Lecture 3: February 11

Dimension and multiplicity. We are going to introduce two important invariants of modules over the Weyl algebra, namely dimension and multiplicity. They are defined using good filtrations. For this, we need to work with the Bernstein filtration on A_n, so in today's lecture, $F_\bullet A_n = F_\bullet^B A_n$ will always mean the Bernstein filtration. Recall that each $F_j^B A_n$ has finite dimension over K.

Let M be a finitely generated A_n-module, where $A_n = A_n(K)$ and K is a field. Choose a good filtration $F_\bullet M$ on M, compatible with the Bernstein filtration $F_\bullet A_n$. We saw last time that the existence of such a filtration is equivalent to M being finitely generated. Since $F_0 A_n = K$, each subspace $F_j M$ in the good filtration is a K-vector space of finite dimension. Consider its dimension

$$\dim_K F_j M = \sum_{i=0}^{j} \dim_K F_i M / F_{i-1} M$$

as a function of $j \geq 0$. Here are some examples:

(1) For $M = A_n$ with the Bernstein filtration, we have

$$F_j A_n = \left\{ \sum c_{\alpha,\beta} x^\alpha \partial^\beta \;\middle|\; |\alpha| + |\beta| \leq j \right\}$$

and therefore

$$\dim F_j A_n = \binom{2n+j}{2n} = \frac{1}{(2n)!} j^{2n} + \cdots$$

is a polynomial of degree $2n$ in the variable j, at least for $j \geq 0$.

(2) For $M = K[x_1, \dots, x_n]$, with the usual filtration by degree, we have

$$\dim F_j M = \binom{n+j}{n} = \frac{1}{n!} j^n + \cdots$$

is a polynomial of degree n in the variable j.

(3) Consider $M = A_n / A_n(x_1, \dots, x_n)$, with the filtration induced by the Bernstein filtration on A_n. As a K-vector space, M is isomorphic to $K[\partial_1, \dots, \partial_n]$, and the filtration is just the filtration by degree. So again,

$$\dim F_j M = \binom{n+j}{n} = \frac{1}{n!} j^n + \cdots$$

(4) Consider the A_1-module $M = K[x, x^{-1}]$, with the filtration $F_j M = F_j A_n \cdot x^{-1}$. Clearly, $F_0 M$ is spanned by x^{-1}, and it is easy to see that $F_j M$ is spanned by $x^{j-1}, x^{j-2}, \dots, x^{-j-1}$ for every $j \geq 0$. So

$$\dim F_j M = 2j + 1$$

for $j \geq 0$, which is again a polynomial of degree 1.

In fact, at least for sufficiently large values of j, the function $\dim_K F_j M$ always grows like a polynomial.

Proposition 3.1. *There is a polynomial $\chi(M, F_\bullet M, t) \in \mathbb{Q}[t]$, called the* Hilbert polynomial *of $(M, F_\bullet M)$, with the property that*

$$\dim_K F_j M = \chi(M, F_\bullet M, j)$$

for all sufficiently large values of j.

Proof. The point is that $\operatorname{gr}^F A_n$ is a polynomial ring in $2n$ variables, and so we can use the theory of Hilbert functions for finitely generated modules over the polynomial ring. (This is explained very well in Eisenbud's book *Commutative Algebra*.) Let me sketch the proof. Set $S = \operatorname{gr}^F A_n$, and recall that this is isomorphic to the polynomial ring in $2n$ variables, with the usual grading by degree. The fact that $F_\bullet M$ is a good filtration means that $\operatorname{gr}^F M$ is a finitely generated graded S-module. By Hilbert's syzygy theorem, every finitely generated graded S-module admits a finite resolution by graded free S-modules; the length of such a resolution is at most the number of variables in the polynomial ring, so $2n$ in our case. Choose such a resolution

$$0 \to E_{2n} \to E_{2n-1} \to \cdots \to E_1 \to E_0 \to \operatorname{gr}^F M \to 0.$$

