# Lecture 12: March 13

**Gabber's theorem.** Last time, we talked about the result that the characteristic variety $\mathrm{Ch}(\mathcal{M})$ of a coherent $\mathscr{D}_X$-module $\mathcal{M}$ is involutive (with respect to the natural symplectic structure on the cotangent bundle). We saw that the ideal

$$\mathrm{Ann}_{\mathrm{gr}^F \mathscr{D}_X} \mathrm{gr}^F \mathcal{M} \subseteq \mathrm{gr}^F \mathscr{D}_X$$

is closed under the Poisson bracket, and that Theorem 11.2 is equivalent to the radical being closed under the Poisson bracket. This is a problem in algebra, albeit a very difficult one, and there is a purely algebraic proof, due to Gabber.

In fact, Gabber works in the following more general setup. Suppose that $R$ is a $\mathbb{Q}$-algebra, with an increasing algebra filtration $F_\bullet R$, such that the associated graded ring $S = \mathrm{gr}^F R$ is commutative and noetherian. This means that if $u \in F_i R$ and $v \in F_j R$, then their commutator $[u,v] = uv - vu \in F_{i+j-1} R$. If we again use the notation $\sigma_i \colon F_i R \to S_i$ for the "symbol" homomorphism, we can therefore define the *Poisson bracket* of two homogeneous elements of $S$ by the formula

$$\{\sigma_i(u), \sigma_j(v)\} = \sigma_{i+j-1}\bigl([u,v]\bigr).$$

After extending this bilinearly, we obtain a Poisson bracket $\{-,-\} \colon S \otimes_{\mathbb{Q}} S \to S$, and one can check that it satisfies the same identities as the Poisson bracket on a symplectic manifold. But note that this is more general than the case $R = D(A)$, because Gabber is not assuming that $S$ is nonsingular.

**Theorem 12.1** (Gabber). *Using the notation from above, suppose that $M$ is a finitely generated $R$-module with a good filtration $F_\bullet M$, and consider the ideal*

$$J = \sqrt{\mathrm{Ann}_{\mathrm{gr}^F R} \mathrm{gr}^F M} \subseteq \mathrm{gr}^F R.$$

*If $P \subseteq \mathrm{gr}^F R$ is minimal among prime ideals containing $J$, then $\{P,P\} \subseteq P$. In particular, one has $\{J,J\} \subseteq J$.*

The minimal primes containing the ideal $J$ correspond, geometrically, to the irreducible components of $\mathrm{Supp}\, \mathrm{gr}^F M$ inside the scheme $\mathrm{Spec}\, S$. So Gabber's theorem is saying that every irreducible component of the support is "involutive", in the sense that its ideal is closed under the Poisson bracket. In the case of $\mathscr{D}$-modules, this is saying that every irreducible component of the characteristic variety of a coherent $\mathscr{D}$-module is involutive.

**Holonomic $\mathscr{D}$-modules.** One consequence of Theorem 11.2 is that Bernstein's inequality holds for algebraic $\mathscr{D}$-modules: If $X$ is a nonsingular algebraic variety of dimension $n$, and $\mathcal{M}$ a coherent $\mathscr{D}_X$-module, then either $\mathcal{M} = 0$, or every irreducible component of $\mathrm{Ch}(\mathcal{M})$ has dimension $\geq n$. As in the case of the Weyl algebra, the most important $\mathscr{D}$-modules are those for which the dimension of the characteristic variety is as small as possible.

**Definition 12.2.** A coherent $\mathscr{D}_X$-module $\mathcal{M}$ is called *holonomic* if $\mathcal{M} \neq 0$ and $\dim \mathrm{Ch}(\mathcal{M}) = n$, or if $\mathcal{M} = 0$.

If $\mathcal{M}$ is nonzero and holonomic, then each irreducible component of its characteristic variety has dimension $n$, and is therefore (by Theorem 11.2) a Lagrangian subvariety of $T^*X$. Since the ideal defining $\mathrm{Ch}(\mathcal{M})$ is homogeneous, these Lagrangians are moreover *conical*, that is, closed under the natural $\mathbb{G}_m$-action on $T^*X$ by rescaling in the fiber direction. Here are some typical examples of conical Lagrangian subvarieties.

