Lecture 2: January 30

Recall that the Weyl algebra $A_n = A_n(K)$ is generated by $x_1, \dots, x_n, \partial_1, \dots, \partial_n$, subject to the relations

$$[x_i, x_j] = 0, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{i,j},$$

Today, we begin studying A_n-modules in detail. One interesting difference between modules over A_n and modules over the polynomial ring $R = K[x_1, \dots, x_n]$ is the absence of nilpotents.

Example 2.1. As a $K[x]$-module, $K[x]/(x^2)$ is not isomorphic to two copies of K, because the action by x is nilpotent but not trivial. On the other hand, it is a fun exercise to show that the left A_1-module A_1/A_1x^2 is actually isomorphic to two copies of A_1/A_1x.

Left and right A_n-modules. The crucial difference between the Weyl algebra and the polynomial ring is that $A_n(K)$ is non-commutative. This means that we need to distinguish between left and right A_n-modules. In fact, there are no interesting two-sided A_n-modules.

Proposition 2.2. *$A_n(K)$ is a simple algebra, meaning that the only two-sided ideals of $A_n(K)$ are the zero ideal and $A_n(K)$.*

Proof. This follows from the commutator relations in A_n. We can write any $P \in A_n$ in multi-index notation as

$$P = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta.$$

One can easily show by induction that

$$[\partial_j, x^\alpha \partial^\beta] = \alpha_j x^{\alpha - e_j} \partial^\beta \quad \text{and} \quad [x_j, x^\alpha \partial^\beta] = -\beta_j x^\alpha \partial^{\beta - e_j},$$

where $e_j \in \mathbb{N}^n$ is the j-th coordinate vector. Now suppose that $I \subseteq A_n$ is a nonzero two-sided ideal. Choose any nonzero $P \in I$, and write it as $P = \sum c_{\alpha,\beta} x^\alpha \partial^\beta$. Let

$$m = \max\{\, \alpha_1 \;\big|\; c_{\alpha,\beta} \neq 0 \,\}$$

be the largest power of x_1 that appears in P. If $m \geq 1$, then by the formulas from above, the commutator

$$[\partial_1, P] = \partial_1 P - P \partial_1$$

is nonzero, and the maximal power of x_1 that appears is now $m - 1$. Because I is a two-sided ideal, we still have $[\partial_1, P] \in I$. After repeating this operation m times, we obtain a nonzero element $P_1 \in I$ in which x_1 does not appear. Continuing in this way, we can successively eliminate $x_1, \dots, x_n$ by taking commutators with $\partial_1, \dots, \partial_n$, and then eliminate $\partial_1, \dots, \partial_n$ by taking commutators with $x_1, \dots, x_n$, until we arrive at a non-zero constant contained in I. But then $I = A_n(K)$. ☐

For reasons of notation, we usually work with left A_n-modules. This is no loss of generality, because one can convert left modules into right modules and vice versa. Before I explain this, let me first show you how to describe left (or right) A_n-modules in very simple terms.

Example 2.3. A left A_n-module is the same thing as a $K[x_1, \dots, x_n]$-module M, together with a family of commuting K-linear endomorphisms $d_1, \dots, d_n \in \operatorname{End}_K(M)$, subject to the condition that

$$d_i(x_j m) - x_j d_i(m) = \delta_{i,j} m$$

for every $m \in M$ and every $i, j = 1, \dots, n$. From this data, we can reconstruct the left A_n-module structure using the generators and relations for the Weyl algebra. Indeed, if we define $\partial_j m = d_j(m)$ for $m \in M$, then the condition on $d_1, \dots, d_n$ says exactly that $[\partial_i, \partial_j]$ and $[\partial_i, x_j] - \delta_{i,j}$ act trivially on M, and so we obtain a left A_n-module.

