Lecture 7: February 25

Review from last time. Let me briefly recall where we are at. The general setting is that R is a (non-commutative) ring with 1, endowed with a filtration $F_\bullet R$, such that the associated graded ring $S = \operatorname{gr}^F R$ is commutative and nonsingular of dimension $\dim S = 2n$. The prototypical example is of course $R = A_n(K)$, with S being the polynomial ring in $2n$ variables. Given a finitely generated left R-module M, together with a good filtration $F_\bullet M$, we are trying to compare

$$\operatorname{Ext}_R^j(M,R) \quad \text{and} \quad \operatorname{Ext}_S^j(\operatorname{gr}^F M, S).$$

More precisely, we want to show that the two integers

$$j(M) = \min\bigl\{\, j \geq 0 \bigm| \operatorname{Ext}_R^j(M,R) \neq 0 \,\bigr\}$$
$$j(\operatorname{gr}^F M) = \min\bigl\{\, j \geq 0 \bigm| \operatorname{Ext}_S^j(\operatorname{gr}^F M, S) \neq 0 \,\bigr\}$$

are always equal to each other. To this end, we had constructed a resolution

$$\cdots \to L_2 \to L_1 \to L_0 \to M \to 0 \tag{7.1}$$

of M by free left R-modules, such that (1) each L_j has a good filtration; (2) the morphisms in the resolution respect the filtrations; (3) the induced complex

$$\cdots \to \operatorname{gr}^F L_2 \to \operatorname{gr}^F L_1 \to \operatorname{gr}^F L_0 \to \operatorname{gr}^F M \to 0 \tag{7.2}$$

is still exact, and therefore gives a resolution of $\operatorname{gr}^F M$ by free S-modules. In fact, each L_j was a direct sum of copies of $R(e)$, for different values of $e \in \mathbb{Z}$, where $R(e) = R$ as a left R-module, but with the good filtration $F_i R(e) = F_{e+i} R$.

Now each $L_j^* = \operatorname{Hom}_R(L_j, R)$ is a right R-module, and the j-th cohomology of the complex of right R-modules

$$0 \to L_0^* \to L_1^* \to L_2^* \to \cdots$$

is equal to $\operatorname{Ext}_R^j(M,R)$. We further showed that each L_j^* again has a good filtration (as a right R-module) – in fact, each L_j^* is again a direct sum of copies of $R(e)$, viewed as a right R-module, by one of the exercises from Lecture 6. One has

$$\operatorname{gr}^F L_j^* \cong \operatorname{Hom}_S\bigl(\operatorname{gr}^F L, S\bigr),$$

and because of the exactness of (7.2), it follows that the j-th cohomology of the complex of graded S-modules

$$0 \to \operatorname{gr}^F L_0^* \to \operatorname{gr}^F L_1^* \to \operatorname{gr}^F L_2^* \to \cdots$$

is equal to $\operatorname{Ext}_S^j(\operatorname{gr}^F M, S)$. So our problem comes down to comparing the cohomology of a filtered complex to the cohomology of the associated graded complex. This can be done using the formalism of spectral sequences.

The spectral sequence of a filtered complex. Generally speaking, a *spectral sequence* is a sequence of complexes

$$\bigl(E_\ell^\bullet, d_\ell\bigr),$$

indexed by $\ell \in \mathbb{N}$. Here each $E_\ell^\bullet$ is a complex of vector spaces, modules, or whatever, and the differentials $d_\ell \colon E_\ell^\bullet \to E_\ell^{\bullet+1}$ are morphisms in the appropriate category. The complex $E_\ell^\bullet$ is often called the "ℓ-th page" of the spectral sequence. What makes a sequence of complexes into a spectral sequence is that each $E_{\ell+1}^\bullet$ is obtained from the previous complex $E_\ell^\bullet$ by taking cohomology:

$$E_{\ell+1}^n \cong H^n\bigl(E_\ell^\bullet\bigr) = \frac{\ker\bigl(d_\ell \colon E_\ell^n \to E_\ell^{n+1}\bigr)}{\operatorname{im}\bigl(d_\ell \colon E_\ell^{n-1} \to E_\ell^n\bigr)}$$

Of course, taking cohomology kills the differentials, and so the new differential $d_{\ell+1}$ has to come from somewhere else.

