LECTURE 24: MAY 6

The Riemann-Hilbert correspondence. Last time, we showed that the solution complex of a regular holonomic $\mathscr{D}$-module of normal crossing type has several special properties: its cohomology sheaves are locally constant on the strata of the divisor, and the dimensions of their supports satisfy a collection of inequalities. This is a special case of the Riemann-Hilbert correspondence, which relates regular holonomic $\mathscr{D}$-modules and constructible sheaves.

Let us begin with a few basic definitions. Let X be a nonsingular algebraic variety over the complex numbers. A *stratification* is a decomposition

$$X = \bigsqcup_{\alpha \in A} X_\alpha$$

into locally closed algebraic subsets, called *strata*, such that each X_α is nonsingular, and such that the Zariski-closure of each X_α is a union of finitely many other strata. The same definition makes sense on complex manifolds, taking each X_α to be a locally closed complex submanifold.

Example 24.1. The divisor $x_1 \cdots x_n = 0$ induces a natural stratification on $\mathbb{A}^n$ with 2^n strata, indexed by subsets $I \subseteq \{1, \dots, n\}$. The stratum corresponding to the subset I consists of those points where $x_i = 0$ for every $i \in I$, and $x_i \neq 0$ for every $i \notin I$.

Example 24.2. We can stratify $\mathbb{A}^2$ according to the singularities of the nodal curve $C = Z(y^2 - x^2 - x^3)$, into a 2-dimensional stratum $\mathbb{A}^2 \setminus C$, a 1-dimensional stratum $C \setminus \{(0,0)\}$, and a 0-dimensional stratum $\{(0,0)\}$.

We also need the notion of constructibility for sheaves. It is necessary to work in the classical (or analytic) topology, because the Zariski topology is too coarse to allow for interesting locally constant sheaves. Given a nonsingular algebraic variety X, we denote by X^{an} the associated complex manifold, with the topology induced by the usual Euclidean topology on $\mathbb{C}^n$. Let F be a sheaf of finite-dimensional $\mathbb{C}$-vector spaces on X^{an}. This means that for every open subset $U \subseteq X^{an}$, the space of sections $\Gamma(U, F)$ is a finite-dimensional $\mathbb{C}$-vector space. We say that F is *constructible* if there is a stratification

$$X = \bigsqcup_{\alpha \in A} X_\alpha$$

such that the restriction of F to each stratum X_α^{an} is a locally constant sheaf. (Constructible sheaves on arbitrary complex manifolds are defined in a similar way.) Every locally constant sheaf is constructible, of course, but the point is that the usual functors on sheaves preserve constructibility. (Going from locally constant sheaves to constructible sheaves is very similar to going from locally free sheaves to coherent sheaves, in that sense.)

Example 24.3. If $f \colon X \to Y$ is a proper morphism between nonsingular algebraic varieties, and if $f^{an} \colon X^{an} \to Y^{an}$ denotes the resulting proper holomorphic mapping between complex manifolds, then the sheaves $R^i f_*^{an} \mathbb{C}_{X^{an}}$ are constructible. The reason is that one can find a stratification for Y, in such a way that the restriction of f to each stratum of Y is a topological fiber bundle.

Example 24.4. If $j \colon U \hookrightarrow X$ is an open embedding, and $j^{an} \colon U^{an} \hookrightarrow X^{an}$ denotes the resulting embedding of complex manifolds, then the sheaves $R^i j_*^{an} \mathbb{C}_{U^{an}}$ are constructible. This is easy to show in the case where $X \setminus U$ is a normal crossing divisor; the general case follows by using resolution of singularities and the result in the previous example.

More generally, one can show that the usual direct and inverse image functors on sheaves preserve constructibility: if $f\colon X \to Y$ in any morphism between nonsingular algebraic varieties, and F any constructible sheaf on X^{an}, then $R^i f_*^{an} F$ is a constructible sheaf on Y^{an}. Likewise, if G is any constructible sheaf on Y^{an}, then $(f^{an})^{-1}G$ is a constructible sheaf on X^{an}. One can say the same thing in the language of derived categories. Denote by $D_c^b(\mathbb{C}_{X^{an}})$ the derived category of (cohomologically) constructible sheaves; its objects are complexes of sheaves of $\mathbb{C}$-vector spaces on X^{an} whose cohomology sheaves are constructible (and zero in all but finitely many degrees). Then if $f\colon X \to Y$ is any morphism between nonsingular algebraic varieties, the usual derived pushforward of sheaves gives an exact functor

$$\mathbf{R}f_*^{an}\colon D_c^b(\mathbb{C}_{X^{an}}) \to D_c^b(\mathbb{C}_{Y^{an}}),$$

and the usual inverse image of sheaves gives an exact functor

$$(f^{an})^{-1}\colon D_c^b(\mathbb{C}_{Y^{an}}) \to D_c^b(\mathbb{C}_{X^{an}}).$$