*Example* 12.3. If $Y \subseteq X$ is a nonsingular subvariety, then the conormal bundle $N^*_{Y|X}$ is a nonsingular Lagrangian subvariety of $T^*X$. Since it is a vector bundle of rank $\dim X - \dim Y$ over $Y$, it is clearly conical. More generally, suppose that $Y \subseteq X$ is an arbitrary reduced and irreducible subvariety. The set of nonsingular points $Y_{\mathrm{reg}}$ is Zariski-open and dense in $Y$, and so the conormal bundle $N^*_{Y_{\mathrm{reg}}|X}$ is locally closed, conical, and Lagrangian. Its Zariski closure

$$T^*_Y X = \overline{N^*_{Y_{\mathrm{reg}}|X}}$$

is therefore a conical Lagrangian subvariety of $T^*X$. It is called the *conormal variety* of $Y$ in $X$.

In fact, every conical Lagrangian subvariety of $T^*X$ is a conormal variety.

**Proposition 12.4.** *Let $W \subseteq T^*X$ be an irreducible subvariety that is conical and Lagrangian. Then $Y = p(W)$ is an irreducible subvariety of $X$, and $W = T^*_Y X$.*

*Proof.* The statement is local, and so we may assume that $X = \operatorname{Spec} A$ is affine and that $T^*X = X \times \mathbb{A}^n_k$. Since $W \subseteq X \times \mathbb{A}^n_k$ is conical, it is defined by an ideal in $A[\xi_1, \dots, \xi_n]$ that is homogeneous in the variables $\xi_1, \dots, \xi_n$. This ideal also defines a closed subvariety $\tilde{W} \subseteq X \times \mathbb{P}^{n-1}_k$, and since the projection $p_1 \colon X \times \mathbb{P}^{n-1}_k \to X$ is proper, it follows that $Y = p(W) = p_1(\tilde{W})$ is an irreducible subvariety of $X$. It remains to show that $W = T^*_Y X$. Since both subvarieties are irreducible of dimension $n$, it will be enough to show that the general point of $W$ is contained in the conormal bundle to $Y_{\mathrm{reg}}$.

Let $(x, \xi) \in W$ be a general nonsingular point. By generic smoothness, we have $x \in Y_{\mathrm{reg}}$ and the map on tangent spaces $T_{(x,\xi)} W \to T_x Y$ is surjective. Choose local coordinates $x_1, \dots, x_n$ in a neighborhood of the point $x$, such that $Y$ is defined by the equations $x_{k+1} = \cdots = x_n = 0$. If we again denote by $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ the resulting coordinates on $T^*X$,then the conormal bundle to $Y_{\mathrm{reg}}$ is defined by the equations $\xi_1 = \cdots = \xi_k = x_{k+1} = \cdots = x_n = 0$. Since $W$ is a Lagrangian subvariety, the subspace

$$T_{(x,\xi)} W \subseteq T_{(x,\xi)}\bigl(T^*X\bigr) = T_x X \oplus (T_x X)^*$$

is $n$-dimensional and Lagrangian. Its image under the projection to $T_x X$ is the subspace $T_x Y$. If we denote vectors in $T_x X \oplus (T_x X)^*$ by $(a_1, \dots, a_n, b_1, \dots, b_n)$, then this image is the set of vectors with $a_{k+1} = \cdots = a_n = 0$. For dimension reasons, $T_{(x,\xi)} W$ must contain an $(n-k)$-dimensional space of vectors of the form $(0, \dots, 0, b_1, \dots, b_n)$, and from the Lagrangian condition, we get $b_1 = \cdots = b_k = 0$.

Now we use the fact that $W$ is conical. Since $(x, \xi) \in W$, the entire line $(x, k \cdot \xi)$ is contained in $W$, and so the tangent vector to the line, which is $(0, \dots, 0, \xi_1, \dots, \xi_n)$, must belong to $T_{(x,\xi)} W$. But as we saw, this implies that $\xi_1 = \cdots = \xi_k = 0$, and so $(x, \xi)$ lies on the conormal bundle to $Y_{\mathrm{reg}}$. Since $(x, \xi)$ was a general point of $W$, we deduce that $W \subseteq T^*_Y X$, which suffices to conclude the proof. $\square$

