Lecture 13: March 25

The transfer module. Last time, we looked at the example of distributions to understand what the pushforward of an algebraic $\mathscr{D}$-module under a closed embedding should be. In the case of $i\colon \mathbb{R}^k \hookrightarrow \mathbb{R}^n$, defined by $i(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$, we concluded that the pushforward of a right $A_k(\mathbb{R})$-module of the form

$$A_k(\mathbb{R})/(P_1, \dots, P_m)A_k(\mathbb{R})$$

should be the right $A_n(\mathbb{R})$-module

$$A_n(\mathbb{R})/(P_1, \dots, P_m, x_{k+1}, \dots, x_n)A_n(\mathbb{R}).$$

Let me know explain how to define the pushforward under a closed embedding in general. Let $i\colon Y \hookrightarrow X$ be a closed embedding, with X nonsingular of dimension n and Y nonsingular of dimension r. The definition uses the *transfer module*

$$\mathscr{D}_{Y\to X} = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{D}_X,$$

which is a $(\mathscr{D}_Y, i^{-1}\mathscr{D}_X)$-bimodule. In other words, $\mathscr{D}_{Y\to X}$ is both a left $\mathscr{D}_Y$-module and a right $i^{-1}\mathscr{D}_X$-module, and the two structures commute with each other. The right $i^{-1}\mathscr{D}_X$-module structure is the obvious one, induced by right multiplication on the second factor of the tensor product. The left $\mathscr{D}_Y$-module structure is less obvious, and involves both factors. Remember that since X and Y are both nonsingular, we have a short exact sequence

$$0 \to \mathscr{T}_Y \xrightarrow{\delta_i} i^*\mathscr{T}_X = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{T}_X \to N_{Y|X} \to 0,$$

where $N_{Y|X}$ is the normal bundle of Y in X, a locally free $\mathscr{O}_Y$-module of rank $\dim X - \dim Y$. Now $\mathscr{T}_Y$ acts on $\mathscr{D}_{Y\to X}$ as follows:

$$\theta \cdot (f \otimes P) = \theta(f) \otimes P + f \cdot \delta_i(\theta) \cdot (1 \otimes P),$$

where $\theta \in \mathscr{T}_Y$, $f \in \mathscr{O}_Y$, and $P \in i^{-1}\mathscr{D}_X$ are local sections. I will leave it as an exercise to show that this extends to a left $\mathscr{D}_Y$-module structure.

Example 13.1. Let us write out everything in local coordinates. Choose local coordinates $x_1, \dots, x_n$ on X, in such a way that Y is defined by the equations $x_{r+1} = \dots = x_n = 0$. We write $\partial_1, \dots, \partial_n$ for the corresponding vector fields on X; then $y_1 = x_1, \dots, y_r = x_r$ are local coordinates on Y, with vector fields $\partial_{y_1}, \dots, \partial_{y_r}$. The morphism $\delta_i\colon \mathscr{T}_Y \to i^*\mathscr{T}_X$ sends ∂_{y_j} to $1 \otimes \partial_j$, and so we get

$$\partial_{y_j} \cdot (f \otimes P) = \partial_{y_j} f \otimes P + f \otimes \partial_j P,$$

where $\partial_j P$ is the product in $\mathscr{D}_X$.

Lemma 13.2. *The transfer module $\mathscr{D}_{Y\to X}$ contains a copy of $\mathscr{D}_Y$ and is a locally free left $\mathscr{D}_Y$-module of infinite rank.*

Proof. Since $\mathscr{D}_{Y\to X} = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{D}_X$, the transfer module has a global section given by $1 \otimes 1$. This embeds a copy of $\mathscr{D}_Y$ into $\mathscr{D}_{Y\to X}$, by letting $\mathscr{D}_Y$ act on $1 \otimes 1$. In local coordinates as above, we have

$$\partial_{y_j} \cdot (1 \otimes 1) = 1 \otimes \partial_j.$$

More generally, for any differential operator $Q = \sum_\alpha f_\alpha \partial_y^\alpha$ on Y, we get

$$Q \cdot (1 \otimes 1) = \sum_\alpha f_\alpha \otimes \partial^\alpha = \sum_\alpha f_\alpha \otimes \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r}.$$

This shows that the resulting morphism $\mathscr{D}_Y \to \mathscr{D}_{Y\to X}$ is injective.

