## Lecture 25: May 8

**Meromorphic connections.** Before the full Riemann-Hilbert correspondence was proved, Deligne established an important special case. It has to do with the relationship between locally constant sheaves and vector bundles with integrable connection. Suppose that $X$ is a nonsingular and proper algebraic variety over the complex numbers. If we are given a vector bundle of rank $r$ with integrable connection, then the subsheaf of flat sections is a locally constant sheaf of rank $m$ (with respect to the analytic topology). Conversely, given a locally constant sheaf of rank $m$, say $E$, we can form the holomorphic vector bundle $\mathscr{E} = \mathscr{O}_X \otimes_{\mathbb{C}} E$, which has the same (locally constant) transition functions as $E$. The formula

$$\nabla(f \otimes s) = df \otimes s$$

defines an integrable connection on $\mathscr{E}$, and the subsheaf of $\nabla$-flat sections is of course isomorphic to $E$. Lastly, $X$ is proper, and so the pair $(\mathscr{E}, \nabla)$ actually comes from an *algebraic* vector bundle with intgrable connection (by a version of Serre's GAGA theorem). The conclusion is that the (a priori topological) object $E$ is actually algebraic.

Deligne's version of the Riemann-Hilbert correspondence generalizes this to not necessarily proper varieties. It goes through an intermediate class of objects, called meromorphic connections. Here is the definition. Let $X$ be a complex manifold, and $D \subseteq X$ a divisor. For simplicity, we are only going to consider the case where $D$ has simple normal crossing singularities: $D$ is a union of nonsingular hypersurfaces meeting transversely. In suitable local coordinates $x_1, \dots, x_n$, the equation defining $D$ is of the form $x_1 \cdots x_r = 0$. We let

$$\mathscr{O}_X(*D)$$

be the sheaf of meromorphic functions on $X$ that are holomorphic on $X \setminus D$; it is naturally a subsheaf of $j_* \mathscr{O}_{X \setminus D}$, where $j \colon X \setminus D \hookrightarrow X$ is the inclusion of the complement. The notation $*D$ is supposed to remind you of the pole order along $D$. Locally, $\mathscr{O}_X(*D)$ is isomorphic to $\mathscr{O}_X[t]/(ht-1)$, where $h$ is a local equation for $D$; it follows that $\mathscr{O}_X(*D)$ is still a coherent sheaf of $\mathscr{O}_X$-algebras.

**Definition 25.1.** A *meromorphic connection* is a coherent $\mathscr{O}_X(*D)$-module $M$, together with an integrable connection

$$\nabla \colon M \to \Omega_X^1 \otimes_{\mathscr{O}_X} M$$

that satisfies the Leibniz rule $\nabla(fs) = df \otimes s + f\nabla s$ and the integrability condition $[\nabla_\theta, \nabla_{\theta'}] = \nabla_{[\theta,\theta']}$.

*Note.* In the Leibniz rule, we are considering only $f \in \mathscr{O}_X$, but the same formula works for every $f \in \mathscr{O}_X(*D)$. To make this precise, define $\Omega_X^1(*D)$ as the sheaf of meromorphic one-forms on $X$ that are holomorphic on $X \setminus D$, so that

$$\Omega_X^1(*D) = \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{O}_X(*D).$$

We can then consider $\nabla$ as a $\mathbb{C}$-linear morphism

$$\nabla \colon M \to \Omega_X^1(*D) \otimes_{\mathscr{O}_X(*D)} M,$$

and now the Leibniz rule makes sense for $f \in \mathscr{O}_X(*D)$.

A meromorphic connection is naturally a left $\mathscr{D}_X$-module, since the two identities imply that the left action by $\mathscr{T}_X$ extends to a left action by $\mathscr{D}_X$ (see the discussion in Lecture 10). On $X \setminus D$, the $\mathscr{D}$-module is coherent, and therefore a holomorphic vector bundle with integrable connection. In that sense, a meromorphic connection is an extension of a vector bundle with integrable connection on $X \setminus D$ to an object on $X$ with singularities along $D$.