This proposition has interesting implications for holonomic $\mathscr{D}$-modules. Suppose that $\mathcal{M}$ is a nonzero holonomic $\mathscr{D}_X$-module. Its characteristic variety is a finite union of conical Lagrangian subvarieties, and so there are finitely many irreducible subvarieties $Y_1, \dots, Y_m \subseteq X$, without loss of generality distinct, such that

$$\operatorname{Ch}(\mathcal{M}) = \bigcup_{i=1}^{m} T^*_{Y_i} X.$$

Now there are two possibilities. If say $Y_1 = X$, then $U = X \setminus (Y_2 \cup \cdots \cup Y_m)$ is a dense Zariski-open subset, and the restriction of $\mathcal{M}$ to $U$ has its characteristic variety equal to the zero section. By Proposition 10.12, it follows that $\mathcal{M}\big|_U$ is locally free of finite rank, and therefore a vector bundle with integrable connection. The connection acquires some kind of singularities at the remaining subvarieties $Y_2, \dots, Y_n$. The other possibility is that $Y_1, \dots, Y_n \neq X$. In that case, the restriction of $\mathcal{M}$ to $X \setminus (Y_1 \cup \cdots \cup Y_n)$ is trivial, which says that $\mathcal{M}$ is supported on the union $Y_1 \cup \cdots \cup Y_n$. Either way, $\mathcal{M}$ is generically a vector bundle with integrable connection.

**Holonomic $\mathscr{D}$-modules and duality.** Our earlier results about duality for holonomic modules still hold in this context; indeed, the assumptions we made in Lecture 6 apply to the case $R = D(A)$. In general, if $\mathcal{M}$ is a coherent left (or right) $\mathscr{D}_X$-module, then each

$$\mathcal{E}xt^j_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X)$$

is again a coherent right (or left) $\mathscr{D}_X$-module. On an affine open subset $U \subseteq X$ with $A = \Gamma(U, \mathscr{O}_X)$, the corresponding $D(A)$-module is of course $\operatorname{Ext}^j_{D(A)}\big(M, D(A)\big)$, where $M = \Gamma(U, \mathcal{M})$. One then has

$$\mathcal{E}xt^j_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X) = 0 \quad \text{for } j \geq n+1,$$

as well as the useful identity

$$\min\Big\{\, j \geq 0 \;\Big|\; \mathcal{E}xt^j_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X) \neq 0 \,\Big\} + \dim \operatorname{Ch}(\mathcal{M}) = 2n.$$

If $\mathcal{M}$ is a nonzero holonomic $\mathscr{D}_X$-module, then $\mathcal{E}xt^j_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X) = 0$ for every $j \neq n$, and one can again define the *holonomic dual* by

$$\mathcal{M}^* = \mathcal{E}xt^n_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X).$$

As before, one has $(\mathcal{M}^*)^* \cong \mathcal{M}$, and $\operatorname{Ch}(\mathcal{M}^*) = \operatorname{Ch}(\mathcal{M})$. The holonomic dual is again an exact contravariant functor from the category of left (or right) holonomic $\mathscr{D}_X$-modules to the category of right (or left) holonomic $\mathscr{D}_X$-modules.

**Direct images under closed embeddings.** In the next few lectures, we are going to look at various operations on algebraic $\mathscr{D}$-modules, such as pushing forward or pulling back along a morphism of algebraic varieties. This will also give us many new examples of $\mathscr{D}$-modules. We will be especially interested in the effect of these functors on holonomic $\mathscr{D}$-modules. Things are somewhat similar to the case of coherent sheaves, formally, but there are also some interesting differences. Let us start with the simplest case, namely pushing forward along a closed embedding.

*Example* 12.5. Consider the closed embedding $i \colon \mathbb{A}^{n-1}_k \to \mathbb{A}^n_k$ defined by the equation $x_n = 0$. If $\mathcal{M}$ is a $\mathscr{D}$-module on $\mathbb{A}^{n-1}_k$, then its pushforward $i_*\mathcal{M}$ is not a $\mathscr{D}$-module on $\mathbb{A}^n_k$. The problem is that $x_1, \dots, x_n$ and $\partial_1, \dots, \partial_{n-1}$ act in a natural way on $i_*\mathcal{M}$, but we don't know what to do with $\partial_n$. In terms of rings and modules, the closed embedding corresponds to the quotient morphism $k[x_1, \dots, x_{n-1}, x_n] \to k[x_1, \dots, x_{n-1}]$, and the $\mathscr{D}$-module to a module $M$ over the Weyl algebra $A_{n-1}(k)$. We can consider $M$ as a module over $k[x_1, \dots, x_n]$, with $x_n$ acting trivially, but we cannot let $\partial_n$ act trivially this would violate the commutator relation $[\partial_n, x_n] = 1$.

