Lecture 14: March 27

Kashiwara's equivalence. Let us start by giving the proof of Kashiwara's equivalence from last time. Here is the statement again.

Theorem (Kashiwara's equivalence)**.** *Let $i\colon Y \hookrightarrow X$ be a closed embedding between nonsingular algebraic varieties. The functor i_+ gives an equivalence between the category of coherent right $\mathscr{D}_Y$-modules and the category of coherent right $\mathscr{D}_X$-modules with support cotained in Y.*

Proof. Recall that if $\mathcal{M}$ is a coherent right $\mathscr{D}_Y$-module, we defined

$$i_+\mathcal{M} = i_*\big(\mathcal{M} \otimes_{\mathscr{D}_Y} \mathscr{D}_{Y\to X}\big),$$

where the transfer module $\mathscr{D}_{Y\to X} = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{D}_X$ is a $(\mathscr{D}_Y, i^{-1}\mathscr{D}_X)$-bimodule. The first step is to construct an inverse for the functor i_+. We have seen that $i_+\mathcal{M}$ always contains a copy of the $\mathscr{O}_X$-module $i_*\mathcal{M}$, and from the local description, it is clear that $i_*\mathcal{M}$ is exactly the subsheaf of $i_+\mathcal{M}$ that is annihilated by the ideal sheaf $\mathcal{I}_Y \subseteq \mathscr{O}_X$. Thus the inverse functor should take a coherent right $\mathscr{D}_X$-module $\mathcal{N}$ to the subsheaf of sections that are annihilated by $\mathcal{I}_Y$. An efficient way to do this is as follows. Given a coherent right $\mathscr{D}_X$-module $\mathcal{N}$, we define

$$i^\sharp\mathcal{N} = \mathcal{H}om_{i^{-1}\mathscr{D}_X}\big(\mathscr{D}_{Y\to X}, i^{-1}\mathcal{N}\big).$$

Here we use the right $i^{-1}\mathscr{D}_X$-module structure on the transfer module for $\mathcal{H}om_{i^{-1}\mathscr{D}_X}$. The left $\mathscr{D}_Y$-module on $\mathscr{D}_{Y\to X}$ then induces a right $\mathscr{D}_Y$-module structure on $i^\sharp\mathcal{N}$. We can rewrite the above definition as

$$i^\sharp\mathcal{N} = \mathcal{H}om_{i^{-1}\mathscr{D}_X}\big(\mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{D}_X, i^{-1}\mathcal{N}\big) \cong \mathcal{H}om_{i^{-1}\mathscr{O}_X}(\mathscr{O}_Y, i^{-1}\mathcal{N}),$$

using the adjunction between $\mathcal{H}om$ and the tensor product. From the short exact sequence $0 \to i^{-1}\mathcal{I}_Y \to i^{-1}\mathscr{O}_X \to \mathscr{O}_Y \to 0$, we obtain an exact sequence

$$0 \to i^\sharp\mathcal{N} \to i^{-1}\mathcal{N} \to \mathcal{H}om_{i^{-1}\mathscr{O}_X}(i^{-1}\mathcal{I}_Y, i^{-1}\mathcal{N})$$

and so $i^\sharp\mathcal{N}$ is exactly the subsheaf of $i^{-1}\mathcal{N}$ annihilated by $i^{-1}\mathcal{I}_Y$. I will leave it as an exercise to check that this isomorphism is compatible with the natural $\mathscr{D}_Y$-module structure on both sides.

Now the claim is that the natural morphism $i^\sharp i_+\mathcal{M} \to \mathcal{M}$ is an isomorphism for every coherent right $\mathscr{D}_Y$-module $\mathcal{M}$, and that the natural morphism $\mathcal{N} \to i_+ i^\sharp\mathcal{N}$ is an isomorphism for every coherent right $\mathscr{D}_X$-module $\mathcal{N}$ such that $\operatorname{Supp}\mathcal{N} \subseteq Y$. This can be checked locally, and so we may assume without loss of generality that $X = \operatorname{Spec} A$ is affine, with coordinates $x_1, \dots, x_n \in A$, and that the closed embedding is defined by the ideal $I = (x_{r+1}, \dots, x_n) \subseteq A$. If we set $B = A/I$, we then have $Y = \operatorname{Spec} B$. In this setting, the pushforward of a right $D(B)$-module M is isomorphic to $M \otimes_k k[\partial_{r+1}, \dots, \partial_n]$, and it is easy to see from this description that the submodule annihilated by the ideal I is exactly $M \otimes 1 \cong M$. This proves the first isomorphism.