**Definition 25.2.** If $(M, \nabla)$ and $(N, \nabla)$ are two meromorphic connections, then a *morphism* $\varphi\colon (M, \nabla) \to (N, \nabla)$ is a morphism of $\mathscr{O}_X(*D)$-module $\varphi\colon M \to N$ that is compatible with the connections, in the sense that

$$\nabla\bigl(\varphi(s)\bigr) = (\mathrm{id} \otimes \varphi)(\nabla s).$$

We denote by $\mathrm{Conn}(X, D)$ the category of meromorphic connections on $(X, D)$. It is an abelian category. There are two simple but useful observations about morphisms in $\mathrm{Conn}(X, D)$. The first says that morphisms are determined by what their restriction to $X \setminus D$.

**Proposition 25.3.** *Let $\varphi\colon (M, \nabla) \to (N, \nabla)$ be a morphism of meromorphic connections. If $\varphi\big|_{X \setminus D}$ is an isomorphism, then $\varphi$ is an isomorphism.*

*Proof.* The kernel and cokernel of $\varphi$ are meromorphic connections whose support is, by construction, contained inside $D$. It is therefore enough to prove that a meromorphic connection $(M, \nabla)$ such that $\operatorname{Supp} M \subseteq D$ must be trivial. Let $s$ be any local section of $M$, and $h$ a local equation for $D$. The subsheaf $\mathscr{O}_X \cdot s \subseteq M$ is coherent over $\mathscr{O}_X$, and its support is contained inside $D$, and so $h^m s = 0$ for $m \gg 0$. But then $s = h^{-m}(h^m s) = 0$, proving that $M = 0$. □

The second observation is useful for functoriality questions. Suppose that $(M, \nabla)$ and $(N, \nabla)$ are two meromorphic connections. Then

$$\mathcal{H}om_{\mathscr{O}_X(*D)}(M, N)$$

is again an $\mathscr{O}_X(*D)$-module in a natural way, and the formula

$$(\nabla \varphi)(s) = (\mathrm{id} \otimes \varphi)(\nabla s) - \nabla\bigl(\varphi(s)\bigr)$$

defines an integrable connection that makes $\mathcal{H}om_{\mathscr{O}_X(*D)}(M, N)$ into a meromorphic connection. You should check that morphisms of meromorphic connections $\varphi\colon (M, \nabla) \to (N, \nabla)$ are exactly the same thing as $\nabla$-flat global sections of $\mathcal{H}om_{\mathscr{O}_X(*D)}(M, N)$.

**Deligne's theorem on meromorphic connections.** Deligne proved that locally constant sheaves on $X \setminus D$ correspond to meromorphic connections on $(X, D)$ that are *regular* along $D$. Regularity was originally defined by restricting to curves, but in the case where $D$ is a normal crossing divisor, we can use another definition that is closer to the Kashiwara-Kawai notion of regularity for $\mathscr{D}$-modules.

**Definition 25.4.** A meromorphic connection $(M, \nabla)$ is called *regular* if there is a locally free $\mathscr{O}_X$-module $L$ with

$$M \cong \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} L,$$

such that in any local trivialization of $L$, the connection has at worst logarithmic poles along $D$.

More precisely, suppose that $e_1, \dots, e_m$ form a local trivialization for $L$. Then the condition is that

$$\nabla e_i = \sum_{j,k} a_{i,j}^k \frac{dx_k}{x_k} \otimes e_j,$$

for certain holomorphic functions $a_{i,j}^k$. Since $L$ is then preserved by the differential operators $x_1 \partial_1, \dots, x_n \partial_n$, this means that $M$, viewed as a left $\mathscr{D}_X$-module, is regular in the sense of Kashiwara and Kawai. The letter L comes from the fact that $L$ is traditionally called a *lattice*.

Keeping the notation from above, we let $A^k \in \mathrm{Mat}_{m \times m}(\mathscr{O}_X)$ be the matrix with entries $a_{i,j}^k$. The restriction of $A^k$ to the divisor $D_k$, defined by the equation $x_k = 0$, is a well-defined endomorphism of the locally free sheaf $L\big|_{D_k}$, called the *residue* of $\nabla$ along $D_k$. We use the symbol

$$\mathrm{Res}^L_{D_k}(\nabla) = A^k\big|_{D_k}$$

to denote the residue. We may drop the superscript $L$ when the lattice is clear from the context.

