Lecture 26: May 10

One-forms on varieties of general type. In the final two lectures, I am going to show you an application of $\mathscr{D}$-module theory to a problem in algebraic geometry. It has to do with holomorphic one-forms and their zero loci. Recall that on a smooth projective curve of genus $g \geq 1$, every holomorphic one-form has exactly $2g-2$ zeros, counted with multiplicity. The situation for surfaces is less clear, but one can still show that every holomorphic one-form on a surface of general type must have a non-empty zero locus. (We'll see a proof of this fact in a second.) This lead Christopher Hacon and Sándor Kovács (and, independently, Tie Luo and Qi Zhang) to conjecture that the same result should hold on any variety of general type; they also proved their conjecture for threefolds. A few years ago, Mihnea Popa and I used $\mathscr{D}$-modules to prove the conjecture in general. The proof I am going to present is a simplified version of our original argument that Chuanhao Wei and I found sometime afterwards.

Theorem 26.1. *Let X be a smooth projective variety over the complex numbers. If X is of general type, then every holomorphic one-form on X has a non-empty zero locus.*

To be precise, for any $\omega \in H^0(X, \Omega_X^1)$, we define the zero locus to be

$$Z(\omega) = \left\{ x \in X \mid \omega(T_x X) = 0 \right\}.$$

Then the theorem is claiming that if X is of general type, in the sense that $\dim H^0(X, \omega_X^m)$ grows like a constant times $m^{\dim X}$, then necessarily $Z(\omega) \neq \emptyset$ for every $\omega \in H^0(X, \Omega_X^1)$. Another motivation for thinking that this might be true is that one-forms are dual to vector fields, and zero loci of vector fields are of course related to the topology of X. (For example, if X admits an everywhere nonzero vector field, then its Euler characteristic must be zero.)

Example 26.2. Let us consider the case of surfaces. Suppose that X is a smooth projective surface of general type. Suppose that there was a holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ with empty zero locus. We will use some of the many numerical identities for surfaces to produce a contradiction.

First, we observe that X must be minimal. Otherwise, X would be the blowup of a smooth projective surface Y at some point, and since $H^0(X, \Omega_X^1) \cong H^0(Y, \Omega_Y^1)$, the one-form ω would be the pullback of a one-form from Y. But then ω has to vanish at some point of the exceptional divisor, contradiction. Now the fact that X is a of general type means that $c_1(X)^2 \geq 1$; together with the Bogomolov-Miyaoka-Yau inequality, we get

$$3c_2(X) \geq c_1(X)^2 \geq 1.$$

But $c_2(X)$ is the topological Euler characteristic of X, and so $e(X) \neq 0$.

Now the contradiction comes from the fact that a surface with a nowhere vanishing holomorphic one-form must have $e(X) = 0$. To see this, consider the complex

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\omega} \Omega_X^1 \xrightarrow{\omega} \Omega_X^2 \longrightarrow 0$$

where the differential is wedge product with ω. This is a Koszul complex, and since $Z(\omega) = \emptyset$, the complex is exact, and so its hypercohomology is trivial. The hypercohomology spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p)$$

therefore converges to zero. This gives

$$e(X) = \sum_{p,q} (-1)^{p+q} \dim H^q(X, \Omega_X^p) = \sum_{p,q} (-1)^{p+q} \dim E_1^{p,q}$$
$$= \sum_{p,q} (-1)^{p+q} \dim E_\infty^{p,q} = 0,$$

since the alternating sum of the dimensions is preserved under taking cohomology.

Let us make a few general observations about Theorem 26.1. The condition that X is of general type can be restated as follows: for any ample line bundle L on X, there is some $m \geq 1$ such that $\omega_X^m \otimes L^{-1}$ has a section.

Example 26.3. In the special case $m = 1$, we can use the Nakano vanishing theorem to give a simple proof of Theorem 26.1. Suppose that $H^0(X, \omega_X \otimes L^{-1}) \neq 0$, and that there is a holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ with $Z(\omega) = \emptyset$. Let $n = \dim X$. As before, the complex

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\omega} \Omega_X^1 \xrightarrow{\omega} \cdots \xrightarrow{\omega} \Omega_X^n \longrightarrow 0$$

is exact, and so the hypercohomology spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p \otimes L^{-1})$$

converges to zero. Since L is ample, the Nakano vanishing theorem tells us that $E_1^{p,q} = 0$ for $p + q < n$. In particular, all the differentials going into the term in the position $(n, 0)$ vanish. But then

$$E_\infty^{n,0} = E_1^{n,0} = H^0(X, \omega_X \otimes L^{-1}) \neq 0,$$

which is a contradiction. Unfortunately, this simple argument totally breaks down once $m \geq 2$. But we will see that it is still basically a vanishing theorem that is responsible for Theorem 26.1.