Typically, there is some quantity that one would like to compute, and the initial page of the spectral sequence is a known (or easily obtained) "approximation" to this quantity. As ℓ gets larger, the approximation gets better and better, and things eventually "converge" to the quantity one is trying to compute. This is of course just a rough description; I am going to make it more precise later on.

In my opinion, the best example for understanding spectral sequences is the spectral sequence of a filtered complex. Suppose then that we have a complex $(K^\bullet, d)$, consisting of vector spaces, modules, or whatever:

$$\cdots \to K^{n-1} \xrightarrow{d} K^n \xrightarrow{d} K^{n+1} \to \cdots$$

We are interested in computing the cohomology

$$H^n(K^\bullet) = \frac{\ker\bigl(d\colon K^n \to K^{n+1}\bigr)}{\operatorname{im}\bigl(d\colon K^{n-1} \to K^n\bigr)}$$

of this complex. Suppose also that the complex is filtered, meaning that each K^n has an increasing filtration $F_\bullet K^n$, possibly infinite in both directions,

$$\cdots \subseteq F_j K^n \subseteq F_{j+1} K^n \subseteq \cdots$$

that is compatible with the differentials in the complex, meaning that $d(F_j K^n) \subseteq F_j K^{n+1}$. We also assume that

$$\bigcup_{j \in \mathbb{Z}} F_j K^n = K^n \quad \text{and} \quad F_j K_n = 0 \text{ for } j \ll 0. \tag{7.3}$$

The compatibility with the differential means that each $F_j K^\bullet$ is a subcomplex of $K^\bullet$, and so we obtain a filtration on the cohomology of $K^\bullet$ by setting

$$F_j H^n(K^\bullet) = \operatorname{im}\Bigl(H^n(F_j K^\bullet) \to H^n(K^\bullet)\Bigr).$$

In fact, it is not hard to see that

$$F_j H^n(K^\bullet) = \frac{F_j K^n \cap \ker d + d(K^{n-1})}{d(K^{n-1})} \cong \frac{F_j K^n \cap \ker d}{F_j K^n \cap d(K^{n-1})}.$$

and hence that that the associated graded object is given by

$$\operatorname{gr}_j^F H^n(K^\bullet) \cong \frac{F_j K^n \cap \ker d}{F_{j-1} K^n \cap \ker d + F_j K^n \cap d(K^{n-1})}.$$

The spectral sequence is going to let us compute not $H^n(K^\bullet)$ itself, but the graded pieces for the above filtration. The first approximation to this – and the starting point for the spectral sequence – is the associated graded complex $\operatorname{gr}^F K^\bullet$, with the induced differential, and terms

$$\cdots \to \operatorname{gr}^F K^{n-1} \xrightarrow{d} \operatorname{gr}^F K^n \xrightarrow{d} \operatorname{gr}^F K^{n+1} \to \cdots$$

Again, it is not hard to show that

$$H^n\bigl(\operatorname{gr}_j^F K^\bullet\bigr) = \frac{\ker\bigl(d\colon \operatorname{gr}_j^F K^n \to \operatorname{gr}_j^F K^{n+1}\bigr)}{\operatorname{im}\bigl(d\colon \operatorname{gr}_j^F K^{n-1} \to \operatorname{gr}_j^F K^n\bigr)} \cong \frac{F_j K^n \cap d^{-1}(F_{j-1} K^{n+1})}{F_{j-1} K^n + d(F_j K^n)}.$$

Note that this is usually not the same as $\operatorname{gr}_j^F H^n(K^\bullet)$.

