Lecture 23: May 1

Regular holonomic $\mathscr{D}$-modules of normal crossing type. Let me briefly recall what we did last time. We first showed that $\mathbb{P}^n$ is $\mathscr{D}$-affine, which meant that the global sections functor

$$\Gamma(\mathbb{P}^n, -)\colon \operatorname{Mod}_{qc}(\mathscr{D}_{\mathbb{P}^n}) \to \operatorname{Mod}\bigl(\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})\bigr)$$

is an equivalence of categories. In other words, algebraic $\mathscr{D}$-modules on $\mathbb{P}^n$ are uniquely determined by their space of global sections, which is a module over the ring $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$. We also showed that the ring of differential operators on $\mathbb{P}^n$ is generated by the $(n+1)^2$ operators $D_{i,j} = x_i \partial_j$, subject to the commutator relations

$$[D_{i,j}, D_{k,\ell}] = \begin{cases} D_{i,i} - D_{j,j} & \text{if } k = j \text{ and } \ell = i, \\ D_{i,\ell} & \text{if } k = j \text{ and } \ell \neq i, \\ -D_{k,j} & \text{if } k \neq j \text{ and } \ell = i, \\ 0 & \text{if } k \neq j \text{ and } \ell \neq i, \end{cases} \tag{23.1}$$

and the extra relation $\theta = D_{0,0} + D_{1,1} + \cdots + D_{n,n} = 0$. We then showed that if $\mathcal{M}$ is a regular holonomic $\mathscr{D}_{\mathbb{P}^n}$-module whose characteristic variety is contained in the set $Z(x_0\xi_0, x_1\xi_1, \dots, x_n\xi_n) \subseteq T^*\mathbb{P}^n$, then we get a decomposition

$$\Gamma(\mathbb{P}^n, \mathcal{M}) = \bigoplus_{\alpha \in A} M_\alpha,$$

where $A = \bigl\{ \alpha \in k^{n+1} \bigm| \alpha_0 + \alpha_1 + \cdots + \alpha_n = 0 \bigr\}$. Here each M_α is a finite-dimensional k-vector space, consisting of those global sections of $\mathcal{M}$ on which the $n+1$ operators $D_{j,j} - \alpha_j$ act nilpotently.

How about the converse? Suppose we are given a collection of finite-dimensional k-vector spaces M_α, indexed by $\alpha \in A$. What extra information is needed to turn the direct sum

$$M = \bigoplus_{\alpha \in A} M_\alpha$$

into (the space of global sections of) a regular holonomic $\mathscr{D}_{\mathbb{P}^n}$-module of normal crossing type? First, M should be a left module over the ring $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$, and so we need to have linear operators

$$D_{i,j}\colon M_\alpha \to M_{\alpha + e_i - e_j}$$

for every $\alpha \in A$ and every $i, j \in \{0, 1, \dots, n\}$. These operators should satisfy the commutator relations above, as well as the identity $D_{0,0} + D_{1,1} + \cdots + D_{n,n} = 0$. We also want M to be finitely generated, which means that finitely many of the M_α should generate M as a $\Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$-module. Finally, the operator $D_{j,j} - \alpha_j$ should act nilpotently on M_α for every $j \in \{0, 1, \dots, n\}$. It is then not hard to show that the corresponding $\mathscr{D}_{\mathbb{P}^n}$-module is regular holonomic of normal crossing type.

Other variants. There are some useful variants of the classification above. One is regular holonomic $\mathscr{D}$-modules of normal crossing type on affine space $\mathbb{A}^n$. Let $\mathcal{M}$ be a holonomic $\mathscr{D}_{\mathbb{A}^n}$-module with the property that

$$\operatorname{Ch}(\mathcal{M}) \subseteq Z(x_1\xi_1, \dots, x_n\xi_n) \subseteq T^*\mathbb{A}^n.$$

