Lecture 4: February 13

Last time, somebody asked what happens to chains of submodules when the dimension is greater than n. Here is an example to show that there can be infinite descending chains. (Since A_n is noetherian, there are no infinite ascending chains in finitely generated A_n-modules.)

Example 4.1. Consider the chain of submodules

$$A_1 \supset A_1 x \supset A_1 x^2 \supset \cdots$$

All modules in this chain are isomorphic to A_1, and all subquotients are isomorphic to A_1/A_1x. What happens is that, in the short exact sequence

$$0 \to A_1 \xrightarrow{x} A_1 \to A_1/A_1x \to 0,$$

the first two modules have dimension 2 and multiplicity 1, whereas the third module has dimension 1 and multiplicity 1.

Distributions and polynomials. Today, we are going to look at an application of holonomic A_n-modules to the study of certain integrals. This was in fact one of the reasons why the theory was developed in the first place. For the time being, we take $K = \mathbb{R}$. Let $p \in \mathbb{R}[x_1, \dots, x_n]$ be a nonzero polynomial with the property that $p(x_1, \dots, x_n) \geq 0$ for every $(x_1, \dots, x_n) \in \mathbb{R}^n$. (We can always achieve this by replacing p by its square.)

Let $S(\mathbb{R}^n)$ be the Schwartz space of all rapidly decreasing functions. A complex-valued function $\varphi \in C^\infty(\mathbb{R}^n)$ is rapidly decreasing if the quantity

$$p_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} \lvert x^\alpha \partial^\beta \varphi(x) \rvert$$

is finite for every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$. Then $S(\mathbb{R}^n)$ is a topological vector space, with the topology defined by the family of semi-norms $p_{\alpha,\beta}$. A *tempered distribution* T is a continuous linear functional $T \colon S(\mathbb{R}^n) \to \mathbb{C}$.

Now fix a rapidly decreasing function $\varphi \in S(\mathbb{R}^n)$, and consider the integral

$$T_s(\varphi) = \int_{\mathbb{R}^n} p(x)^s \varphi(x) \, d\mu(x),$$

as a function of the complex parameter $s \in \mathbb{C}$. For $\operatorname{Re} s > 0$, the integral makes sense and has a finite value, due to the fact that φ is rapidly decreasing (and p only takes nonnegative real values). Differentiation under the integral sign shows that $T_s(\varphi)$ is actually a holomorphic function of s for $\operatorname{Re} s > 0$.

Example 4.2. The Gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx$$

is a typical example of such an integral. The integral only makes sense for $\operatorname{Re} s > 0$, but in fact, $\Gamma(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$ with simple poles along $\{0, -1, -2, \dots\}$. This is done step by step, using integration by parts. One has

$$\frac{d}{dx}\bigl(x^s e^{-x}\bigr) = s x^{s-1} e^{-x} - x^s e^{-x},$$

and therefore

$$s\Gamma(s) = x^s e^{-x} \Big|_0^\infty + \int_0^\infty x^s e^{-x} \, dx = \Gamma(s+1)$$

for $\operatorname{Re} s > 0$; now the identity $\Gamma(s) = \Gamma(s+1)/s$ provides an extension of the Gamma function to $\operatorname{Re} s > -1$, with a simple pole at $s = 0$.

Now the question is whether $T_s(\varphi)$ can always be extended to a meromorphic function on the entire complex plane. Bernstein discovered that the answer is yes. The reason is that one always has a functional equation of the form

$$D(s)p(x)^{s+1} = b(s)p(x)^s, \tag{4.3}$$

where $b(s) \in \mathbb{R}[s]$ is a monic polynomial, and $D(s) \in A_n\big(\mathbb{R}[s]\big)$ is a differential operator with coefficients in the ring $\mathbb{R}[s]$. This sort of relation gives the desired meromorphic extension, again step by step. Indeed, after substituting into the integral and integrating by parts, we get

$$b(s)T_s(\varphi) = \int_{\mathbb{R}^n} D(s)p(x)^{s+1}\varphi(x)\,d\mu = \int_{\mathbb{R}^n} p(x)^{s+1}\sigma\big(D(s)\big)\varphi(x)\,d\mu,$$