Example 7.4. Here is a typical example of a filtered complex. Let $(A, \mathfrak{m})$ be a local ring, and suppose that $K^\bullet$ is a complex of free A-modules of finite rank. We can filter each K^n by powers of the maximal ideal,

$$K^n \supseteq \mathfrak{m} K^n \supseteq \mathfrak{m}^2 K^n \supseteq \cdots,$$

which amounts to setting $F_0 K^n = K^n$ and $F_{-j} K^n = \mathfrak{m}^j K^n$ for $j \geq 0$. Here the second condition in (7.3) does not hold, but it turns out that one can weaken this to the condition that

$$\bigcap_{j \in \mathbb{Z}} (F_j K^n + L) = L$$

for every submodule $L \subseteq K_n$, which does hold in this example (by Krull's theorem). In particular, the intersection of all $F_j K^n$ equals zero, which makes sense if we think of elements of $\mathfrak{m}^j$ as functions that vanish to order j; going further down in the filtration on K^n therefore means getting closer to zero.

Example 7.5. The long exact sequence in cohomology is a toy example of a spectral sequence. Suppose that we just have one subcomplex $K_0^\bullet \subseteq K^\bullet$. Together with the quotient complex, this makes a short exact sequence

$$0 \to K_0^\bullet \to K^\bullet \to K_1^\bullet \to 0$$

and so we get a long exact sequence in cohomology:

$$\cdots \to H^{n-1}\big(K_1^\bullet\big) \to H^n\big(K_0^\bullet\big) \to H^n\big(K^\bullet\big) \to H^n\big(K_1^\bullet\big) \to H^{n+1}\big(K_0^\bullet\big) \to \cdots$$

This tells us how the cohomology of $K^\bullet$ is related to the cohomology of the subcomplex and the quotient complex: there are additional maps $H^n(K_1^\bullet) \to H^{n+1}(K_0^\bullet)$, and the two graded pieces of $H^n(K^\bullet)$ are the cokernel respectively kernel of these maps. If the filtration is longer, then the picture is still similar, but it takes more steps to get from the cohomology of the associated graded complex to the associated graded of the cohomology of $K^\bullet$.

As explained above, we may think of elements of $F_j K^n$ as being "close to zero" when $j \ll 0$. The idea behind the spectral sequence is to "approximate" the condition $x \in F_j K^n$ and $dx = 0$ by the weaker condition $dx \in F_{j-\ell} K^n$, and then increasing the value of $\ell \geq 0$. In other words, we are approximating $F_j K^n \cap \ker d$ by the decreasing sequence of submodules $F_j K^n \cap d^{-1}(F_{j-\ell} K^{n+1})$ for $\ell \geq 0$; this makes sense because of the condition in (7.3). With this in mind, we can now give the precise definition of the spectral sequence of a filtered complex.

For each $n, j \in \mathbb{Z}$ and each $\ell \in \mathbb{N}$, we define

$$Z_{\ell,j}^n = F_j K^n \cap d^{-1}\big(F_{j-\ell} K^{n+1}\big).$$

In other words, an element $x \in F_j K^n$ belongs to $Z_{\ell,j}^n$ iff $dx \in F_{j-\ell} K^{n+1}$. By construction, the differential $d\colon K^n \to K^{n+1}$ induces a morphism

$$d_\ell\colon Z_{\ell,j}^n \to Z_{\ell,j-\ell}^{n+1}, \quad x \mapsto dx.$$

Similarly, for each $n, j \in \mathbb{Z}$ and each $\ell \in \mathbb{N}$, we define

$$\begin{aligned} B_{\ell,j}^n &= Z_{\ell,j}^n \cap \Big(F_{j-1} K^n + d\big(F_{j+\ell-1} K^{n-1}\big)\Big) \\ &= F_{j-1} K^n \cap d^{-1}\big(F_{j-\ell} K^{n+1}\big) + F_j K^n \cap d\big(F_{j+\ell-1} K^{n-1}\big) \\ &= Z_{\ell-1,j-1}^n + d\big(Z_{\ell-1,j+\ell-1}^{n-1}\big). \end{aligned}$$

We can then form the quotient

$$E_{\ell,j}^n = Z_{\ell,j}^n / B_{\ell,j}^n,$$

and observe that d_ℓ maps $B_{\ell,j}^n$ into $B_{\ell,j-\ell}^{n+1}$, and therefore induces a morphism

$$d_\ell\colon E_{\ell,j}^n \to E_{\ell,j-\ell}^{n+1}$$

with the property that $d_\ell \circ d_\ell = 0$.