The proof of the second isomorphism is more interesting. Suppose that N is a right $D(A)$-module with $\operatorname{Supp} N$ contained in the closed subscheme $V(I)$. This means that every $s \in N$ is annihilated by a sufficiently large power of I. Our goal is to prove that $N \cong N_0 \otimes_k k[\partial_{r+1}, \dots, \partial_n]$, where $N_0 = \big\{\, s \in N \bigm| sI = 0 \,\big\}$. For this, we consider the effect of the operators

$$T_j = x_j \partial_j$$

on the module N. The point is that

$$T_j \cdot \partial_{r+1}^{e_{r+1}} \cdots \partial_n^{e_n} = \partial_{r+1}^{e_{r+1}} \cdots \partial_n^{e_n} \cdot (T_j - e_j),$$

and since T_j acts trivially on the submodule N_0, we have

$$s \otimes \partial_{r+1}^{e_{r+1}} \cdots \partial_n^{e_n} \cdot (T_j - e_j) = 0$$

for every $s \in N_0$. This means that we can read off the exponents of each monomial from the eigenvalues of the operators $T_{r+1}, \dots, T_n$.

Now let us make this precise. The operators $T_{r+1}, \dots, T_n$ commute, and a short calculation shows that

$$T_j(T_j - 1) \cdots (T_j - e) = x_j^{e+1} \partial_j^{e+1}$$

for every $e \geq 0$. For any $s \in N$, we have $s x_j^{e+1} = 0$ for $e \gg 0$, and therefore

$$s T_j(T_j - 1) \cdots (T_j - e) = s x_j^{e+1} \partial_j^{e+1} = 0.$$

This means that s can be written as a sum of eigenvectors of T_j with eigenvalues in $\mathbb{N}$. Since $T_{r+1}, \dots, T_n$ commute, we therefore obtain a decomposition

$$N = \bigoplus_{e_{r+1}, \dots, e_n \in \mathbb{N}} N_{e_{r+1}, \dots, e_n}$$

into simultaneous eigenspaces, where T_j acts on $N_{e_{r+1}, \dots, e_n}$ as multiplication by e_j. Now the claim is that $N_{0,\dots,0} = N_0$, and that this decomposition gives us an isomorphism $N \cong N_0 \otimes_k k[\partial_{r+1}, \dots, \partial_n]$ between N and the pushforward of N_0.

To simplify the notation, let me assume that $r = n - 1$, meaning that $I = (x_n)$ is principal. Then the eigenspace decomposition becomes

$$N = \bigoplus_{e \in \mathbb{N}} N_e,$$

where the operator $T_n = x_n \partial_n$ acts on N_e as multiplication by e. Since T_n commutes with $x_1, \dots, x_{n-1}, \partial_1, \dots, \partial_{n-1}$, each N_e is a $D(B)$-module. Suppose that we have $s \in N_e$. Then we get $s\partial_n \in N_{e+1}$, because

$$s\partial_n T_n = s(\partial_n x_n)\partial_n = s(x_n \partial_n + 1)\partial_n = s\partial_n(e+1);$$

likewise, we get $s x_n \in N_{e-1}$, because

$$s x_n T_n = s x_n (x_n \partial_n) = s x_n (\partial_n x_n - 1) = s x_n e - s x_n = s x_n (e - 1).$$