**Lemma 25.5.** *Let $(M,\nabla)$ be a meromorphic connection with lattice $L$.*

*(a) On $D_k \cap D_\ell$, the residues $\mathrm{Res}_{D_k}(\nabla)$ and $\mathrm{Res}_{D_\ell}(\nabla)$ commute.*

*(b) The eigenvalues of $\mathrm{Res}_{D_\ell}(\nabla)$ are locally constant along $D_\ell$.*

*Proof.* In the notation from above, we have

$$\nabla e_i = \sum_{j,k} a^k_{i,j} \frac{dx_k}{x_k} \otimes e_j,$$

and $A^k$ is the $m \times m$-matrix with entries $a^k_{i,j}$. With respect to the trivialization $e_1, \dots, e_m$, we therefore have $\nabla_{\partial_k} = A^k/x_k$. The integrability condition for the connection is $[\nabla_{\partial_k}, \nabla_{\partial_\ell}] = 0$, which expands out to

$$\frac{\partial}{\partial x_k}\left(\frac{A^\ell}{x_\ell}\right) + \frac{A^\ell}{x_\ell}\frac{A^k}{x_k} = \frac{\partial}{\partial x_\ell}\left(\frac{A^k}{x_k}\right) + \frac{A^k}{x_k}\frac{A^\ell}{x_\ell}.$$

After rearranging the terms, this becomes

$$x_k \partial_k(A^\ell) + A^\ell A^k = x_\ell \partial_\ell(A^k) + A^k A^\ell,$$

and so the restriction of the two matrices $A^k$ and $A^\ell$ to the set $x_k = x_\ell = 0$ commute with each other.

For the proof of the second assertion, denote by $\bar{L}$ the restriction of $L$ to the divisor $D_\ell$; similarly, $\bar{A}^k$ is the restriction of $A^k$, and so on. The formula

$$\nabla \bar{e}_i = \sum_{j,k\neq\ell} \bar{a}^k_{i,j} \frac{dx_k}{x_k} \otimes \bar{e}_j$$

defines an integrable connection with logarithmic poles on $\bar{L}$, and one checks that $\bar{A}^\ell$ is a horizontal section of $\mathcal{H}om_{\mathscr{O}_{D_\ell}}(\bar{L}, \bar{L})$. It follows that the eigenvalues of $\bar{A}^\ell$ must be locally constant. □

Deligne's main theorem is that every bundle with integrable connection on $U$ can be uniquely extended to a regular meromorphic connection on $(X, D)$; in fact, even the lattice is more or less unique, except for a small ambiguity in the eigenvalues of the residues.

**Theorem 25.6.** *Let $X$ be a complex manifold, and $D \subseteq X$ a divisor with simple normal crossing singularities. Set $U = X \setminus D$, and fix a section $\tau\colon \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. Given $(M,\nabla) \in \mathrm{Conn}(U)$, there is a unique locally free sheaf $L_\tau$ on $X$ with the following three properties:*

*(a) One has $L_\tau\big|_U = M$.*

*(b) The connection $\nabla\colon M \to \Omega^1_U \otimes_{\mathscr{O}_U} M$ extends to*

$$\nabla\colon M_\tau \to \Omega^1_X \otimes_{\mathscr{O}_X} M_\tau,$$

*where $M_\tau = \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} L_\tau$.*

*(c) At each irreducible component of $D$, the residue of $\nabla$ has eigenvalues in the set $\tau(\mathbb{C}/\mathbb{Z}) \subseteq \mathbb{C}$.*

*Moreover, with the above choice of $L_\tau$, the restriction mapping*

$$\Gamma(X, M_\tau)^\nabla \to \Gamma(U, M)^\nabla$$

*from $\nabla$-flat sections of $M_\tau$ to $\nabla$-flat sections of $M$ is an isomorphism.*

**Proof of Deligne's theorem.** The proof of Deligne's theorem has two parts. The first part is to prove that $L_\tau$ is unique (up to isomorphism). The second part is to construct a suitable lattice $L_\tau$ locally on $X$; the local objects can then be glued together into a global lattice using uniqueness.