To obtain a complex $(E_\ell^\bullet, d_\ell)$, we consider the graded modules

$$E_\ell^n = \bigoplus_{j\in\mathbb{Z}} E_{\ell,j}^n.$$

By construction, the differential $d_\ell \colon E_\ell^n \to E_\ell^{n+1}$ reduces the degree by ℓ.

Example 7.6. For $\ell = 0$, we have

$$Z_{0,j}^n = F_j K^n \quad \text{and} \quad B_{0,j}^n = F_{j-1} K^n,$$

since $d(F_j K^n) \subseteq F_j K^{n+1}$ by assumption. Consequently,

$$E_{0,j}^n = \frac{F_j K_n}{F_{j-1} K_n} = \mathrm{gr}_j^F K^n,$$

with differential d_0 induced by d. Given (7.3), it also makes sense to set

$$Z_{\infty,j}^n = F_j K^n \cap \ker d \quad \text{and} \quad B_{\infty,j}^n = F_{j-1} K^n \cap \ker d + F_j K^n \cap d\big(K^{n-1}\big),$$

which extends the above notation (formally) to $\ell = \infty$. Then

$$E_{\infty,j}^n = \frac{F_j K^n}{F_{j-1} K^n \cap \ker d + F_j K^n \cap d(K^{n-1})} \cong \mathrm{gr}_j^F H^n(K^\bullet),$$

according to our earlier calculation.

Now let us show that the complexes $(E_\ell^\bullet, d_\ell)$ really form a spectral sequence.

Proposition 7.7. *For each $n, j \in \mathbb{Z}$ and each $\ell \in \mathbb{N}$, one has*

$$E_{\ell+1,j}^n \cong H^n\big(E_{\ell,j}^\bullet, d_\ell\big).$$

Proof. Set $H_{\ell,j}^n = H^n(E_{\ell,j}^\bullet)$, and recall that this is the cohomology of the complex

$$Z_{\ell,j+\ell}^{n-1}/B_{\ell,j+\ell}^{n-1} \xrightarrow{d_\ell} Z_{\ell,j}^n/B_{\ell,j}^n \xrightarrow{d_\ell} Z_{\ell,j-\ell}^{n+1}/B_{\ell,j-\ell}^{n+1}.$$

We start by defining a function

$$\phi \colon E_{\ell+1,j}^n \to H_{\ell,j}^n.$$

Suppose that $x \in Z_{\ell+1,j}^n$. Then also $x \in Z_{\ell,j}^n$ and

$$d_\ell x = dx \in d\big(Z_{\ell+1,j}^n\big) \subseteq B_{\ell,j-\ell}^{n+1},$$

and so x defines a class $\phi(x) \in H_{\ell,j}^n$. This class does not depend on the choice of representative, because

$$B_{\ell+1,j}^n = Z_{\ell+1,j}^n \cap \Big(B_{\ell,j}^n + d\big(Z_{\ell,j+\ell}^{n-1}\big)\Big)$$

by the lemma below. Indeed, we see that $x \in B_{\ell+1,j}^n$ if and only if its image in $H_{\ell,j}^n$ is zero, and so ϕ is well-defined and injective.

