Lecture 17: April 10

Direct images in general. We are now going to define the direct image functor for (right) $\mathscr{D}$-modules for an arbitrary morphism $f\colon X \to Y$ between nonsingular algebraic varieties. Let $\mathcal{M}$ be a right $\mathscr{D}_X$-module. By analogy with the case of closed embeddings, the direct image should be

$$f_*\big(\mathcal{M} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\big).$$

Recall that the transfer module $\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$ is a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule, and so the direct image is again a right $\mathscr{D}_Y$-module. The problem with this definition is that the resulting functor is neither right nor left exact, and therefore not suitable from a homological algebra standpoint. (The reason is that we are mixing the right exact functor $\otimes$ with the left exact functor f_*.) This problem can be fixed by working in the derived category; in fact, Sato, who founded algebraic analysis, independently invented the theory of derived categories for his needs.

Derived categories. Let me very briefly review some basic facts. Let X be a topological space, and $\mathscr{R}_X$ a sheaf of (maybe noncommutative) rings on X. We denote by $\mathrm{Mod}(\mathscr{R}_X)$ the category of (sheaves of) left $\mathscr{R}_X$-modules; this is an abelian category. Note that right $\mathscr{R}_X$-modules are the same thing as left modules over the opposite ring $\mathscr{R}_X^{op}$. We use the notation

$$D^b(\mathscr{R}_X)$$

for the *derived category* of cohomologically bounded complexes of left $\mathscr{R}_X$-modules. The objects of this category are complexes of left $\mathscr{R}_X$-modules, with the property that only finitely many of the cohomology sheaves are nonzero. The set of morphisms between two objects takes more time to describe, and this is where the action is happening. Recall that when we compute a derived functor, we have to replace a sheaf (or complex of sheaves) by a suitable resolution: injective resolutions in the case of pushforward, flat resolutions in the case of tensor product, etc. The reason for introducing the derived category is that one wants to have a place where a sheaf (or complex of sheaves) is isomorphic to any of its resolutions.

Example 17.1. Suppose that we choose an injective resolution

$$0 \to \mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

for a sheaf of $\mathscr{O}_X$-modules, say. Homological algebra shows that any two such resolutions are the same up to homotopy, meaning that if $\mathcal{J}^\bullet$ is another injective resolution of $\mathscr{F}$, then there is a morphism of complexes $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$, unique up to homotopy; and its composition with the morphism going the other way is homotopic to the identity morphism. But $\mathscr{F}$ is not isomorphic to the complex $\mathcal{I}^\bullet$; all one has is a *quasi-isomorphism*, meaning a morphism of complexes that induces isomorphisms on cohomology sheaves. So if we want $\mathscr{F}$ to be isomorphic to $\mathcal{I}^\bullet$, then we need to work up to homotopy and somehow create an inverse for the morphism $\mathscr{F} \to \mathcal{I}^\bullet$.

Back to $D^b(\mathscr{R}_X)$. The set of morphisms between two objects is obtained by a two-step procedure: starting from all morphisms of complexes, one first identifies morphisms that are homotopy equivalent, and then one formally adjoins inverses for all quasi-isomorphisms. As I said, this construction makes sure that a sheaf (or complex of sheaves) is isomorphic to any of its resolutions by a unique isomorphism.

Concerning the existence of resolutions, one has the following basic fact:

(1) Every $\mathscr{R}_X$-module can be embedded into an injective $\mathscr{R}_X$-module.
(2) Every $\mathscr{R}_X$-module is a quotient of a flat $\mathscr{R}_X$-module.

One can then use the Cartan-Eilenberg construction to show that every cohomologically bounded complex of $\mathscr{R}_X$-modules has both injective and flat resolutions.