Another observation is that holomorphic one-forms are closely related to abelian varieties. Indeed, we always have the Albanese mapping

$$\mathrm{alb}\colon X \to \mathrm{Alb}(X) = H^0(X, \Omega_X^1)^*/H_1(X, \mathbb{Z})$$

to an abelian variety of dimension $h^0(X, \Omega_X^1)$, and by construction,

$$H^0(X, \Omega_X^1) \cong H^0\big(\mathrm{Alb}(X), \Omega_{\mathrm{Alb}(X)}^1\big).$$

It thus makes sense to consider more generally an arbitrary morphism $f\colon X \to A$ to an abelian variety A, and to ask about the zero loci of the holomorphic one-forms $f^*\omega$, for $\omega \in H^0(A, \Omega_A^1)$. Of course, we should replace the assumption "X of general type" by the condition that $\omega_X^m \otimes f^*L^{-1}$ has sections for $m \gg 1$, where L is an ample line bundle on A. This suggests the following more general result.

Theorem 26.4. *Let $f\colon X \to A$ be a morphism from a smooth projective variety to an abelian variety. If $H^0(X, \omega_X^m \otimes f^*L^{-1}) \neq 0$ for some $m \geq 1$ and some ample line bundle L on A, then one has $Z(f^*\omega) \neq \emptyset$ for every $\omega \in H^0(A, \Omega_A^1)$.*

Set $W = H^0(A, \Omega_A^1)$, and consider the incidence variety

$$Z_f = \big\{ (x, \omega) \in X \times W \bigm| x \in Z(f^*\omega) \big\} \subseteq X \times W.$$

The theorem is claiming that the second projection $p_2\colon Z_f \to W$ is surjective. Since A is an abelian variety, we have $T^*A = A \times W$, and so the usual diagram of morphisms between cotangent bundles becomes:

$$\begin{array}{ccc} X \times W & \xrightarrow{df} & T^*X \\ \big\downarrow{\scriptstyle f \times \mathrm{id}} & & \\ A \times W & & \end{array}$$

With this notation, we have $Z_f = df^{-1}(0)$. When we looked at direct images for $\mathscr{D}$-modules (in Lecture 13), we encountered the set

$$S_f = (f \times \mathrm{id})\bigl(df^{-1}(0)\bigr) = (f \times \mathrm{id})(Z_f).$$

It contains the characteristic varieties of the direct image $\mathscr{D}$-modules $\mathcal{H}^j f_+ \omega_X$. (In Lecture 13, we proved this for closed embeddings.) Concretely,

$$S_f = \bigl\{ (a, \omega) \in A \times W \bigm| f^{-1}(a) \cap Z(f^* \omega) \neq \emptyset \bigr\},$$

and so $Z(f^*\omega) \neq \emptyset$ for every $\omega \in W$ is equivalent to the surjectivity of $p_2 \colon S_f \to W$. This suggests the following strategy for proving Theorem 26.4: find a $\mathscr{D}_A$-module whose characteristic variety $\mathrm{Ch}(\mathcal{M})$ is contained in the set S_f, and then use results about $\mathscr{D}$-modules to show that $p_2 \colon \mathrm{Ch}(\mathcal{M}) \to W$ must be onto.

We could not actually get this idea to work, but we found a good replacement for it, based on work of Viehweg and Zuo. Here is a rought outline for the proof of Theorem 26.4. On the cotangent bundle $T^*A = A \times W$, we construct a morphism $\mathscr{F} \to \mathscr{G}$ between two coherent sheaves, with the following three properties:

(a) The support of $\mathscr{F}$ is contained in the set S_f.
(b) The induced morphism $H^0(A \times W, \mathscr{F}) \to H^0(A \times W, \mathscr{G})$ is nontrivial.
(c) The coherent sheaf $(p_2)_* \mathscr{G}$ on W is torsion-free.

