Lecture 18: April 15

Direct images and coherence. Last time, we defined the direct image functor (for right $\mathscr{D}$-modules) as the composition

$$D^b(\mathscr{D}_X^{op}) \xrightarrow{\overset{\mathbf{L}}{\otimes}\mathscr{D}_{X\to Y}} D^b(f^{-1}\mathscr{D}_Y^{op}) \xrightarrow{\mathbf{R}f_*} D^b(\mathscr{D}_Y^{op}), \qquad f_+ = \mathbf{R}f_* \circ (- \overset{\mathbf{L}}{\otimes} \mathscr{D}_{X\to Y})$$

where $f: X \to Y$ is any morphism between nonsingular algebraic varieties. We also showed that $g_+ \circ f_+ \cong (g \circ f)_+$.

Today, our first task is to prove that direct images preserve quasi-coherence and, in the case when f is proper, coherence. The definition of the derived category $D^b(\mathscr{D}_X^{op})$ did *not* include any quasi-coherence assumptions. We are going to denote by $D^b_{qc}(\mathscr{D}_X^{op})$ the full subcategory of $D^b(\mathscr{D}_X^{op})$, consisting of those complexes of right $\mathscr{D}_X$-modules whose cohomology sheaves are quasi-coherent as $\mathscr{O}_X$-modules. Recall that we included the condition of quasi-coherence into our definition of algebraic $\mathscr{D}$-modules in Lecture 10. Similarly, we denote by $D^b_{coh}(\mathscr{D}_X^{op})$ the full subcategory of $D^b(\mathscr{D}_X^{op})$, consisting of those complexes of right $\mathscr{D}_X$-modules whose cohomology sheaves are coherent $\mathscr{D}_X$-modules (and therefore quasi-coherent $\mathscr{O}_X$-modules). This category is of course contained in $D^b_{qc}(\mathscr{D}_X^{op})$.

Theorem 18.1. *Let $f: X \to Y$ be a morphism between nonsingular algebraic varieties. Then the functor f_+ takes $D^b_{qc}(\mathscr{D}_X^{op})$ into $D^b_{qc}(\mathscr{D}_Y^{op})$. When f is proper, it also takes $D^b_{coh}(\mathscr{D}_X^{op})$ into $D^b_{coh}(\mathscr{D}_Y^{op})$.*

We are going to deduce this from the analogous result for $\mathscr{O}_X$-modules. Recall that if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, then the higher direct image sheaves $R^j f_*\mathscr{F}$ are again quasi-coherent $\mathscr{O}_Y$-modules. Moreover, if $\mathscr{F}$ is coherent and f is a proper morphism, then each $R^j f_*\mathscr{F}$ is a coherent $\mathscr{O}_Y$-module. The first result is fairly elementary; the second one, due to Grauert in the analytic setting and to Grothendieck in the algebraic setting, takes more work to prove.

To go from $\mathscr{O}_X$-modules to $\mathscr{D}_X$-modules, we work with "induced $\mathscr{D}$-modules". The construction is straightforward. Given any $\mathscr{O}_X$-module $\mathscr{F}$, the tensor product

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X$$

is a right $\mathscr{D}_X$-module in the obvious way. Right $\mathscr{D}_X$-modules of this form are called *induced $\mathscr{D}$-modules*. If $\mathscr{F}$ is quasi-coherent, then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is quasi-coherent as an $\mathscr{O}_X$-module; if $\mathscr{F}$ is coherent, then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is a coherent $\mathscr{D}_X$-module.

Lemma 18.2. *Every (quasi)coherent $\mathscr{D}_X$-module admits a resolution by (quasi)-coherent induced $\mathscr{D}_X$-modules. The same thing is true for complexes.*

Proof. The point is that every (quasi)coherent $\mathscr{D}_X$-module is the quotient of a (quasi)coherent induced $\mathscr{D}_X$-module. Indeed, if $\mathcal{M}$ is a right $\mathscr{D}_X$-module that is quasi-coherent over $\mathscr{O}_X$, then we can use the obvious surjection

$$\mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathcal{M}.$$

If $\mathcal{M}$ is a coherent right $\mathscr{D}_X$-module, we showed in Lecture 11 that there exists a coherent $\mathscr{O}_X$-module $\mathscr{F} \subseteq \mathcal{M}$ with the property that $\mathscr{F} \cdot \mathscr{D}_X = \mathcal{M}$. This says that

