Lecture 5: February 18

Basic facts about algebraic geometry. The goal of today's class is to give a geometric interpretation for the dimension $d(M)$ from last time. Suppose for the time being that K is an algebraically closed field (such as $\mathbb{C}$). We can then think of the polynomial ring $K[x_1, \ldots, x_n]$ as being the ring of algebraic functions on the affine space K^n. If $A_n = A_n(K)$ is the Weyl algebra, and $F_\bullet A_n$ is either the Bernstein filtration or the degree filtration, then $\operatorname{gr}^F A_n \cong K[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$, where ξ_j is the class of ∂_j. We can think of this polynomial ring in $2n$ variables as the ring of algebraic functions on $K^{2n} = K^n \times K^n$, viewed as the cotangent bundle of K^n. The additional variables $\xi_1, \ldots, \xi_n$, are linear functions on the fibers of the cotangent bundle. We will see below that $d(M)$ can be interpreted as the "dimension" of a certain subset of K^{2n}, called the *characteristic variety* of M.

Since algebraic geometry language will be useful for this, we start with a brief review of the basic correspondence between closed algebraic subsets of K^n and ideals in the polynomial ring $K[x_1, \ldots, x_n]$. To any ideal $I \subseteq K[x_1, \ldots, x_n]$, we can associate a closed subset

$$Z(I) = \bigl\{ (a_1, \ldots, a_n) \in K^n \bigm| f(a_1, \ldots, a_n) = 0 \text{ for every } f \in I \bigr\}$$

Since the polynomial ring is noetherian, every ideal is finitely generated, and so every closed subset of this type can in fact be defined by finitely many polynomial equations. Conversely, to a closed subset $Z \subseteq K^n$ defined by polynomial equations, we can associate the ideal

$$I_Z = \bigl\{ f \in K[x_1, \ldots, x_n] \bigm| f(a_1, \ldots, a_n) = 0 \text{ for every } (a_1, \ldots, a_n) \in Z \bigr\}$$

of all polynomials that vanish on Z. If $f^m \in I_Z$ for some $m \geq 1$, then of course also $f \in I_Z$ (because K is a field), and so I_Z is always a radical ideal. Here the radical of an ideal I is defined as

$$\sqrt{I} = \bigl\{ f \in K[x_1, \ldots, x_n] \bigm| f^m \in I \text{ for some } m \geq 1 \bigr\},$$

and an ideal is called a *radical ideal* if $I = \sqrt{I}$. One can show that

$$Z(I_Z) = Z \quad \text{and} \quad I_{Z(I)} = \sqrt{I}.$$

The second assertion is usually called the *Nullstellensatz*. One can summarize this by saying that $I \mapsto Z(I)$ and $Z \mapsto I_Z$ sets up a one-to-one correspondence

$$\bigl(\text{closed algebraic subsets of } K^n\bigr) \longleftrightarrow \bigl(\text{radical ideals in } K[x_1, \ldots, x_n]\bigr)$$

This correspondence reverses the order, meaning that $I_1 \subseteq I_2$ iff $Z(I_2) \subseteq Z(I_1)$. The quotient ring $K[x_1, \ldots, x_n]/I_Z$ can be viewed as the ring of algebraic functions on the algebraic variety Z, where a polynomial determines a function on Z by restriction (and I_Z is the ideal of functions whose restriction to Z is zero).

Since K is algebraically closed, every maximal ideal in $K[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $(a_1, \ldots, a_n) \in K^n$, and so under the above correspondence, maximal ideals in the polynomial ring correspond to points of K^n. More generally, prime ideals correspond to irreducible algebraic subsets, where irreducible means that the set cannot be written as a union of two strictly smaller algebraic sets. One can define the *dimension* of a closed algebraic subset $Z \subseteq K^n$ in two equivalent ways: geometrically, as the length of the longest strictly decreasing chain of irreducible closed algebraic subsets

$$Z \supseteq Z_0 \supset Z_1 \supset \cdots \supset Z_d$$

contained in Z; algebraically, as the length of the longest strictly increasing chain of prime ideals