In that case, we say that $\mathcal{M}$ is *of normal crossing type*. Recall that $\mathcal{M}$ is regular, in the sense of Kashiwara and Kawai, if the direct image $j_+\mathcal{M}$ is regular on $\mathbb{P}^n$, where $j\colon \mathbb{A}^n \hookrightarrow \mathbb{P}^n$ is the open embedding. One can show that if $\mathcal{M}$ is regular holonomic of normal crossing type on $\mathbb{A}^n$, then $j_+\mathcal{M}$ is regular holonomic of normal crossing type on $\mathbb{P}^n$. Thus we obtain a decomposition

$$\Gamma(\mathbb{A}^n, \mathcal{M}) = \Gamma(\mathbb{P}^n, j_+\mathcal{M}) = \bigoplus_{\alpha \in k^n} M_\alpha,$$

which we are now indexing by $\alpha \in k^n$. (This is okay because $\alpha_0 = -(\alpha_1 + \dots + \alpha_n)$, so there is no loss of information.) Again, each M_α is a finite-dimensional k-vector space, consisting of all global sections of $\mathcal{M}$ on which the n commuting operators $x_j \partial_j - \alpha_j$ act nilpotently. This time, we have

$$x_j \colon M_\alpha \to M_{\alpha + e_j} \quad \text{and} \quad \partial_j \colon M_\alpha \to M_{\alpha - e_j}$$

for every $j = 1, \dots, n$; this follows from the commutator relation $[\partial_j, x_j] = 1$. Conversely, given a collection of finite-dimensional k-vector spaces M_α, indexed by $\alpha \in k^n$, and a collection of linear operators $x_j \colon M_\alpha \to M_{\alpha+e_j}$ and $\partial_j \colon M_\alpha \to M_{\alpha - e_j}$ subject to the relations $[\partial_i, x_j] = \delta_{i,j}$, the direct sum

$$M = \bigoplus_{\alpha \in k^n} M_\alpha$$

becomes a module over the Weyl algebra $\Gamma(\mathbb{A}^n, \mathscr{D}_{\mathbb{A}^n})$; if this module is finitely generated, and if each $x_j \partial_j - \alpha_j$ acts nilpotently on M_α, then the corresponding $\mathscr{D}_{\mathbb{A}^n}$-module is regular holonomic of normal crossing type.

There is also a local analytic version of the classification, for $k = \mathbb{C}$. Let $\mathscr{D}_{\mathbb{C}^n,0}$ denote the ring of linear differential operators with holomorphic coefficients that are defined in some neighborhood of the origin in $\mathbb{C}^n$. We say that a holonomic $\mathscr{D}_{\mathbb{C}^n,0}$-module $\mathcal{M}$ is of normal crossing type if its characteristic variety $\mathrm{Ch}(\mathcal{M})$ is contained in the set $Z(x_1\xi_1, \dots, x_n\xi_n)$. We say that $\mathcal{M}$ is regular if it satisfies the condition from Lecture 21, meaning if there exists a good filtration $F_\bullet \mathcal{M}$ such that each $F_k \mathcal{M}$ is a finitely generated $\mathscr{O}_{\mathbb{C}^n,0}$-module stable under the action by $x_1\partial_1, \dots, x_n\partial_n$. Define

$$M_\alpha = \left\{ s \in \mathcal{M} \;\middle|\; (x_j \partial_j - \alpha_j)^m s = 0 \text{ for } j = 0, 1, \dots, n \text{ and } m \gg 0 \right\}.$$

Each M_α is a finite-dimensional $\mathbb{C}$-vector space, and their direct sum

$$M = \bigoplus_{\alpha \in \mathbb{C}^n} M_\alpha$$

is a regular holonomic module over the Weyl algebra $A_n(\mathbb{C})$, of normal crossing type. Then one can show (with a lot of extra work) that

$$\mathcal{M} \cong \mathscr{D}_{\mathbb{C}^n,0} \otimes_{A_n(\mathbb{C})} M.$$

In other words, the $\mathscr{D}_{\mathbb{C}^n,0}$-module structure on $\mathcal{M}$ is completely determined by the much simpler algebraic $\mathscr{D}$-module M. Note that this result is only true in the local analytic setting. The following example explains why.