We can now state the first general result about solution complexes of regular holonomic $\mathscr{D}$-modules. Let $\mathscr{D}_X$ be the usual sheaf of differential operators on X, and denote by $\mathscr{D}_{X^{an}}$ the sheaf of differential operators with holomorphic coefficients on the complex manifold X^{an}. Given a coherent $\mathscr{D}_X$-module $\mathcal{M}$, we denote by $\mathcal{M}^{an}$ the associated analytic $\mathscr{D}_{X^{an}}$-module; this can be constructed using local presentations of $\mathcal{M}$, for example. The following result was proved by Kashiwara in his thesis; it is usually called "Kashiwara's constructibility theorem".

Theorem 24.5. *Let X be a nonsingular algebraic variety, and $\mathcal{M}$ a holonomic left $\mathscr{D}_X$-module. Then the solution complex*

$$\operatorname{Sol}(\mathcal{M}) = \mathbf{R}\mathcal{H}om_{\mathscr{D}_{X^{an}}}\big(\mathcal{M}^{an}, \mathscr{O}_{X^{an}}\big)$$

is constructible, hence an object of $D_c^b(\mathbb{C}_{X^{an}})$.

In fact, Kashiwara proves this result for holonomic $\mathscr{D}$-modules on complex manifolds. One consequence is that one has an exact (contravariant) solutions functor

$$\operatorname{Sol}\colon D_h^b(\mathscr{D}_X) \to D_c^b(\mathbb{C}_{X^{an}})^{op}$$

that associates to every complex of $\mathscr{D}_X$-modules with holonomic cohomology a constructible complex of solutions. We saw a very special case of this result last time, namely solutions of regular holonomic $\mathscr{D}$-modules of normal crossing type.

The solution functor is contravariant, but there is also a covariant version of Kashiwara's theorem. Recall that the Spencer complex

$$\operatorname{Sp}(\mathscr{D}_X) = \Big[\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{T}_X \to \dots \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^2 \mathscr{T}_X \to \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{T}_X \to \mathscr{D}_X\Big]$$

is a resolution of $\mathscr{O}_X$ by locally free left $\mathscr{D}_X$-modules; likewise, $\operatorname{Sp}(\mathscr{D}_{X^{an}})$ is a resolution of $\mathscr{O}_{X^{an}}$ by locally free left $\mathscr{D}_{X^{an}}$-modules. Thus

$$\begin{aligned}\operatorname{Sol}(\mathcal{M}) &= \mathbf{R}\mathcal{H}om_{\mathscr{D}_{X^{an}}}\big(\mathcal{M}^{an}, \mathscr{O}_{X^{an}}\big)\\ &\cong \mathbf{R}\mathcal{H}om_{\mathscr{D}_{X^{an}}}\big(\mathcal{M}^{an}, \operatorname{Sp}(\mathscr{D}_{X^{an}})\big)\\ &\cong \mathbf{R}\mathcal{H}om_{\mathscr{D}_{X^{an}}}\big(\mathcal{M}^{an}, \mathscr{D}_{X^{an}}\big) \otimes_{\mathscr{D}_{X^{an}}} \operatorname{Sp}(\mathscr{D}_{X^{an}}).\end{aligned}$$

Now suppose that $\mathcal{M}$ is holonomic. Then the complex $\mathbf{R}\mathcal{H}om_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X)$ only has cohomology in degree n, and

$$\mathcal{E}xt^n_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X) = \mathcal{M}^*$$

is the holonomic dual (which is a holonomic right $\mathscr{D}_X$-module). Consequently,

$$\mathbf{R}\mathcal{H}om_{\mathscr{D}_{X^{an}}}\big(\mathcal{M}^{an}, \mathscr{D}_{X^{an}}\big) \cong \mathcal{M}^{*,an}[-n],$$

and after plugging this into the relation from above, we get

$$\mathrm{Sol}(\mathcal{M}) \cong \mathcal{M}^{*,an}[-n] \otimes_{\mathscr{D}_{X^{an}}} \mathrm{Sp}(\mathscr{D}_{X^{an}}) \cong \mathrm{Sp}(\mathcal{M}^{*,an})[-n]. \tag{24.6}$$