Denoting by $S(q)$ the graded S-module with $S(q)_i = S_{q+i}$, we have

$$E_p = \bigoplus_{q \in \mathbb{N}} S(-q)^{\oplus b_{p,q}}$$

for certain natural numbers $b_{p,q} \in \mathbb{N}$, all but finitely many of which are of course zero. By counting monomials, we have

$$\dim S_i = \binom{i+2n-1}{2n-1}$$

for $i \geq 0$, and so if we take dimensions in the resolution from above, we get

$$\dim F_i M / F_{i-1} M = \sum_{p=0}^{2n} (-1)^p \sum_q b_{p,q} \dim S_{i-q} = \sum_{p=0}^{2n} (-1)^p \sum_q b_{p,q} \binom{i-q+2n-1}{2n-1}.$$

At least for $i \gg 0$, this is a polynomial of degree at most $2n-1$ in the variable i, whose coefficients are rational numbers. It follows that

$$\dim F_j M = \sum_{i=0}^{j} \dim F_i M / F_{i-1} M$$

is a polynomial of degree at most $2n$ in the variable j, at least for $j \gg 0$. □

If $M \neq 0$, then the Hilbert polynomial is not the zero polynomial; let $d \geq 0$ be its degree. The proof shows that $d \leq 2n$. Since $\dim F_j M$ is of course always a non-negative integer, it is not hard to see that the leading coefficient of the polynomial $\chi(M, F_\bullet M, t)$ must be of the form

$$\frac{m}{d!}$$

for some integer $m \geq 1$. (See the exercises.) Both d and m are actually invariants of the module M itself.

Lemma 3.2. *The two numbers d and m only depend on M, but they do not depend on the choice of good filtration on M.*

Proof. Let $\chi_F(t) = \chi(M, F_\bullet M, t)$ be the Hilbert polynomial for the good filtration $F_\bullet M$. Suppose that $G_\bullet M$ is another good filtration, with Hilbert polynomial $\chi_G(t) = \chi(M, G_\bullet M, t)$. By Corollary 2.15, there is an integer $k \geq 0$ such that

$$F_{j-k} M \subseteq G_j M \subseteq F_{j+k} M$$

for every $j \geq 0$. This gives

$$\dim F_{j-k} M \leq \dim G_j M \leq \dim F_{j+k} M,$$

and therefore we obtain the inequality

$$\chi_F(t-k) \leq \chi_G(t) \leq \chi_F(t+k)$$

for the Hilbert polynomials. Since $\chi_F(t \pm k)$ has the same leading term as $\chi_F(t)$, it follows that $\chi_G(t)$ is also a polynomial of degree d with leading coefficient $m/d!$. □

The number $d = d(M)$ is called the *dimension* of the A_n-module M, and the number $m = m(M)$ is called the *multiplicity*. As long as $M \neq 0$, we have $d(M) \geq 0$ and $m(M) \geq 1$. If $M = 0$, we use the convention that $m(M) = 0$. We will see later what the geometric significance of these two numbers is. Going back to the four examples from above, we see that A_n has dimension $2n$ and multiplicity 1; both $K[x_1, \ldots, x_n]$ and $A_n/A_n(x_1, \ldots, x_n)$ have dimension n and multiplicity 1; and the A_1-module $K[x, x^{-1}]$ has dimension 1 and multiplicity 2.

Let us investigate the behavior of dimension and multiplicity for submodules and quotient modules. Recall that a short exact sequence of A_n-modules

$$0 \to M' \to M \to M'' \to 0$$

means that M' is a submodule of M, and that M'' is isomorphic to the quotient module M/M'. Given a filtration $F_\bullet M$, we can induce filtrations on M' and M'' by setting

$$F_j M' = M' \cap F_j M \quad \text{and} \quad F_j M'' = \operatorname{im}(F_j M \to M'').$$

With this definition, the associated graded modules form a short exact sequence

$$0 \to \operatorname{gr}^F M' \to \operatorname{gr}^F M \to \operatorname{gr}^F M'' \to 0,$$

now in the category of $\operatorname{gr}^F A_n$-modules.

Proposition 3.3. *Let M be a finitely generated A_n-module, and $F_\bullet M$ a good filtration. Suppose that*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of A_n-modules. Then the induced filtration $F_\bullet M'$ and $F_\bullet M''$ are both good, and

$$0 \to \operatorname{gr}^F M' \to \operatorname{gr}^F M \to \operatorname{gr}^F M'' \to 0$$

is a short exact sequence of finitely generated graded $\operatorname{gr}^F A_n$-modules. Moreover:

(a) One has $\chi(M, F_\bullet M, t) = \chi(M', F_\bullet M', t) + \chi(M'', F_\bullet M'', t)$.
(b) One has $d(M) = \max\{d(M'), d(M'')\}$.
(c) If $d(M') = d(M'')$, then $m(M) = m(M') + m(M'')$.