Since we are working locally, every differential operator P on X can be written uniquely in the form $P = \sum_\beta g_\beta \partial^\beta$, where $\beta \in \mathbb{N}^n$. By restriction, each $g_\beta \in \Gamma(X, \mathscr{O}_X)$ defines an element $\bar{g}_\beta \in \Gamma(Y, \mathscr{O}_Y)$, and we have

$$f \otimes P = \sum_\beta f \otimes g_\beta \partial^\beta = \sum_{\beta_{r+1},\dots,\beta_n} \left(\sum_{\beta_1,\dots,\beta_r} f \bar{g}_\beta \otimes \partial_1^{\beta_1} \cdots \partial_r^{\beta_r} \right) \cdot \partial_{r+1}^{\beta_{r+1}} \cdots \partial_n^{\beta_n}.$$

This shows that the morphism $\mathscr{D}_Y \otimes_k k[\partial_{r+1}, \dots, \partial_n] \to \mathscr{D}_{Y \to X}$, given by multiplication, is an isomorphism. More formally, consider the subalgebra

$$\mathscr{D}_X^Y = \bigoplus_{\alpha \in \mathbb{N}^r} \mathscr{O}_X \cdot \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r} \subseteq \mathscr{D}_X.$$

Then we have $\mathscr{D}_X \cong \mathscr{D}_X^Y \otimes_k k[\partial_{r+1}, \dots, \partial_n]$, and therefore

$$\mathscr{D}_{Y \to X} \cong (\mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{D}_X^Y) \otimes_k k[\partial_{r+1}, \dots, \partial_n],$$

and the discussion above shows that $\mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{D}_X^Y$ identifies with the copy of $\mathscr{D}_Y$ inside $\mathscr{D}_{Y \to X}$. □

Definition 13.3. The *pushforward* of a right $\mathscr{D}_Y$-module is defined as

$$i_+\mathcal{M} = i_*\big(\mathcal{M} \otimes_{\mathscr{D}_Y} \mathscr{D}_{Y \to X}\big);$$

it becomes a right $\mathscr{D}_X$-module through the morphism $\mathscr{D}_X \to i_* i^{-1} \mathscr{D}_X$.

Note that the pushforward is an exact functor, in the sense that if

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

is a short exact sequence of right $\mathscr{D}_Y$-modules, then

$$0 \to i_+\mathcal{M}' \to i_+\mathcal{M} \to i_+\mathcal{M}'' \to 0$$

is a short exact sequence of right $\mathscr{D}_X$-modules. The reason is that the tensor product over $\mathscr{D}_Y$ is exact (because $\mathscr{D}_{Y \to X}$ is locally free as a left $\mathscr{D}_Y$-module) and that i_* is exact (because $i\colon Y \hookrightarrow X$ is a closed embedding).

The inclusion $\mathscr{D}_Y \hookrightarrow \mathscr{D}_{Y \to X}$ induces an inclusion of $i_*\mathcal{M}$ into the pushforward $i_+\mathcal{M}$. In local coordinates as in the lemma, we get

$$i_+\mathcal{M} \cong i_*\mathcal{M} \otimes_k k[\partial_{r+1}, \dots, \partial_n],$$

and so the problem that $i_*\mathcal{M}$ is not a $\mathscr{D}_X$-module is solved by simply creating a new copy of $i_*\mathcal{M}$ for every monomial in $\partial_{r+1}, \dots, \partial_n$. Note the the submodule $i_*\mathcal{M}$ is annihilated by the equations $x_{r+1}, \dots, x_n$ of Y, but because of the relation $[\partial_j, x_j] = 1$, this is no longer true for $i_+\mathcal{M}$. In general, every section of $i_*\mathcal{M}$ is annihilated by the ideal sheaf $\mathcal{I}_Y \subseteq \mathscr{O}_X$, and every section of $i_+\mathcal{M}$ is annihilated by some power of $\mathcal{I}_Y$.

Example 13.4. Let's compute the pushforward of $\mathscr{D}_Y$. We have

$$i_+\mathscr{D}_Y = i_*\big(\mathscr{D}_Y \otimes_{\mathscr{D}_Y} \mathscr{D}_{Y \to X}\big) = i_*\mathscr{D}_{Y \to X} = i_*\big(\mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{D}_X\big).$$

The natural morphism $\mathscr{D}_X \to i_+\mathscr{D}_Y$, given by sending $P \in \mathscr{D}_X$ to $1 \otimes P$, is clearly surjective, and its kernel is exactly the right ideal $\mathcal{I}_Y \mathscr{D}_X$. Thus $i_+\mathscr{D}_Y \cong \mathscr{D}_X / \mathcal{I}_Y \mathscr{D}_X$.