Under the conversion between right and left $\mathscr{D}$-modules, the Spencer complex of a right $\mathscr{D}$-module goes to the de Rham complex of a left $\mathscr{D}$-module. This leads to the following equivalent formulation of Kashiwara's constructibility theorem: If $\mathcal{M}$ is a holonomic left $\mathscr{D}_X$-module on a nonsingular algebraic variety X, then the de Rham complex

$$\mathrm{DR}(\mathcal{M}^{an}) = \Big[\mathcal{M}^{an} \to \Omega^1_{X^{an}} \otimes \mathcal{M}^{an} \to \cdots \to \Omega^n_{X^{an}} \otimes \mathcal{M}^{an}\Big],$$

placed in degrees $-n, \dots, 0$, is constructible. More generally, the de Rham functor

$$\mathrm{DR}\colon D^b_h(\mathscr{D}_X) \to D^b_c(\mathbb{C}_{X^{an}})$$

is an exact covariant functor.

Kashiwara's theorem makes no assumptions about regularity, but the price to pay is that many different $\mathscr{D}$-modules can have the same solution complex.

Example 24.7. Here is the simplest example of this phenomenon. On $\mathbb{A}^1$, consider the family of $\mathscr{D}_{\mathbb{A}^1}$-modules $\mathcal{M}_\lambda = \mathscr{D}_{\mathbb{A}^1}/\mathscr{D}_{\mathbb{A}^1}(\partial - \lambda)$, indexed by $\lambda \in \mathbb{C} \setminus \{0\}$. We have already seen that these $\mathscr{D}$-modules have an irregular singularity at infinity. The solution complex of $\mathcal{M}_\lambda$ is

$$\mathscr{O}_{\mathbb{C}} \xrightarrow{\partial-\lambda} \mathscr{O}_{\mathbb{C}}.$$

The kernel of $\partial - \lambda$ is clearly spanned by the function $e^{\lambda x}$, while the cokernel is trivial; this means that the solution complex is always isomorphic to the constant sheaf $\mathbb{C}$, independently of λ. On the other hand, $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ are not isomorphic as $\mathscr{D}_{\mathbb{A}^1}$-modules for $\lambda \neq \mu$.

If one imposes the condition of regularity, then this problem goes away, and the solutions functor (as well as the de Rham functor) becomes an equivalence of categories. This is the content of the famous Riemann-Hilbert correspondence.

Theorem 24.8. *Let X be a nonsingular algebraic variety. Then the functors*

$$\mathrm{Sol}\colon D^b_h(\mathscr{D}_X) \to D^b_c(\mathbb{C}_{X^{an}})^{op}$$
$$\mathrm{DR}\colon D^b_h(\mathscr{D}_X) \to D^b_c(\mathbb{C}_{X^{an}})$$

are equivalences of categories.

This result again holds more generally on complex manifolds. There are three proofs: an analytic proof by Kashiwara; a more algebraic proof by Mebkhout; and a completely algebraic proof by Bernstein (which only works on algebraic varieties). The Riemann-Hilbert correspondence also respects the various functors on both sides: for example,

$$\mathrm{DR} \circ f_+ \cong \mathbf{R}f_* \circ \mathrm{DR} \quad \text{and} \quad \mathrm{DR} \circ \mathbf{L}j^* \cong j^{-1} \circ \mathrm{DR}.$$

These isomorphisms do not hold without the assumption of regularity. The Riemann-Hilbert correspondence therefore establishes a direct link between algebraic objects (regular holonomic $\mathscr{D}$-modules) and topological objects (constructible sheaves).