The direct image functor. We can now define the direct image functor for an arbitrary morphism $f\colon X \to Y$ between nonsingular algebraic varieties. The construction is done in two stages. First, we have a functor

$$D^b(\mathscr{D}_X^{op}) \to D^b(f^{-1}\mathscr{D}_Y^{op}), \quad \mathcal{M}^\bullet \mapsto \mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y},$$

obtained by taking the derived tensor product with the transfer module $\mathscr{D}_{X\to Y}$. Concretely, this means that we choose a flat resolution for the complex of right $\mathscr{D}_X$-modules $\mathcal{M}^\bullet$, and then tensor this resolution with $\mathscr{D}_{X\to Y}$. For the time being, we do not make any quasi-coherence assumptions. Second, we have a functor

$$D^b(f^{-1}\mathscr{D}_Y^{op}) \to D^b(\mathscr{D}_Y^{op}), \quad \mathcal{N}^\bullet \mapsto \mathbf{R}f_*\mathcal{N}^\bullet,$$

obtained by applying the derived pushforward functor for sheaves. Concretely, this means that we choose an injective resolution for the complex of right $f^{-1}\mathscr{D}_Y$-modules $\mathcal{N}^\bullet$, and then apply the usual pushforward functor f_* to each sheaf in the complex. Each sheaf in the resulting complex is naturally a right $\mathscr{D}_Y$-module through the morphism $\mathscr{D}_Y \to f_*f^{-1}\mathscr{D}_Y$.

One has to show that both functors are well-defined and "exact", meaning that they preserve distinguished triangles (which are the derived category version of short exact sequences of complexes). We define the pushforward functor as the composition of the two functors above.

Definition 17.2. Let $f\colon X \to Y$ be a morphism between nonsingular algebraic varieties. The *pushforward* is the exact functor

$$f_+\colon D^b(\mathscr{D}_X^{op}) \to D^b(\mathscr{D}_Y^{op}), \quad f_+\mathcal{M}^\bullet = \mathbf{R}f_*\big(\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\big)$$

between derived categories.

Note that the general definition involves first choosing a flat resolution for the complex $\mathcal{M}^\bullet$, and then a second injective resolution for $\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}$. Of course, this is only for theoretical purposes; in practice, we factor f into a closed embedding followed by a projection, and there are simple formulas for computing the pushforward in both cases.

Example 17.3. Another word about resolutions. In the case of $\mathscr{D}_X$-modules, one can use results about $\mathscr{O}_X$-modules to get resolutions very easily. For example, suppose that we want to represent a quasi-coherent right $\mathscr{D}_X$-module $\mathcal{M}$ as a quotient of a flat $\mathscr{D}_X$-module. Pick a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F} \subseteq \mathcal{M}$ that generates $\mathcal{M}$ as a $\mathscr{D}_X$-module. If $\mathcal{M}$ is a coherent $\mathscr{D}_X$-module, we can choose $\mathscr{F}$ to be a coherent $\mathscr{O}_X$-module; in general, $\mathscr{F} = \mathcal{M}$ will always do the job. Now pick a flat $\mathscr{O}_X$-module $\mathscr{E}$ that maps onto $\mathscr{F}$. Then the composition

$$\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathcal{M}$$

is surjective, and $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is flat as a right $\mathscr{D}_X$-module.

Here are some concrete examples of the pushforward functor.

Example 17.4. Suppose that $i\colon X \hookrightarrow Y$ is a closed embedding. In this case, the transfer module $\mathscr{D}_{X\to Y}$ is locally free (as a left $\mathscr{D}_X$-module), and tensoring with $\mathscr{D}_{X\to Y}$ is therefore exact. The pushforward functor i_* is also exact, and so we have

$$i_+\mathcal{M}^\bullet = i_*\big(\mathcal{M}^\bullet \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\big).$$

This agrees with our earlier definition in the case of a single $\mathscr{D}_X$-module; in the case of a complex, we simply apply the naive pushforward functor for a closed embedding term by term.

Example 17.5. Suppose that $j\colon U \hookrightarrow Y$ is an open embedding. Then

$$\mathscr{D}_{U\to Y} = \mathscr{O}_U \otimes_{j^{-1}\mathscr{O}_Y} j^{-1}\mathscr{D}_Y \cong \mathscr{D}_U,$$

by the basic properties of $\mathscr{D}_Y$ from Lecture 9. This shows that the pushforward functor agrees with $\mathbf{R}j_*$ in this case. Generally speaking, j_* is exact when the complement $Y \setminus U$ is a divisor; otherwise, there might be higher derived functors. The localization $k[x_1,\dots,x_n,p^{-1}]$ that we analyzed in Lecture 3 is a concrete example, namely the pushforward of $k[x_1,\dots,x_n]$ along the open embedding $\mathbb{A}^n \setminus Z(p) \hookrightarrow \mathbb{A}^n$.