Since N_e is trivial for $e \leq -1$, we conclude that $N_0 = \{ s \in N \mid s x_n = 0 \}$; moreover, we see that for $e \geq 0$, the morphism

$$N_0 \to N_e, \quad s \mapsto s\partial_n^e,$$

is an isomorphism of $D(B)$-modules. It is now easy to check that

$$N_0 \otimes_k k[\partial_n] \to N, \quad \sum_{e \in \mathbb{N}} s_e \otimes \partial_n^e \mapsto \sum_{e \in \mathbb{N}} s_e \partial_n^e,$$

is an isomorphism of $D(A)$-modules. This proves the second isomorphism. □

Example 14.1. Kashiwara's equivalence implies that $\mathscr{D}$-modules, unlike $\mathscr{O}$-modules, never have nontrivial nilpotents. For example, the A_1-module $A_1/x^3 A_1$ is isomorphic to three copies of $A_1/x A_1$.

Kashiwara's equivalence suggests the following definition of the category of algebraic $\mathscr{D}$-modules on a singular algebraic variety. Suppose that X is a nonsingular algebraic variety, and $Y \subseteq X$ any closed subvariety. Then an algebraic $\mathscr{D}_Y$-module is defined to be an algebraic $\mathscr{D}_X$-module whose support is contained in Y. One can use Kashiwara's equivalence to show that the resulting category is, up to equivalence, independent of the choice of nonsingular ambient variety X.

Pulling back. Suppose that $f\colon X \to Y$ is a morphism between two nonsingular algebraic varieties. It is not hard to construct a pullback functor from algebraic $\mathscr{D}_Y$-modules to algebraic $\mathscr{D}_X$-modules. Recall that we have a natural morphism

$$\delta_f\colon \mathscr{T}_X \to f^*\mathscr{T}_Y = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{T}_Y,$$

dual to the pullback morphism $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ on Kähler differentials. Now if $\mathcal{M}$ is any left $\mathscr{D}_Y$-module, then this morphism gives

$$f^*\mathcal{M} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathcal{M}$$

the structure of a left $\mathscr{D}_X$-module. The formula is the same as in the case of the transfer module: one has

$$\theta \cdot (g \otimes u) = \theta(g) \otimes u + g \cdot \delta_f(\theta) \cdot (1 \otimes u),$$

where $\theta \in \mathscr{T}_X$, $g \in \mathscr{O}_X$, and $u \in f^{-1}\mathcal{M}$ are local sections. We can say this more compactly by noting that

$$f^*\mathcal{M} \cong \left(\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y\right) \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}\mathcal{M} = \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}\mathcal{M}.$$

The transfer module $\mathscr{D}_{X\to Y}$ is a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule, and $f^*\mathcal{M}$ becomes a left $\mathscr{D}_X$-module through the left $\mathscr{D}_X$-module structure on $\mathscr{D}_{X\to Y}$. Since the pullback of a quasi-coherent $\mathscr{O}_Y$-module is a quasi-coherent $\mathscr{O}_X$-module, it is clear that $f^*\mathcal{M}$ is again an algebraic $\mathscr{D}_X$-module.

Now the functor f^{-1} is exact, but tensor product is only right-exact, and so makes sense to consider also the right derived functors.

Definition 14.2. We define the *inverse image* of a left $\mathscr{D}_Y$-module $\mathcal{M}$ by the formula $f^*\mathcal{M} = \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}\mathcal{M}$. For $j \geq 0$, we define $L^{-j}f^*\mathcal{M}$ as the j-th right derived functor of f^*.

As usual, $L^{-j}f^*\mathcal{M}$ is computed by choosing a resolution of $\mathcal{M}$ by $\mathscr{D}_Y$-modules that are locally free (or flat) over $\mathscr{O}_Y$; alternatively, we can choose a resolution of $\mathscr{D}_{X\to Y}$.

Example 14.3. Suppose that $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module with an integrable connection $\nabla\colon \mathscr{E} \to \Omega^1_{Y/k} \otimes_{\mathscr{O}_Y} \mathscr{E}$, viewed as a left $\mathscr{D}_Y$-module. The inverse image is then simply the usual pullback $f^*\mathscr{E}$, together with the integrable connection

$$f^*\nabla\colon f^*\mathscr{E} \to f^*\Omega^1_{Y/k} \otimes_{\mathscr{O}_X} f^*\mathscr{E} \to \Omega^1_{X/k} \otimes_{\mathscr{O}_X} f^*\mathscr{E},$$

viewed as a left $\mathscr{D}_X$-module.

