Lecture 21: April 24

Regularity for holonomic $\mathscr{D}$-modules. Last time, we considered differential equations of the form $Pu = 0$, where $P = a_0(x)\partial^m + a_1(x)\partial^{m-1} + \cdots + a_m(x)$ is a differential operator of order m with holomorphic coefficients, such that $a_0(0) = 0$. We showed that all multi-valued solutions have moderate growth near the origin iff

$$x^m P = (x\partial)^m + b_1(x)(x\partial)^{m-1} + \cdots + b_m(x), \tag{21.1}$$

with $b_1(x), \dots, b_m(x)$ holomorphic. In that case, one says that the equation $Pu = 0$ has a regular singularity at the origin. Let us now reformulate this algebraic condition in terms of the left $\mathscr{D}_X$-module $\mathcal{M} = \mathscr{D}_X/\mathscr{D}_X P$. For the time being, $\mathscr{D}_X$ again means the sheaf of linear differential operators with holomorphic coefficients.

We first observe that the characteristic variety of $\mathcal{M}$ is defined by the principal symbol $\sigma_m(P) = a_0(x)\xi^m$, where x and ξ are the natural coordinates on the cotangent bundle. Since $a_0(0) = 0$, it follows that $\mathrm{Ch}(\mathcal{M})$ is the subset defined by the equation $x\xi = 0$. This means that if $F_\bullet\mathcal{M}$ is any good filtration of $\mathcal{M}$, for example the one induced by the order filtration on $\mathscr{D}_X$, then some power of $x\xi$ annihilates $\mathrm{gr}^F\mathcal{M}$. Let me now show you how (21.1) can be used to construct a particular good filtration with better properties.

Suppose that we have (21.1) with $b_1(x), \dots, b_m(x)$ holomorphic. Then we can define a good filtration $F_\bullet\mathcal{M}$ by setting

$$F_k\mathcal{M} = \sum_{j=0}^{m-1} F_k\mathscr{D}_X \cdot (x\partial)^j + \mathscr{D}_X P.$$

It is not hard to see that this is indeed a good filtration; moreover,

$$x\partial \cdot F_k\mathcal{M} \subseteq F_k\mathcal{M}$$

for every $k \in \mathbb{N}$, by virtue of (21.1). This means that $\mathrm{gr}^F\mathcal{M}$ is annihilated by the first power of $x\xi$.

Kashiwara and Kawai introduced the notion of holonomic $\mathscr{D}$-modules with regular singularities as a generalization of this case. From now on, we let X be a nonsingular algebraic variety (over a field k of characteristic zero). For a coherent left $\mathscr{D}_X$-module $\mathcal{M}$, we denote by $\mathcal{I}_{\mathrm{Ch}(\mathcal{M})} \subseteq \mathscr{O}_{T^*X}$ the ideal sheaf of the characteristic variety. Recall that

$$\mathcal{I}_{\mathrm{Ch}(\mathcal{M})} = \sqrt{\mathrm{Ann}_{\mathrm{gr}^F\mathscr{D}_X}\,\mathrm{gr}^F\mathcal{M}},$$

where $F_\bullet\mathcal{M}$ is any good filtration. It follows that there is some (usually large) integer N such that $\mathcal{I}^N_{\mathrm{Ch}(\mathcal{M})} \cdot \mathrm{gr}^F\mathcal{M} = 0$. Roughly speaking, we say that $\mathcal{M}$ is regular if we can find a good filtration for which $N = 1$. For technical reasons, we have to be slightly more careful. Suppose first that X is proper over $\mathrm{Spec}\, k$.

Definition 21.2. Let X be a nonsingular algebraic variety that is proper over $\mathrm{Spec}\, k$. A holonomic left $\mathscr{D}_X$-module $\mathcal{M}$ is called *regular* (in the sense of Kashiwara and Kawai) if it admits a good filtration $F_\bullet\mathcal{M}$ such that $\mathcal{I}_{\mathrm{Ch}(\mathcal{M})} \cdot \mathrm{gr}^F\mathcal{M} = 0$.

If $P \in F_k\mathscr{D}_X$ is a differential operator of order k, then $\sigma_k(P)$ belongs to $\mathcal{I}_{\mathrm{Ch}(\mathcal{M})}$ if and only if $\sigma_k(P)$ vanishes along the characteristic variety of $\mathcal{M}$. The condition in the definition is therefore saying that whenever P is a differential operator of order k such that $\sigma_k(P)$ vanishes along $\mathrm{Ch}(\mathcal{M})$, then

$$P \cdot F_j\mathcal{M} \subseteq F_{j+k-1}\mathcal{M}$$

for every $j \in \mathbb{Z}$.