Suppose that $i \colon X \to Y$ is a closed embedding between two nonsingular algebraic varieties, and $\mathcal{M}$ an algebraic $\mathscr{D}_X$-module. For the same reason as above, $i_*\mathcal{M}$ is not in general a $\mathscr{D}_Y$-module. To motivate the correct definition, let us first look at the example of distributions.

*Example* 12.6. Consider the closed embedding

$$i \colon \mathbb{R}^k \to \mathbb{R}^n, \quad i(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0).$$

Suppose that we have a distribution $D$ on $\mathbb{R}^k$; recall that $D$ is a continuous linear functional on the space of compactly supported smooth functions $C^\infty_0(\mathbb{R}^k)$, and that $\langle D, \varphi \rangle$ denotes the real number obtained by evaluating $D$ on a test function $\varphi$. The pushforward distribution $i_*D$ is defined in the obvious way:

$$\langle i_*D, \psi \rangle = \langle D, \psi\big|_{\mathbb{R}^k} \rangle,$$

for any $\psi \in C_0^\infty(\mathbb{R}^n)$. The point is of course that we know how to pull back functions. Now suppose that $D$ satisfies a system of partial differential equations. Can we figure out the partial differential equations satisfied by $i_*D$?

Recall that the Weyl algebra $A_k(\mathbb{R})$ acts on the space of distributions by formal integration by parts: if $\varphi \in C_0^\infty(\mathbb{R}^k)$ and $P \in A_k(\mathbb{R})$, then

$$\langle D \cdot P, \varphi \rangle = \langle D, P\varphi \rangle.$$

Therefore $D$ determines a right ideal

$$I(D) = \left\{ P \in A_k(\mathbb{R}) \;\middle|\; D \cdot P = 0 \right\} \subseteq A_k(\mathbb{R}),$$

and also a right $A_k(n)$-module $A_k(n)/I(D)$. In these terms, we are trying to find the right ideal $I(i_*D)$ from $I(D)$. This is actually fairly easy.

First, the functions $x_{k+1}, \dots, x_n$ vanish on $\mathbb{R}^k$, and so every differential operator of the form $Q = x_{k+1}Q_{k+1} + \dots + x_n Q_n \in A_n(\mathbb{R})$ annihilates $i_*D$, because

$$\big\langle i_*D \cdot Q, \psi \big\rangle = \sum_{j=k+1}^{n} \big\langle i_*D \cdot x_j Q_j, \psi \big\rangle = \sum_{j=k+1}^{n} \big\langle D, x_j Q_j \psi\big|_{\mathbb{R}^k} \big\rangle = 0.$$

We can write any $Q \in A_n(\mathbb{R})$ in the form

$$Q = x_{k+1}Q_{k+1} + \dots + x_n Q_n + \sum_{\alpha \in \mathbb{N}^{n-k}} P_\alpha \partial_{k+1}^{\alpha_{k+1}} \cdots \partial_n^{\alpha_n}$$

where $P_\alpha \in A_k(\mathbb{R})$ only involves $x_1, \dots, x_k, \partial_1, \dots, \partial_k$. Suppose that $Q \in I(i_*D)$. If we act on a test function of the form $\varphi\eta$, with $\varphi \in C_0^\infty(\mathbb{R}^k)$ and $\eta \in C_0^\infty(\mathbb{R}^{n-k})$, we obtain

$$\big\langle i_*D \cdot Q, \varphi\eta \big\rangle = \sum_{\alpha \in \mathbb{N}^{n-k}} \frac{\partial^{\alpha_{k+1} + \dots + \alpha_n} \eta}{\partial x_{k+1}^{\alpha_{k+1}} \cdots \partial x_n^{\alpha_n}}(0) \cdot \big\langle D, P_\alpha \varphi \big\rangle.$$

By choosing $\eta$ appropriately, we can pick out the individual terms, and so

$$0 = \langle D, P_\alpha \varphi \rangle = \langle D \cdot P_\alpha, \varphi \rangle$$

for every $\alpha \in \mathbb{N}^{n-k}$ and every $\varphi \in C_0^\infty(\mathbb{R}^k)$. In other words, each $P_\alpha$ belongs to $I(D)$. It is easy to see that the converse is also true, and so we conclude that

$$I(i_*D) = (x_{k+1}, \dots, x_n)A_n(\mathbb{R}) + I(D)A_n(\mathbb{R}).$$

Here is another way to put this. Remembering that right (and left) ideals in the Weyl algebra are finitely generated, we have $I(D) = (P_1, \dots, P_r)A_k(\mathbb{R})$, and so the right $A_k(\mathbb{R})$-module determined by the distribution $D$ is

$$A_k(\mathbb{R})/(P_1, \dots, P_r)A_k(\mathbb{R}).$$

Then the right $A_n(\mathbb{R})$-module determined by the distribution $i_*D$ is

$$A_n(\mathbb{R})/(P_1, \dots, P_r, x_{k+1}, \dots, x_n)A_n(\mathbb{R}).$$

This is much larger than the other module, but has a natural action by $A_n(\mathbb{R})$.