Example 14.4. Consider the left A_1-module $M = A_1/A_1x$ and its pullback to the origin in $\mathbb{A}^1_k$. The corresponding morphism of k-algebras is $k[x] \to k$; using the free resolution

$$k[x] \xrightarrow{x} k[x]$$

for k, the derived functors of the pullback are computed by the complex

$$A_1/A_1x \xrightarrow{x} A_1/A_1x,$$

where the map is $P \mapsto xP$. The kernel is isomorphic to k, generated by the image of $1 \in A_1$; the cokernel is trivial, because $1 = -x\partial$ modulo A_1x. Thus $L^0i^*M = 0$ and $L^{-1}i^*M = k$.

In Lecture 12, I said that the definition of the pushforward functor (in the case of a closed embedding) was motivated by the pushforward of distributions. So why do I not talk about pulling back functions before introducing the pullback functor? The reason is that pulling back $\mathscr{D}$-modules does not correspond to pulling back functions; as we will see next week, the actual meaning is much more interesting. For now, let me just point out one difference between the two functors: pulling back does not necessarily preserve coherence.

Example 14.5. Consider the embedding $\operatorname{Spec} k \hookrightarrow \mathbb{A}^1_k$ of the origin, corresponding to the morphism of k-algebras $k[x] \to k$. The pullback of $\mathscr{D}_{\mathbb{A}^1_k}$ is the k-module $k \otimes_{k[x]} A_1(k) = A_1(k)/xA_1(k)$. This is infinite-dimensional, because the elements $1, \partial, \partial^2, \dots$ are all linearly independent, and in particular, it is not coherent over k.

In general, the pullback of a $\mathscr{D}_X$-module of the form $\mathscr{D}_X/\mathscr{D}_X(P_1, \dots, P_m)$ is not coherent, and so we cannot interpret it as pulling back functions and looking at the differential equations they satisfy.

The following lemma is obvious from the definition.

Lemma 14.6. *If $f \colon X \to Y$ and $g \colon Y \to Z$ are morphisms between nonsingular algebraic varieties, then one has a natural isomorphism of functors $(g \circ f)^* = f^* g^*$.*

We can factor any morphism $f \colon X \to Y$ through its graph as

$$X \xrightarrow{i_f} X \times Y \xrightarrow{p_2} Y$$

as a closed embedding i_f followed by a smooth morphism p_2 (actually, a projection in a product). Because of the lemma, this means that it suffices to understand the pullback functor in the case of closed embeddings and smooth morphisms.

Non-characteristic inverse image. I am now going to describe a condition under which f^* preserves coherence. This will also help us understand what the pullback functor is doing in terms of differential equations. To do this, we revisit a very pretty classical result about differential equations, called the *Cauchy-Kovalevskaya theorem.* Let's begin with the case of ordinary differential equations.

Theorem 14.7 (Cauchy-Kovalevskaya)**.** *Consider the initial value problem*

$$\frac{du}{dt} = F(u), \quad u(0) = 0,$$

for a real function u. If $F \colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ is real-analytic near 0, then the solution u is also real-analytic near 0.

Proof. Although it is not directly connected with $\mathscr{D}$-modules, let me show you the proof, because it is very beautiful. The proof is basically Cauchy's original proof. How do we show that u is real-analytic? We have to prove that the Taylor series

$$\sum_{n=0}^{\infty} u^{(n)}(0) \frac{t^n}{n!}$$

converges in a neighborhood of 0, and for that, we need to compute the values of all the derivatives $u^{(n)}(0)$. The differential equation gives

$$\begin{aligned} u' &= F(u) \\ u'' &= F'(u)u' = F'(u)F(u) \\ u''' &= F''(u)u'F(u) + (F'(u))^2 u' = F''(u)(F(u))^2 + (F'(u))^2 F(u). \end{aligned}$$