The original definition by Kashiwara and Kawai is only asking that a good filtration with $\mathcal{I}_{\mathrm{Ch}(\mathcal{M})} \cdot \mathrm{gr}^F \mathcal{M} = 0$ should exist locally on X; but they show that $\mathcal{M}$ then actually has a globally defined good filtration with this property.

One can prove (with a lot of work) that direct images by proper morphisms, and inverse images by arbitrary morphisms, preserve regularity. If we used the above definition to define regularity when X is not proper, we would run into the problem that direct images by open embeddings do not necessarily preserve regularity.

Example 21.3. Consider the holonomic A_1-module $M = A_1/A_1(\partial - 1)$. The filtration induced by the order filtration certainly has the property in the definition (and the differential equation $\partial u = u$ has a regular singularity at the origin). The problem occurs near the point at infinity. Indeed, if we consider the open embedding $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, and look at M in the other affine chart with coordinate $y = x^{-1}$, we get $\partial_x - 1 = -y^2\partial_y - 1$. The A_1-module

$$A_1/A_1(y^2\partial_y + 1)$$

is *not* regular in the above sense; indeed, the differential equation $y^2\partial_y u + u = 0$ does not satisfy the condition in Theorem 20.4.

Example 21.4. A more well-behaved example is $M = A_1/A_1(x\partial - \alpha)$, for $\alpha \in k$. Since $x\partial x = -y\partial_y$, this becomes $A_1/A_1(y\partial_y + \alpha)$ in the chart at infinity, which again has a regular singularity.

Since we would like direct images by arbitrary morphisms to preserve regularity, we need to include open embeddings into the definition. Let X be a nonsingular algebraic variety. Since k has characteristic zero, Nagata's theorem implies that we can always embed X into a nonsingular algebraic variety $\bar{X}$ that is proper over $\operatorname{Spec} k$. We can always arrange that $\bar{X} \setminus X$ is a divisor; using embedded resolution of singularities, we can moreover achieve that this divisor only has normal crossing singularities. In either case, $j \colon X \hookrightarrow \bar{X}$ is an affine morphism, and so if $\mathcal{M}$ is a holonomic left $\mathscr{D}_X$-module, the direct image $j_+\mathcal{M} = j_*\mathcal{M}$ is again a holonomic left $\mathscr{D}_{\bar{X}}$-module.

Definition 21.5. Let X be a nonsingular algebraic variety. A holonomic left $\mathscr{D}_X$-module $\mathcal{M}$ is called *regular* (in the sense of Kashiwara and Kawai) if, for any affine open embedding $j \colon X \hookrightarrow \bar{X}$ into a nonsingular algebraic variety $\bar{X}$ that is proper over $\operatorname{Spec} k$, the direct image $j_+\mathcal{M}$ is regular on $\bar{X}$.

In fact, it suffices to check this for a single embedding $j \colon X \hookrightarrow \bar{X}$. Here is why. Given any two affine open embeddings $j \colon X \hookrightarrow \bar{X}$ and $j' \colon X \hookrightarrow \bar{X}'$, one can take the closure of the image of $(j, j') \colon X \hookrightarrow \bar{X} \times \bar{X}'$, and resolve the resulting singularities to obtain a third embedding $j'' \colon X \hookrightarrow \bar{X}''$ such that $j = f \circ j''$ and $j' = f' \circ j''$ for two proper morphisms $f \colon \bar{X}'' \to X$ and $f' \colon \bar{X}'' \to \bar{X}'$. Since direct images by proper morphisms preserve regularity, it follows that $j_+\mathcal{M}$ is regular on $\bar{X}$ if and only if $j'_+\mathcal{M}$ is regular on $\bar{X}'$.

Regularity and solutions. Over the complex numbers, one can also detect regularity by looking at solutions. The idea is that a left $\mathscr{D}_X$-module $\mathcal{M}$ is regular if and only if all formal power series solutions of $\mathcal{M}$ are convergent. Let us make this precise. We now assume that X is a complex manifold of dimension n, and we denote by $\mathscr{D}_X$ the sheaf of differential operators with holomorphic coefficients. If $\mathcal{M}$ is a holonomic left $\mathscr{D}_X$-module, we can define regularity as above by the (local) existence of a good filtration such that $\mathcal{I}_{\mathrm{Ch}(\mathcal{M})} \cdot \mathrm{gr}^F \mathcal{M} = 0$. Fix a point $x \in X$, and denote by $\mathscr{O}_{X,x}$ the local ring of holomorphic functions that are defined in some neighborhood of x, and by $\hat{\mathscr{O}}_{X,x}$ its completion with respect to the maximal ideal. Concretely, $\hat{\mathscr{O}}_{X,x}$ are formal power series in local coordinates $x_1, \dots, x_n$, and the subring $\mathscr{O}_{X,x}$ consists of those power series that actually converge in a neighorhood of the given point. The stalk $\mathcal{M}_x$ is a holonomic left $\mathscr{D}_{X,x}$-module. In particular, it is coherent, and so we can think of $\mathcal{M}_x$ as being obtained from a system of linear partial differential equations (by choosing a presentation of $\mathcal{M}_x$). As we discussed in Lecture 1, the space of holomorphic solutions to the system can be described as