Example 23.2. Consider the $\mathscr{D}_{\mathbb{A}^1}$-module $\mathcal{M} = \mathscr{D}_{\mathbb{A}^1}/\mathscr{D}_{\mathbb{A}^1}(\partial - 1)$. It is easy to see that $\mathrm{Ch}(\mathcal{M})$ is the zero section, and that $\mathcal{M}$ is actually a line bundle with integrable connection. Except for regularity at infinity, $\mathcal{M}$ is therefore regular holonomic of normal crossing type. But it is not true, not even Zariski-locally, that $\Gamma(\mathbb{A}^1, \mathcal{M}) = A_1/A_1(\partial - 1)$ has a decomposition into generalized eigenspaces for $x\partial$; in fact, you can check for yourself that $x\partial$ does not have any nontrivial eigenvectors. What goes wrong is that we need a solution to $\partial u = u$ to get an isomorphism between $\mathcal{M}$ and $\mathscr{O}_{\mathbb{A}^1}$. But the solution is $u = e^x$, which is not an algebraic function, because it has an essential singularity at infinity. Another way to say this is that $\mathcal{M}$ is *not* regular at infinity.

Solutions. Let us discuss a few more properties of the classification on $\mathbb{A}^n$. For simplicity, I will assume from now on that $k = \mathbb{C}$. Consider a regular holonomic $\mathscr{D}$-module of normal crossing type, with decomposition

$$M = \bigoplus_{\alpha \in \mathbb{C}^n} M_\alpha.$$

Here each M_α is a finite-dimensional $\mathbb{C}$-vector space. By construction, $x_j\partial_j - \alpha_j$ acts nilpotently on M_α, and so $x_j\partial_j$ is an isomorphism as long as $\alpha_j \neq 0$. Consequently,

$$\partial_j \colon M_\alpha \to M_{\alpha - e_j} \quad \text{and} \quad x_j \colon M_{\alpha - e_j} \to M_\alpha$$

are injective respectively surjective for $\alpha_j \neq 0$. Likewise, $\partial_j x_j - \alpha_j - 1$ acts nilpotently on M_α, and so $\partial_j x_j$ is an isomorphism as long as $\alpha_j \neq -1$. Thus

$$\partial_j \colon M_{\alpha + e_j} \to M_\alpha \quad \text{and} \quad x_j \colon M_\alpha \to M_{\alpha + e_j}$$

are surjective respectively injective for $\alpha_j \neq -1$. We can summarize this by saying that $\partial_j \colon M_\alpha \to M_{\alpha - e_j}$ is an isomorphism for $\alpha_j \neq 0$, and that $x_j \colon M_\alpha \to M_{\alpha + e_j}$ is an isomorphism for $\alpha_j \neq -1$.

This implies of course that those vector spaces M_α with

$$-1 \leq \operatorname{Re} \alpha_j \leq 0 \quad \text{for every } j = 1, \dots, n$$

determine all the others. Since M is finitely generated over $A_n(\mathbb{C})$, the set

$$F = \left\{ \alpha \in \mathbb{C}^n \;\middle|\; M_\alpha \neq 0 \text{ and } -1 \leq \operatorname{Re} \alpha_j \leq 0 \text{ for all } j \right\}$$

must be finite. Thus M is generated as an $A_n(\mathbb{C})$-module by the direct sum of those M_α with $\alpha \in F$.

Recall that any holonomic A_n-module has finite length, meaning that it has a finite composition series whose subquotients are simple. Let us describe more explicitly what simple regular holonomic $\mathscr{D}$-modules of normal crossing type look like. Suppose that M is simple but nonzero. Choose some $\alpha \in F$, so that $M_\alpha \neq 0$ and $-1 \leq \operatorname{Re} \alpha_j \leq 0$ for all j. Since each $x_j\partial_j - \alpha_j$ acts nilpotently on M_α, we can find a common eigenvector $s \in M_\alpha$ such that $x_j\partial_j s = \alpha_j s$ for every $j = 1, \dots, n$. Since M is simple, we must have $A_n s = M$. Because s is an eigenvector, it is not hard to see that $A_n s$ intersects M_α exactly in the subspace $\mathbb{C}s$. Thus $M_\alpha = \mathbb{C}s$ is one-dimensional. Now there are two special cases:

(1) One case is that $\alpha_j = 0$. Then $x_j\partial_j s = 0$, and so the submodule $A_n(\mathbb{C})\partial_j s$ does not contain s. Since M is simple, this forces $\partial_j s = 0$.