Example 2.4. A right A_n-module is a $K[x_1, \dots, x_n]$-module M, together with a family of commuting K-linear endomorphisms $d_1, \dots, d_n \in \mathrm{End}_K(M)$, such that

$$d_i(x_j m) - x_j d_i(m) = -\delta_{i,j} m$$

for every $m \in M$ and every $i, j = 1, \dots, n$. From this data, we can reconstruct the right A_n-module structure by setting $m\partial_j = d_j(m)$ for $m \in M$. As before, the condition on $d_1, \dots, d_n$ says that $[\partial_i, \partial_j]$ and $[\partial_i, x_j] - \delta_{i,j}$ act trivially on M, and so we obtain a right A_n-module.

Since the only difference in the two descriptions is the minus sign, we can easily convert left A_n-modules into right A_n-modules (and back) by changing the sign.

Example 2.5. Suppose that M is a left A_n-module. Define $d_1, \dots, d_n \in \mathrm{End}_K(M)$ by setting $d_i(m) = -\partial_i m$ for $m \in M$. The sign change means that

$$d_i(x_j m) - x_j d_i(m) = -\partial_i(x_j m) + x_j \partial_i m = -[\partial_i, x_j] = -\delta_{i,j} m,$$

and so this defines a right A_n-module structure on M. Concretely, a differential operator $P = \sum c_{\alpha,\beta} x^\alpha \partial^\beta$ now acts on an element $m \in M$ as

$$mP = \sigma(P)m,$$

where $\sigma(P) = \sum (-1)^{|\beta|} c_{\alpha,\beta} x^\alpha \partial^\beta$ and $|\beta| = \beta_1 + \dots + \beta_n$. The resulting involution $\sigma \colon A_n \to A_n$ also swaps the left and right module structure on A_n itself.

Filtrations on algebras. Recall that the order of a partial differential operator $P = \sum c_{\alpha,\beta} x^\alpha \partial^\beta \in A_n(K)$ is the maximal number of partial derivatives that appear in P; in symbols,

$$\mathrm{ord}(P) = \max\left\{ \beta_1 + \dots + \beta_n \;\middle|\; c_{\alpha,\beta} \neq 0 \right\}$$

Because of the relation $[\partial_i, x_i] = \delta_{i,j}$, the commutator between a differential operator of order d and a differential operator of order e is a differential operator of order at most $d + e - 1$. In this sense, the Weyl algebra is only mildly non-commutative. In fact, A_n is an example of a filtered algebra, in the following sense.

Definition 2.6. Let R be a K-algebra, not necessarily commutative. A filtration $F_\bullet = F_\bullet R$ on R is a sequence of linear subspaces

$$\{0\} = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \dots \subseteq R,$$

such that $F_j \cdot F_k \subseteq F_{j+k}$ and $R = \bigcup F_k$.

In particular, $F_0 R$ is a subalgebra of R, and each $F_k R$ is a left (and right) module over $F_0 R$. In many cases of interest, the $F_k R$ are finitely generated as $F_0 R$-modules.

Example 2.7. The *order filtration* on A_n is defined by

$$F_j^{\mathrm{ord}} A_n = \left\{ P = \sum c_{\alpha,\beta} x^\alpha \partial^\beta \;\middle|\; \mathrm{ord}(P) = |\beta| \leq j \right\}$$

In this case, $F_0^{\mathrm{ord}} A_n = K[x_1, \dots, x_n]$, and each $F_j^{\mathrm{ord}} A_n$ is a finitely generated $K[x_1, \dots, x_n]$-module. Note that we have $F_j^{\mathrm{ord}} \cdot F_k^{\mathrm{ord}} = F_{j+k}^{\mathrm{ord}}$ for every $j, k \geq 0$.

Example 2.8. The *Bernstein filtration* on A_n is defined by

$$F_j^B A_n = \left\{ P = \sum c_{\alpha,\beta} x^\alpha \partial^\beta \;\middle|\; |\alpha| + |\beta| \leq j \right\}.$$

In this case, $F_0^B A_n = K$, and each $F_j^B A_n$ is a K-vector space of finite dimension. Note that we have $F_j^B \cdot F_k^B = F_{j+k}^B$ for every $j, k \geq 0$.