Proof. The short exact sequence follows from the definition of the filtrations on M' and M''. Since $F_\bullet M$ is a good filtration, $\operatorname{gr}^F M$ is finitely generated over the polynomial ring $\operatorname{gr}^F A_n$. The polynomial ring is commutative and noetherian, and so both the submodule $\operatorname{gr}^F M'$ and the quotient module $\operatorname{gr}^F M''$ are again finitely generated, which means that $F_\bullet M'$ and $F_\bullet M''$ are also good filtrations. Taking dimensions in the short exact sequence, we get the relation

$$\chi(M, F_\bullet M, t) = \chi(M', F_\bullet M', t) + \chi(M'', F_\bullet M'', t)$$

among the three Hilbert polynomials. The other two assertions are obvious consequences. □

Example 3.4. The calculation in the proposition explains for example why the multiplicity of the A_1-module $K[x, x^{-1}]$ should be 2. Indeed, we have a short exact sequence

$$0 \to K[x] \to K[x, x^{-1}] \to K[x, x^{-1}]/K[x] \to 0.$$

The class of x^{-1} generates the quotient module, but since $x \cdot x^{-1} = 1$, it is also annihilated by x, and so the quotient module is actually isomorphic to $A_1/A_1(x)$. Both the submodule and the quotient module have multiplicity 1, and therefore $K[x, x^{-1}]$ must have multiplicity 2.

Bernstein's inequality. In our discussion of Hilbert functions, we have only used properties of the polynomial ring $\mathrm{gr}^F A_n$. Now comes the first place where A_n-modules are genuinely different from modules over the polynomial ring. The following important result is due to Joseph Bernstein.

Theorem 3.5 (Bernstein's inequality)**.** *Let $M \neq 0$ be a finitely generated A_n-module. Then $d(M) \geq n$.*

Choose a filtration $F_\bullet M$, compatible with the Bernstein filtration on A_n; after a shift in the indexing, we can assume that $F_0 M \neq 0$.

Lemma 3.6. *The multiplication map*

$$F_j^B A_n \to \mathrm{Hom}_K(F_j M, F_{2j} M), \quad P \mapsto (m \mapsto Pm),$$

is injective for every $j \geq 0$.

Proof. We argue by induction on $j \geq 0$. For $j = 0$, the statement is clearly true: $F_0^B A_n = K$, and since $F_0 M \neq 0$, the multiplication map $K \to \mathrm{Hom}_K(F_0 M, F_0 M)$ is obviously injective. Now suppose that the result is known for $j - 1 \geq 0$. Assume for the sake of contradiction that there is a nonzero differential operator $P \in F_j^B A_n$ that lies in the kernel of the multiplication map, so that $Pm = 0$ for every $m \in F_j M$. Clearly, P cannot be constant (because $F_j M$ is nonzero), and so P has to contain x_i or ∂_i for some $i = 1, \dots, n$. If x_i appears in P, then by a calculation we did in Lecture 1, the commutator $[P, \partial_i] \in F_{j-1}^B A_n$ is still nonzero. But then

$$[P, \partial_i] m = P(\partial_i m) - \partial_i (Pm) = 0$$

for every $m \in F_{j-1} M$; indeed, both m and $\partial_i m$ belong to $F_j M$, and P annihilates $F_j M$ by assumption. This contradicts the inductive hypothesis. If ∂_i appears in P, then we use the same argument with $[P, x_i]$ instead. □

Now suppose that $F_\bullet M$ is a good filtration, and let $\chi(t) = \chi(M, F_\bullet M, t)$ be the Hilbert polynomial. The lemma gives

$$\dim F_j^B A_n \leq \dim \mathrm{Hom}_K(F_j M, F_{2j} M) = \dim F_j M \cdot \dim F_{2j} M,$$

and therefore

$$\binom{j+2n}{2n} \leq \chi(j) \cdot \chi(2j)$$

for all sufficiently large values of j. Since $\chi(t)$ is a polynomial of degree $d(M)$, we conclude that $2n \leq 2d(M)$, or $n \leq d(M)$. This proves Bernstein's inequality.

Holonomic modules. Bernstein's inequality suggests the following definition.