It remains to argue that ϕ is also surjective. Any class in $H_{\ell,j}^n$ can be represented by an element $x \in Z_{\ell,j}^n$ with $d_\ell x \in B_{\ell,j-\ell}^{n+1}$. After unwinding the definitions, this is saying that $x \in F_j K^n$ and $dx \in F_{j-\ell} K^{n+1}$ and

$$dx = dx' + y$$

for some $x' \in F_{j-1} K^n$ with $dx' \in F_{j-\ell} K^{n+1}$ and some $y \in F_{j-\ell-1} K^{n+1}$. Thus

$$x - x' \in F_j K^n \cap d^{-1}\big(F_{j-\ell-1} K^{n+1}\big) = Z_{\ell+1,j}^n,$$

and after replacing x by $x - x'$, we can assume from the beginning that $x \in Z_{\ell+1,j}^n$. But this means exactly that the given class is in the image of ϕ. □

Lemma 7.8. *One has*

$$B_{\ell+1,j}^n = Z_{\ell+1,j}^n \cap \Big(B_{\ell,j}^n + d\big(Z_{\ell,j+\ell}^{n-1}\big)\Big)$$

for every $j, n \in \mathbb{Z}$ and every $\ell \in \mathbb{N}$.

Proof. Unwinding the definitions shows that

$$B^n_{\ell,j} + d\bigl(Z^{n-1}_{\ell,j+\ell}\bigr) = F_{j-1}K^n \cap d^{-1}\bigl(F_{j-\ell}K^{n+1}\bigr) + F_jK^n \cap d\bigl(F_{j+\ell}K^{n-1}\bigr)$$

and so the intersection with $Z^n_{\ell+1,j} = F_jK^n \cap d^{-1}(F_{j-\ell-1}K^{n+1})$ equals

$$F_{j-1}K^n \cap d^{-1}\bigl(F_{j-\ell-1}K^{n+1}\bigr) + F_jK^n \cap d\bigl(F_{j+\ell}K^{n-1}\bigr) = B^n_{\ell+1,j}. \qquad \square$$

In what sense does the spectral sequence of a filtered complex "converge"? Note that the $Z^n_{\ell,j}$ form a decreasing chain of submodules of F_jK^n with

$$Z^n_{\infty,j} = \bigcap_{\ell\in\mathbb{N}} Z^n_{\ell,j}.$$

Proposition 7.7 shows that $E^n_{\ell+1,j}$ is a subquotient of $E^n_{\ell,j}$, but there is in general no natural morphism from one to the other, which means that one cannot take a (direct or inverse) limit in the algebraic sense. Fortunately, what happens almost always in practice is that, for each fixed $j, n \in \mathbb{Z}$, the modules $E^n_{\ell,j}$ stabilize for sufficiently large ℓ. In fact, one has the following necessary and sufficient condition for stabilization, in terms of the filtration on the complex.

Proposition 7.9. *Fix some $n \in \mathbb{Z}$. The differential $d_\ell \colon E^n_\ell \to E^{n+1}_\ell$ vanishes for every $\ell \geq \ell_0$ if, and only if, the filtration satisfies*

$$F_jK^{n+1} \cap d\bigl(K^n\bigr) = F_jK^{n+1} \cap d\bigl(F_{j+\ell_0-1}K^n\bigr)$$

for every $j \in \mathbb{Z}$.

Proof. The differential $d_\ell \colon E^n_\ell \to E^{n+1}_\ell$ vanishes for every $\ell \geq \ell_0$ exactly when $d(Z^n_{\ell,j}) \subseteq B^{n+1}_{\ell,j-\ell}$ for every $\ell \geq \ell_0$ and every $j \in \mathbb{Z}$. After replacing j by $j+\ell$, this translates into the condition that

$$\begin{aligned} &F_jK^{n+1} \cap d\bigl(F_{j+\ell}K^n\bigr) \\ &\qquad \subseteq F_{j-1}K^{n+1} \cap d^{-1}\bigl(F_{j-\ell}K^{n+2}\bigr) + F_jK^{n+1} \cap d\bigl(F_{j+\ell-1}K^n\bigr), \end{aligned}$$

or after intersecting with $d(F_{j+\ell}K^n)$,

$$F_jK^{n+1} \cap d\bigl(F_{j+\ell}K^n\bigr) = F_{j-1}K^{n+1} \cap d\bigl(F_{j+\ell}K^n\bigr) + F_jK^{n+1} \cap d\bigl(F_{j+\ell-1}K^n\bigr).$$