Here $p_1 \colon A \times W \to A$ and $p_2 \colon A \times W \to W$ are the two projections. We will see next time that $\mathscr{G}$ is (almost) the coherent sheaf coming from a $\mathscr{D}_A$-module $\mathcal{M}$ with a good filtration $F_\bullet \mathcal{M}$.

Lemma 26.5. *Such a morphism $\mathscr{F} \to \mathscr{G}$ can only exist if $p_2(S_f) = W$.*

Proof. Consider the induced morphism

$$(p_2)_* \mathscr{F} \to (p_2)_* \mathscr{G}.$$

Both sheaves are coherent (by properness of p_2), and the support of $(p_2)_* \mathscr{F}$ is contained in the set $p_2(S_f)$. Now suppose that $p_2(S_f) \neq W$. Then $(p_2)_* \mathscr{F}$ is a torsion sheaf, and so the morphism to the torsion-free sheaf $(p_2)_* \mathscr{G}$ must be trivial. Taking global sections, we find that

$$H^0(A \times W, \mathscr{F}) = H^0\bigl(W, (p_2)_* \mathscr{F}\bigr) \to H^0\bigl(W, (p_2)_* \mathscr{G}\bigr) = H^0(A \times W, \mathscr{G})$$

is trivial; but this is a contradiction. □

Filtered $\mathscr{D}$-modules and the Rees construction. For the proof of Theorem 26.4, it is important to work with pairs $(\mathcal{M}, F_\bullet \mathcal{M})$, where $\mathcal{M}$ is a coherent $\mathscr{D}$-module, and $F_\bullet \mathcal{M}$ a good filtration. Here the filtration is not just a tool to study $\mathscr{D}$-modules, but an essential piece of data. One can define the direct image and duality functors for filtered $\mathscr{D}$-modules by analogy with the unfiltered case, as follows.

Let X be a nonsingular algebraic variety over a field k (of characteristic zero). We can combine $\mathscr{D}_X$ with its order filtration $F_\bullet \mathscr{D}_X$ into a single sheaf of algebras

$$\tilde{\mathscr{D}}_X = \bigoplus_{k=0}^{\infty} F_k \mathscr{D}_X,$$

called the *Rees algebra* of $\mathscr{D}_X$. This is a sheaf of non-commutative graded algebras, with multiplication defined in the obvious way. We denote by $z \in \tilde{\mathscr{D}}_{X,1}$ the image of $1 \in F_1 \mathscr{D}_X$; then $\tilde{\mathscr{D}}_X$ contains a copy of $\mathscr{O}_X[z]$. It is easy to see that

$$\tilde{\mathscr{D}}_X / \tilde{\mathscr{D}}_X (z - z_0) \cong \mathscr{D}_X$$

for every $z_0 \neq 0$, because in the quotient, each $F_k \mathscr{D}_X$ gets identified with its image in $F_{k+1} \mathscr{D}_X$. Likewise,

$$\tilde{\mathscr{D}}_X / \tilde{\mathscr{D}}_X z \cong \mathrm{gr}^F \mathscr{D}_X,$$

because in the quotient, the image of $F_k\mathscr{D}_X$ in $F_{k+1}\mathscr{D}_X$ goes to zero. We can therefore think of the Rees algebra $\tilde{\mathscr{D}}_X$ as a family of algebras over the affine line $\operatorname{Spec} k[z]$, in which $\mathscr{D}_X$ deforms into $\operatorname{gr}^F \mathscr{D}_X$.

Given a coherent left (or right) $\mathscr{D}_X$-module $\mathcal{M}$ and a good filtration $F_\bullet\mathcal{M}$, we can form the *Rees module*

$$\tilde{\mathcal{M}} = R_F\mathcal{M} = \bigoplus_{k\in\mathbb{Z}} F_k\mathcal{M}.$$

This is a graded left (or right) module over $\tilde{\mathscr{D}}_X$ in the obvious way; since the filtration is good, $\tilde{\mathcal{M}}$ is coherent over $\tilde{\mathscr{D}}_X$. As before, one checks that

$$\tilde{\mathcal{M}}/(z-z_0)\tilde{\mathcal{M}} \cong \mathcal{M}$$

for every $z_0 \neq 0$, whereas

$$\tilde{\mathcal{M}}/z\tilde{\mathcal{M}} \cong \operatorname{gr}^F\mathcal{M}.$$

An important point is that not every graded $\tilde{\mathscr{D}}_X$-module comes from a filtered $\mathscr{D}_X$-module.