Example 17.6. Let's consider the case where $f\colon X \to \operatorname{Spec} k$ is the morphism to a point. In this case, the pushforward $f_+\mathcal{M}$ should be viewed as something like the cohomology of X with coefficients in a right $\mathscr{D}_X$-module $\mathcal{M}$. The transfer module

$$\mathscr{D}_{X\to\operatorname{Spec} k} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_{\operatorname{Spec} k}} f^{-1}\mathscr{D}_{\operatorname{Spec} k} \cong \mathscr{O}_X$$

is just $\mathscr{O}_X$ in this case; it has the structure of a left $\mathscr{D}_X$-module (and a right k-module). To compute the pushforward

$$f_+\mathcal{M} = \mathbf{R}f_*\big(\mathcal{M} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{O}_X\big),$$

we can use a resolution of $\mathscr{O}_X$ by left $\mathscr{D}_X$-modules. Such a resolution is furnished by the *Spencer complex*

$$\operatorname{Sp}(\mathscr{D}_X) = \Big[\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{T}_X \to \dots \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^2 \mathscr{T}_X \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{T}_X \to \mathscr{D}_X\Big],$$

which lives in degrees $-n,\dots,-1,0$. The Spencer complex maps to $\mathscr{O}_X$ via the $\mathscr{D}_X$-linear map $\mathscr{D}_X \to \mathscr{O}_X$ that takes $P \in \mathscr{D}_X$ to $P(1) \in \mathscr{O}_X$. This is surjective, and the kernel is generated by $\mathscr{T}_X$. The general formula for the differentials

$$d\colon \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^{k+1} \mathscr{T}_X \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^k \mathscr{T}_X$$

in the Spencer complex is as follows:

$$\begin{aligned} d\big(P \otimes \theta_0 \wedge \theta_1 \wedge \dots \wedge \theta_k\big) &= \sum_{i=0}^k (-1)^i (P\theta_i) \otimes \theta_0 \wedge \dots \wedge \widehat{\theta_i} \wedge \dots \wedge \theta_k \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} P \otimes [\theta_i,\theta_j] \wedge \theta_0 \wedge \dots \wedge \widehat{\theta_i} \wedge \dots \wedge \widehat{\theta_j} \wedge \dots \wedge \theta_k \end{aligned}$$

In local coordinates $x_1,\dots,x_n$, the tangent sheaf is a free $\mathscr{O}_X$-module with basis $\partial_1,\dots,\partial_n$, and the above formula simplifies to

$$d\big(P \otimes \partial_{i_0} \wedge \partial_{i_1} \wedge \dots \wedge \partial_{i_k}\big) = \sum_{j=0}^k (-1)^j (P\partial_{i_j}) \otimes \partial_{i_0} \wedge \dots \wedge \widehat{\partial_{i_j}} \wedge \dots \wedge \partial_{i_k}.$$

Except for the fact that $\mathscr{D}_X$ is noncommutative, this is the same formula as for the differentials in a Koszul complex. Let us check that the Spencer complex resolves $\mathscr{O}_X$. From the formula for the differentials, it is clear that we can filter $\operatorname{Sp}(\mathscr{D}_X)$ by the family of subcomplexes

$$F_p \operatorname{Sp}(\mathscr{D}_X) = \Big[F_{p-n}\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{T}_X \to \dots \to F_{p-1}\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{T}_X \to F_p\mathscr{D}_X\Big].$$

The description of the differential in local coordinates shows that the associated graded complex

$$\operatorname{gr}^F_\bullet \operatorname{Sp}(\mathscr{D}_X) = \Big[\operatorname{gr}^F_{\bullet-n}\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{T}_X \to \dots \to \operatorname{gr}^F_{\bullet-1}\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{T}_X \to \operatorname{gr}^F_\bullet \mathscr{D}_X\Big]$$

identifies with the Koszul complex for the regular sequence $\partial_1, \dots, \partial_n \in \operatorname{gr}_1^F \mathscr{D}_X$, and is therefore a resolution of $\mathscr{O}_X$ as a graded $\operatorname{gr}^F \mathscr{D}_X$-module. This proves that the Spencer complex resolves $\mathscr{O}_X$ as a left $\mathscr{D}_X$-module.