Example 13.5. Let us compare the definition with the calculation from last time. Consider the closed embedding $i\colon \mathbb{A}_k^r \hookrightarrow \mathbb{A}_k^n$, corresponding to the quotient morphism $k[x_1, \dots, x_n] \to k[x_1, \dots, x_r]$. Let's compute the pushforward of the right A_r-module $M = A_r/(P_1, \dots, P_m)A_r$. By the previous example, the pushforward of A_r itself is given by $A_n/(x_{r+1}, \dots, x_n)A_n$. Using the presentation

$$A_r^{\oplus m} \xrightarrow{(P_1, \dots, P_m)} A_r$$

for M and the exactness of i_+, we see that the pushforward of M is the cokernel of the induced morphism

$$\big(A_n/(x_{r+1}, \dots, x_n)A_n\big)^{\oplus m} \longrightarrow A_n/(x_{r+1}, \dots, x_n)A_n.$$

One then checks that for the endomorphism of A_r given by left multiplication by a differential operator $P \in A_r$, the induced endomorphism of $A_n/(x_{r+1}, \dots, x_n)A_n$ is still left multiplication by P. Thus that the pushforward of M is isomorphic to

$$A_n/(P_1, \dots, P_m, x_{r+1}, \dots, x_n)A_n,$$

in agreement with the calculation we did for distributions last time.

Coherence and characteristic variety. Now let us study the effect of the pushforward functor on coherence and on the characteristic variety.

Lemma 13.6. *If $\mathcal{M}$ is a coherent right $\mathscr{D}_Y$-module, then $i_+\mathcal{M}$ is a coherent right $\mathscr{D}_X$-module.*

Proof. Since $\mathcal{M}$ is coherent over $\mathscr{D}_Y$, we can find a coherent $\mathscr{O}_Y$-module $\mathscr{F} \subseteq \mathcal{M}$ such that $\mathscr{F} \cdot \mathscr{D}_Y = \mathcal{M}$. Using the embedding of $i_*\mathcal{M}$ into $i_+\mathcal{M}$, the coherent $\mathscr{O}_X$-module $i_*\mathscr{F}$ embeds into $i_+\mathcal{M}$, and one checks in local coordinates that it generates $i_+\mathcal{M}$ as a right $\mathscr{D}_X$-module. Therefore $i_+\mathcal{M}$ is coherent. $\square$

To understand the effect of pushing forward on the characteristic variety, we need to investigate in more detail what happens to a good filtration. Suppose that $\mathcal{M}$ is a coherent right $\mathscr{D}_Y$-module, and choose a good filtration $F_\bullet\mathcal{M}$, so that each $F_j\mathcal{M}$ is a coherent $\mathscr{O}_Y$-module. Using the embedding of $i_*\mathcal{M}$ into the pushforward $i_+\mathcal{M} = i_*(\mathcal{M} \otimes_{\mathscr{D}_Y} \mathscr{D}_{Y\to X})$, each $F_j\mathcal{M}$ therefore defines a subsheaf $i_*(F_j\mathcal{M}) \subseteq i_+\mathcal{M}$. To get a filtration that is compatible with the $\mathscr{D}_X$-module structure, we now define

$$F_j(i_+\mathcal{M}) = i_*(F_j\mathcal{M}) + i_*(F_{j-1}\mathcal{M}) \cdot F_1\mathscr{D}_X + i_*(F_{j-2}\mathcal{M}) \cdot F_2\mathscr{D}_X + \cdots \tag{13.7}$$

Since $F_j\mathcal{M} = 0$ for $j \ll 0$, there are only finitely many terms, and so each $F_j(i_+\mathcal{M})$ is a coherent $\mathscr{O}_X$-module. To check that this gives a good filtration, we work in local coordinates. So let $U \subseteq X$ be an affine open subset, with local coordinates $x_1, \dots, x_n \in A = \Gamma(U, \mathscr{O}_X)$, such that Y is defined by the ideal $I = (x_{r+1}, \dots, x_n)$. Set $B = A/I$, and let $M = \Gamma(U \cap Y, \mathscr{D}_Y)$; this is a right $D(B)$-module, of course, but we may also consider it as an A-module on which I acts trivially. From our earlier discussion,

$$\Gamma(U, i_+\mathcal{M}) \cong M \otimes_k k[\partial_{r+1}, \dots, \partial_n] \underset{\text{def}}{=} \tilde{M},$$

and the above filtration is given by

$$F_j\tilde{M} = F_jM \otimes 1 + (F_{j-1}M \otimes 1) \cdot F_1D(A) + (F_{j-2}M \otimes 1) \cdot F_2D(A) + \cdots .$$

We can write this in more compact notation as

$$F_j\tilde{M} = \sum_\alpha F_{j-|\alpha|}M \otimes \partial_{r+1}^{\alpha_{r+1}} \cdots \partial_n^{\alpha_n}.$$

