Homework 9 Solutions

§14.3

6. The boundary circle is given by

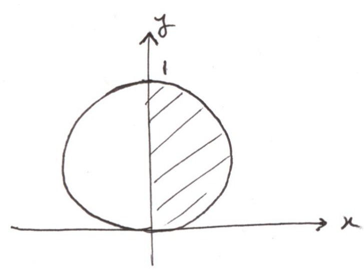
$$(x-0)^2 + (y-2)^2 = 2^2$$
$$(r\cos\theta)^2 + (r\sin\theta - 2)^2 = 4$$

Simplifying the last equation gives

$$r = 4\sin\theta$$

So the region is $R = \{(r,\theta) : 0 \le r \le 4\sin\theta,\ 0 \le \theta \le \pi\}$

10. The region lies in the first quadrant as shown below.

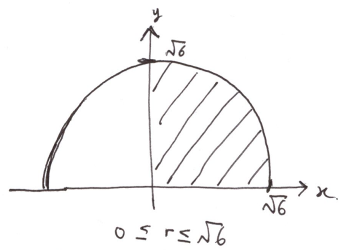


$0 \le r \le \sin\theta$

$0 \le \theta \le \pi/2$

$$\int_0^{\pi/2}\int_0^{\sin\theta} r^2\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^3}{3}\Big|_0^{\sin\theta}\right)d\theta = \frac{1}{3}\int_0^{\pi/2}\sin^3\theta\,d\theta$$
$$= \frac{1}{3}\int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\,d\theta$$
$$= \frac{1}{3}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]\Big|_0^{\pi/2}$$
$$= \frac{2}{9}$$

26. The region we integrate over is the region bounded by the coordinate axes and the circle with radius $\sqrt{6}$ and center 0. We can convert the integral into polar coordinates as follows.



$$\int_0^{\sqrt{6}}\int_0^{\sqrt{6-x^2}} \sin\sqrt{x^2+y^2}\,dydx = \int_0^{\pi/2}\int_0^{\sqrt{6}} (\sin r) r dr d\theta$$

Now integrating by parts, we get

$$\begin{aligned}\int_0^{\pi/2}\int_0^{\sqrt{6}} (\sin r) r dr d\theta &= \frac{\pi}{2}[(-r\cos r)\Big|_0^{\sqrt{6}} + \int_0^{\sqrt{6}} \cos r dr] \\ &= \frac{\pi}{2}(\sin\sqrt{6} - \sqrt{6}\cos\sqrt{6})\end{aligned}$$

36. We want to find the volume of the solid bounded by the graphs of the following equations: $z = \ln(x^2+y^2)$, $z = 0$, $x^2+y^2 \geq 1$, $x^2+y^2 \leq 4$. Denote the volume of the solid by V. Then,

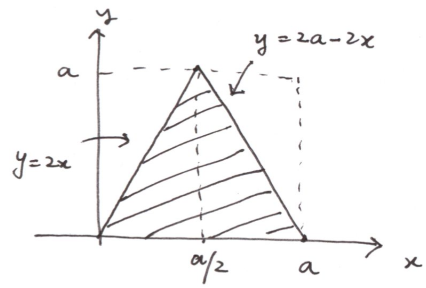
$$V = \int_0^{2\pi}\int_1^2 (\ln r^2) r dr d\theta = 2\pi\frac{1}{2}[r^2\ln(r^2) - r^2]\Big|_1^2 = \pi[4\ln 4 - 4 + 1]$$
$$= \pi(8\ln 2 - 3)$$

44. The shaded region is $\{(r,\theta) : 0 \leq r \leq 2+\sin\theta,\ 0 \leq \theta \leq 2\pi\}$ So it has area

$$A = \int_0^{2\pi}\int_0^{2+\sin\theta} r dr d\theta = \frac{1}{2}\int_0^{2\pi} r^2\Big|_0^{2+\sin\theta} d\theta = \frac{1}{2}\int_0^{2\pi}(2+\sin\theta)^2 d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}(4 + 4\sin\theta + \sin^2\theta)d\theta$$
$$= 4\pi + 2[-\cos\theta]\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}\frac{1-\cos 2\theta}{2}d\theta = \frac{9\pi}{2}$$

§14.4

10. The region R is the triangle with vertices $(0,0)$, $(a/2,a)$, $(a,0)$. To find the center of mass of the lamina corresponding to R, we need to find the mass m, the moments of mass with respect to the x and $y-$axes, M_x and M_y.