where $\sigma\big(D(s)\big)$ is the differential operator obtained from $D(s)$ by the left-to-right transformation in Lecture 2. (The reason is that each time we integrate by parts to move ∂_j from the first to the second factor, we get an additional minus sign.) The new integral is again holomorphic for $\operatorname{Re} s > -1$, and after dividing by $b(s)$, we obtain a meromorphic extension of $T_s(\varphi)$ to the half plane $\operatorname{Re} s > -1$, possibly with poles along the zero set of $b(s)$. Continuing in this manner, we can extend $T_s(\varphi)$ to a meromorphic function on the entire complex plane, with poles contained in the set

$$\big\{\, s \in \mathbb{C} \;\big|\; b(s+k) = 0 \text{ for some } k \geq 0 \,\big\}.$$

For this reason, we obviously want to choose the polynomial $b(s)$ in (4.3) to be of minimal degree.

Example 4.4. In the case of the Gamma function, we have $p(x) = x$, and the desired relation is simply that $\partial x^{s+1} = (s+1)x^s$.

Berstein polynomials. Let us now investigate the existence of the relation in (4.3). This works over any field K, and so we relax the assumptions and allow $p \in K[x_1, \dots, x_n]$ to be any nonzero polynomial. Set $m = \deg p$. Since we are going to work algebraically, we let s be an independent variable, and consider the field of rational functions $K(s)$, and the Weyl algebra $A_n\big(K(s)\big)$ with coefficients in $K(s)$. We now endow the $K(s)$-vector space

$$M = K(s)[x_1, \dots, x_n, p^{-1}]$$

with the structure of a left $A_n\big(K(s)\big)$-module, as follows. Multiplication by polynomials with coefficients in $K(s)$ is defined as usual; and

$$\partial_j\big(fp^{-\ell}\big) = \frac{\partial f}{\partial x_j}p^{-\ell} + (s-\ell)f\frac{\partial p}{\partial x_j}p^{-(\ell+1)}.$$

One can check, based on the discussion in Lecture 2, that this defines a left action by the Weyl algebra with coefficients in $K(s)$. The formulas are easier to remember if we introduce a formal symbol p^s, with the property that

$$\partial_j p^s = sp^{-1}\frac{\partial p}{\partial x_j} \cdot p^s,$$

and write elements of Mp^s in the form $fp^{s-\ell}$. Then the formula from above is simply the (formally correct) differentiation rule

$$\partial_j\big(fp^{s-\ell}\big) = \frac{\partial f}{\partial x_j}p^{s-\ell} + (s-\ell)f\frac{\partial p}{\partial x_j}p^{s-(\ell+1)}. \tag{4.5}$$

The same calculation as in Lecture 3 shows that the filtration

$$F_j M = \big\{\, fp^{-\ell} \;\big|\; \deg f \leq (m+1)\ell \,\big\}$$

is compatible with the Bernstein filtration on $A_n\big(K(s)\big)$, and

$$\dim_{K(s)} F_j M = \binom{(m+1)\ell + n}{n}.$$

According to Lemma 3.11, M is therefore a holonomic module, of multiplicity at most $(m+1)^n$.

Now consider, for $k \geq 0$, the submodule $M_k \subseteq M$ generated by p^k; concretely,

$$M_k = A_n\big(K(s)\big) \cdot p^k \subseteq M.$$

Clearly $M_0 \supseteq M_1 \supseteq M_2 \supseteq$, and because M is holonomic, each M_k is holonomic, and the chain has to stabilize after at most $m(M)$ many steps. So there exists some $k \geq 0$ such that $M_{k+1} = M_k$. This means concretely that there is a differential operator $Q(s) \in A_n\big(K(s)\big)$ with the property that $Q(s)p^{k+1} = p^k$. Note that $Q(s)$ has coefficients in the field of rational functions $K(s)$, so there may be denominators. Let $d(s) \in K[s]$ be a nonzero polynomial such that $R(s) = d(s)Q(s)$ has coefficients in $K[s]$. Then we get $R(s)p^{k+1} = d(s)p^k$, which we can write symbolically as

$$R(s)p^{s+k+1} = d(s)p^{s+k}.$$

After replacing s by $s-k$ everywhere (which is compatible with the differentiation rule in (4.5), and therefore okay), we obtain the identity

$$R(s-k)p^{s+1} = d(s-k)p^s,$$

which has the same shape as (4.3). Now let $b(s) \in K[s]$ be the monic polynomial of minimal degree that satisfies a relation of the form

$$D(s)p^{s+1} = b(s)p^s$$

for some differential operator $D(s) \in A_n\big(K[s]\big)$.