Lemma 26.6. *A graded $\tilde{\mathscr{D}}_X$-module $\tilde{\mathcal{M}}$ is the Rees module of a filtered $\mathscr{D}_X$-module if and only if it has no z-torsion.*

Graded $\tilde{\mathscr{D}}_X$-modules without z-torsion are called *strict*. Since $\operatorname{Spec} k[z]$ is one-dimensional, this condition is equivalent to flatness over $k[z]$.

Proof. It is easy to see that a graded $\tilde{\mathscr{D}}_X$-module of the form $R_F\mathcal{M}$ does not have any z-torsion. Let us prove the converse. Suppose for the time being that $\tilde{\mathcal{M}}$ is any graded left $\tilde{\mathscr{D}}_X$-module. Define

$$\mathcal{M} = \tilde{\mathcal{M}}/(z-1)\tilde{\mathcal{M}},$$

which is a left module over $\tilde{\mathscr{D}}_X/\tilde{\mathscr{D}}_X(z-1) \cong \mathscr{D}_X$. The image of the k-th graded piece $\tilde{\mathcal{M}}_k$ defines a subsheaf $F_k\mathcal{M} \subseteq \mathcal{M}$, with the property that $F_j\mathscr{D}_X \cdot F_k\mathcal{M} \subseteq F_{j+k}\mathcal{M}$. It follows that the Rees module $R_F\mathcal{M}$ is a graded $\tilde{\mathscr{D}}_X$-module without z-torsion.

Now we have a morphism of graded $\tilde{\mathscr{D}}_X$-modules

$$\varphi\colon \tilde{\mathcal{M}} \to R_F\mathcal{M},$$

that takes $\tilde{\mathcal{M}}_k$ to $F_k\mathcal{M}$; by construction, this morphism is surjective. One checks that $\ker\varphi$ consists exactly of those sections of $\tilde{\mathcal{M}}$ that are killed by some power of z. In particular, φ is an isomorphism whenever $\tilde{\mathcal{M}}$ does not have any z-torsion. □

Functors for Rees modules. One can define all the usual functors for $\mathscr{D}$-modules also for modules over the larger algebra $\tilde{\mathscr{D}}$. The two functor we need are the direct image functor and the duality functor. Given a morphism $f\colon X \to Y$, we define the *transfer module*

$$\tilde{\mathscr{D}}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\tilde{\mathscr{D}}_Y$$

by the same formula as for $\mathscr{D}$-modules. It is again a $(\tilde{\mathscr{D}}_X, f^{-1}\tilde{\mathscr{D}}_Y)$-bimodule. We can then define the direct image functor

$$f_+(-) = \mathbf{R}f_*\big(- \overset{\mathbf{L}}{\otimes}_{\tilde{\mathscr{D}}_X} \tilde{\mathscr{D}}_{X\to Y}\big)\colon D^b_{g,qc}(\tilde{\mathscr{D}}_X^{op}) \to D^b_{g,qc}(\tilde{\mathscr{D}}_Y^{op})$$

between the derived categories of quasi-coherent graded right $\tilde{\mathscr{D}}$-modules. As in the case of $\mathscr{D}$-modules, one can use induced $\tilde{\mathscr{D}}$-modules to show that the direct image by a proper morphism preserves coherence.

If we specialize to $z = 1$, for example, by taking the (derived) tensor product with $\tilde{\mathscr{D}}/\tilde{\mathscr{D}}(z-1)$, we recover the usual direct image functor for right $\mathscr{D}$-modules.

On the other hand, we can specialize to $z = 0$, by taking the (derived) tensor product with $\tilde{\mathscr{D}}/\tilde{\mathscr{D}}z$. This gives us a functor

$$\mathrm{gr}^F\colon D^b_{g,qc}(\tilde{\mathscr{D}}_X^{op}) \to D^b_{g,qc}(\mathrm{gr}^F \mathscr{D}_X),$$

with takes a Rees module of the form $R_F\mathcal{M}$ to the associated graded module $\mathrm{gr}^F\mathcal{M}$. By computing what happens to the transfer module, one checks that the following diagram is commutative:

$$\begin{array}{ccc} D^b_{g,qc}(\tilde{\mathscr{D}}_X^{op}) & \xrightarrow{f_+} & D^b_{g,qc}(\tilde{\mathscr{D}}_Y^{op}) \\ \big\downarrow{\scriptstyle \mathrm{gr}^F} & & \big\downarrow{\scriptstyle \mathrm{gr}^F} \\ D^b_{g,qc}(\mathrm{gr}^F \mathscr{D}_X) & \longrightarrow & D^b_{g,qc}(\mathrm{gr}^F \mathscr{D}_Y) \end{array}$$