(2) The other case is that $\alpha_j = -1$. Then $\partial_j x_j s = 0$, and for the same reason as before, this forces $x_j s = 0$.

We conclude that M is generated as an A_n-module by $s \in M_\alpha$, and that s is annihilated by $(x_j\partial_j - \alpha_j)$ for $\alpha_j \neq -1, 0$, by ∂_j for $\alpha_j = 0$, and by x_j for $\alpha_j = -1$. It is easy to see that there cannot be any other relations, and so we get

$$M \cong A_n/I,$$

where $I_\alpha \subseteq A_n$ is the left ideal generated by the n differential operators

$$\begin{cases} x_j\partial_j - \alpha_j & \text{for } \alpha_j \neq -1, 0, \\ \partial_j & \text{for } \alpha_j = 0, \\ x_j & \text{for } \alpha_j = -1. \end{cases}$$

We see that M is supported on the linear subspace

$$\operatorname{Supp} M_\alpha = \bigcap_{\alpha_j = -1} Z(x_j),$$

and so by Kashiwara's equivalence, it is the pushforward of a regular holonomic $\mathscr{D}$-module of normal crossing type on $\operatorname{Supp} M_\alpha$. Outside of the union of the hyperplanes $Z(x_j)$ with $\alpha_j \neq -1, 0$, the latter is a line bundle with integrable connection; this connection has a regular singularity at each of the hyperplanes in question, with monodromy $e^{2\pi i \alpha_j}$.

Now let see what we can say about the solutions of regular holonomic $\mathscr{D}$-modules of normal crossing type on $\mathbb{C}^n$. Since algebraic differential equations typically do not have algebraic solutions, we need to work in the analytic topology; we use the notation $\mathscr{O}_{\mathbb{C}^n}$ for the sheaf of holomorphic functions on $\mathbb{C}^n$, and the notation $\mathscr{D}_{\mathbb{C}^n}$ for the sheaf of differential operators with holomorphic coefficients. Let us write $\mathcal{M} = \mathscr{D}_{\mathbb{C}^n} \otimes_{A_n} M$ for the analytic $\mathscr{D}_{\mathbb{C}^n}$-module determined by the $A_n(\mathbb{C})$-module M. Recall that we have the (derived) solutions functor

$$\mathrm{Sol}(\mathcal{M}) = \mathbf{R}\mathcal{H}om_{\mathscr{D}_{\mathbb{C}^n}}\big(\mathcal{M}, \mathscr{O}_{\mathbb{C}^n}\big).$$

It can be computed for example by choosing a resolution of $\mathcal{M}$ by free $\mathscr{D}_{\mathbb{C}^n}$-modules, and then applying the usual solutions functor term by term. For simple modules of normal crossing type, this is easily done. Fix a multi-index $\alpha \in F$ as above. To keep the notation simple, let me set

$$P_j = \begin{cases} x_j\partial_j - \alpha_j & \text{if } \alpha_j \neq -1, 0, \\ \partial_j & \text{if } \alpha_j = 0, \\ x_j & \text{if } \alpha_j = -1. \end{cases}$$

Then our simple $\mathscr{D}_{\mathbb{C}^n}$-module has the form

$$\mathcal{M}_\alpha = \mathscr{D}_{\mathbb{C}^n}/\mathscr{D}_{\mathbb{C}^n}(P_1, \dots, P_n),$$

The Koszul complex for $P_1, \dots, P_n$ gives a resolution by free $\mathscr{D}_{\mathbb{C}^n}$-modules:

$$\mathscr{D}_{\mathbb{C}^n} \to \mathscr{D}_{\mathbb{C}^n}^{\oplus n} \to \cdots \to \mathscr{D}_{\mathbb{C}^n}^{\oplus\binom{n}{2}} \to \mathscr{D}_{\mathbb{C}^n}^{\oplus n} \to \mathscr{D}_{\mathbb{C}^n}$$

