Homework 7 solutions

§13.7

12. Introduce a new variable z and a function $F(x,y,z) = f(x,y) - z = x^2 - 2xy + y^2 - z$. Then the graph $z = f(x,y)$ of the function f is the same thing as the level surface $F(x,y,z) = 0$ of F. Now the tangent plane of the level surface $F(x,y,z) = 0$ at the point $(1,2,1)$ is defined by the equation

$$\nabla F(1,2,1) \cdot \langle x-1, y-2, z-1 \rangle = 0.$$

Computation shows that the gradient of F is $\nabla F = \langle 2x-2y, -2x+2y, -1 \rangle$. Hence $\nabla F(1,2,1) = \langle -2, 2, -1 \rangle$. Hence the equation above becomes $-2(x-1)+2(y-2)-(z-1) = 0$. After simplification, it is $2x - 2y + z = -1$.

16. Define a function $F(x,y,z) = x^2 - y^2 + 2z^2$. Its gradient is $\nabla F = \langle 2x, -2y, 4z \rangle$. Hence, $\nabla F(1,3,-2) = \langle 2, -6, -8 \rangle$ and the defining equation of the tangent plane at $(1,3,-2)$ is

$$\langle 2, -6, -8 \rangle \cdot \langle x-1, y-3, z+2 \rangle = 0.$$

Simplifying it, we get $2x - 6y - 8z = 0$.

26. Note that the given equation can be simplified as $y(\ln x + 2\ln z) = 2$. Define a function $F(x,y,z) = y(\ln x + 2\ln z)$. Then our surface is the level surface $F(x,y,z) = 2$.

(a) The gradient of F is

$$\nabla F = \left\langle \frac{y}{x}, \ln x + 2\ln z, \frac{2y}{z} \right\rangle.$$

The defining equation of the tangent plane at $(e,2,1)$ is

$$\nabla F(e,2,1) \cdot \langle x-e, y-2, z-1 \rangle = 0.$$

After computation, we get $\frac{2}{e}x + y + 4z = 8$.

(b) The normal line has a directional vector $\nabla F(e,2,1) = \langle \frac{2}{e}, 1, 4 \rangle$ and it passes through $(e,2,1)$. Thus, its defining equation is

$$\frac{x-e}{\frac{2}{e}} = y - 2 = \frac{z-1}{4}.$$

40. Letting $F(x,y,z) = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4 - z$, the graph of the given equation is the the level surface $F(x,y,z) = 0$. Hence, the normal vector of the tangent plane is its gradient vector $F(x,y,z)$, which is

$$\nabla F = \langle 8x + 4y + 8, 4x - 4y - 5, -1 \rangle.$$

Note that the tangent plane is horizontal if and only if both x and y coordinates of ∇F vanish. This happens when $8x + 4y + 8 = 4x - 4y - 5 = 0$. Solving the equation gives us $x = -\frac{1}{4}, y = -\frac{3}{2}$. The z-coordinate can be computed by substituting $(x,y) = \left(-\frac{1}{4}, -\frac{3}{2}\right)$ into the relation $z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$. This gives $z = -\frac{17}{4}$. Hence the desired point is $\left(-\frac{1}{4}, -\frac{3}{2}, -\frac{17}{4}\right)$.

50. Define $F(x, y, z) = x^2 + 4y^2 - z^2$. Let (a, b, c) be a point lying on the level surface $F(x, y, z) = 1$. The gradient vector $\nabla F(a, b, c) = \langle 2a, 8b, -2c\rangle$ represents the normal vector of the tangent plane at (a, b, c). Hence, the tangent plane is parallel to the plane $x + 4y - z = 0$ when

$$\langle 2a, 8b, -2c\rangle = \lambda\langle 1, 4, -1\rangle.$$

Solving the equation, we get $a = b = c = \frac{\lambda}{2}$. Note that our point (a, b, c) lies on the level surface $F(x, y, z) = 0$. This implies an additional relation $a^2 + 4b^2 - c^2 = 1$, and thus we can conclude $\lambda = \pm 1$. This means $(a, b, c) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ or $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$.