Here the arrow on the bottom is the functor

$$\mathbf{R}f_*\big(- \overset{\mathbf{L}}{\otimes}_{\mathrm{gr}^F\mathscr{D}_X} f^*(\mathrm{gr}^F \mathscr{D}_Y)\big)\colon D^b_{g,qc}(\mathrm{gr}^F \mathscr{D}_X) \to D^b_{g,qc}(\mathrm{gr}^F \mathscr{D}_Y).$$

If we forget about the grading, then quasi-coherent sheaves of $\mathrm{gr}^F\mathscr{D}_X$-modules are the same thing as quasi-coherent sheaves of $\mathscr{O}_{T^*X}$-modules on the cotangent bundle. The geometric interpretation of the above functor is then

$$\mathbf{R}(p_2)_* \circ \mathbf{L}(df)^*\colon D^b_{qc}(\mathscr{O}_{T^*X}) \to D^b_{qc}(\mathscr{O}_{T^*Y}),$$

where the morphisms between cotangent bundles are as in the diagram below.

$$\begin{array}{ccc} X \times_Y T^*Y & \xrightarrow{df} & T^*X \\ \big\downarrow{\scriptstyle p_2} & & \\ T^*Y & & \end{array}$$

The direct image functor for Rees modules therefore interpolates between the usual direct image functor for $\mathscr{D}$-modules, and the natural functor on the level of cotangent bundles. One subtle point is that even if we start from a Rees module $R_F\mathcal{M}$, the direct image

$$f_+(R_F\mathcal{M}) \in D^b_{g,qc}(\tilde{\mathscr{D}}_Y^{op})$$

might have z-torsion (= not be strict). If that happens, it means that $f_+(R_F\mathcal{M})$ has more cohomology that the complex of right $\mathscr{D}_Y$-modules $f_+\mathcal{M}$. (The extra cohomology is z-torsion, of course.) Equivalently, it means that the complex of graded $\mathrm{gr}^F\mathscr{D}_Y$-modules

$$\mathbf{R}f_*\big(\mathrm{gr}^F\mathcal{M} \overset{\mathbf{L}}{\otimes}_{\mathrm{gr}^F\mathscr{D}_X} f^*(\mathrm{gr}^F \mathscr{D}_Y)\big)$$

has some additional cohomology that is not visible to the direct image $f_+\mathcal{M}$ of the underlying $\mathscr{D}$-module.

One can also define a duality functor for $\tilde{\mathscr{D}}$-modules. As with $\mathscr{D}$-modules, the tensor product $\omega_X \otimes_{\mathscr{O}_X} \tilde{\mathscr{D}}_X$ has two commuting structures of right $\tilde{\mathscr{D}}_X$-modules. If $\tilde{\mathcal{M}}$ is a right $\tilde{\mathscr{D}}_X$-module, then

$$\mathcal{H}om_{\tilde{\mathscr{D}}_X}(\tilde{\mathcal{M}}, \omega_X \otimes_{\mathscr{O}_X} \tilde{\mathscr{D}}_X)$$

still has the structure of a right $\tilde{\mathscr{D}}_X$-module. Passing to derived categories, we obtain the (contravariant) duality functor

$$\mathbb{D}_X = \mathbf{R}\mathcal{H}om_{\tilde{\mathscr{D}}_X}(-, \omega_X \otimes_{\mathscr{O}_X} \tilde{\mathscr{D}}_X)[n]\colon D^b_{g,qc}(\tilde{\mathscr{D}}_X^{op}) \to D^b_{g,qc}(\tilde{\mathscr{D}}_X^{op})^{op}.$$

Here $[n]$ means shifting to the left by $n = \dim X$ steps. If we specialize to $z = 1$, we recover the usual duality functor for $\mathscr{D}_X$-modules; if we specialize instead to $z = 0$, we obtain the functor

$$\mathbf{R}\mathcal{H}om_{\mathrm{gr}^F\mathscr{D}_X}(-, \omega_X \otimes_{\mathscr{O}_X} \mathrm{gr}^F \mathscr{D}_X)[n]\colon D^b_{g,qc}(\mathrm{gr}^F \mathscr{D}_X) \to D^b_{g,qc}(\mathrm{gr}^F \mathscr{D}_X)^{op}.$$

