular algebraic variety with a symplectic form $\sigma$. Then $\sigma_x$ induces an isomorphism between the tangent space $T_x X$ and the cotangent space $(T_x X)^*$ at every closed point $x \in X$, and this allows us to convert one-forms into vector fields and vice versa. In particular, every function $f \in \Gamma(U, \mathscr{O}_X)$ determines a vector field $H_f \in \Gamma(U, \mathscr{T}_X)$, with the property that $df = \sigma(-, H_f)$ as one-forms on $U$. The *Poisson bracket* of two functions $f, g \in \Gamma(U, \mathscr{O}_X)$ is defined by

$$\{f, g\} = H_f(g) = dg(H_f) = \sigma(H_f, H_g) \in \Gamma(U, \mathscr{O}_X).$$

If $d\sigma = 0$, then one has $[H_f, H_g] = H_{\{f,g\}}$.

*Example* 11.7. In local coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ on the cotangent bundle, the Hamiltonian vector field of a function $f$ is given by

$$H_f = \sum_{i=1}^n \left( \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i} \right),$$

and consequently, the Poisson bracket can be calculated as

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} \right).$$

We can extend the notion of involutive (or Lagrangian or isotropic) to subvarieties of $X$ by looking at their tangent spaces at nonsingular points. Thus a reduced algebraic subvariety $Y \subseteq X$ is called *involutive* (or *Lagrangian* or *isotropic*) if at every nonsingular closed point $x \in Y$, the tangent space $T_x Y \subseteq T_x X$ is involutive (or Lagrangian or isotropic).

*Example* 11.8. In the case of the cotangent bundle $T^*X$, the conormal bundle of a nonsingular subvariety $Z \subseteq X$ is a nonsingular Lagrangian subvariety. At a closed point $x \in Z$, the fiber of the conormal bundle consists of all those cotangent vectors in $(T_x X)^*$ that vanish on the subspace $T_x Z$. As a subspace of

$$T_{(x,\xi)}\big(T^*X\big) = T_x X \oplus (T_x X)^*,$$

the tangent space to the conormal bundle is therefore

$$T_x Z \oplus \ker\big((T_x X)^* \to (T_x Z)^*\big),$$

and this is a Lagrangian subspace by Example 11.5 from above. If we choose local coordinates $x_1, \dots, x_n$ on $X$ such that $Z$ is defined by $x_{k+1} = \cdots = x_n = 0$, then the conormal bundle is defined by $\xi_1 = \cdots = \xi_k = x_{k+1} = \cdots = x_n = 0$ in the corresponding coordinates on the cotangent bundle.

The following lemma gives a way to check whether a reduced subvariety $Y \subseteq X$ is involutive by using the ideal sheaf $\mathcal{I}_Y \subseteq \mathscr{O}_X$.

**Lemma 11.9.** *Let $X$ be a nonsingular algebraic variety with a symplectic form, and $Y \subseteq X$ a reduced algebraic subvariety. The following conditions are equivalent:*

*(a) The subvariety $Y$ is involutive.*
*(b) The ideal sheaf $\mathcal{I}_Y$ is closed under the Poisson bracket, $\{\mathcal{I}_Y, \mathcal{I}_Y\} \subseteq \mathcal{I}_Y$.*

*Proof.* Without loss of generality, we may assume that $X$ is affine, and that $Y$ is the closed subvariety defined by an ideal $I \subseteq \Gamma(X, \mathscr{O}_X)$. Note that $Y$ is assumed to be reduced. We start with a general observation. Let $x \in Y$ be a nonsingular point, and let $\sigma_x$ be the symplectic form on $T_x X$. Then

$$(T_x Y)^\perp = \big\{\, v \in T_x X \;\big|\; \sigma_x(v, w) = 0 \text{ for every } w \in T_x Y \,\big\}$$

is spanned by the values at $x$ of the Hamiltonian vector fields $H_f$, as $f$ ranges over the elements of the ideal $I$. Indeed, since $x \in Y$ is a nonsingular point, a tangent vector $v \in T_x X$ belongs to the subspace $T_x Y$ exactly when $df(v) = 0$ for every $f \in I$. Under the Hamiltonian isomorphism, this condition becomes

$$\sigma_x(v, H_f) = df(v) = 0,$$

whence the claim.

Now let us show that $\{I, I\} \subseteq I$ implies that $Y$ is involutive. If $x \in Y$ is a nonsingular point, we need to argue that $(T_x Y)^\perp \subseteq T_x Y$. In light of the observation from above, this amounts to saying that, for every $f, g \in I$, the function $dg(H_f)$ vanishes at the point $x$. But this is the case, because $dg(H_f) = H_f(g) = \{f, g\} \in I$.