Since each term of the Spencer complex is a locally free $\mathscr{D}_X$-module, we get

$$f_+ \mathcal{M} \cong \mathbf{R} f_* \big(\mathcal{M} \otimes_{\mathscr{D}_X} \operatorname{Sp}(\mathscr{D}_X) \big) = \mathbf{R} f_* \operatorname{Sp}(\mathcal{M}),$$

where the Spencer complex of $\mathcal{M}$ is defined analogously by

$$\operatorname{Sp}(\mathcal{M}) = \Big[\mathcal{M} \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{T}_X \to \cdots \to \mathcal{M}_X \otimes_{\mathscr{O}_X} \bigwedge^2 \mathscr{T}_X \to \mathcal{M}_X \otimes_{\mathscr{O}_X} \mathscr{T}_X \to \mathcal{M} \Big],$$

with the same formula for the differentials. The pushforward of a right $\mathscr{D}_X$-module is therefore equal to the hypercohomology of its Spencer complex $\operatorname{Sp}(\mathcal{M})$.

Example 17.7. In the case of ω_X, you can check that the Spencer complex $\operatorname{Sp}(\omega_X)$ is isomorphic to the algebraic de Rham complex

$$\operatorname{DR}(\mathscr{O}_X) = \Big[\mathscr{O}_X \to \Omega^1_{X/k} \to \cdots \to \Omega^n_{X/k} \Big].$$

The j-th hypercohomology group of the de Rham complex is denoted by $H^j_{dR}(X/k)$ and is called the *j-th algebraic de Rham cohomology* of X. When X is defined over the complex numbers, Grothendieck's comparison theorem tells us that $H^j_{dR}(X/\mathbb{C}) \cong H^j(X, \mathbb{C})$ is isomorphic to the singular cohomology of X, considered as a complex manifold.

Let us check that the pushforward functor is compatible with composition of morphisms.

Proposition 17.8. *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be morphisms between nonsingular algebraic varieties. Then one has $g_+ \circ f_+ \cong (g \circ f)_+$, as functors from $D^b(\mathscr{D}_X^{op})$ to $D^b(\mathscr{D}_Z^{op})$.*

Proof. Let $\mathcal{M}^\bullet \in D^b(\mathscr{D}_X^{op})$ be any complex of right $\mathscr{D}_X$-modules. By definition,

$$\begin{aligned} g_+ \big(f_+ \mathcal{M}^\bullet \big) &= \mathbf{R} g_* \Big(\mathbf{R} f_* \big(\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Y} \big) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y \to Z} \Big) \\ (g \circ f)_+ \mathcal{M}^\bullet &= \mathbf{R} (g \circ f)_* \big(\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Z} \big). \end{aligned}$$

We clearly need a relation among the three transfer modules to compare these two expressions. Here is the relevant computation:

$$\begin{aligned} \mathscr{D}_{X \to Z} &= \mathscr{O}_X \otimes_{(g \circ f)^{-1} \mathscr{O}_Z} (g \circ f)^{-1} \mathscr{D}_Z \\ &\cong \mathscr{O}_X \otimes_{f^{-1} \mathscr{O}_Y} \big(f^{-1} \mathscr{O}_Y \otimes_{f^{-1} g^{-1} \mathscr{O}_Z} f^{-1} g^{-1} \mathscr{D}_Z \big) \\ &\cong \mathscr{O}_X \otimes_{f^{-1} \mathscr{O}_Y} f^{-1} \big(\mathscr{O}_Y \otimes_{g^{-1} \mathscr{O}_Z} g^{-1} \mathscr{D}_Z \big) \\ &= \mathscr{O}_X \otimes_{f^{-1} \mathscr{O}_Y} f^{-1} \mathscr{D}_{Y \to Z} \\ &\cong \big(\mathscr{O}_X \otimes_{f^{-1} \mathscr{O}_Y} f^{-1} \mathscr{D}_Y \big) \otimes_{f^{-1} \mathscr{D}_Y} f^{-1} \mathscr{D}_{Y \to Z} \\ &= \mathscr{D}_{X \to Y} \otimes_{f^{-1} \mathscr{D}_Y} f^{-1} \mathscr{D}_{Y \to Z} \end{aligned}$$