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathcal{M}$$

is surjective. The kernel of the morphism is again either quasi-coherent or coherent, and so we can iterate the construction to produce the desired resolution

$$\cdots \to \mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{F}_0 \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathcal{M} \to 0.$$

Keep in mind that the morphisms $\mathscr{F}_k \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{F}_{k-1} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ are typically *not* induced by morphisms of $\mathscr{O}_X$-modules $\mathscr{F}_k \to \mathscr{F}_{k-1}$.

To deduce the result for complexes, one can then apply the usual Cartan-Eilenberg construction. □

Direct images of induced $\mathscr{D}$-modules are very easy to compute. Indeed,

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y} \cong \mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \mathscr{D}_{X\to Y} = \mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \big(\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y\big)$$
$$\cong \mathscr{F} \overset{\mathbf{L}}{\otimes}_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y = \mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y,$$

due to the fact that $\mathscr{D}_Y$ is locally free, hence flat, over $\mathscr{O}_Y$. Now the usual projection formula (for $\mathscr{O}_Y$-modules) gives

$$f_+(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \cong \mathbf{R}f_*\big(\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y\big) \cong \mathbf{R}f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{D}_Y.$$

All cohomology modules of this complex are therefore again induced $\mathscr{D}_Y$-modules of the form $R^j f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{D}_Y$. They are quasi-coherent as $\mathscr{O}_Y$-modules if $\mathscr{F}$ is quasi-coherent; and coherent as $\mathscr{D}_Y$-modules if $\mathscr{F}$ is coherent and f is proper. This proves the theorem for all induced $\mathscr{D}$-modules.

Proof of Theorem 18.1. Let us first prove the assertion about quasi-coherence. By the lemma, every object in $D^b_{qc}(\mathscr{D}_X^{op})$ is isomorphic to a complex of of quasi-coherent induced $\mathscr{D}_X$-modules, of the form

$$\cdots \to \mathscr{F}^p \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{F}^{p+1} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \cdots$$

let me stress again that the differentials in this complex are $\mathscr{D}_X$-linear, but not induced by $\mathscr{O}_X$-linear morphisms from $\mathscr{F}^p$ to $\mathscr{F}^{p+1}$. If we apply the direct image functor f_+ to this complex, and use our calculation for induced $\mathscr{D}$-modules from above, we obtain a spectral sequence with

$$E_1^{p,q} = (R^q f_*\mathscr{F}^p) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y$$

that converges to the cohomology sheaves of $f_+\big(\mathscr{F}^\bullet \otimes_{\mathscr{O}_X} \mathscr{D}_X\big)$. Each $E_1^{p,q}$ is quasi-coherent as an $\mathscr{O}_Y$-module, and so the cohomology sheaves of the direct image are also quasi-coherent as $\mathscr{O}_Y$-modules.

The proof for coherence is similar. By the lemma, every object in $D^b_{coh}(\mathscr{D}_X^{op})$ is isomorphic to a complex of coherent induced $\mathscr{D}_X$-modules; this means that we can choose all the $\mathscr{F}^p$ as coherent $\mathscr{O}_X$-modules. If $f\colon X \to Y$ is proper, then each $R^q f_*\mathscr{F}^p$ is a coherent $\mathscr{O}_Y$-module. But then each $E_1^{p,q}$ is a coherent $\mathscr{D}_Y$-module, and the spectral sequence implies that the cohomology sheaves of the direct image are also coherent $\mathscr{D}_Y$-modules. □

Example 18.3. Suppose that X is proper over $\operatorname{Spec} k$. Then Theorem 18.1 says in particular that the hypercohomology groups of $\operatorname{Sp}(\mathcal{M})$ are finite-dimensional k-vector spaces for every coherent right $\mathscr{D}_X$-module $\mathcal{M}$. In particular, the algebraic de Rham cohomology groups $H^j_{dR}(X/k)$ are finite-dimensional whenever X is proper over $\operatorname{Spec} k$. (We will see later that this is actually true without properness!)