(a)$\rho = k$

The mass $m = k \cdot \text{Area(R)} = ka^2/2$ is evident. The moment of mass with respect to the x-axis is

$$M_x = \iint_R ky dA = k \int_0^a \int_{y/2}^{a-y/2} ydxdy = k \int_0^a y(a-y)dy = \frac{ka^3}{6}$$

Calculations for M_y is similar.

$$M_y = \iint_R kydA = k \int_0^a \int_{y/2}^{a-y/2} xdxdy = k \int_0^a \left. \frac{x^2}{2} \right|_{y/2}^{a-y/2} dy$$
$$= \frac{k}{2} \int_0^a (a^2 - ay)dy = \frac{ka^3}{4}$$

So the center of mass is

$$(\bar{x}, \bar{y}) = (\frac{M_y}{m}, \frac{M_x}{m}) = (\frac{ka^3/4}{ka^2/2}, \frac{ka^3/6}{ka^2/2}) = (\frac{a}{2}, \frac{a}{3})$$

(b)$\rho = kxy$

$$m = \int_0^a \int_{y/2}^{a-y/2} kxydxdy = \frac{k}{2} \int_0^a y(a^2 - ya)dy = \frac{k}{2} (\frac{a^2y^2}{2} - \frac{y^3 a}{3}) \Big|_0^a = \frac{ka^4}{12}$$

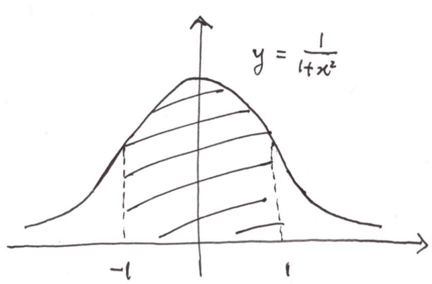
$$M_x = \int_0^a \int_{y/2}^{a-y/2} y \cdot kxydxdy = \frac{k}{2}\int_0^a y^2(a^2 - ya) = \frac{k}{2}(a^2\frac{y^3}{3}\Big|_0^a - a\frac{y^4}{4}\Big|_0^a)$$
$$= \frac{ka^5}{24}$$

$$M_y = \int_0^a \int_{y/2}^{a-y/2} x \cdot kxydxdy = k\int_0^a y\frac{x^3}{3}\Big|_{y/2}^{a-y/2} dy$$
$$= \frac{k}{3}\int_0^a y(a^3 - \frac{3a^2y}{2} + \frac{3ay^2}{4} - \frac{y^3}{4})dy$$
$$= \frac{11ka^5}{240}$$

So the center of mass is

$$(\bar{x}, \bar{y}) = (\frac{M_y}{m}, \frac{M_x}{m}) = (\frac{11a}{20}, \frac{a}{2})$$

16. The region is as below. We wish to find the center of mass with density $\rho = k$.



As before, we first calculate the mass.

$$m = k\iint_R dA = 2k\int_0^1 \frac{1}{1+x^2}dx = 2k\arctan x\Big|_0^1 = \frac{k\pi}{2}$$

$$M_x = k\iint_R dA = \int_{-1}^1 \int_0^{1/(1+x^2)} ydydx = \frac{1}{2}\int_{-1}^1 (\frac{1}{1+x^2})^2dx = \int_0^1 (\frac{1}{1+x^2})^2dx$$