and so on. In principle, we can compute $u^{(n)}(0)$ for every $n \geq 0$, but the formulas get very complicated, and so trying to prove the convergence of the series looks pretty hopeless. Still, what we get is that

$$u^{(n)} = P_n\big(F(u), F'(u), \dots, F^{(n-1)}(u)\big),$$

where P_n is a polynomial with nonnegative integer coefficients. These polynomials are universal, in the sense that they do not depend on the given function F. For example, $P_2(x, y) = yx$ and $P_3(x, y, z) = zx^2 + y^2 x$. Because P_n has nonnegative coefficients, this gives us an upper bound

$$|u^{(n)}(0)| \leq P_n\big(|F(0)|, |F'(0)|, \dots, |F^{(n-1)}(0)|\big)$$

on the derivatives of u, using the initial condition $u(0) = 0$. Now Cauchy makes the following brilliant observation. Suppose that we have another function G with the property that $|F^{(n)}(0)| \leq G^{(n)}(0)$ for every $n \geq 0$. Then

$$|u^{(n)}(0)| \leq P_n\bigl(G(0), G'(0), \dots, G^{(n-1)}(0)\bigr) = v^{(n)}(0),$$

where v is the solution to the initial value problem

$$\frac{dv}{dt} = G(v), \quad v(0) = 0.$$

The reason is again that P_n has nonnegative coefficients, and that the same polynomial P_n works for both F and G. Such a function G is called a "majorant", and the proof is known as the *method of majorants*. Suppose that we manage to find G in such a way that the function v is real-analytic. Then the Taylor series

$$\sum_{n=0}^{\infty} v^{(n)}(0) \frac{t^n}{n!}$$

has a positive radius of convergence, and since $|u^{(n)}(0)| \leq v^{(n)}(0)$ for every $n \geq 0$, the same is true for the series

$$\sum_{n=0}^{\infty} |u^{(n)}(0)| \frac{t^n}{n!}.$$

This is sufficient to conclude that u is real-analytic in a neighborhood of 0.

It remains to construct a suitable majorant G. By assumption, F is real-analytic near 0, and so its Taylor series

$$\sum_{n=0}^{\infty} F^{(n)}(0) \frac{t^n}{n!}$$

has a positive radius of convergence. By comparing this series with a geometric series, we find that there are constants $C > 0$ and $r > 0$ such that $|F^{(n)}(0)| \leq Cn!/r^n$ for every $n \geq 0$. We can then take

$$G(t) = C \sum_{n=0}^{\infty} \left(\frac{t}{r}\right)^n = \frac{Cr}{r-t},$$

because $G^{(n)}(0) = Cn!/r^n \geq |F^{(n)}(0)|$ by construction. The solution of the corresponding initial value problem

$$\frac{dv}{dt} = \frac{Cr}{r-v}, \quad v(0) = 0,$$

is easily found using separation of variables; the result is that $v = r - r\sqrt{1 - 2Ct/r}$. This is evidently real-analytic for $|t| < r/2C$, and so we are done. □

Exercises.

Exercise 14.1. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, where $B = A/I$ for an ideal $I \subseteq A$ and both A and B are nonsingular. Let N be a right $D(A)$-module.

(a) Show that $N_0 = \{\, s \in N \mid sI = 0 \,\}$ is a B-module, and that the map

$$N_0 \otimes_B T_B \to N_0, \quad s \otimes \theta \mapsto s \cdot \delta(\theta),$$

makes N_0 into a right $D(B)$-module, where $\delta\colon \operatorname{Der}_k(B) \to B \otimes_A \operatorname{Der}_k(A)$ is the induced morphism between derivations.

(b) Check that the isomorphism of B-modules

$$\operatorname{Hom}_{D(A)}\bigl(B \otimes_A D(A), N\bigr) \cong \operatorname{Hom}_A(B, N) \cong N_0$$

is actually an isomorphism of right $D(B)$-modules.

Exercise 14.2. If $T = x\partial$, prove the identities

$$T\partial^e = \partial^e(T - e) \quad \text{and} \quad T(T-1)\cdots(T-e) = x^{e+1}\partial^{e+1}$$

for every $e \geq 0$.