The associated graded module is therefore given by

$$\operatorname{gr}^F \tilde{M} = \operatorname{gr}^F M \otimes_k k[\partial_{r+1}, \dots, \partial_n], \tag{13.8}$$

with the grading in which every ∂_j has degree 1. Concretely,

$$\operatorname{gr}_j^F \tilde{M} = \bigoplus_\alpha \operatorname{gr}_{j-|\alpha|}^F M \otimes \partial_{r+1}^{\alpha_{r+1}} \cdots \partial_n^{\alpha_n}.$$

Now $\operatorname{gr}^F \tilde{M}$ is a graded module over $\operatorname{gr}^F D(A) \cong A[\partial_1, \ldots, \partial_n]$. Let us describe the module structure in more detail. Recall that $\operatorname{gr}^F M$ is a finitely generated graded module over $\operatorname{gr}^F D(B) \cong B[\partial_1, \ldots, \partial_r]$. From (13.8), we get

$$\operatorname{gr}^F \tilde{M} \cong \operatorname{gr}^F M \otimes_{B[\partial_1, \ldots, \partial_r]} B[\partial_1, \ldots, \partial_n],$$

and since $I \subseteq A$ acts trivially on $\operatorname{gr}^F \tilde{M}$ by construction, this is actually an isomorphism of $A[\partial_1, \ldots, \partial_n]$-modules. Since $\operatorname{gr}^F M$ is finitely generated over $B[\partial_1, \ldots, \partial_r]$, this shows that $\operatorname{gr}^F \tilde{M}$ is finitely generated over $\operatorname{gr}^F D(A)$, and so the filtration in (13.7) is indeed good.

The calculation we have just done has the following geometric interpretation. The closed embedding $i \colon Y \hookrightarrow X$ gives rise to two morphisms between the cotangent bundles of X and Y:

$$\begin{tikzcd} Y \times_X T^*X \arrow[r, "di"] \arrow[d, "p_2"] & T^*Y \\ T^*X & \end{tikzcd}$$

Here the morphism $di \colon Y \times_X T^*X \to T^*Y$ corresponds to the pullback morphism $i^*\Omega^1_{X/k} \to \Omega^1_{Y/k}$ between Kähler differentials, and is therefore a morphism of vector bundles, with kernel the conormal bundle of Y in X. In particular, it is a smooth morphism of relative dimension $\dim X - \dim Y$. If we denote by $\widetilde{\operatorname{gr}^F \mathcal{M}}$ the coherent $\mathscr{O}_{T^*Y}$-module corresponding to $\operatorname{gr}^F \mathcal{M}$, then the above isomorphism takes the form

$$\widetilde{\operatorname{gr}^F(i_+\mathcal{M})} \cong (p_2)_* di^* \widetilde{\operatorname{gr}^F \mathcal{M}}. \tag{13.9}$$

The reason is that, in local coordinates, the morphisms of k-algebras corresponding to the morphisms between cotangent bundles are

$$\begin{tikzcd} B[\partial_1, \ldots, \partial_n] & B[\partial_1, \ldots, \partial_r] \arrow[l] \\ A[\partial_1, \ldots, \partial_n] \arrow[u] & \end{tikzcd}$$

and so pulling back via di corresponds to tensoring the $B[\partial_1, \ldots, \partial_r$-module $\operatorname{gr}^F M$ by $B[\partial_1, \ldots, \partial_n]$, and pushing forward via p_2 corresponds to consider the result as a module over $A[\partial_1, \ldots, \partial_n]$. The calculation from above shows that the result is isomorphic to $\operatorname{gr}^F \tilde{M}$. Let us summarize the conclusion.

Proposition 13.10. *Let $i \colon Y \to X$ be a closed embedding, and $\mathcal{M}$ a coherent right $\mathscr{D}_Y$-module. Then the pushforward $i_+\mathcal{M}$ satisfies*

$$\operatorname{Ch}(i_+\mathcal{M}) = p_2\big(di^{-1} \operatorname{Ch}(\mathcal{M})\big),$$

and so $\dim \operatorname{Ch}(i_+\mathcal{M}) = \dim \operatorname{Ch}(\mathcal{M}) + \dim X - \dim Y$.