$$I_Z \subseteq P_0 \subset P_1 \subset \cdots \subset P_d$$

containing I_Z. This notion of dimension is known as the *Krull dimension*, and is denoted by $\dim Z$. The geometric picture of the chain is that Z_0 has dimension d, Z_1 has dimension $d-1$, and so on, down to Z_d, which has dimension 0 (and hence is a point). Since ideals in $K[x_1, \dots, x_n]$ containing I_Z are in one-to-one correspondence with ideals in the quotient ring $K[x_1, \dots, x_n]/I_Z$, one also has

$$\dim Z = \dim\bigl(K[x_1, \dots, x_n]/I_Z\bigr),$$

where the dimension $\dim R$ of a commutative ring R is by definition the length of the longest strictly increasing chain of prime ideals in R. The polynomial ring $K[x_1, \dots, x_n]$ has dimension n, of course.

We shall also need the notion of the support of a module. Let M be a finitely generated module over $K[x_1, \dots, x_n]$. Then

$$\operatorname{Supp} M \subseteq K^{2n}$$

is the set of all points $(a_1, \dots, a_n) \in K^n$ such that the localization of M at the maximal ideal $(x_1 - a_1, \dots, x_n - a_n)$ is nontrivial. The geometric picture is that M corresponds to a (coherent) sheaf on K^n, and the support of M is the set of points where the stalk of this sheaf is nontrivial. (In other words, the complement of $\operatorname{Supp} M$ is the largest open set on which the sheaf is trivial.) The support of M is a closed algebraic subset, defined by the annihilator ideal

$$\operatorname{Ann} M = \operatorname{Ann}_{K[x_1,\dots,x_n]} M = \bigl\{\, f \in K[x_1, \dots, x_n] \bigm| fm = 0 \text{ for every } m \in M \,\bigr\}.$$

We have $\dim \operatorname{Supp} M = \dim K[x_1, \dots, x_n]/\operatorname{Ann} M$.

Characteristic varieties. Now we return to modules over the Weyl algebra. Let M be a finitely generated left A_n-module. If we choose a good filtration $F_\bullet M$, compatible with the Bernstein filtration on A_n, then the associated graded module $\operatorname{gr}^F M$ is finitely generated over $\operatorname{gr}^F A_n$, the polynomial ring in $2n$ variables. One of the basic facts about Hilbert polynomials is that the degree $d(M)$ of the Hilbert polynomial of $\operatorname{gr}^F M$ is equal to the dimension of $\operatorname{Supp} \operatorname{gr}^F M$; in symbols,

$$d^B(M) = \dim \operatorname{Supp}(\operatorname{gr}^F M) = \dim \operatorname{gr}^F A_n / \operatorname{Ann}(\operatorname{gr}^F M).$$

I have added the superscript B to indicate that this notion of dimension is related to the Bernstein filtration on A_n. We would now like to have an analogous definition for the degree filtration on the Weyl algebra, since that is the case that generalizes to arbitrary $\mathscr{D}$-modules.

From now on, we use the notation $F_\bullet A_n$ for the filtration by the degree of differential operators. Let M be a finitely generated left A_n-module, and choose a good filtration $F_\bullet M$ compatible with the degree filtration on A_n. We define

$$I(M, F_\bullet) = \operatorname{Ann}_{\operatorname{gr}^F A_n}(\operatorname{gr}^F M)$$

as the annihilator of $\operatorname{gr}^F M$, and use the notation

$$J(M) = \sqrt{I(M, F_\bullet M)}$$

for the radical ideal. We will see in a moment that $J(M)$ only depends on M, but not on the particular good filtration chosen, justifying the notation. As we said earlier, the closed subset of K^{2n} corresponding to the radical ideal $J(M)$ is the support of the module $\operatorname{gr}^F M$.

Definition 5.1. The *characteristic variety* $\operatorname{Ch}(M)$ is the closed algebraic subset of K^{2n} corresponding to the radical ideal $J(M)$. Let

$$d^{\deg}(M) = \dim \operatorname{Ch}(M) = \dim\bigl(\operatorname{gr}^F A_n / J(M)\bigr)$$

be the dimension of the characteristic variety.

Examples show that the ideal $I(M, F_\bullet M)$ depends on the filtration. Nevertheless, the radical ideal $J(M)$ and the characteristic variety $\mathrm{Ch}(M)$ only depend on M.

