Homework 5 solutions

§13.3

20. $z = 7ye^{\frac{y}{x}}$.

$$\frac{\partial z}{\partial x} = 7ye^{\frac{y}{x}} \cdot \left(-\frac{y}{x^2}\right) = -7\frac{y^2}{x^2}e^{\frac{y}{x}}, \qquad \frac{\partial z}{\partial y} = 7e^{\frac{y}{x}} + 7\frac{y}{x}e^{\frac{y}{x}}.$$

30. $f(x,y) = \sqrt{2x+y^3} = (2x+y^3)^{\frac{1}{2}}$.

$$\frac{\partial f}{\partial x} = (2x+y^3)^{-\frac{1}{2}}, \qquad \frac{\partial f}{\partial y} = \frac{3}{2}y^2(2x+y^3)^{-\frac{1}{2}}.$$

40. $f(x,y) = \int_x^y (2t+1)dt + \int_y^x (2t-1)dt = \int_x^y [(2t+1)-(2t-1)]\,dt = \int_x^y 2dt = 2y-2x$. Thus $\frac{\partial f}{\partial x} = -2$ and $\frac{\partial f}{\partial y} = 2$.

50. Recall the formula $\frac{d}{dt}\arccos t = \frac{1}{\sqrt{1-t^2}}$. Using the formula and the chain rule, we get

$$f_x = \frac{\partial f}{\partial x} = \frac{y}{\sqrt{1-x^2y^2}}, \qquad f_y = \frac{\partial f}{\partial y} = \frac{x}{\sqrt{1-x^2y^2}}.$$

At $(x,y) = (1,1)$, both f_x and f_y does not exist.

56. The graph of the given function $h(x,y) = x^2 - y^2$ is a surface, and the slope of this surface in x- and y-directions are just $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$, respectively. Now we have

$$\frac{\partial h}{\partial x} = 2x, \qquad \frac{\partial h}{\partial y} = -2y.$$

Thus the answers are $\frac{\partial h}{\partial x}(-2,1) = -4$ and $\frac{\partial h}{\partial y}(-2,1) = -2$.

§13.4

4. The given function is $z = z(x,y) = 2x^3y - 8xy^4$. Computing the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we get the total differential

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (6x^2y - 8y^4)dx + (2x^3 - 32xy^3)dy.$$

8. The given function is $w = w(x,y,z) = \frac{x+y}{z-3y}$. Computing all the partial derivatives $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$, we get the total differential

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz = \frac{1}{z-3y}dx + \frac{z+3x}{(z-3y)^2}dy - \frac{x+y}{(z-3y)^2}dz.$$

12. The given function is $z = f(x,y) = \frac{y}{x}$.

(a) $f(2,1)=\frac{1}{2}$ and $f(2.1,1.05)=\frac{1}{2}$. Hence $\Delta z=0$.

(b) The computations of partial derivatives of f yields $df=-\frac{y}{x^2}dx+\frac{1}{x}dy$. This implies we have an approximation

$$\Delta z \approx -\frac{y}{x^2}\Delta x+\frac{1}{x}\Delta y.$$

Substituting $x=2, y=1, \Delta x=0.1, \Delta y=0.05$, we get $\Delta z\approx 0$.

18. Define a function $f(x,y)=\sqrt{x^2+y^2}$ and let $z=f(x,y)$. Then our desired value is just

$$\sqrt{4.03^2+3.1^2}-\sqrt{4^2+3^2}=f(4.03,3.1)-f(4,3)=\Delta z,$$

at the point $(x,y)=(4,3)$ with $(\Delta x,\Delta y)=(0.03,0.1)$. Now we have the total differential $dz=\frac{y}{\sqrt{x^2+y^2}}dx+\frac{x}{\sqrt{x^2+y^2}}dy$ and this means we can approximate

$$\Delta z\approx \frac{y}{\sqrt{x^2+y^2}}\Delta x+\frac{x}{\sqrt{x^2+y^2}}\Delta y.$$

Substituting $x=4, y=3, \Delta x=0.03, \Delta y=0.1$, we get $\Delta z\approx 0.084$.

24. Define a volume function $V(r,h)=\pi r^2 h$. One can easily compute the total differential $dV=2\pi rh dr+\pi r^2 dh$. Thus $\Delta V\approx 2\pi rh\Delta r+\pi r^2\Delta h$. Substituting $r=3, h=10, \Delta r=\Delta h=0.05$, we get the propagated error $\Delta V\approx 10.83$. The relative error is $\frac{\Delta V}{V}=0.0383$, or 3.83%.

§13.5

6. The chain rule in this case is written as

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}.$$

From the given w, x and y, one can compute

$$\frac{\partial w}{\partial x}=-\frac{1}{x},\quad \frac{\partial w}{\partial y}=\frac{1}{y},\quad \frac{dx}{dt}=-\sin t,\quad \frac{dy}{dt}=\cos t.$$

Substituting the values, we have $\frac{dw}{dt}=\frac{\sin t}{x}+\frac{\cos t}{y}=\tan t+\cot t$. At $t=\frac{\pi}{4}$, we have $\frac{dw}{dt}=2$.

14. The distance between the two points is

$$\begin{aligned} z &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(48\sqrt{2}t-48\sqrt{3}t)^2+(48\sqrt{2}t-16t^2-48t+16t^2)^2}=48\sqrt{8-2\sqrt{6}-2\sqrt{2}}t. \end{aligned}$$

Hence $\frac{dz}{dt}=48\sqrt{8-2\sqrt{6}-2\sqrt{2}}$, regardless of the value of t.

18. The two chain rules in this case are read off as

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s},\qquad \frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

From the given expressions of w, x and y, we have

$$\frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = -2y, \quad \frac{\partial x}{\partial s} = \cos t, \quad \frac{\partial y}{\partial s} = -s \sin t.$$

Thus we can compute $\frac{\partial w}{\partial s} = 2s(\cos^2 t - \sin^2 t) = 2s\cos(2t)$. Similarly, $\frac{\partial w}{\partial t} = -4s^2 \sin t \cos t = -2s^2 \sin(2t)$. At $(s,t) = (3, \frac{\pi}{4})$, we have $\frac{\partial w}{\partial s} = 0$ and $\frac{\partial w}{\partial t} = -18$.

34. The given relation of x,y and z is $x \ln y + y^2 z + z^2 = 8$. Write $z = z(x,y)$ and take the partial derivate of the whole relation with respect to x. The result is a new relation

$$\ln y + y^2 \frac{\partial z}{\partial x} + 2z \frac{\partial z}{\partial x} = 0,$$

giving us the desired result $\frac{\partial z}{\partial x} = -\frac{\ln y}{y^2 + 2z}$. Similarly, taking the partial derivate with respect to y gives us

$$\frac{x}{y} + \left(2yz + y^2 \frac{\partial z}{\partial y}\right) + 2z \frac{\partial z}{\partial y} = 0,$$

yielding $\frac{\partial z}{\partial y} = -\frac{\frac{x}{y} + 2yz}{y^2 + 2z}$.

38. We have $w = w(x,y,z) = \sqrt{x-y} + \sqrt{y-z}$. The three partial derivatives are

$$\frac{\partial w}{\partial x} = \frac{1}{2\sqrt{x-y}}, \quad \frac{\partial w}{\partial y} = -\frac{1}{2\sqrt{x-y}} + \frac{1}{2\sqrt{y-z}}, \quad \frac{\partial w}{\partial z} = -\frac{1}{2\sqrt{y-z}}.$$