Consequently, $\mathrm{Sol}(\mathcal{M}_\alpha)$ is represented by the complex

$$\mathscr{O}_{\mathbb{C}^n} \to \mathscr{O}_{\mathbb{C}^n}^{\oplus n} \to \mathscr{O}_{\mathbb{C}^n}^{\oplus\binom{n}{2}} \to \cdots \to \mathscr{O}_{\mathbb{C}^n}^{\oplus n} \to \mathscr{O}_{\mathbb{C}^n}, \tag{23.3}$$

placed in degrees $0, 1, \dots, n$, and with a Koszul-type differential, induced by the n operators $f \mapsto T_j f$. We are interested in computing the cohomology sheaves of this complex.

Example 23.4. For $n = 1$, there are three cases. If $\alpha = 0$, the complex looks like

$$\mathscr{O}_{\mathbb{C}} \xrightarrow{\partial} \mathscr{O}_{\mathbb{C}}.$$

By the holomorphic Poincaré lemma (or by a direct computation with power series), this complex only has cohomology in degree 0, where we get the constant sheaf $\mathbb{C}$. If $\alpha = -1$, the complex looks like

$$\mathscr{O}_{\mathbb{C}} \xrightarrow{x} \mathscr{O}_{\mathbb{C}}.$$

It only has cohomology in degree 1, where we get a one-dimensional skyscraper sheaf at the origin. Lastly, if $\alpha \neq -1, 0$, the complex looks like

$$\mathscr{O}_{\mathbb{C}} \xrightarrow{x\partial - \alpha} \mathscr{O}_{\mathbb{C}}.$$

This only has cohomology in degree 0. Away from the origin, the multi-valued holomorphic function x^α solves the equation $(x\partial - \alpha)f = 0$, and so we get a locally constant sheaf on $\mathbb{C}^*$, with monodromy $e^{2\pi i\alpha}$. At the origin, the function x^α does not make sense, and in fact, the equation $(x\partial - \alpha)f = 0$ does not have a solution that is holomorphic in a neighborhood of the origin. So in this case, the 0-th cohomology sheaf of the complex is a so-called constructible sheaf: it is locally constant on $\mathbb{C}^*$, but with a different stalk at the origin. Note that in each case, exactly one cohomology sheaf is nonzero; and if the nonzero cohomology sheaf occurs in degree 0, it is supported on all of $\mathbb{C}$; if it occurs in degree 1, then it is supported at the origin.

By working with power series, one can show that the complex in (23.3) is (locally) quasi-isomorphic to a product; thus its cohomology is described by what happens for each of the n operators T_j individually. The conclusion is that (23.3) has exactly one nonzero cohomology sheaf, say in degree k (where k is the number of j such that $\alpha_j = -1$); moreover, that cohomology sheaf is supported on the linear subspace

$$\bigcap_{\alpha_j=-1} Z(x_j),$$

whose codimension is exactly k. It is also a constructible sheaf, meaning locally constant (of rank 0 or 1) on each stratum of the natural stratification on $\mathbb{C}^n$.

From this, we can deduce what happens for $\operatorname{Sol}(\mathcal{M})$ in general. Recall that $\mathcal{M}$ has a finite composition series whose subquotients $\mathcal{M}_1, \dots, \mathcal{M}_r$ are simple.

Example 23.5. Suppose that $\mathcal{M}$ has a composition series of length two:

$$0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$$

Since the solutions functor is contravariant, we obtain a long exact sequence

$$\mathcal{H}^{i-1}\operatorname{Sol}(\mathcal{M}_1) \to \mathcal{H}^{i}\operatorname{Sol}(\mathcal{M}_2) \to \mathcal{H}^{i}\operatorname{Sol}(\mathcal{M}) \to \mathcal{H}^{i}\operatorname{Sol}(\mathcal{M}_1) \to \mathcal{H}^{i+1}\operatorname{Sol}(\mathcal{M}_2)$$