Proposition 5.2. *The ideal $J(M)$ only depends on M, but not on the choice of good filtration $F_\bullet M$. The same is therefore true for $\mathrm{Ch}(M)$.*

Proof. We first need to describe the annihilator of $\mathrm{gr}^F M$ more concretely. For a differential operator $P \in F_k A_n$ of order exactly k, we denote by $[P]$ its image in $\mathrm{gr}_k^F A_n$; this is usually called the (principal) symbol of P. Likewise, if $m \in F_j M$, we write $[m] \in \mathrm{gr}_j^F M$ for its image in the associated graded module. The module structure on $\mathrm{gr}^F M$ is then defined by setting

$$[P] \cdot [m] = [Pm] \in \mathrm{gr}_{k+j}^F M$$

for $[P] \in \mathrm{gr}_k^F A_n$ and $[m] \in \mathrm{gr}_j^F M$. Thus $[P] \cdot [m] = 0$ means that $Pm \in F_{k+j-1} M$ (but it does not mean that $Pm = 0$). Since $\mathrm{gr}^F M$ is a graded module, the annihilator ideal $\mathrm{Ann}(\mathrm{gr}^F M)$ is a homogeneous ideal; by what we just said, it is generated by all those homogeneous elements $[P] \in \mathrm{gr}_k^F A_n$ with the property that

$$P \cdot F_j M \subseteq F_{k+j-1} M$$

for every $j \geq 0$. The radical ideal $\sqrt{I(M, F_\bullet M)}$ is therefore generated by those homogeneous elements $[P] \in \mathrm{gr}_k^F A_n$ such that, for some $m \geq 1$, one has

$$P^m \cdot F_j M \subseteq F_{mk+j-1} M \tag{5.3}$$

for every $j \geq 0$.

Now let $G_\bullet M$ be another good filtration. By Corollary 2.15, the two good filtrations are comparable, and so there is some $j_0 \geq 0$ such that

$$F_j M \subseteq G_{j+j_0} M \quad \text{and} \quad G_j M \subseteq F_{j+j_0} M$$

for every $j \geq 0$. Suppose that $[P] \in \mathrm{gr}_k^F A_n$ belongs to the radical of $I(M, F_\bullet M)$, hence that we have (5.3) for some $m \geq 1$. Let $\ell \geq 1$ be any integer. We have

$$P^{\ell m} \cdot G_j M \subseteq P^{\ell m} \cdot F_{j+j_0} M \subseteq F_{\ell mk+j+j_0-\ell} M \subseteq G_{\ell mk+j+2j_0-\ell} M.$$

If we take $\ell = 2j_0 + 1$ and $m' = \ell m$, then we have

$$P^{m'} \cdot G_j M \subseteq G_{m'k+j-1} M$$

for every $j \geq 0$, and so P belongs to the radical of $I(M, G_\bullet M)$. Since the situation is symmetric, we conclude that $\sqrt{I(M, G_\bullet)} = \sqrt{I(M, F_\bullet M)}$, and hence that $J(M)$ is independent of the choice of good filtration. □

Example 5.4. One can tell from the characteristic variety whether or not a finitely generated A_n-module M is actually finitely generated over the polynomial ring $K[x_1, \dots, x_n]$. Suppose that M is finitely generated over $K[x_1, \dots, x_n]$. Then setting $F_{-1} M = \{0\}$ and $F_j M = M$ for $j \geq 0$ defines a good filtration, and since $\mathrm{gr}_j^F M = 0$ for $j \neq 0$, every element in $\mathrm{gr}^F A_n$ of strictly positive degree annihilates $\mathrm{gr}^F M$. This means that $\mathrm{Ch}(M)$ is defined by the ideal $(\xi_1, \dots, \xi_n)$ in the polynomial ring $\mathrm{gr}^F A_n = K[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$; in other words, $\mathrm{Ch}(M)$ is the "zero section".