Proof. Since the characteristic variety of $\mathcal{M}$ is the support of $\widetilde{\operatorname{gr}^F \mathcal{M}}$, the formula for the characteristic variety is an immediate consequence of (13.9). Because di is a smooth morphism of relative dimension $\dim Y - \dim X$, whereas p_2 is a closed embedding, the asserted formula for the dimension of the characteristic variety follows from this. $\square$

The formula for the characteristic variety of the pushforward has several useful consequences. Firstly, it implies that $\mathcal{M}$ is holonomic if and only if $i_+\mathcal{M}$ is holonomic. The reason is of course that $\dim \operatorname{Ch}(i_+\mathcal{M}) - \dim X = \dim \operatorname{Ch}(\mathcal{M}) - \dim Y$. Secondly, it gives another proof for Bernstein's inequality $\dim \operatorname{Ch}(\mathcal{M}) \geq \dim X$, independently of symplectic geometry. Recall that, back in Lecture 3, we proved Bernstein's inequality for finitely generated modules over the Weyl algebra, by looking at Hilbert functions. We can now deduce from this that Bernstein's inequality holds for all algebraic $\mathscr{D}$-modules. Suppose then that $\mathcal{M}$ is a finitely generated right $\mathscr{D}_X$-module, where X is a nonsingular algebraic variety. Since the question is local, we may assume that X is affine. We can then choose a closed embedding $i\colon X \hookrightarrow \mathbb{A}_k^m$ into affine space. By Proposition 13.10, we have

$$\dim \mathrm{Ch}(\mathcal{M}) - \dim X = \dim \mathrm{Ch}(i_+\mathcal{M}) - m \geq 0,$$

where the inequality is a consequence of Bernstein's inequality for the Weyl algebra. Thus $\dim \mathrm{Ch}(\mathcal{M}) \geq \dim X$.

Kashiwara's equivalence. Let $i\colon Y \hookrightarrow X$ be a closed embedding. We had already noted that

$$i_+\colon (\text{coherent right } \mathscr{D}_Y\text{-modules}) \to (\text{coherent right } \mathscr{D}_X\text{-modules})$$

is an exact functor. One of the first results that Kashiwara proved in his thesis is a description of the image of this functor. Clearly, every right $\mathscr{D}_X$-module of the form $i_+\mathcal{M}$ is supported on Y, in the following sense.

Definition 13.11. The *support* of a coherent right $\mathscr{D}_X$-module $\mathcal{N}$ is defined as

$$\operatorname{Supp} \mathcal{N} = p\bigl(\mathrm{Ch}(\mathcal{N})\bigr),$$

where $p\colon T^*X \to X$ is the projection.

Since $\mathrm{Ch}(\mathcal{N})$ is conical, its image in X is always a closed algebraic subset. It follows that $\operatorname{Supp} \mathcal{N}$ is the complement of the largest Zariski-open subset $U \subseteq X$ such that $\mathcal{N}\big|_U$ is trivial. Since every section of $i_+\mathcal{M}$ is annihilated by a sufficiently large power of $\mathcal{I}_Y$, it is clear that $\operatorname{Supp}(i_+\mathcal{M}) \subseteq Y$. (This allows follows from Proposition 13.10, of course.)

Theorem 13.12 (Kashiwara's equivalence)**.** *The functor i_+ is an equivalence of categories between the category of (coherent) right $\mathscr{D}_Y$-modules and the category of (coherent) right $\mathscr{D}_X$-modules with support contained in Y.*

We will give the proof next time.

Exercises.

Exercise 13.1. Suppose that $X = \operatorname{Spec} A$ is affine, and that Y is the closed subscheme defined by an ideal $I \subseteq A$, so that $Y = \operatorname{Spec} B$ for $B = A/I$. Show that the morphism $\mathrm{Der}_k(B) \to B \otimes_A \mathrm{Der}_k(A)$ puts a left $D(B)$-module structure on $B \otimes_A D(A)$, and that it commutes with the natural right $D(A)$-module structure.

Exercise 13.2. Let $X = \operatorname{Spec} A$, with local coordinates $x_1, \dots, x_n \in A$, and let $I = (x_{r+1}, \dots, x_n)$. Show that if M is a finitely generated right $D(B)$-module, where $B = A/I$, then $M \otimes_k k[\partial_{r+1}, \dots, x_n]$ is finitely generated as a right $D(A)$-module.

Exercise 13.3. Let M be a graded $B[\partial_1, \dots, \partial_r]$-module. Show that

$$\begin{aligned}\operatorname{Ann}_{A[\partial_1,\dots,\partial_n]}&\bigl(M \otimes_k k[\partial_{r+1}, \dots, \partial_n]\bigr)\\ &= (x_{r+1}, \dots, x_n) + A[\partial_1, \dots, \partial_n] \cdot \operatorname{Ann}_{B[\partial_1,\dots,\partial_r]} M,\end{aligned}$$

as ideals in $A[\partial_1, \dots, \partial_n]$.