We can again express this in geometric terms: if $\mathscr{G}$ denotes the coherent sheaf on T^*X corresponding to $\operatorname{gr}^F \mathcal{M}$, then the above functor is

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_{T^*X}}(\mathscr{G}, p^*\omega_X)[n],$$

where $p\colon T^*X \to X$ is the projection. As before, $\mathbb{D}_X(R_F\mathcal{M})$ can acquire z-torsion. For instance, suppose that $\mathcal{M}$ is a holonomic right $\mathscr{D}_X$-module. Then

$$\mathbf{R}\mathcal{H}om_{\mathscr{D}_X}(\mathcal{M}, \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X)[n]$$

only has cohomology in degree zero (where we get the holonomic dual $\mathcal{M}^*$). But the complex $\mathbb{D}_X(R_F\mathcal{M})$ might have cohomology in other degrees as well (which will then be z-torsion). In fact, one can show that $\mathbb{D}_X(R_F\mathcal{M})$ is again strict if and only if the complex

$$\mathbf{R}\mathcal{H}om_{\operatorname{gr}^F\mathscr{D}_X}(\operatorname{gr}^F\mathcal{M}, \omega_X \otimes_{\mathscr{O}_X} \operatorname{gr}^F\mathscr{D}_X)[n]$$

only has cohomology in degree zero; in commutative algebra terminology, this is equivalent to $\operatorname{gr}^F\mathcal{M}$ being a Cohen-Macaulay module over $\operatorname{gr}^F\mathscr{D}_X$.

Hodge modules. You can think of *Hodge modules* as being a special class of filtered $\mathscr{D}$-modules that behave well under the various functors. More precisely, a Hodge module on a nonsingular algebraic variety X is a (regular holonomic) right $\mathscr{D}_X$-module $\mathcal{M}$ together with a good filtration $F_\bullet\mathcal{M}$. There is some extra data, too, and several very restrictive conditions have to be satisfied, which make sure that the pair $(\mathcal{M}, F_\bullet\mathcal{M})$ comes from a polarizable variation of Hodge structure.

Example 26.7. The pair $(\omega_X, F_\bullet\omega_X)$, with the filtration defined by $F_{-n-1}\omega_X = 0$ and $F_{-n}\omega_X = \omega_X$, is an example of a Hodge module. That this is so is a deep theorem by Morihiko Saito, who created this theory.

For our purposes, the following three facts are important. (Again, all three are difficult theorems due to Saito.) First, if $(\mathcal{M}, F_\bullet\mathcal{M})$ is a Hodge module on X, and if $f\colon X \to Y$ is a *proper* morphism between nonsingular algebraic varieties, then all cohomology modules of the complex $f_+(R_F\mathcal{M})$ are strict, and the resulting filtered $\mathscr{D}_Y$-modules are again Hodge modules on Y. In particular, we can compute their associated graded modules:

$$\operatorname{gr}^F\mathcal{H}^j f_+\mathcal{M} \cong R^j f_*\big(\operatorname{gr}^F\mathcal{M} \overset{\mathbf{L}}{\otimes}_{\operatorname{gr}^F\mathscr{D}_X} f^*(\operatorname{gr}^F\mathscr{D}_Y)\big).$$

Second, the duality functor preserves Hodge modules: the complex $\mathbb{D}_X(R_F\mathcal{M})$ only has cohomology in degree zero, which is strict, and the resulting filtered $\mathscr{D}_X$-module $(\mathcal{M}', F_\bullet\mathcal{M}')$ is again a Hodge module on X. Once again, this means that we can compute the associated graded module:

$$\operatorname{gr}^F\mathcal{M}' \cong R^n\mathcal{H}om_{\operatorname{gr}^F\mathscr{D}_X}(\operatorname{gr}^F\mathcal{M}, \omega_X \otimes_{\mathscr{O}_X} \operatorname{gr}^F\mathscr{D}_X).$$