The advantage of the Bernstein filtration is that each F_j^B is finite dimensional. The advantage of the order filtration is that it generalizes to the case of $\mathscr{D}$-modules on arbitrary smooth algebraic varieties (whereas the Bernstein filtration only makes sense on affine space).

Definition 2.9. Given a filtration $F_\bullet R$ on a K-algebra R, the *associated graded algebra* is defined as

$$\operatorname{gr}^F R = \bigoplus_{j=0}^{\infty} F_j/F_{j-1}.$$

It inherits a multiplication from R in the natural way: for $r \in F_j$ and $s \in F_k$, the product $(r + F_{j-1}) \cdot (s + F_{k-1}) = rs + F_{j+k-1}$ is well-defined.

For both the order filtration and the Bernstein filtration, the associated graded algebra of A_n is simply the polynomial ring in $2n$ variables. In particular, the associated graded algebra is commutative.

Proposition 2.10. *Let $A_n = A_n(K)$.*

(a) If $F_\bullet A_n$ is the Bernstein filtration, then

$$\operatorname{gr}^F A_n \cong K[x_1, \dots, x_n, \partial_1, \dots, \partial_n],$$

with the usual grading by the total degree in $x_1, \dots, x_n, \partial_1, \dots, \partial_n$.

(b) If $F_\bullet A_n$ is the order filtration, then

$$\operatorname{gr}^F A_n \cong K[x_1, \dots, x_n, \partial_1, \dots, \partial_n],$$

with the grading by the total degree in $\partial_1, \dots, \partial_n$.

Proof. We prove this only for the Bernstein filtration, the other case being similar. From the definition of the Bernstein filtration as

$$F_j = \left\{ P = \sum c_{\alpha,\beta} x^\alpha \partial^\beta \;\middle|\; |\alpha| + |\beta| \le j \right\},$$

it is obvious that $x_1, \dots, x_n, \partial_1, \dots, \partial_n \in F_1$. For clarity, we use $\bar{x}_1, \dots, \bar{x}_n, \bar{\partial}_1, \dots, \bar{\partial}_n$ to denote their images in F_1/F_0. It is also obvious that F_j/F_{j-1} is generated by all monomials of degree j in $\bar{x}_1, \dots, \bar{x}_n, \bar{\partial}_1, \dots, \bar{\partial}_n$. It remains to analyze the relations. Obviously, $\bar{x}_1, \dots, \bar{x}_n$ commute, and $\bar{\partial}_1, \dots, \bar{\partial}_n$ commute. Since

$$\partial_i x_j - x_j \partial_i = [\partial_i, x_j] = \delta_{i,j} \in F_0,$$

we have $\bar{\partial}_i \bar{x}_j - \bar{x}_j \bar{\partial}_i = 0$ as elements of F_2/F_1. Therefore, all $2n$ elements commute with each other; as there are no further relations, we obtain the desired isomorphism with the polynomial ring. □

Filtrations on A_n-modules. For the time being, we only consider left A_n-modules. Let $F_\bullet A_n$ be either the Bernstein filtration or the order filtration.

Definition 2.11. Let M be a left A_n-module. A *compatible filtration $F_\bullet M$* on M is a sequence of linear subspaces

$$\{0\} F_{-1} M \subseteq F_0 M \subseteq F_1 M \subseteq \cdots \subseteq M,$$

with $F_j A_n \cdot F_k M \subseteq F_{j+k} M$ and $M = \bigcup F_k M$, such that each $F_k M$ is finitely generated as an $F_0 A_n$-module.