Recursively applying this identity (for $\ell \geq \ell_0$), and using the fact that the filtration on K^n is exhaustive, we can rewrite this in the equivalent form

$$F_jK^{n+1} \cap d\bigl(K^n\bigr) = F_{j-1}K^{n+1} \cap d\bigl(K^n\bigr) + F_jK^{n+1} \cap d\bigl(F_{j+\ell_0-1}K^n\bigr).$$

According to (7.3), there is some $j_0 \in \mathbb{Z}$ with $F_{j_0}K^{n+1} = 0$. We now get the desired conclusion by recursively applying the identity above (for $j \geq j_0$). $\square$

Corollary 7.10. *If there is some $\ell_0 \in \mathbb{N}$ with the property that*

$$\begin{aligned} F_jK^{n+1} \cap d\bigl(K^n\bigr) &= F_jK^{n+1} \cap d\bigl(F_{j+\ell_0-1}K^n\bigr) \\ F_jK^n \cap d\bigl(K^{n-1}\bigr) &= F_jK^n \cap d\bigl(F_{j+\ell_0-1}K^{n-1}\bigr) \end{aligned}$$

for every $j \in \mathbb{Z}$, then one has $E^n_{\ell_0} = E^n_\infty$.

For example, one has $E^n_1 = E^n_\infty$ exactly when the differential d is strictly compatible with the filtration, in the sense that $F_jK^n \cap d(K^{n-1}) = d(F_jK^{n-1})$ (and the same condition with $n+1$ in place of n).

Note. I have been using the "natural" indexing for the spectral sequence, where n is the position in the complex $K^\bullet$, and j the degree with respect to the filtration on K^n. For historical reasons, people usually index their spectral sequences differently, and our $E^n_{\ell,j}$ is usually denoted by $E^{-j,n+j}_\ell$. (This looks more natural in the special case of a double complex.)

Application to our problem. Now we return to the case of a finitely generated left R-module M, endowed with a good filtration $F_\bullet M$. If we apply the spectral sequence formalism to the complex of right R-modules

$$0 \to L_0^* \to L_1^* \to L_2^* \to \cdots ,$$

with the good filtration $F_\bullet L_j^*$ constructed earlier, we obtain a spectral sequence with $E_0^j = \mathrm{gr}^F L_j^*$ and with differential d_0 induced by the differential in the original complex. It follows that

$$E_1^j = \mathrm{Ext}_S^j\big(\mathrm{gr}^F M, S\big),$$

because the complex in (7.2) is a free resolution of $\mathrm{gr}^F M$. On the other hand, the complex in (7.1) is a free resolution of M, and so we get

$$E_\infty^j = \mathrm{gr}^F \mathrm{Ext}_R^j(M, R).$$

Recall that we are trying to prove the identity $j(M) = j(\mathrm{gr}^F M)$. The first thing we should do is check that the spectral sequence converges, in the sense that each E_ℓ^j stabilizes for $\ell \gg 0$. This is a consequence of the following lemma about good filtrations.

Lemma 7.11. *Let $(K^\bullet, d)$ be a complex of left (or right) R-modules, and suppose that each K^n has a good filtration $F_\bullet K^n$ such that $d(F_j K^n) \subseteq F_j K^{n+1}$ for every $j, n \in \mathbb{Z}$. Then for every $n \in \mathbb{Z}$, there is some $j_0 \in \mathbb{N}$ such that*

$$F_j K^{n+1} \cap d\big(K^n\big) = F_j K^{n+1} \cap d\big(F_{j+j_0} K^n\big).$$

Proof. On the submodule $d(K^n) \subseteq K^{n+1}$, we have two good filtrations, one induced by the good filtration on K^{n+1}, the other by the good filtration on K^n. Let us denote these by

$$F_j d(K^n) = F_j K^{n+1} \cap d(K^n) \quad \text{and} \quad G_j d(K^n) = d(F_j K^n).$$