The example suggest that pushing forward works naturally for right $\mathscr{D}$-modules. The reason is that distributions give rise to right $\mathscr{D}$-modules, whereas functions give rise to left $\mathscr{D}$-modules, and one can push forward distributions, but not functions. It also suggests how to define the pushforward, at least in the special case of modules over the Weyl algebra.

**The transfer module.** Let me now show you the actual definition. Suppose that $i\colon X \to Y$ is a closed embedding between two nonsingular algebraic varieties; since $X$ and $Y$ are both nonsingular, $X$ is locally a complete intersection in $Y$. We will see next time that

$$\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}\mathscr{D}_Y$$

is a $(\mathscr{D}_X, i^{-1}\mathscr{D}_Y)$-bimodule, which is to say that it has both a left action by $\mathscr{D}_X$ and a right action by $i^{-1}\mathscr{D}_Y$, and the two actions commute. The right action by $i^{-1}\mathscr{D}_Y$ is the obvious one; the left action by $\mathscr{D}_X$ is less obvious and involves both factors in the tensor product. Given a right $\mathscr{D}_X$-module $\mathcal{M}$, one then defines its pushforward as

$$i_+\mathcal{M} = i_*\bigl(\mathcal{M} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\bigr);$$

this becomes a right $\mathscr{D}_Y$-module through the natural morphism $\mathscr{D}_Y \to i_* i^{-1}\mathscr{D}_Y$. We will see next time that, in local coordinates, this definition agrees with what happens for distributions.

**Exercises.**

*Exercise* 12.1. Let $\mathcal{M}$ be a left $\mathscr{D}_X$-module and $\mathcal{N}$ a right $\mathscr{D}_X$-module. Show that the tensor product $\mathcal{N} \otimes_{\mathscr{O}_X} \mathcal{M}$ is naturally a right $\mathscr{D}_X$-module.

*Exercise* 12.2. Recall that the canonical line bundle $\omega_X$ is a right $\mathscr{D}_X$-module. Show that the tensor product $\mathscr{D}_X^\omega = \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is a right $\mathscr{D}_X$-module in *two* different ways. Show that the two right $\mathscr{D}_X$-module structures commute with each other, and that there is an automorphism of $\mathscr{D}_X^\omega$ that interchanges them.

*Exercise* 12.3. The previous exercise gives a way to convert left $\mathscr{D}$-modules into right $\mathscr{D}$-modules and back. Show that if $\mathcal{M}$ is a left $\mathscr{D}_X$-module, then

$$\mathscr{D}_X^\omega \otimes_{\mathscr{D}_X} \mathcal{M}$$

is a right $\mathscr{D}_X$-module; here one right $\mathscr{D}_X$-module structure on $\mathscr{D}_X^\omega$ is used to define the tensor product, and the other one is used to turn the tensor product into a right $\mathscr{D}_X$-module. Conversely, show that if $\mathcal{N}$ is a right $\mathscr{D}_X$-module, then

$$\mathcal{H}om_{\mathscr{D}_X}\bigl(\mathscr{D}_X^\omega, \mathcal{N}\bigr)$$

is a left $\mathscr{D}_X$-module; here one right $\mathscr{D}_X$-module structure on $\mathscr{D}_X^\omega$ is used to define $\mathcal{H}om_{\mathscr{D}_X}$, and the other one is used to turn $\mathcal{H}om_{\mathscr{D}_X}$ into a left $\mathscr{D}_X$-module. Finally, show that the obvious morphism

$$\mathcal{M} \to \mathcal{H}om_{\mathscr{D}_X}\bigl(\mathscr{D}_X^\omega, \mathscr{D}_X^\omega \otimes_{\mathscr{D}_X} \mathcal{M}\bigr)$$

is an isomorphism of left $\mathscr{D}_X$-modules.