Given a compatible filtration on M, one forms the associated graded module

$$\operatorname{gr}^F M = \bigoplus_{k=0}^{\infty} F_k M / F_{k-1} M,$$

which again inherits the structure of a graded module over $\operatorname{gr}^F A_n$ by defining $(r + F_{j-1} A_n) \cdot (m + F_{k-1} M) = rm + F_{j+k-1} M$. Since $\operatorname{gr}^F A_n$ is a polynomial ring in $2n$-variables, this puts us back in the world of commutative algebra. At least for finitely generated modules, one can use this device to transfer properties of modules over the polynomial ring to modules over the Weyl algebra.

Definition 2.12. A compatible filtration $F_\bullet M$ is called *good* if $\operatorname{gr}^F M$ is finitely generated over $\operatorname{gr}^F A_n$.

The following proposition gives a useful necessary and sufficient criterion for a filtration to be good.

Proposition 2.13. *Let M be a left A_n-module. A compatible filtration $F_\bullet M$ is good if, and only if, there exists $j_0 \geq 0$ such that $F_i A_n \cdot F_j M = F_{i+j} M$ for every $i \geq 0$ and every $j \geq j_0$.*

Proof. To simplify the notation, we put

$$\operatorname{gr}_j^F A_n = F_j A_n / F_{j-1} A_n \quad \text{and} \quad \operatorname{gr}_k^F M = F_k M / F_{k-1} M.$$

Let us first prove that the condition is sufficient. Taking $j = j_0$ and $i = j - j_0$, we see that $F_j M = F_{j-j_0} A_n \cdot F_{j_0} M$ for every $j \geq j_0$. This implies almost immediately that $\operatorname{gr}^F M$ is generated, over $\operatorname{gr}^F A_n$, by the direct sum of all $\operatorname{gr}_j^F M$ with $j \leq j_0$. Now each $F_j M$ is finitely generated over $F_0 A_n$, which means that $\operatorname{gr}_j^F M$ is finitely generated over $\operatorname{gr}_0^F A_n$. In total, we therefore get a finite number of elements that generate $\operatorname{gr}^F M$ as a $\operatorname{gr}^F A_n$-module.

The more interesting part is to show that the condition is sufficient. Here it is enough to prove the existence of an integer $j_0 \geq 0$ such that $F_j M = F_{j-j_0} A_n \cdot F_{j_0} M$ for every $j \geq j_0$; the general case follows from this by induction on $j \geq j_0$. Since everything is graded, the fact that $\operatorname{gr}^F M$ is finitely generated over $\operatorname{gr}^F A_n$ implies that it can be generated by finitely many homogeneous elements; let j_0 be the maximum of their degrees. For every $j \geq j_0$, we then have

$$\operatorname{gr}_j^F M = \sum_{i=0}^{j_0} \operatorname{gr}_{j-i}^F A_n \cdot \operatorname{gr}_i^F M,$$

which translates into the relation

$$F_j M = F_{j-1} M + \sum_{i=0}^{j_0} F_{j-i} A_n \cdot F_i M = F_{j-1} M + F_{j-j_0} A_n \cdot F_{j_0} M,$$

using the fact that $F_{j-i} A_n = F_{j-j_0} A_n \cdot F_{j_0-i} A_n$. At this point, we can prove the desired equality $F_j M = F_{j-j_0} A_n \cdot F_{j_0} M$ by induction on $j \geq j_0$. □

We can now show that the existence of a good filtration characterizes finitely generated A_n-modules.

Corollary 2.14. *Let M be a left A_n-module. Then M admits a good filtration if, and only if, it is finitely generated over A_n.*

Proof. Suppose that M is generated, over A_n, by finitely many elements $m_1, \dots, m_k$. Then we can define a compatible filtration $F_\bullet M$ by setting

$$F_j M = F_j A_n \cdot m_1 + \cdots + F_j A_n \cdot m_k.$$

Note that each $F_j M$ is finitely generated over $F_0 A_n$, due to the fact that $F_j A_n$ is finitely generated over $F_0 A_n$. With this definition, we have $F_j M = F_j A_n \cdot F_0 M$ for every $j \geq 0$, and therefore the filtration is good by Proposition 2.13.

