Lecture 8: February 27

Holonomic modules and duality. Recall that R is a filtered ring, whose associated graded ring $S = \mathrm{gr}^F R$ is commutative, noetherian, and nonsingular of dimension $\dim S = 2n$. Last time, we proved the following theorem about finitely generated (left or right) R-modules.

Theorem. *Let M be a finitely generated R-module with a good filtration $F_\bullet M$.*

(a) One has $j(\mathrm{gr}^F M) = j(M)$, and thus $\mathrm{Ext}^j_R(M,R) = 0$ for $j < j(\mathrm{gr}^F M)$.

(b) One has $d\big(\mathrm{Ext}^j_R(M,R)\big) \leq 2n - j$ for every $j \geq 0$.

(c) One has $d\big(\mathrm{Ext}^{j(M)}_R(M,R)\big) = 2n - j(M)$.

As I explained before, the fact that $j(\mathrm{gr}^F M) = j(M)$, together with the identity $d(\mathrm{gr}^F M) + j(M) = 2n$, implies that

$$d(M) + j(M) = 2n$$

for every finitely generated R-module.

Example 8.1. In the case of the Weyl algebra A_n, this says that the two notions of dimension (with respect to the Bernstein filtration and the degree filtration) are the same. Since we know from Bernstein's inequality that $d^B(M) \geq n$ for every nonzero finitely generated A_n-module M, it follows that also $d^{\deg}(M) \geq n$.

Let us now assume that Bernstein's inequality holds: Every finitely generated left or right R-module M satisfies $d(M) \geq n$, provided that $M \neq 0$. We saw earlier that this holds when $R = A_n$. An equivalent formulation is that every finitely generated left or right R-module satisfies $j(M) \leq n$, meaning that $\mathrm{Ext}^j_R(M,R) \neq 0$ for some $j \leq n$, again provided that $M \neq 0$. Bernstein's inequality, together with the above theorem, has some remarkable consequences.

Corollary 8.2. *If M is a finitely generated R-module, then $\mathrm{Ext}^j_R(M,R) = 0$ for $j > n$.*

Proof. Let M be a finitely generated left (or right) R-module. Then each $E^j = \mathrm{Ext}^j_R(M,R)$ is a finitely generated right (or left) R-module, and the theorem gives $d(E^j) \leq 2n - j$. But Bernstein's inequality says that $d(E^j) \geq n$ whenever $E^j \neq 0$, and so the conclusion is that $E^j = 0$ for $j > n$. □

Note that this is completely false for finitely generated S-modules, where Ext^j can be nonzero in the range $0 \leq j \leq 2n$.

The most interesting R-modules are clearly those for which the dimension $d(M)$ is minimal (or where the quantity $j(M) = 2n - d(M)$ is maximal). By analogy with the case $R = A_n$, we call such modules holonomic.

Definition 8.3. A finitely generated left (or right) R-module M is called *holonomic* if either $M = 0$, or $M \neq 0$ and $d(M) = n$.

An equivalent definition is that M is holonomic if either $M = 0$, or $M \neq 0$ and $j(M) = n$. Since $\mathrm{Ext}^j_R(M,R) = 0$ for $j > n$, we obtain the following alternative characterization of holonomic R-modules.

Corollary 8.4. *A finitely generated R-module M is holonomic if and only if $\mathrm{Ext}^j_R(M,R) = 0$ for every $j \neq n$.*

Given any holonomic left (or right) R-module M, we therefore get another right (or left) R-module

$$M^* = \mathrm{Ext}^j_R(M,R).$$

This is called the *holonomic dual*. Let us investigate the properties of M^*.