Definition 3.7. A finitely generated A_n-module M is called *holonomic* if either $M \neq 0$ and $d(M) = n$, or if $M = 0$.

Holonomic modules are those for which the dimension takes the minimal value allowed by Bernstein's inequality. We also consider the zero module to be holonomic for convenience. In the special case of holonomic modules, Proposition 3.3 has many nice consequences. The following result would be cumbersome to state if we did not consider the zero module to be holonomic.

Corollary 3.8. *Suppose that*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of A_n-modules. Then M is holonomic if and only if M' and M'' are holonomic. In particular, submodules and quotient modules of holonomic modules are again holonomic.

Proof. This follows from the fact that $d(M) = \max\{d(M'), d(M'')\}$ and Bernstein's inequality. □

Now suppose that M is a nonzero holonomic module, with a certain multiplicity $m(M) \geq 1$. If we have any chain of submodules

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M_\ell \subseteq M,$$

then each M_j is again holonomic, hence of dimension n. By Proposition 3.3, the multiplicities add, and so

$$m(M) = m(M_1) + m(M/M_1) = m(M_1) + m(M_2/M_1) + \cdots + m(M_\ell/M_{\ell-1}).$$

If the chain is strictly increasing, then each term in the sum is ≥ 1, and so we get $\ell \leq m(M)$. In other words, the length of any strictly increasing (or decreasing) chain of submodules is bounded by $m(M)$.

Corollary 3.9. *Let M be a holonomic A_n-module.*

(a) M is both noetherian and artininian, meaning that every increasing or decreasing chain of submodules stabilizes.
(b) M has finite length, meaning that it admits a finite filtration whose subquotients are simple A_n-modules.

Proof. The first assertion follows from the calculation we just did. For the second assertion, see the exercises. □

We have already seen a few simple examples of holonomic modules; for instance, $K[x_1, \dots, x_n]$ is a holonomic A_n-module, and $K[x, x^{-1}]$ is a holonomic A_1-module. Here is a more interesting class of holonomic A_n-modules.

Proposition 3.10. *Let $p \in K[x_1, \dots, x_n]$ be a nonzero polynomial. Then*

$$M = K[x_1, \dots, x_n, p^{-1}],$$

with the structure of left A_n-module given by formal differentiation, is a holonomic A_n-module.

Unlike the example of $K[x, x^{-1}]$, it is not even obvious that M is finitely generated. Fortunately, we can use the following numerical criterion for holonomicity.

Lemma 3.11. *Let M be a A_n-module, and $F_\bullet M$ a filtration compatible with the Bernstein filtration on A_n. If*

$$\dim_K F_j M \leq \frac{c}{n!} j^n + c_1 (j+1)^{n-1}$$

for some constants $c, c_1 \geq 1$, then M is holonomic and $m(M) \leq c$. In particular, M is finitely generated.

Proof. The idea is to study finitely generated submodules of M. These are easy to construct: simply take any finite number of elements of M and look at the submodule they generate. Let $N \subseteq M$ be any nonzero finitely generated submodule, and $F_\bullet N$ a good filtration of N. The filtration $N \cap F_\bullet M$ is compatible with the Bernstein filtration, but of course not necessarily good. Still, according to Corollary 2.15, there is an integer $k \geq 0$ such that

$$F_j N \subseteq N \cap F_{j+k} M \subseteq F_{j+k} M$$

for every $j \geq 0$. Taking dimensions, we get

$$\dim F_j N \leq \dim F_{j+k} M \leq \frac{c}{n!}(j+k)^n + c_1 (j+k+1)^{n-1},$$

and therefore $d(N) \leq n$. Since $d(N) \geq n$ by Bernstein's inequality, we see that $d(N) = n$, and so N is holonomic. It also follows that $m(N) \leq c$, by looking at the leading terms on both sides. Therefore any finitely generated submodule of M is holonomic and has multiplicity at most c.