Example 24.9. The holonomic dual also has a natural interpretation in terms of the Riemann-Hilbert correspondence. On $D^b_c(\mathbb{C}_{X^{an}})$, one has Verdier's duality functor

$$\mathbb{D}_{X^{an}}\colon D^b_c(\mathbb{C}_{X^{an}}) \to D^b_c(\mathbb{C}_{X^{an}})^{op}, \quad F \mapsto \mathbf{R}\mathcal{H}om_{\mathbb{C}_{X^{an}}}\big(F, \mathbb{C}_{X^{an}}[2n]\big),$$

where $n = \dim X$. One can show that, for any holonomic $\mathscr{D}_X$-module $\mathcal{M}$, one has an isomorphism

$$\mathbb{D}_{X^{an}}\big(\mathrm{DR}(\mathcal{M}^{an})\big) \cong \mathrm{Sp}(\mathcal{M}^{*,an})$$

which means that the Riemann-Hilbert correspondence turns holonomic duality into Verdier duality.

Perverse sheaves. The Riemann-Hilbert correspondence works on the level of the derived category. Where do regular holonomic $\mathscr{D}$-modules go under the equivalence of categories? We saw last time that the solution complex of a regular holonomic $\mathscr{D}$-module of normal crossing type satisfies a collection of inequalities: the j-th cohomology sheaf of $\mathrm{Sol}(\mathcal{M})$ is supported on a union of strata of codimension at least j. Kashiwara proved that this is true for arbitrary holonomic $\mathscr{D}$-modules: if $\mathcal{M}$ is a holonomic $\mathscr{D}$-module on a nonsingular algebraic variety (or, more generally, on a complex manifold), then

$$\operatorname{codim} \operatorname{Supp} R^j \operatorname{Sol}(\mathcal{M}) \geq j$$

for every $j \in \mathbb{Z}$. Using the identity in (24.6), an equivalent formulation is that

$$\dim \operatorname{Supp} \mathcal{H}^j \operatorname{DR}(\mathcal{M}^{an}) \leq -j$$

for every $j \in \mathbb{Z}$. One gets a similar collection of inequalities also for the Verdier dual $\mathbb{D}_{X^{an}} \operatorname{DR}(\mathcal{M}^{an})$, because of the identity in Example 24.9. This motivates the following definition.

Definition 24.10. A complex $F \in D_c^b(\mathbb{C}_{X^{an}})$ is called a *perverse sheaf* if

$$\dim \operatorname{Supp} \mathcal{H}^j F \leq -j \quad \text{and} \quad \dim \operatorname{Supp} \mathcal{H}^j \mathbb{D}_{X^{an}}(F) \leq -j$$

for every $j \in \mathbb{Z}$.

Example 24.11. If $\mathcal{M}$ is a holonomic $\mathscr{D}_X$-module, then $\operatorname{DR}(\mathcal{M}^{an})$ is a perverse sheaf. This is simply a rewording of Kashiwara's theorem. Note that regularity is not needed here.

The definition (and the somewhat strange name) of perverse sheaves is due to Beilinson, Bernstein, Deligne, and Gabber. They showed that the collection of perverse sheaves forms an abelian category contained in $D_c^b(\mathbb{C}_{X^{an}})$. The collection of inequalities in the definition had actually appeared in two completely independent places: once in Kashiwara's study of holonomic $\mathscr{D}$-modules, and then again in the intersection homology theory of Goresky and Macpherson. This circumstance is of course explained by the Riemann-Hilbert correspondence. In fact, once Theorem 24.8 is known, purely formal reasoning implies that the de Rham functor

$$\operatorname{DR}\colon D_h^b(\mathscr{D}_X) \to D_c^b(\mathbb{C}_{X^{an}})$$

takes the abelian category of regular holonomic $\mathscr{D}_X$-modules isomorphically to the abelian category of perverse sheaves. Unfortunately, I cannot offer you any good explanation of what perverse sheaves really are, other than saying that they are the image of the regular holonomic $\mathscr{D}$-modules under the Riemann-Hilbert correspondence. From this point of view, the crucial result is the equivalence between the two derived categories; the collection of inequalities is just what one gets when one goes from one side to the other.

Exercises.

Exercise 24.1. Show that if $\lambda \neq \mu$, then $\mathcal{M}_\lambda = \mathscr{D}_{\mathbb{A}^1}/\mathscr{D}_{\mathbb{A}^1}(\partial - \lambda)$ is not isomorphic to $\mathcal{M}_\mu$ as a $\mathscr{D}_{\mathbb{A}^1}$-module.