The first filtration is good because $\mathrm{gr}^F d(K^n)$ is a submodule of the finitely generated S-module $\mathrm{gr}^F K^{n+1}$; the second filtration is good because $\mathrm{gr}^G d(K^n)$ is a quotient module of the finitely generated S-module $\mathrm{gr}^F K^n$. In both cases, we are using the fact that S is noetherian. By Corollary 2.15, there is an integer $j_0 \geq 0$ such that

$$F_j d(K^n) \subseteq G_{j+j_0} d(K^n)$$

for every $j \in \mathbb{Z}$. We get the result by intersecting both sides with $F_j K^{n+1}$. □

Together with the convergence criterion in Corollary 7.10, this shows that $E_\ell^n = E_\infty^n$ for $\ell \gg 0$, and so our spectral sequence does indeed converge. Now recall that

$$E_1^j = \mathrm{Ext}_S^j(\mathrm{gr}^F M, S).$$

We can use the results about E_1^j from Proposition 6.4, plus the spectral sequence, to prove the following theorem.

Theorem 7.12. *Let M be a finitely generated R-module with a good filtration $F_\bullet M$.*

(a) One has $j(\mathrm{gr}^F M) = j(M)$, and thus $\mathrm{Ext}_R^j(M, R) = 0$ for $j < j(\mathrm{gr}^F M)$.

(b) One has $d\big(\mathrm{Ext}_R^j(M, R)\big) \leq 2n - j$ for every $j \geq 0$.

(c) One has $d\big(\mathrm{Ext}_R^{j(M)}(M, R)\big) = 2n - j(M)$.

Proof. To simplify the notation, let me set $j_0 = j(\mathrm{gr}^F M)$, which means that $E_1^j = 0$ for all $j < j_0$. According to Proposition 6.4, we have

$$d(E_1^j) \leq 2n - j$$

for every $j \geq 0$, with equality for $j = j_0$. Here $d(M) = \dim S/J(M)$ is the dimension of the support.

Since $E_{\ell+1}^j$ is a subquotient of E_ℓ^j, it follows that $E_\ell^j = 0$ for $j < j_0$ and $\ell \geq 1$. But $E_\infty^j = E_\ell^j$ for $\ell \gg 0$, and so $E_\infty^j = 0$ for $j < j_0$. Remembering that

$$E_\infty^j = \operatorname{gr}^F \operatorname{Ext}_R^j(M, R),$$

we deduce that $\operatorname{Ext}_R^j(M, R) = 0$ for $j < j_0$, and hence that $j(M) \geq j_0$. This gives us one half of (a), namely

$$j(M) \geq j(\operatorname{gr}^F M).$$

By the same reasoning, $d(E_1^j) \leq 2n - j$ implies that $d(E_\infty^j) \leq 2n - j$, and therefore

$$d\big(\operatorname{Ext}_R^j(M, R)\big) \leq 2n - j$$

for every $j \geq 0$, which is (b). Lastly, we have $d(E_1^{j_0}) = 2n - j_0$, but $E_1^{j_0 - 1} = 0$ and $d(E_1^{j_0+1}) \leq 2n - j_0 - 1$. Therefore

$$E_2^{j_0} \cong \ker\big(d_1 \colon E_1^{j_0} \to E_1^{j_0+1}\big),$$

and since $d(E_1^{j_0+1}) \leq 2n - j_0 - 1$, we see that $d(E_2^{j_0}) = 2n - j_0$. Continuing in this way, we get $d(E_\ell^{j_0}) = 2n - j_0$ for every $\ell \geq 1$, and therefore

$$d\big(\operatorname{Ext}_R^{j_0}(M, R)\big) = 2n - j_0.$$

In particular, $\operatorname{Ext}_R^{j_0}(M, R) \neq 0$, and so $j_0 \geq j(M)$. This gives us the other inequality

$$j(\operatorname{gr}^F M) \geq j(M),$$

and so (a) and (c) are proved. □

Exercises.

Exercise 7.1. Generalize the proof of Proposition 7.9 to the case where the filtration on each module K^n in the complex satisfies

$$\bigcap_{j \in \mathbb{Z}} (F_j K^n + L) = L$$

for every submodule $L \subseteq K_n$.