$$\operatorname{Hom}_{\mathscr{D}_{X,x}}(\mathcal{M}_x, \mathscr{O}_{X,x}).$$

Roughly speaking, regularity of $\mathcal{M}$ means that the natural morphism

$$\operatorname{Hom}_{\mathscr{D}_{X,x}}(\mathcal{M}_x, \mathscr{O}_{X,x}) \hookrightarrow \operatorname{Hom}_{\mathscr{D}_{X,x}}(\mathcal{M}_x, \hat{\mathscr{O}}_{X,x})$$

is an isomorphism. In other words, every convergent power series solution actually converges. This is not quite true, but it becomes true if we replace the naive solution functor by its derived version

$$\mathbf{R}\operatorname{Hom}_{\mathscr{D}_{X,x}}(\mathcal{M}_x, \mathscr{O}_{X,x}).$$

Concretetly, this is computed by choosing a resolution of $\mathcal{M}_x$ by free $\mathscr{D}_{X,x}$-modules of finite rank, and then applying the functor $\operatorname{Hom}_{\mathscr{D}_{X,x}}(-, \mathscr{O}_{X,x})$.

Theorem 21.6 (Kashiwara-Kawai). *Let X be a complex manifold, and $\mathcal{M}$ a holonomic left $\mathscr{D}_X$-module. Then $\mathcal{M}$ is regular, in the sense that it (locally) admits a good filtration $F_\bullet\mathcal{M}$ with $\mathcal{I}_{\mathrm{Ch}(\mathcal{M})} \cdot \mathrm{gr}^F \mathcal{M} = 0$, iff the morphism*

$$\mathbf{R}\operatorname{Hom}_{\mathscr{D}_{X,x}}(\mathcal{M}_x, \mathscr{O}_{X,x}) \to \mathbf{R}\operatorname{Hom}_{\mathscr{D}_{X,x}}(\mathcal{M}_x, \hat{\mathscr{O}}_{X,x})$$

is an isomorphism in the derived category, for every point $x \in X$.

We do not have the tools to prove this, so let me instead illustrate the result by a simple example.

Example 21.7. On $X = \mathbb{C}$, consider the left $\mathscr{D}$-module $\mathcal{M} = \mathscr{D}/\mathscr{D}(x^2\partial - 1)$, which is clearly not regular at the point $x = 0$. Let us see how the solution functor detects this. A free resolution of $\mathcal{M}$ is given by

$$\mathscr{D} \xrightarrow{x^2\partial - 1} \mathscr{D}$$

and so we need to compare the cohomology of the two complexes

$$\begin{array}{ccc} \mathscr{O} & \xrightarrow{x^2\partial-1} & \mathscr{O} \\ \big\downarrow & & \big\downarrow \\ \hat{\mathscr{O}} & \xrightarrow{x^2\partial-1} & \hat{\mathscr{O}} \end{array}$$

The horizontal differential takes a (convergent) power series $\sum_{n=0}^\infty a_n x^n$ to the (convergent) power series

$$(x^2\partial - 1)\sum_{n=0}^\infty a_n x^n = \sum_{n=0}^\infty \big((n-1)a_{n-1} - a_n\big)x^n$$

where $a_{-1} = 0$ (to simplify the notation). It is easy to see that the kernel of $x^2\partial - 1$ is trivial: from the relations $(n-1)a_{n-1} - a_n = 0$ for every $n \in \mathbb{N}$, one obtains $a_0 = a_1 = a_2 = \cdots = 0$.

The behavior of the cokernel is more interesting. On $\hat{\mathscr{O}}$, the operator $x^2\partial - 1$ is surjective. Indeed, if $\sum_{n=0}^\infty b_n x^n$ is any formal power series, then the equation

$$\sum_{n=0}^\infty b_n x^n = (x^2\partial - 1)\sum_{n=0}^\infty a_n x^n$$

means that $(n-1)a_{n-1} - a_n = b_n$, and this can be solved recursively. But on $\mathscr{O}$, the operator is no longer surjective. For instance, if we try to solve

$$x = (x^2\partial - 1)\sum_{n=0}^{\infty} a_n x^n,$$

we obtain $a_0 = 0$, $a_1 = -1$, and $a_n = (n-1)a_{n-1}$ for $n \geq 2$, from which it follows that $a_n = -(n-1)!$ for $n \geq 1$. The resulting series

$$-\sum_{n=1}^{\infty}(n-1)! \cdot x^n$$

clearly has radius of convergence equal to zero.