Third, Hodge modules on projective varieties satisfy a vanishing theorem similar to the Kodaira vanishing theorem. Given a Hodge module $(\mathcal{M}, F_\bullet\mathcal{M})$, we can form the Spencer complex

$$\operatorname{Sp}(\mathcal{M}) = \Big[\mathcal{M} \otimes \bigwedge^n \mathscr{T}_X \to \dots \to \mathcal{M} \otimes \mathscr{T}_X \to \mathcal{M}\Big]$$

which lives in degrees $-n, \dots, 0$. (Since $\mathcal{M}$ is regular holonomic, $\operatorname{Sp}(\mathcal{M})$ is actually a perverse sheaf, by Kashiwara's theorem.) The Spencer complex is filtered by the family of subcomplexes

$$F_k \operatorname{Sp}(\mathcal{M}) = \Big[F_{k-n}\mathcal{M} \otimes \bigwedge^n \mathscr{T}_X \to \dots \to F_{k-1}\mathcal{M} \otimes \mathscr{T}_X \to F_k\mathcal{M}\Big],$$

and the k-th subquotient

$$\operatorname{gr}_k^F \operatorname{Sp}(\mathcal{M}) = \Big[\operatorname{gr}_{k-n}^F \mathcal{M} \otimes \bigwedge^n \mathscr{T}_X \to \cdots \to \operatorname{gr}_{k-1}^F \mathcal{M} \otimes \mathscr{T}_X \to \operatorname{gr}_k^F \mathcal{M}\Big]$$

is a complex of coherent $\mathscr{O}_X$-modules. For example, for the pair $(\omega_X, F_\bullet \omega_X)$, the Spencer complex is the holomorphic de Rham complex, and the $(-p)$-th subquotient is Ω_X^p, placed in degree $n-p$.

Theorem 26.8 (Saito's vanishing theorem)**.** *Let X be a nonsingular projective variety, and L an ample line bundle. If $(\mathcal{M}, F_\bullet \mathcal{M})$ is a Hodge module on X, then*

$$H^i\big(X, \operatorname{gr}_k^F \operatorname{Sp}(\mathcal{M}) \otimes L\big) = 0 \quad \textit{for every } i > 0,$$
$$H^i\big(X, \operatorname{gr}_k^F \operatorname{Sp}(\mathcal{M}) \otimes L^{-1}\big) = 0 \quad \textit{for every } i < 0.$$

Hodge modules on abelian varieties. Let us now return to abelian varieties. Suppose that A is an abelian variety and L an ample line bundle on A. Since the tangent bundle of A is trivial, one can prove a much stronger vanishing theorem. Let me explain how this works. Fix a Hodge module $(\mathcal{M}, F_\bullet \mathcal{M})$ on A, and for simplicity, suppose that $F_{-1}\mathcal{M} = 0$ and $F_0 \mathcal{M} \neq 0$. Then

$$\operatorname{gr}_0^F \operatorname{Sp}(\mathcal{M}) = \operatorname{gr}_0^F \mathcal{M},$$

and so Saito's vanishing theorem gives

$$H^i(A, \operatorname{gr}_0^F \mathcal{M} \otimes L) = 0 \quad \text{for all } i > 0. \tag{26.9}$$

The next subquotient of the Spencer complex is

$$\operatorname{gr}_1^F \operatorname{Sp}(\mathcal{M}) = \Big[\operatorname{gr}_0^F \mathcal{M} \otimes \mathscr{T}_A \to \operatorname{gr}_1^F \mathcal{M}\Big].$$

Since $\mathscr{T}_A \cong \mathscr{O}_A^{\oplus g}$, where $g = \dim A$, the term $\operatorname{gr}_0^F \mathcal{M} \otimes \mathscr{T}_A$ has no higher cohomology by (26.9). On the other hand, the vanishing theorem says that

$$H^i\big(A, \operatorname{gr}_1^F \operatorname{Sp}(\mathcal{M}) \otimes L\big) = 0 \quad \text{for all } i > 0.$$

If we put these two facts together, we find that

$$H^i(A, \operatorname{gr}_1^F \mathcal{M} \otimes L) = 0 \quad \text{for all } i > 0. \tag{26.10}$$

Continuing in this manner, we arrive at the conclusion that

$$H^i(A, \operatorname{gr}_k^F \mathcal{M} \otimes L) = 0 \quad \text{for all } i > 0, \tag{26.11}$$

and so all graded quotients $\operatorname{gr}_k^F \mathcal{M}$ satisfy the same Kodaira-type vanishing theorem.