§13.8

8. The given function can be rewritten as $f(x, y) = -(x - 5)^2 - (y - 6)^2 - 3$. In this form, it is clear that $f(x, y) \leq -3$ for all (x, y). The value -3 can actually be achieved at $(x, y) = (5, 6)$. Hence the absolute maximum is -3.

On the other hand, one can compute the gradient $\nabla f = \langle -2(x-5), -2(y-6)\rangle$. Solving the equation $\nabla f = \langle 0, 0\rangle$ yields the critical point $(x, y) = (5, 6)$. Indeed, this point was the absolute maximum point.

20. One can compute the gradient $\nabla h = \left\langle \frac{2}{3}x(x^2 + y^2)^{-\frac{2}{3}}, \frac{2}{3}y(x^2 + y^2)^{-\frac{2}{3}} \right\rangle$. Recall that the critical point is either a solution of $\nabla h = \langle 0, 0\rangle$, or a point where ∇h is undefined. Here the equation $\nabla h = \langle 0, 0\rangle$ does not have a solution, but ∇h is undefined at $(0, 0)$. Hence the critical point is $(0, 0)$.

Now compute all the second partial derivatives. These are

$$f_{xx} = \frac{2}{3}(x^2 + y^2)^{-\frac{2}{3}} - \frac{8}{3}x^2(x^2 + y^2)^{-\frac{5}{3}}, \quad f_{yy} = \frac{2}{3}(x^2 + y^2)^{-\frac{2}{3}} - \frac{8}{3}y^2(x^2 + y^2)^{-\frac{5}{3}},$$
$$f_{xy} = -\frac{8}{3}xy(x^2 + y^2)^{-\frac{5}{3}}.$$

These second partial derivatives are again undefined at $(0, 0)$. Thus the second partials test does not apply, and the test is inconclusive.

Note that from the given form of the function $h(x, y) = (x^2 + y^2)^{\frac{1}{3}} + 2$, it is clear that the point $(0, 0)$ yields an absolute minimum 2. This tells us that even though $(0, 0)$ is a relative minimum point, it is possible that the second partials test cannot detect the answer.

34. The value $d = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 = 100$ is positive, and $f_{xx}(x_0, y_0)$ is also positive. Hence the situation describes a relative minimum point.

46. Define a new function $g(t) = \frac{2t}{t^2+1}$. Then the given function f is just $f(x, y) = g(x)g(y)$. We first compute the absolute minimum and maximum of $g(t)$ for $0 \leq t \leq 1$. This is a single variable function, so one can apply various methods to determine its extrema (that we have learned in Calculus I).

One way to compute the extrema of g is the following. The function g has a derivative $g'(t) =$

$\frac{-t^2+1}{(t^2+1)^2}$. Since we have a restriction $0 \leq t \leq 1$, we conclude $g'(t) \geq 0$ and hence g is an increasing function on $[0,1]$. It follows $g(t)$ has the absolute minimum $g(0) = 0$ and maximum $g(1) = 1$.

Returning to the original problem, recall that $f(x,y) = g(x)g(y)$ and $0 \leq x, y \leq 1$. Since we know $0 \leq g(t) \leq 1$ when $0 \leq t \leq 1$, we can conclude $0 \leq g(x)g(y) \leq 1$. Hence, the absolute minimum of $f(x,y)$ is 0 and it occurs when $x = 0$ or $y = 0$. The absolute maximum of $f(x,y)$ is 1 and it occurs only when $x = y = 1$.

48. The gradient of the function is

$$\nabla f = \langle -2x(y-1)^2(z+2)^2, -2x^2(y-1)(z+2)^2, -2x^2(y-1)^2(z+2) \rangle.$$

To compute critical points, we need to solve the equation $\nabla f = \langle 0, 0, 0 \rangle$. This gives us the whole locus of critical points $x = 0$ or $y = 1$ or $z = -2$. In fact, these are all absolute maximum points. From the form of the given function $f(x,y,z) = 9 - x^2(y-1)^2(z+2)^2$, we can clearly see $f(x,y,z) \leq 9$ and the equality holds exactly when $x = 0$ or $y = 1$ or $z = -2$. This was exactly the critical locus.