Let us start with the local existence, since that is easier. Since we are working locally, we can assume that $X = \Delta^n$, where $\Delta \subseteq \mathbb{C}$ is an open disk containing the origin. The divisor $D$ will be given by the equation $x_1 \cdots x_r = 0$, and so $U = (\Delta^*)^r \times \Delta^{n-r}$. By the correspondence between vector bundles with integrable connection and locally constant sheaves, $(M, \nabla) \in \operatorname{Conn}(U)$ corresponds to a locally constant sheaf on $U$, hence to a representation $\pi_1(U) \to \mathrm{GL}_m(\mathbb{C})$, where $m$ is the rank of $M$. Since the fundamental group of $U$ is abelian, this is equivalent to giving $r$ commuting matrices $C^1, \dots, C^r \in \mathrm{GL}_m(\mathbb{C})$. (These are the monodromy matrices of the locally constant sheaf.)

It is a simple exercise to show that there are matrices $\Gamma_1, \dots, \Gamma_r \in \mathrm{Mat}_{m\times m}(\mathbb{C})$, uniquely determined by the following three conditions:

(1) $e^{2\pi i\, \Gamma^j} = C^j$,
(2) the eigenvalues of $\Gamma^j$ lie in the set $\tau(\mathbb{C}/\mathbb{Z})$,
(3) $\Gamma^1, \dots, \Gamma^r$ commute.

We can now define $L_\tau = \mathscr{O}_X^{\oplus m}$, and put a meromorphic connection on the free $\mathscr{O}_X(*D)$-module $M_\tau = \mathscr{O}_X(*D)^{\oplus m}$ by the formula

$$\nabla e_i = \sum_{j,k} \Gamma^k_{i,j} \frac{dx_k}{x_k} \otimes e_j.$$

From the construction, it is clear that this has the three properties in the statement of the theorem. What about flat sections? A $\nabla$-flat section of $M$ is the same thing as a monodromy invariant vector $v \in \mathbb{C}^m$, meaning one with $C^1 v = \cdots = C^r v = v$. This is equivalent to $\Gamma^1 v = \cdots = \Gamma^r v = 0$, and so $v$ also represents a $\nabla$-flat section of $M_\tau$.

The more demanding part of the proof is the uniqueness of $L_\tau$. You will see that the argument is very similar to what we did for the theorem of Fuchs (in Lecture 20). The problem is local, and so we continue to assume that $X = \Delta^n$, with coordinates $x_1, \dots, x_n$, and $D$ defined by $x_1 \cdots x_r = 0$. Suppose that $L$ and $L'$ are two lattices that both have the three properties stated in the theorem. Denote by $\nabla$ and $\nabla'$ the logarithmic connections on $L$ and $L'$. With respect to a trivialization $e_1, \dots, e_m$ for $L$, we can write

$$\nabla e_i = \sum_{j,k} a^k_{i,j} \frac{dx_k}{x_k} \otimes e_j,$$

where $a^k_{i,j}$ are holomorphic functions on $X$; we set

$$\omega = \sum_k A^k \frac{dx_k}{x_k},$$

which is an $m \times m$-matrix of logarithmic one-forms. We use primes to denote the corresponding objects for $(L', \nabla')$.

By assumption, $(L, \nabla)\big|_U \cong (L', \nabla')\big|_U$. After a short calculation, the isomorphism between the two bundles with connection translates into the existence of an invertible matrix $S \in \mathrm{GL}_m(\mathscr{O}_U)$ such that

$$dS = S\omega - \omega' S.$$

The entries of $S$ are holomorphic functions on $U = X \setminus D$, possibly with essential singularities along $D$. To prove the uniqueness statement, it is enough to show that $S \in \mathrm{GL}_m(\mathscr{O}_X)$, meaning that the entries of $S$ should extend to holomorphic functions on $X$. By Hartog's theorem, holomorphic functions extend over subsets of codimension $\geq 2$, and so we only need to to prove that the entries of $S$ extend over the generic point of each irreducible component of $D$. To keep the notation simple, we will check this at points of

$$D_1 \setminus \bigcup_{k\neq 1} D_k,$$

meaning at points where $x_1 = 0$ but $x_2 \cdots x_r \neq 0$. Write

$$\omega = A^1 \frac{dx_1}{x_1} + \sum_{k\geq 2} A^k \frac{dx_k}{x_k}$$
$$\omega' = A'^1 \frac{dx_1}{x_1} + \sum_{k\geq 2} A'^k \frac{dx_k}{x_k}$$