This implies now that M itself must be finitely generated, hence holonomic. To see this, choose any nonzero element $m_1 \in M$, and let N_1 be the submodule generated by m_1. If $N_1 = M$, then we are done; otherwise, choose an element $m_2 \in M \setminus N_1$, and let N_2 be the submodule generated by m_1 and m_2. If $N_2 = M$, then we are done; otherwise, choose an element $m_3 \in M \setminus N_2$, and let N_3 be the submodule generated by m_1, m_2, m_3. Continuing in this way, we produce an chain of submodules $N_1 \subset N_2 \subset N_3 \subset \cdots$. Because each N_j is holonomic with $m(N_j) \leq c$, this chain has to stabilize after at most c steps, and so M is in fact generated by at most c elements. In particular, M is holonomic and $m(M) \leq c$. $\square$

Note that the filtration $F_\bullet M$ is not necessarily good. The lemma is quite remarkable: it allows us to prove that M is finitely generated simply by computing the dimensions of $F_j M$.

Now we apply this to study the A_n-module $M = K[x_1, \dots, x_n, p^{-1}]$. The action by A_n is by formal differentiation:

$$\partial_j(fp^{-\ell}) = -\ell f \frac{\partial p}{\partial x_j} p^{-(\ell+1)} + \frac{\partial f}{\partial x_j} p^{-\ell} = \left(-\ell f \frac{\partial p}{\partial x_j} + p \frac{\partial f}{\partial x_j} \right) p^{-(\ell+1)}.$$

Let $m = \deg p$, and consider the filtration

$$F_j M = \left\{ fp^{-\ell} \;\middle|\; \deg f \leq (m+1)\ell \right\}.$$

Each $F_j M$ is a finite-dimensional K-vector space. If $fp^{-\ell} \in F_j M$, then $\deg f \leq (m+1)\ell$, and so $x_j fp^{-\ell}$ and $\partial_j(fp^{-\ell})$ again belong to $F_{j+1} M$ (by the above formula). In other words, the filtration is compatible with the Bernstein filtration on A_n. Lastly, we have $M = \bigcup F_j M$; indeed, given any element $fp^{-\ell} \in M$, we have

$$fp^{-\ell} = (fp^k) p^{-(\ell+k)},$$

and since $\deg(fp^k) = \deg f + km \leq (m+1)(\ell+k)$ for sufficiently large k, the element eventually belongs to $F_{\ell+k} M$. Taking dimensions, we have

$$\dim F_j M = \binom{(m+1)j + n}{n},$$

which is a polynomial of degree n in j with leading coefficient $(m+1)^n/n!$. So the lemma shows that M is holonomic with $m(M) \leq (m+1)^n$.

Exercises.

Exercise 3.1. Suppose that $\chi(t) \in \mathbb{Q}[t]$ has the property that $\chi(j) \in \mathbb{Z}$ for all sufficiently large values of $j \in \mathbb{Z}$. Show that $\chi(t)$ can be written as a linear combination, with integer coefficients, of the polynomials

$$\chi_n(t) = \frac{t(t-1)\cdots(t-n+1)}{n!}$$

for $n \geq 0$. Conclude that the leading coefficient of $\chi(t)$ has the form $m/d!$ for some $m \in \mathbb{Z}$, where d is the degree of $\chi(t)$.

Exercise 3.2. Show that $A_1/A_1 P$ is holonomic for every nonzero $P \in A_1$.

Exercise 3.3. Recall that a (left) A_n-module M is said to be *simple* if it does not have any A_n-submodules besides $\{0\}$ and M. Show that every simple A_n-module is *cyclic*, meaning that it be generated by a single element.

Exercise 3.4. The goal of this exercise is to prove that every holonomic A_n-module is cyclic. This phenomenon is very different from the case of modules over the polynomial ring.

(a) Let M be a nonzero holonomic A_n-module. Show that M has finite length, meaning that it admits a filtration by A_n-submodules whose subquotients are simple modules. Let $\ell \geq 1$ be the length of such a filtration.
(b) Show that the result is true if $\ell = 1$.
(c) If $\ell \geq 2$, let $N \subseteq M$ be a simple submodule, generated by some $m_0 \in N$. By induction, M/N is cyclic, so let $m \in M$ be any element that maps to a generator of M/N. Show that the left ideal $I = \left\{ P \in A_n \;\middle|\; Pm = 0 \right\}$ is nonzero.
(d) Show that there is some $Q \in A_n$ such that IQ is not contained in the left ideal $\left\{ P \in A_n \;\middle|\; Pm_0 = 0 \right\}$. (Hint: A_n is a simple algebra.)
(e) Now choose $P \in I$ such that $PQm_0 \neq 0$. Show that the element $m + Qm_0$ generates M as a left A_n-module.