Conversely, if $\mathrm{Ch}(M)$ is the zero section, then M is actually finitely generated over $K[x_1, \dots, x_n]$. Here is the reason. Choose a good filtration $F_\bullet M$, so that $\mathrm{gr}^F M$ is finitely generated over $\mathrm{gr}^F A_n = K[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$. By assumption, some power of each ξ_j belongs to the annihilator, which means that $\xi_1^{e_1} \cdots \xi_n^{e_n}$ acts trivially on $\mathrm{gr}^F M$ as long as $e_1 + \cdots + e_n$ is sufficiently large. Thus the finitely many generators of $\mathrm{gr}^F M$ over $\mathrm{gr}^F A_n$, together with their finitely many images under the elements $\xi_1^{e_1} \cdots \xi_n^{e_n}$ for $e \in \mathbb{N}^n$, generate $\mathrm{gr}^F M$ over $K[x_1, \dots, x_n]$. But this implies that M itself is finitely generated over $K[x_1, \dots, x_n]$.

Equality of dimensions. In the next few lectures, we are going to prove that the two notions of dimension (with respect to the Bernstein filtration and with respect to the degree filtration) agree: for any finitely generated A_n-module, one has

$$d^B(M) = d^{\deg}(M).$$

This will tell us in particular that the Bernstein inequality $d(M) \geq n$ also holds with respect to the degree filtration. The geometric interpretation is that the characteristic variety $\operatorname{Ch}(M)$ always has dimension at least n. The strategy for proving this is to relate two kinds of dimension to a third invariant of M, which is of a more homological nature and can be defined without reference to good filtrations. The invariant is defined in terms of the Ext-modules $\operatorname{Ext}_R^j(M, R)$, namely

$$j(M) = \min\bigl\{\, j \geq 0 \bigm| \operatorname{Ext}_R^j(M, R) \neq 0 \,\bigr\}.$$

The precise result that we are going to prove is that

$$d^B(M) = 2n - j(M) = d^{\deg}(M).$$

Let me end with a brief reminder about Ext-modules. Recall that if R is any ring, and if M and N are two left R-modules, we can form the group

$$\operatorname{Hom}_R(M, N)$$

of all left R-linear morphisms from M to R. This defines a contravariant functor $\operatorname{Hom}_R(-, N)$ from left R-modules to groups, and $\operatorname{Ext}_R^j(M, N)$ is by definition the j-th derived functor. Concretely, one computes $\operatorname{Ext}_R^j(M, N)$ by choosing a resolution of M by free left R-modules,

$$\dotsb \to L_2 \to L_1 \to L_0 \to M \to 0,$$

and then applying the functor $\operatorname{Hom}_R(-, N)$ to this resolution. Thus $\operatorname{Ext}_R^j(M, N)$ is the j-th cohomology group of the complex

$$0 \to \operatorname{Hom}_R(L_0, N) \to \operatorname{Hom}_R(L_1, N) \to \operatorname{Hom}_R(L_2, N) \to \dotsb$$

In particular, $\operatorname{Ext}_R^0(M, N) = \operatorname{Hom}_R(M, N)$. Note that unless R is commutative, $\operatorname{Hom}_R(M, N)$ typically no longer has the structure of a left or right R-module. But in the special case where $N = R$, we can use the right R-module structure on the ring R to endow $\operatorname{Hom}_R(M, R)$ with the structure of a right R-module. Concretely, for $f \in \operatorname{Hom}_R(M, R)$, and for $r \in R$, we define $f \cdot r \in \operatorname{Hom}_R(M, R)$ by the formula

$$(f \cdot r)(x) = f(x)r.$$

Since the multiplication in R is associative, $f \cdot r$ is again left R-linear. Using a resolution as above, it follows that each $\operatorname{Ext}_R^j(M, R)$ is naturally a right R-module. (Similar comments apply if we work with right R-modules.)

Exercises.

Exercise 5.1. Let $M = A_1/A_1(x)$ be the left A_1-module related to the δ-function. Show that the image of $1 \in A_1$ and the image of $\partial \in A_1$ both generate M, but that the two resulting good filtrations $F_\bullet M$ and $G_\bullet M$ give rise to different annihilator ideals: $I(M, F_\bullet M) \neq I(M, G_\bullet M)$.

Exercise 5.2. Let $I \subseteq A_n$ be a left ideal, and let $F_j I = I \cap F_j A_n$ be the induced filtration. Describe the ideal $\operatorname{Ann}(\operatorname{gr}^F I)$ inside $\operatorname{gr}^F A_n$ in concrete terms.