In fact, since $\mathscr{D}_Z$ is locally free as an $\mathscr{O}_Z$-module, the higher derived functors of all the tensor products in the above calculation are trivial, and we even have

$$\mathscr{D}_{X \to Z} \cong \mathscr{D}_{X \to Y} \overset{\mathbf{L}}{\otimes}_{f^{-1} \mathscr{D}_Y} f^{-1} \mathscr{D}_{Y \to Z}. \tag{17.9}$$

Because $\mathbf{R}(g \circ f)_* \cong \mathbf{R} g_* \circ \mathbf{R} f_*$, it will therefore be enough to show that

$$\mathbf{R} f_* \big(\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Y} \big) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y \to Z} \to \mathbf{R} f_* \big(\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Y} \overset{\mathbf{L}}{\otimes}_{f^{-1} \mathscr{D}_Y} f^{-1} \mathscr{D}_{Y \to Z} \big)$$

is an isomorphism (in the derived category of right $g^{-1}\mathscr{D}_Z$-modules). Setting

$$A = \mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y} \in D^b(f^{-1}\mathscr{D}_Y^{op}) \quad \text{and} \quad B = \mathscr{D}_{Y\to Z} \in D^b(\mathscr{D}_Y),$$

this is a consequence of the "projection formula" in the following lemma. □

Lemma 17.10. *If $A \in D^b(f^{-1}\mathscr{D}_Y^{op})$ and $B \in D^b(\mathscr{D}_Y)$, then*

$$\mathbf{R}f_* A \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} B \to \mathbf{R}f_*\big(A \overset{\mathbf{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}B\big)$$

is an isomorphism.

Proof. This is a local question, and so we can assume that Y is affine. We can then resolve B by a complex of free $\mathscr{D}_Y$-modules, and thereby reduce the problem to the case where B is a free $\mathscr{D}_Y$-module. But the result is obvious in that case because all the functors preserve direct sums. □

Exercises.

Exercise 17.1. The de Rham complex of a left $\mathscr{D}_X$-module $\mathcal{M}$ is defined as

$$\mathrm{DR}(\mathcal{M}) = \Big[\mathcal{M} \to \Omega^1_{X/k} \otimes_{\mathscr{O}_X} \mathcal{M} \to \cdots \to \Omega^n_{X/k} \otimes_{\mathscr{O}_X} \mathcal{M}\Big],$$

with differentials given in local coordinates $x_1, \ldots, x_n$ by the formula

$$d(\alpha \otimes m) = d\alpha \otimes m + (-1)^{\deg \alpha} \sum_{j=1}^n dz_j \wedge \alpha \otimes (\partial_j m).$$

Here $n = \dim X$. Recall from Lecture 12 that $\mathscr{D}_X^\omega \otimes_{\mathscr{D}_X} \mathcal{M} \cong \omega_X \otimes_{\mathscr{O}_X} \mathcal{M}$ has the structure of a right $\mathscr{D}_X$-module. Show that the Spencer complex of $\mathscr{D}_X^\omega \otimes_{\mathscr{D}_X} \mathcal{M}$ is isomorphic to the de Rham complex of $\mathcal{M}$.

Exercise 17.2. Continuing from the previous exercise, show that

$$\mathbb{H}^{-n}\,\mathrm{DR}(\mathcal{M}) = \big\{\, s \in \Gamma(X, \mathcal{M}) \;\big|\; \partial_1 s = \cdots = \partial_n s = 0 \,\big\}$$

is the space of global sections of $\mathcal{M}$ that are annihilated by all vector fields.