Example 18.4. Our calculation for induced $\mathscr{D}$-modules shows that the direct image of a coherent $\mathscr{D}_X$-module by a non-proper morphism is usually not coherent. For example, if $f\colon X \to \operatorname{Spec} k$ is not proper, the j-th cohomology module of $f_+\mathscr{D}_X$ is isomorphic to $H^j(X, \mathscr{O}_X)$, which is typically not finite-dimensional over k.

Preservation of holonomicity. The direct and inverse image functors

$$f_+ \colon D^b_{qc}(\mathscr{D}_X^{op}) \to D^b_{qc}(\mathscr{D}_Y^{op}) \quad \text{and} \quad \mathbf{L}f^* \colon D^b_{qc}(\mathscr{D}_Y) \to D^b_{qc}(\mathscr{D}_X)$$

only preserve coherence with some extra assumptions. For $\mathbf{L}f^*$, we need the non-characteristic property; for f_+, we need properness. A small miracle of the theory is that both functors nevertheless preserve the most interesting class of $\mathscr{D}$-modules, namely the holonomic ones. We have already seen one special case of this phenomenon back in Lecture 3, namely that the localization $k[x_1, \dots, x_n, p^{-1}]$ along a nonzero polynomial $P \in k[x_1, \dots, x_n]$ is holonomic over the Weyl algebra $A_n(k)$.

By analogy with quasi-coherent and coherent $\mathscr{D}$-modules, we use the notation $D^b_h(\mathscr{D}_X)$ for the full subcategory of $D^b_{coh}(\mathscr{D}_X)$, whose objects are those complexes of $\mathscr{D}_X$-modules whose cohomology sheaves are holonomic. This category contains all bounded complexes of holonomic $\mathscr{D}_X$-modules, of course, but also injective or flat resolutions of such complexes; we need to work in this larger category in order to define f_+ or $\mathbf{L}f^*$. Fortunately, Beilinson has shown that the inclusion functor

$$D^b\big(\mathrm{Mod}_h(\mathscr{D}_X)\big) \to D^b_h(\mathscr{D}_X)$$

is an equivalence of categories. This means concretely that every complex of $\mathscr{D}_X$-modules with holonomic cohomology sheaves is isomorphic, in $D^b_h(\mathscr{D}_X)$, to a bounded complex of holonomic $\mathscr{D}_X$-modules.

Theorem 18.5. *Let $f \colon X \to Y$ be a morphism of nonsingular algebraic varieties.*
(a) The functor f_+ takes $D^b_h(\mathscr{D}_X^{op})$ into $D^b_h(\mathscr{D}_Y^{op})$.
(b) The functor $\mathbf{L}f^*$ takes $D^b_h(\mathscr{D}_Y)$ into $D^b_h(\mathscr{D}_X)$.

Let me remind you about the case of closed embeddings.

Lemma 18.6. *Let $i \colon X \hookrightarrow Y$ be a closed embedding, and $\mathcal{M}^\bullet \in D^b_{coh}(\mathscr{D}_X^{op})$. Then one has $\mathcal{M}^\bullet \in D^b_h(\mathscr{D}_X^{op})$ if and only if $i_+\mathcal{M}^\bullet \in D^b_h(\mathscr{D}_Y^{op})$.*

Proof. The naive direct image functor $i_+\mathcal{M} = i_*\big(\mathcal{M} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\big)$ is exact, and so

$$\mathcal{H}^k(i_+\mathcal{M}^\bullet) \cong i_+(\mathcal{H}^k\mathcal{M}^\bullet).$$

This reduces the problem to the case of a single coherent right $\mathscr{D}_X$-module $\mathcal{M}$. We showed back in Lecture 13 that $i_+\mathcal{M}$ is a coherent right $\mathscr{D}_Y$-module, and that

$$\dim \mathrm{Ch}(i_+\mathcal{M}) = \dim \mathrm{Ch}(\mathcal{M}) + \dim Y - \dim X.$$

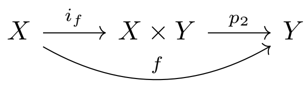
It follows that $\mathcal{M}$ is holonomic if and only if $i_+\mathcal{M}$ is holonomic. □

The proof of Theorem 18.5 is done in two stages. First, there are a certain number of (formal) steps that reduce the general problem to the case of modules over the Weyl algebra. Second, one uses the Bernstein filtration to do the required work for modules over the Weyl algebra. Let me go over the reduction steps rather quickly, without paying too much attention to the details.