Conversely, suppose that M admits a good filtration $F_\bullet M$. By Proposition 2.13, there is an integer $j_0 \geq 0$ such that $F_j M = F_{j-j_0} A_n \cdot F_{j_0} M$ for every $j \geq j_0$. Since $M = \bigcup F_j M$, and since $F_{j_0} M$ is finitely generated over $F_0 A_n$, it follows pretty directly that M is finitely generated over A_n. □

The following result is useful for comparing different good filtrations.

Corollary 2.15. *Let M be a left A_n-module with a good filtration $F_\bullet M$. Then for every compatible filtration $G_\bullet M$, there exists some $j_1 \geq 0$ such that $F_j M \subseteq G_{j+j_1} M$ for all $j \geq 0$.*

Proof. As before, choose $j_0 \geq 0$ with the property that $F_j M = F_{j-j_0} A_n \cdot F_{j_0} M$ for every $j \geq j_0$. Since $F_{j_0} M$ is finitely generated over the commutative noetherian ring $F_0 A_n$, and since $G_\bullet M$ is an exhaustive filtration of M by finitely generated $F_0 A_n$-modules, there is some $j_1 \geq 0$ such that $F_{j_0} M \subseteq G_{j_1} M$. But then

$$F_j M \subseteq F_{j+j_0} M = F_j A_n \cdot F_{j_0} M \subseteq F_j A_n \cdot G_{j_1} M \subseteq G_{j+j_1} M,$$

as claimed. □

Let us conclude the discussion of good filtrations by proving that the Weyl algebra is left noetherian. Notice how, during the proof, passing to the associated graded algebra/module allows us to transfer the noetherian property from the commutative ring $\operatorname{gr}^F A_n$ to the non-commutative ring A_n.

Proposition 2.16. *Let M be a finitely generated left A_n-module. Then every submodule of M is again finitely generated. In particular, A_n itself is left noetherian.*

Proof. Let $N \subseteq M$ be a left A_n-submodule. Since M is finitely generated, it admits a good filtration $F_\bullet M$. If we define

$$F_j N = N \cap F_j M,$$

then it is easy to see that $F_i A_n \cdot F_j N \subseteq F_{i+j} N$. Moreover, each $F_j N$ is finitely generated over $F_0 A_n$: this follows from the fact that $F_j M$ is finitely generated over $F_0 A_n$ because $F_0 A_n$ is commutative and noetherian. Therefore $F_\bullet N$ is a good filtration. By construction, we have

$$\operatorname{gr}_j^F N \subseteq \operatorname{gr}_j^F M,$$

which says that $\operatorname{gr}^F N$ is a submodule of $\operatorname{gr}^F M$. Since the original filtration was good, $\operatorname{gr}^F M$ is a finitely generated module over the commutative noetherian ring $\operatorname{gr}^F A_n$, and so $\operatorname{gr}^F N$ is also finitely generated over $\operatorname{gr}^F A_n$. This proves that N is finitely generated over A_n. □

Exercises.

Exercise 2.1. Consider the left A_1-module $M = A_1/A_1 x$. As a K-vector space, M is isomorphic to $K[\partial]$. Write down a formula for the resulting A_1-action on $K[\partial]$.

Exercise 2.2. Show that the left A_1-module $A_1/A_1 x^2$ is isomorphic to the direct sum of two copies of $A_1/A_1 x$.

Exercise 2.3. $M = K[x, x^{-1}]$ is a left A_1-module, with the usual differentiation rule $\partial \cdot x^k = k x^{k-1}$. Show that M is generated, as an A_1-module, by x^{-1}. What does the associated graded module for the good filtration $F_j M = F_j A_1 \cdot x^{-1}$ look like?