The relation $dS = S\omega - \omega' S$ gives

$$x_1 \frac{\partial S}{\partial x_1} = SA^1 - A'^1 S, \tag{25.7}$$

and after taking the matrix norm of both sides, we obtain

$$|x_1| \cdot \left\| \frac{\partial S}{\partial x_1} \right\| \leq C \cdot \|S\|,$$

where $C > 0$ is a constant that depends on the size of the (holomorphic) entries of the two matrices $A^1$ and $A'^1$. As in Lecture 20, we can now apply Grönwall's inequality to deduce that the entries of $S$ have moderate growth near $x_1$, hence are meromorphic functions on the set where $x_2 \cdots x_r \neq 0$.

It remains to show that the entries of $S$ are actually holomorphic functions for $x_2 \cdots x_r \neq 0$. Consider the Laurent expansion

$$S = \sum_{j=p}^{\infty} S_j x_1^j,$$

where $S_p \neq 0$ is the leading term. After substituting this into (25.7), we get

$$\sum_{j=p}^{\infty} j S_j x_1^j = \sum_{j=p}^{\infty} \bigl(S_j A^1 - A'^1 S_j\bigr) x_1^j.$$

The coefficients at $x_1^p$ equate to

$$pS_p = S_p \cdot A^1\big|_{x_1=0} - A'^1\big|_{x_1=0} \cdot S_p = S_p \cdot \operatorname{Res}_{D_1}^L(\nabla) - \operatorname{Res}_{D_1}^{L'}(\nabla') \cdot S_p.$$

Since both $\operatorname{Res}_{D_1}^L(\nabla)$ and $\operatorname{Res}_{D_1}^{L'}(\nabla')$ have their eigenvalues contained in the set $\tau(\mathbb{C}/\mathbb{Z})$, this relation forces $p = 0$. Indeed, suppose that $v$ is a nontrivial eigenvector for $\operatorname{Res}_{D_1}^L(\nabla)$, with eigenvalue $\lambda$. Then

$$p(S_p v) = \lambda(S_p v) - \operatorname{Res}_{D_1}^{L'}(\nabla')(S_p v),$$

and so $S_p v$ is an eigenvector for $\operatorname{Res}_{D_1}^{L'}(\nabla')$, with eigenvalue $\lambda - p$. (Since $S$ is invertible, we must have $S_p v \neq 0$). As the difference of the two eigenvalues is an integer, this can only happen for $p = 0$. The conclusion is that $S$ extends holomorphically to all of $X$, proving the desired uniqueness.

**Deligne's Riemann-Hilbert correspondence.** We are now ready for Deligne's version of the Riemann-Hilbert correspondence. Let $\mathrm{Loc}(X \setminus D)$ denote the category of locally constant sheaves (of finite-dimensional $\mathbb{C}$-vector spaces) on $X \setminus D$.

**Theorem 25.8.** *Let $X$ be a complex manifold, and $D \subseteq X$ a divisor with simple normal crossing singularities. Then the restriction functor*

$$\mathrm{Conn}(X, D)^{\mathrm{reg}} \to \mathrm{Loc}(X \setminus D)$$

*is an equivalence of categories.*

Here we associate to a meromorphic connection $(M, \nabla) \in \mathrm{Conn}(X, D)$ the locally constant sheaf of $\nabla$-flat sections of $M\big|_U$, where $U = X \setminus D$. The proof is very easy at this point. First, every locally constant sheaf on $X \setminus D$ is the sheaf of $\nabla$-flat sections of some $(M, \nabla) \in \mathrm{Conn}(U)$. By Theorem 25.6, there is an extension of $(M, \nabla)$ to a regular meromorphic connection on $(X, D)$: for any choice of $\tau$, the pair $(M_\tau, \nabla)$ will do. This shows that the restriction functor is essentially surjective.