For the converse, suppose that $Y$ is involutive, so that $(T_x Y)^\perp \subseteq T_x Y$ at every nonsingular point $x \in Y$. Then we again have $\{f, g\} = dg(H_f) = 0$ at every nonsingular point of $Y$, and hence on all of $Y$ because $\{f, g\}$ is a regular function and the set of nonsingular points is Zariski-open and dense in $Y$. Because $Y$ is reduced, it follows that $\{f, g\} \in I$. □

**Involutivity of the characteristic variety.** We return to the characteristic varieties of coherent $\mathscr{D}_X$-modules. If $p\colon T^*X \to X$ is the cotangent bundle, then

$$p_* \mathscr{O}_{T^*X} \cong \operatorname{gr}^F \mathscr{D}_X,$$

and one can use this isomorphism to describe the Poisson bracket in terms of differential operators. For each $j \geq 0$, we denote by

$$\sigma_j\colon F_j \mathscr{D}_X \to \operatorname{gr}_j^F \mathscr{D}_X$$

the "principal symbol" operator. If $P$ is a local section of $F_i\mathscr{D}_X$, and $Q$ a local section of $F_j\mathscr{D}_X$, then their commutator $[P,Q]$ is a local section of $F_{i+j-1}\mathscr{D}_X$. One can show, using the description of the Poisson bracket in local coordinates, that

$$\{\sigma_i(P), \sigma_j(Q)\} = \sigma_{i+j-1}\big([P,Q]\big).$$

Now suppose that $\mathcal{M}$ is a coherent left $\mathscr{D}_X$-module, and that $F_\bullet\mathcal{M}$ is a good filtration. It is easy to see, using the alternative description of the Poisson bracket, that the ideal

$$\mathrm{Ann}_{\mathrm{gr}^F\mathscr{D}_X}\,\mathrm{gr}^F\mathcal{M} \subseteq \mathrm{gr}^F\mathscr{D}_X$$

is closed under the Poisson bracket. This is a local question, and so we may restrict everything to an affine open subset $U \subseteq X$. If we set $A = \Gamma(U, \mathscr{O}_X)$ and $R = D(A)$, we then have a finitely generated left $R$-module $M$, together with a good filtration $F_\bullet M$, such that $\mathrm{gr}^F M$ is finitely generated over $S = \mathrm{gr}^F R$. The claim is that the homogeneous ideal

$$I = I(M, F_\bullet M) = \mathrm{Ann}_S\,\mathrm{gr}^F M$$

is closed under the Poisson bracket on $S$. Suppose that we have two elements $P \in F_iR$ and $Q \in F_jR$ such that $\sigma_i(P)$ and $\sigma_j(Q)$ belong to the ideal $I$. Recall from Lecture 5 that this is equivalent to having

$$P \cdot F_kM \subseteq F_{i+k-1}M \quad \text{and} \quad Q \cdot F_kM \subseteq F_{j+k-1}M$$

for every $k \in \mathbb{Z}$. But then

$$[P,Q] \cdot F_kM \subseteq P \cdot F_{j+k-1}M + Q \cdot F_{i+k-1}M \subseteq F_{(i+j-1)+k-1}M,$$

and therefore $\sigma_{i+j-1}([P,Q]) \in I$. This shows that $\{I, I\} \subseteq I$.

Why does this argument not prove Theorem 11.2? The issue is that the ideal of the characteristic variety is not $I$ itself, but $\sqrt{I}$, because the characteristic variety is by definition reduced. For non-reduced ideals, being closed under the Poisson bracket does not correspond to the geometric notion of being involutive, because all points of a nonreduced subscheme can be singular. And the fact that an ideal is closed under the Poisson bracket does not imply the same property for its radical. This is what makes Theorem 11.2 nontrivial.

**Exercises.**

*Exercise* 11.1. Let $X$ be a nonsingular affine variety with a symplectic form. Prove the following three identities involving the Poisson bracket: for all $f, g, h \in \Gamma(X, \mathscr{O}_X)$,

$$\{f,g\} + \{g,f\} = 0$$
$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$$
$$\{f,gh\} = \{f,g\}h + g\{f,h\}.$$

The first two identities are saying that $\Gamma(X, \mathscr{O}_X)$ is a Lie algebra under the operation $(f,g) \mapsto \{f,g\}$. The third identity is saying that $\{f, -\}$ is a derivation of $\Gamma(X, \mathscr{O}_X)$.

*Exercise* 11.2. Show that if $d\sigma = 0$, then one has $[H_f, H_g] = H_{\{f,g\}}$.

*Exercise* 11.3. Let $X$ be a nonsingular affine variety with local coordinates $x_1, \dots, x_n$. Use the description of the Poisson bracket on $T^*X$ to prove that

$$\{\sigma_i(P), \sigma_j(Q)\} = \sigma_{i+j-1}\big([P,Q]\big),$$

for every $P \in F_iD(A)$ and every $Q \in F_jD(A)$, where $A = \Gamma(X, \mathscr{O}_X)$.