Definition 4.6. The polynomial $b(s) \in K[s]$ is called the *Bernstein polynomial* of $p \in K[x_1, \ldots, x_n]$, and $D(s) \in A_n\big(K[s]\big)$ is called a *Bernstein operator* for p.

In fact, the set of all polynomials for which such a relation holds is closed under addition and multiplication by elements of $K[s]$, and therefore an ideal in $K[s]$. The Bernstein polynomial is then simply the unique monic generator of this ideal, keeping in mind that $K[s]$ is a principal ideal domain.

Note. The relation $D(s)p = b(s)$ in the module M implies (by induction on the exponent of p in the denominator) that $M_0 = M$, in the notation from above. Here is another way of looking at the Bernstein polynomial: Multiplication by s defines an endomorphism of the quotient module

$$M_0/M_1 = M/A_n\big(K(s)\big)p,$$

and $b(s)$ is the minimal polynomial for this endomorphism.

Let us finish by computing a few examples of Bernstein polynomials.

Example 4.7. In one variable, let $p = x$. Here $\partial x^{s+1} = (s+1)x^s$, and so we have $b(s) = s+1$ and $D(s) = \partial$.

Example 4.8. Still in one variable, take $p = x^2$. Now $\partial p^{s+1} = (s+1)2xp^s$, and

$$\partial^2 p^{s+1} = (s+1)\big(2p^s + 4x^2 sp^{s-1}\big) = (s+1)(2p^s + 4sp^s) = (s+1)(4s+2)p^s,$$

and therefore $b(s) = (s+1)(s+\frac{1}{2})$.

Example 4.9. The previous example generalizes to $p = x^m$; after applying ∂^m, one finds that $b(s) = (s+1)(s+\frac{m-1}{m})\cdots(s+\frac{1}{m})$.

Example 4.10. In n variables, we can take $p = x_1^{m_1} \cdots x_n^{m_n}$, and after applying the differential operator $\partial_1^{m_1} \cdots \partial_n^{m_n}$, we get

$$b(s) = \prod_{j=1}^{n} \prod_{k=1}^{m_j} \left(s - \frac{k}{m_j} \right)$$

Example 4.11. Another case that can be computed by hand is $p = x_1^2 + \cdots + x_n^2$. Here we again have

$$\partial_j^2 p^{s+1} = (s+1)\big(2p^s + 4x_j^2 s p^{s-1}\big)$$

by the calculation in the second example, and therefore

$$(\partial_1^2 + \cdots + \partial_n^2) p^{s+1} = (s+1)(2n+4s)p^s.$$

So the Bernstein polynomial in this case is $b(s) = (s+1)(s+\frac{n}{2})$.

These examples suggest that $s = -1$ is always a root of the Bernstein polynomial. It can be proved (using resolution of singularities) that all roots of the Berstein polynomial are negative rational numbers. In general, the Bernstein polynomial can be found using computer algebra systems (such as *Macaulay 2*); except when p is homogeneous, the shape of the Bernstein operator $D(s)$ is not easy to guess in advance, however. Here is a more complicated example for algebraic geometers.

Example 4.12. Consider the polynomial $p = x_1^2 + x_2^3$; this has a so-called cusp singularity at the origin. One can show that

$$\left(\frac{1}{27} \partial_2^3 + \frac{x_2}{6} \partial_1^2 \partial_2 + \frac{x_1}{8} \partial_1^3 \right) p^{s+1} = (s + \tfrac{5}{6})(s+1)(s+\tfrac{7}{6})p^s,$$

and so the Bernstein polynomial is $b(s) = (s+\frac{5}{6})(s+1)(s+\frac{7}{6})$.

The Bernstein polynomial is of interest in the study of hypersurface singularities. Indeed, the zero set of the polynomial p defines a hypersurface in affine space, to use the terminology from algebraic geometry, and many invariants of its singularities are related to the roots of the Bernsteint polynomial. For example, the largest root of the Bernstein polynomial is the so-called "log canonical threshold" of p.