The crucial observation is that (a) follows from the special case of a coordinate projection $\mathbb{A}^{n+1}_k \to \mathbb{A}^n_k$. Let me explain how this works. First, we observe that it is enough to consider a single holonomic $\mathscr{D}_X$-module $\mathcal{M}$. The reason is that, as with any complex, one has a convergent spectral sequence

$$E_2^{p,q} = \mathcal{H}^p f_+(\mathcal{H}^q\mathcal{M}^\bullet) \Longrightarrow \mathcal{H}^{p+q} f_+\mathcal{M}^\bullet,$$

and as long as each $\mathcal{H}^p f_+(\mathcal{H}^q\mathcal{M}^\bullet)$ is holonomic, it follows that all cohomology sheaves of $f_+\mathcal{M}^\bullet$ are holonomic. Second, we can factor any morphism as



into a closed embedding followed by a projection. Since we already know that $(i_f)_+\mathcal{M}$ is again holonomic, we only need to consider the case where $X = Z \times Y$ and $f\colon Z \times Y \to Y$ is the second projection.

Third, we can further reduce the problem to the case where $X = Z \times Y$ and Y are both affine. Since the statement is local on Y, we can obviously assume that Y is affine. Choose an affine open covering $Z = Z_1 \cup \cdots \cup Z_n$, such that each $Z \setminus Z_j$ is a nonsingular divisor in Z. Set $U_j = Z_j \times Y$, and for each subset $\alpha \subseteq \{1, \dots, n\}$, denote the resulting open embedding by

$$j_\alpha\colon U_\alpha = \bigcup_{j\in\alpha} U_j \hookrightarrow X.$$

For any sheaf of $\mathscr{O}_X$-modules, and in particular for our holonomic right $\mathscr{D}_X$-module $\mathcal{M}$, we have the Cech resolution

$$0 \to \mathcal{C}^0(M) \to \mathcal{C}^1(M) \to \cdots,$$

whose terms are given by

$$\mathcal{C}^k(M) = \bigoplus_{|\alpha|=k} (j_\alpha)_*\big(\mathcal{M}\big|_{U_\alpha}\big).$$

Since j_α is an affine morphism, we have

$$(j_\alpha)_*\big(\mathcal{M}\big|_{U_\alpha}\big) = \mathbf{R}(j_\alpha)_*\big(\mathcal{M}\big|_{U_\alpha}\big) \cong (j_\alpha)_+\big(\mathcal{M}\big|_{U_\alpha}\big),$$

and so the Cech complex is actually a resolution of $\mathcal{M}$ by right $\mathscr{D}_X$-modules. It is therefore enough to show that each

$$f_+(j_\alpha)_+\big(\mathcal{M}\big|_{U_\alpha}\big) \cong (f \circ j_\alpha)_+\big(\mathcal{M}\big|_{U_\alpha}\big)$$

is a complex of $\mathscr{D}_Y$-modules with holonomic cohomology sheaves. Since the restriction of $\mathcal{M}$ to the affine open subset U_α is holonomic, this reduces the problem to the case of a morphism between nonsingular affine varieties.

Fourth, the result for coordinate projections on affine space implies the result for all morphisms $f\colon X \to Y$ between nonsingular affine varieties. To see this, let us choose closed embeddings $i_X\colon X \hookrightarrow \mathbb{A}^m$ and $i_Y\colon Y \hookrightarrow \mathbb{A}^n$. We then have a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\big\downarrow{\scriptstyle i_f} & & \\
X \times Y & & \big\downarrow{\scriptstyle i_Y} \\
\big\downarrow{\scriptstyle i_X \times i_Y} & & \\
\mathbb{A}^m \times \mathbb{A}^n & \xrightarrow{p_2} & \mathbb{A}^n
\end{array}$$

where all vertical morphisms are closed embeddings. The lemma says that $f_+\mathcal{M}$ belongs to $D_h^b(\mathscr{D}_Y^{op})$ if and only if $(i_Y \circ f)_+\mathcal{M}$ belongs to $D_h^b(\mathscr{D}_{\mathbb{A}^n}^{op})$. Since we already know that the closed embeddings i_f and $i_X \times i_Y$ preserve holonomicity, we only have to consider what happens for $p_2\colon \mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^m$. This can be factored as a composition of m coordinate projections, and so we have successfully reduced the proof of (a) to the special case of a coordinate projection $\mathbb{A}^{n+1} \to \mathbb{A}^n$.