Now recall that $T^*A = A \times W$, where $W = H^0(A, \Omega_A^1)$. The vanishing theorem can be used to produce torsion-free sheaves on W. Suppose that $(\mathcal{M}, F_\bullet \mathcal{M})$ is a Hodge module on A. Denote by $\mathscr{G}$ the coherent sheaf on the cotangent bundle corresponding to the associated graded module $\operatorname{gr}^F \mathcal{M}$. Also let $p_1 \colon A \times W \to A$ and $p_2 \colon A \times W \to W$ be the two projections.

Lemma 26.12. *If L is an ample line bundle on A, then $(p_2)_*\big(\mathscr{G} \otimes p_1^* L^{-1}\big)$ is a torsion-free coherent sheaf on W.*

Proof. Coherence is clear (because p_2 is proper). Let us first analyze what happens when we tensor by L instead of L^{-1}. The higher direct images sheaves

$$R^i(p_2)_*(\mathscr{G} \otimes p_1^* L)$$

are coherent, and since W is affine, we have

$$H^0\big(W, R^i(p_2)_*(\mathscr{G} \otimes p_1^* L)\big) = H^i\big(A \times W, \mathscr{G} \otimes p_1^* L\big) = H^i\big(A, (p_1)_* \mathscr{G} \otimes L\big).$$

This vanishes for every $i > 0$ because of (26.11) and the fact that $(p_1)_* \mathscr{G} = \operatorname{gr}^F \mathcal{M}$. The conclusion is that the complex

$$\mathbf{R}(p_2)_*(\mathscr{G} \otimes p_1^* L)$$

is actually a single coherent sheaf in degree zero.

Now let us turn to the sheaf $(p_2)_*(\mathscr{G} \otimes p_1^* L^{-1})$. If we apply Grothendieck duality for the proper morphism p_2, we get

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_W}\big(\mathbf{R}(p_2)_*(\mathscr{G} \otimes p_1^* L^{-1}), \mathscr{O}_W\big) \cong \mathbf{R}(p_2)_* \mathbf{R}\mathcal{H}om_{\mathscr{O}_{A\times W}}\big(\mathscr{G} \otimes p_1^* L^{-1}, p_1^* \omega_A[g]\big),$$

since the relative dualizing sheaf is $\omega_{A\times W/W} = p_1^* \omega_A$. We can rewrite the right-hand side in the more compact form

$$\mathbf{R}(p_2)_*(\mathscr{G}' \otimes p_1^* L),$$

where we have introduced the new complex

$$\mathscr{G}' = \mathbf{R}\mathcal{H}om_{\mathscr{O}_{A\times W}}(\mathscr{G}, p_1^* \omega_A)[g].$$

We can now use the results about the duality functor. They imply that $\mathscr{G}'$ is actually a coherent sheaf; more precisely, we have $\mathbb{D}_X(R_F\mathcal{M}) = R_F\mathcal{M}'$ for a Hodge module $(\mathcal{M}', F_\bullet\mathcal{M}')$, and $\mathscr{G}'$ is the coherent sheaf associated to $\operatorname{gr}^F \mathcal{M}'$. According to the discussion above,

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_W}\big(\mathbf{R}(p_2)_*(\mathscr{G} \otimes p_1^* L^{-1}), \mathscr{O}_W\big) \cong (p_2)_*\big(\mathscr{G}' \otimes p_1^* L\big)$$

is therefore a coherent sheaf in degree zero. After dualizing again, we get

$$(p_2)_*(\mathscr{G} \otimes p_1^* L^{-1}) \cong \mathcal{H}om_{\mathscr{O}_W}\big((p_2)_*(\mathscr{G}' \otimes p_1^* L), \mathscr{O}_W\big),$$

which is reflexive, hence torsion-free. □

Exercise.

Exercise 26.1. Let $\tilde{\mathcal{M}}$ be a coherent graded left $\tilde{\mathscr{D}}_X$-module. Define $\mathcal{M} = \tilde{\mathcal{M}}/(z-1)\tilde{\mathcal{M}}$, and let $F_k\mathcal{M}$ be the image of $\tilde{\mathcal{M}}_k$.

(a) Show that $F_\bullet\mathcal{M}$ is a good filtration.

(b) Show that the kernel of the morphism $\varphi\colon \tilde{\mathcal{M}} \to R_F\mathcal{M}$ consists exactly of those sections that are killed by some power of z.