Lemma 8.5. *If M is holonomic, then M^* is also holonomic.*

Proof. Since $j(M) = n$, the theorem from last time shows that

$$d(M^*) = d\big(\operatorname{Ext}_R^{j(M)}(M,R)\big) = 2n - j(M) = n.$$

This says that M^* is again holonomic. □

The association $M \mapsto M^*$ is contravariant functor from the category of holonomic left (or right) R-modules to the category of holonomic right (or left) R-modules. Indeed, given a morphism of left R-modules $f\colon M \to N$ between two holonomic R-modules M and N, the functoriality of Ext shows that we have a morphism of right R-modules

$$f^*\colon \operatorname{Ext}_R^n(N,R) \to \operatorname{Ext}_R^n(M,R)$$

in the opposite direction, and it is not hard to see that $(f \circ g)^* = g^* \circ f^*$. As a contravariant functor, the holonomic dual is also exact: if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of holonomic left (or right) R-modules, then the long exact sequence for $\operatorname{Ext}_R^j(-,R)$ becomes a short exact sequence

$$0 \to \operatorname{Ext}_R^n(M_3,R) \to \operatorname{Ext}_R^n(M_2,R) \to \operatorname{Ext}_R^n(M_1,R) \to 0,$$

due to the vanishing of $\operatorname{Ext}_R^j(M_i,R)$ for $j \neq n$. In other words,

$$0 \to M_3^* \to M_2^* \to M_1^* \to 0$$

is again a short exact sequence.

Proposition 8.6. *We have $M \cong M^{**}$ for every holonomic left (or right) R-module M, and hence the holonomic dual gives an equivalence of categories*

$$(\textit{holonomic left } R\textit{-modules}) \cong (\textit{holonomic right } R\textit{-modules})^{\mathrm{op}}.$$

Proof. Let M be a holonomic left R-module. Choose a free resolution

$$\cdots \to L_2 \to L_1 \to L_0 \to M \to 0$$

by free left R-modules of finite rank. The complex of right R-modules

$$0 \to L_0^* \to L_1^* \to L_2^* \to \cdots$$

is then exact except in degree n, where the cohomology is $M^* = \operatorname{Ext}_R^n(M,R)$. Choose another free resolution

$$\cdots \to K_2 \to K_1 \to K_0 \to M^* \to 0$$

by free right R-modules of finite rank. By a general lemma in homological algebra, there is a morphism of complexes of right R-modules

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K_1 & \xrightarrow{d} & K_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow & & \\ \cdots & \longrightarrow & L_{n-1}^* & \xrightarrow{d} & L_n^* & \xrightarrow{d} & L_{n+1}^* & \longrightarrow & \cdots \end{array}$$

that induces an isomorphism on cohomology. (Such morphisms are called quasi-isomorphisms.) Let me briefly recall the construction. Since M^* is the cohomology in degree n of the complex, we have $M^* = \ker d / \operatorname{im} d$, and so the submodule $\ker d \subseteq L_n^*$ maps onto M^*. Because K_0 is a free R-module, we can find a lifting

$$\begin{array}{ccc} K_0 & & \\ \downarrow & \searrow & \\ \ker d & \twoheadrightarrow & M^* \end{array}$$

indicated by the dashed arrow, and we denote by $f_0\colon K_0 \to L_n^*$ the composition. By construction, $d \circ f_0 = 0$, and so the first square in the diagram below commutes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K_1 & \xrightarrow{d} & K_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow{\scriptstyle f_0} & & \downarrow & & \\ \cdots & \longrightarrow & L^*_{n-1} & \xrightarrow{d} & L^*_n & \xrightarrow{d} & L^*_{n+1} & \longrightarrow & \cdots \end{array}$$

Since the composition $K_1 \to K_0 \to M^*$ is zero, the morphism $f_0 \circ d$ maps K_1 into the submodule $\operatorname{im} d \subseteq \ker d \subseteq L_n^*$. This submodule is the image of L_{n-1}^*, and because K_1 is a free R-module, and so we can again find a lifting

$$\begin{array}{ccc} K_1 & & \\ \downarrow{\scriptstyle f_1} & \searrow & \\ L^*_{n-1} & \xrightarrow{d} & \operatorname{im} d \end{array}$$

which now makes the second square in the diagram commute:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K_1 & \xrightarrow{d} & K_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow & & \\ \cdots & \longrightarrow & L^*_{n-1} & \xrightarrow{d} & L^*_n & \xrightarrow{d} & L^*_{n+1} & \longrightarrow & \cdots \end{array}$$