The second observation is that the statement for the inverse image functor in (b) is a formal consequence of (a). As before, we only have to consider a single holonomic left $\mathscr{D}_Y$-module $\mathcal{M}$, and since we know that pulling back along a smooth morphism preserves holonomicity, the general problem reduces to the case of closed embeddings. Locally, we can factor any closed embedding as a composition of closed embeddings of codimension one, and so we only have to prove that if $\mathcal{M}$ is a holonomic left $\mathscr{D}_Y$-module, and $i\colon X \hookrightarrow Y$ a closed embedding of codimension one, then $\mathbf{L}i^*\mathcal{M} \in D^b_h(\mathscr{D}_X)$. Let $j\colon U \hookrightarrow Y$ be the open embedding of the complement $U = Y \setminus X$. Ignoring the difference between left and right $\mathscr{D}$-modules,

$$j_*(\mathcal{M}\big|_U) \cong j_+(\mathcal{M}\big|_U)$$

is again a $\mathscr{D}_Y$-module, due to the fact that j is affine. Provided that we know (a) for the open embedding $j\colon U \hookrightarrow Y$, it follows that $j_*(\mathcal{M}\big|_U)$ is affine. We will show next time that we have an exact sequence of $\mathscr{D}_Y$-modules

$$0 \to i_+(L^{-1}i^*\mathcal{M}) \to \mathcal{M} \to j_*(\mathcal{M}\big|_U) \to i_+(L^0 i^*\mathcal{M}) \to 0,$$

where I am again ignoring the difference between left and right $\mathscr{D}$-modules. It follows that each $i_+(L^{-j}i^*\mathcal{M})$ is a holonomic $\mathscr{D}_Y$-module, and by the case of closed embeddings, this implies that $L^{-j}i^*\mathcal{M}$ is a holonomic $\mathscr{D}_X$-module. This is what we wanted to show.

Excercises.

Exercise 18.1. Morihiko Saito observed that every right $\mathscr{D}_X$-module $\mathcal{M}$ has a *canonical* resolution by induced $\mathscr{D}_X$-modules. Recall that the Spencer complex $\operatorname{Sp}(\mathscr{D}_X)$ is a resolution of $\mathscr{O}_X$ by locally free left $\mathscr{D}_X$-modules.

(a) Show that each term of the complex

$$\operatorname{Sp}(\mathcal{M}) \otimes_{\mathscr{O}_X} \mathscr{D}_X$$

has the structure of a right $\mathscr{D}_X$-module. (Hint: See Lecture 12.)

(b) Construct an isomorphism of right $\mathscr{D}_X$-modules

$$\mathcal{M} \otimes_{\mathscr{O}_X} \Big(\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^k \mathscr{T}_X\Big) \cong \Big(\mathcal{M} \otimes_{\mathscr{O}_X} \bigwedge^k \mathscr{T}_X\Big) \otimes_{\mathscr{O}_X} \mathscr{D}_X$$

to show that each term in above complex is an induced $\mathscr{D}_X$-module.

(c) Show that the above complex is a resolution of $\mathcal{M}$ by induced $\mathscr{D}_X$-modules.

Exercise 18.2. Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{O}_X$-modules. We have a morphism

$$\operatorname{Hom}_{\mathscr{D}_X}\big(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{D}_X\big) \to \operatorname{Hom}_{\mathscr{O}_X}\big(\mathscr{F}, \mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{D}_X\big) \to \operatorname{Hom}_k(\mathscr{F}, \mathscr{G}),$$

obtained by composing with $\mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{G}$, $u \otimes P \mapsto u \cdot P(1)$. Show that this morphism is injective. The image is called the space of *differential morphisms* from $\mathscr{F}$ to $\mathscr{G}$.