Since $\operatorname{Sol}(\mathcal{M}_1)$ and $\operatorname{Sol}(\mathcal{M}_2)$ each have only a single nonzero cohomology sheaf, it follows that $\operatorname{Sol}(\mathcal{M})$ can have at most two nonzero cohomology sheaves, both constructible with respect to the natural stratification on $\mathbb{C}^n$. Moreover, $\dim \operatorname{Supp} \mathcal{H}^i \operatorname{Sol}(\mathcal{M}) \geq i$. The inequality can be strict, for example if $\mathcal{H}^i\operatorname{Sol}(\mathcal{M}_2) \neq 0$ and $\mathcal{H}^{i-1}\operatorname{Sol}(\mathcal{M}_1) \neq 0$; then $\mathcal{H}^i\operatorname{Sol}(\mathcal{M})$ is a quotient of the constructible sheaf $\mathcal{H}^i\operatorname{Sol}(\mathcal{M}_2)$, whose support is a linear subspace of codimension i. It follows that $\mathcal{H}^i\operatorname{Sol}(\mathcal{M})$ is still constructible, but its support may be smaller than than of $\mathcal{H}^i\operatorname{Sol}(\mathcal{M}_2)$.

In general, we have a spectral sequence

$$E_1^{p,q} = \mathcal{H}^{p+q}\operatorname{Sol}(\mathcal{M}_p) \Longrightarrow \mathcal{H}^{p+q}\operatorname{Sol}(\mathcal{M}).$$

Each $\operatorname{Sol}(\mathcal{M}_p)$ has exactly one nonzero cohomology sheaf, which is constructible for the natural stratification on $\mathbb{C}^n$; if $\mathcal{H}^j\operatorname{Sol}(\mathcal{M}_p) \neq 0$, then it is supported on a linear subspace of codimension j. Since kernels and cokernels of morphisms between constructible sheaves are again constructible, we see that all cohomology sheaves of $\operatorname{Sol}(\mathcal{M})$ are constructible; it also follows, as in the example, that

$$\operatorname{codim} \operatorname{Supp} \mathcal{H}^j \operatorname{Sol}(\mathcal{M}) \geq j.$$

Exercises.

Exercise 23.1. Suppose that we are given a family of k-vector spaces M_α, indexed by $\alpha \in A$, and a family of linear mappings $D_{i,j} \colon M_\alpha \to M_{\alpha+e_i-e_j}$.

(1) Show that if the relations in (23.1) hold, and $D_{0,0} + D_{1,1} + \cdots + D_{n,n} = 0$, then the direct sum
$$M = \bigoplus_{\alpha \in A} M_\alpha$$
becomes a left module over $R = \Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$.

(2) Suppose that M is finitely generated as an R-module, and that each operator $D_{j,j} - \alpha_j$ acts nilpotently on M_α. Show that the characteristic variety of $\mathcal{M} = \mathscr{D}_{\mathbb{P}^n} \otimes_R M$ is contained in the set $Z(x_0\xi_0, x_1\xi_1, \dots, x_n\xi_n)$.

(3) Show that $\mathcal{M}$ is a regular holonomic $\mathscr{D}_{\mathbb{P}^n}$-module of normal crossing type.

Exercise 23.2. Find the decomposition of $\Gamma(\mathbb{P}^n, \mathcal{M})$ in the following cases:

(1) $\mathcal{M} = \mathscr{O}_{\mathbb{P}^n}$
(2) $\mathcal{M} = j_+\mathscr{O}_U$, where $U = \mathbb{P}^n \setminus Z(x_0 x_1 \cdots x_n)$
(3) $\mathcal{M} = i_+\mathscr{O}_{\mathbb{P}^{n-1}}$, where $\mathbb{P}^{n-1} = Z(x_0)$.

Exercise 23.3. Let $\mathcal{M}$ be a regular holonomic $\mathscr{D}_{\mathbb{P}^n}$-module of normal crossing type. Given the decomposition for $\Gamma(\mathbb{P}^n, \mathcal{M})$, determine the resulting decomposition for the holonomic dual of $\mathcal{M}$.