It remains to prove that it is also fully faithful. The functor of $\nabla$-flat sections gives an equivalence of categories between $\mathrm{Conn}(U)$ and $\mathrm{Loc}(U)$, and so it suffices to prove that $\mathrm{Conn}(X, D)^{\mathrm{reg}} \to \mathrm{Conn}(U)$ is fully faithful. Let $(M, \nabla)$ and $(N, \nabla)$ be meromorphic connections, and set $H = \mathcal{H}om_{\mathscr{O}_X(*D)}(M, N)$; recall that $(H, \nabla)$ is again a meromorphic connection. As we saw earlier, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Conn}(X,D)}\Big((M, \nabla), (N, \nabla)\Big) \cong \Gamma(X, H)^{\nabla}$$

between the set of morphisms in the category $\mathrm{Conn}(X, D)$ and the set of $\nabla$-flat sections of $H$. Similarly,

$$\mathrm{Hom}_{\mathrm{Conn}(U)}\Big((M, \nabla)\big|_U, (N, \nabla)\big|_U\Big) \cong \Gamma(U, H)^{\nabla},$$

and so the problem reduces to showing that

$$\Gamma(X, H)^{\nabla} \to \Gamma(U, H)^{\nabla}$$

is an isomorphism.

**Lemma 25.9.** *Let $(M, \nabla) \in \mathrm{Conn}(X, D)$ be a regular meromorphic connection. Then the restriction morphism*

$$\Gamma(X, M)^{\nabla} \to \Gamma(U, M)^{\nabla}$$

*is an isomorphism, where $U = X \setminus D$.*

*Proof.* Since $(M, \nabla)$ is regular, there is a lattice $L$ with $M \cong \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} L$, such that $\nabla$ has logarithmic poles. Pick any section $\tau\colon \mathbb{C}/\mathbb{Z} \to \mathbb{C}$, for example with $\mathrm{Re}\,\tau \in [0, 1)$. By Theorem 25.6, there exists $L_\tau$ with $(L, \nabla)\big|_U \cong (L_\tau, \nabla)\big|_U$. Arguing as in the proof of Theorem 25.6, we find that the isomorphism is locally given by a matrix with meromorphic entries, and hence that $(M, \nabla)$ is isomorphic to $(M_\tau, \nabla)$ as a meromorphic connection. Now the assertion about flat sections follows from the last sentence of Theorem 25.6. □

Deligne's Riemann-Hilbert correspondence again leads to an interesting algebraicity result. Suppose that $X$ is a nonsingular proper variety. Then every locally constant sheaf on $X \setminus D$ comes from a meromorphic connection on $(X, D)$, and hence (by a version of Serre's GAGA theorem) from an algebraic object. Since we have resolution of singularities, we can write every nonsingular algebraic variety in the form $X \setminus D$. Thus every locally constant sheaf on a nonsingular algebraic variety comes from an algebraic vector bundle with integrable connection.

**Exercises.**

*Exercise* 25.1. Let $(M,\nabla)$ and $(N,\nabla)$ be meromorphic connections. Check that $\big(\mathcal{H}om_{\mathscr{O}_X(*D)}(M,N),\nabla\big)$ is a meromorphic connection, and that $\varphi\colon (M,\nabla)\to(N,\nabla)$ is a morphism of meromorphic connections if and only if, when viewed as a global section of $\mathcal{H}om_{\mathscr{O}_X(*D)}(M,N)$, it satisfies $\nabla\varphi=0$.

*Exercise* 25.2. Let $C\in \mathrm{GL}_m(\mathbb{C})$. Show that there is a unique $\Gamma\in\mathrm{Mat}_{m\times m}(\mathbb{C})$ such that $e^{2\pi i\,\Gamma}=C$ and such that the eigenvalues of $\Gamma$ lie in the set $\tau(\mathbb{C}/\mathbb{Z})$.