Continuing in this manner produces the desired morphism of complexes. If we now apply the functor $\operatorname{Hom}_R(-, R)$ a second time, we obtain a morphism of complexes of left R-modules

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & L_{n+1} & \longrightarrow & L_n & \longrightarrow & L_{n-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & K_0^* & \longrightarrow & K_1^* & \longrightarrow & \cdots \end{array}$$

One can show that this morphism still induces an isomorphism on cohomology. Now the complex in the first row is a resolution of M, and therefore only has cohomology at L_0. Likewise, because M^* is holonomic, the complex in the second row only has cohomology at K_n^*, where the cohomology is M^{**}. In this way, we obtain a morphism of left R-modules $M \to M^{**}$, which is an isomorphism by the comment above. $\square$

We can use this result to compare the characteristic varieties of M and M^*.

Corollary 8.7. *If M is holonomic, then* $\operatorname{Ch}(M) = \operatorname{Ch}(M^*)$.

Proof. Choose a good filtration $F_\bullet M$ and recall that $\operatorname{Ch}(M)$ is the closed subset of $\operatorname{Spec} S$ defined by the radical of $\operatorname{Ann}_S(\operatorname{gr}^F M)$, or equivalently, the support of the finitely generated S-module $\operatorname{gr}^F M$. The filtered free resolution from last time induces a good filtration on $M^* = \operatorname{Ext}^n_R(M, R)$; in fact, using the spectral sequence from last time, $E_\infty^n = \operatorname{gr}^F \operatorname{Ext}^n_R(M, R) = \operatorname{gr}^F M^*$. Since the spectral sequence converges, E_∞^n is a subquotient of $E_1^n = \operatorname{Ext}^n_S(\operatorname{gr}^F M, S)$, and therefore

$$\operatorname{Ch}(M^*) = \operatorname{Supp} E_\infty^n \subseteq \operatorname{Supp} E_1^n \subseteq \operatorname{Supp}(\operatorname{gr}^F M) = \operatorname{Ch}(M).$$

But then we also have $\operatorname{Ch}(M) = \operatorname{Ch}(M^{**}) \subseteq \operatorname{Ch}(M^*)$, and so the two characteristic varieties are in fact equal. $\square$

The existence of the holonomic dual gives another explanation for the fact that the category of holonomic A_n-modules is both artinian and noetherian. In fact, recall that we showed earlier, using the notion of multiplicity, that every ascending or descending chain of submodules of a holonomic A_n-module M has finite length (bounded by the multiplicity of M). Since the holonomic dual takes ascending chains of submodules of M to descending chains of submodules of M^*, both chain conditions are equivalent in this case. This is again unlike the commutative case.

Exercises.

Exercise 8.1. Let R be a ring with 1. Let $A_\bullet$ and $B_\bullet$ be two complexes of free R-modules of finite rank. Suppose that we have a morphism of complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\ \cdots & \longrightarrow & B_{n-1} & \longrightarrow & B_n & \longrightarrow & B_{n+1} & \longrightarrow & \cdots \end{array}$$

that induces isomorphisms on cohomology. Show that the same thing is true after applying the functor $(-)^* = \mathrm{Hom}_R(-, R)$: the induced morphism of complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & B^*_{n+1} & \longrightarrow & B^*_n & \longrightarrow & B^*_{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^*_{n+1}} & & \downarrow{\scriptstyle f^*_n} & & \downarrow{\scriptstyle f^*_{n-1}} & & \\ \cdots & \longrightarrow & A^*_{n+1} & \longrightarrow & A^*_n & \longrightarrow & A^*_{n+1} & \longrightarrow & \cdots \end{array}$$

is again a quasi-isomorphism. (Hint: Use the mapping cone. Show that the mapping cone of f is an exact complex of free R-modules, and therefore homotopic to zero. Show that this property is preserved by the functor $\mathrm{Hom}_R(-, R)$, and conclude that the morphism between the dual complexes is also a quasi-isomorphism.)