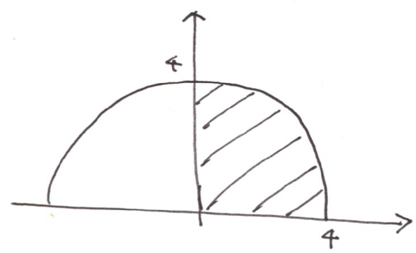
Let $x = \tan\theta$, so $dx = \sec^2\theta d\theta$. Then M_x becomes

$$M_x = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}d\theta = \int_0^{\pi/4} \cos^2\theta d\theta = k(\frac{\pi}{8} + \frac{1}{4})$$

$M_y = 0$ since the lamina is symmetric with respect to the y-axis and the density is constant (which can be checked by direct computation).

$$(\bar{x}, \bar{y}) = (\frac{M_y}{m}, \frac{M_x}{m}) = (0, \frac{1}{4} + \frac{1}{2\pi})$$

24. The region is as shown, with density given to be $\rho = k(x^2 + y^2)$



$$m = \int_0^{\pi/2} \int_0^4 kr^2 r dr d\theta = \frac{\pi k}{2} \frac{r^4}{4} \Big|_0^4 = 32\pi k$$

$$M_x = \iint_R y \cdot k(x^2 + y^2) dx dy = \int_0^{\pi/2} \int_0^4 r \sin\theta \cdot kr^3 dr d\theta$$

$$= (-\cos\theta) \Big|_0^{\pi/2} \cdot \frac{kr^5}{5} \Big|_0^4 = \frac{k4^5}{5}$$

Note that $M_y = M_x$ by symmetry. So, the center of mass

$$(\bar{x}, \bar{y}) = (\frac{32}{5\pi}, \frac{32}{5\pi})$$

34. Assume the lamina has constant density $\rho = 1 g/cm^2$. We want to find the moment of inertia and the radius of gyration with respect to both axes. First, we look at

$$I_x = \iint_R y^2 dA = 4 \int_0^1 \int_0^{\sqrt{1 - x^2/a^2}} y^2 dy dx = \frac{4}{3} b^3 \int_0^1 (1 - \frac{x^2}{a^2})^{\frac{3}{2}}$$

Now we make a change of variable $x = a \sin\theta$, then

$$I_x = \frac{4}{3} b^3 \int_0^{\pi/2} a \cos^4\theta d\theta = \frac{4ab^3}{3} \int_0^{\pi/2} (\frac{1 + \cos 2\theta}{2})^2 d\theta = \frac{\pi}{4} ab^3$$

Switching the role of a, b and x, y, one sees that

$$I_y = \frac{\pi}{4} a^3 b$$

So,

$$I_0 = I_x + I_y = \frac{1}{4}\pi ab(a^2 + b^2)$$

The mass $m = \rho \cdot \text{Area(R)} = \pi ab$. The radii of gyrations with respect to both axes are

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \frac{1}{2}a$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \frac{1}{2}b$$

46. The height at which a vertical gate in a dam should be hinged so that there is no moment causing rotation is given to be

$$y_a = \bar{y} - \frac{I_{\bar{y}}}{hA}$$

Observe that $\bar{y} = 0$, $h = d + a$ and $A = \pi a^2$. From Q34, we also know that $I_{\bar{y}} = \frac{\pi}{4} a^4$. By the model above, we get

$$y_a = -\frac{a^2}{4(d+a)}$$

§14.5

6. We will find the surface area of the plane $z = f(x, y)$, where $f(x, y) = 12 + 2x - 3y$ over the region $R = (x, y) : x^2 + y^2 \leq 9$. The partial derivatives are

$$f_x = 2, \quad f_y = 3$$

So the surface area is

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA = \iint_R \sqrt{14} dA = 9\sqrt{14}\pi$$

16. Here $f(x,y) = \sqrt{a^2 - x^2 - y^2}$ and $R = \{(x,y) : x^2 + y^2 \leq a^2\}$. As in Q6, we calculate the first partials.

$$f_x = -x(a^2 - x^2 - y^2)^{-1/2}, \quad f_y = -y(a^2 - x^2 - y^2)^{-1/2}$$

So the surface area is

$$S = \int_0^{\pi} \int_0^{a} \frac{a}{\sqrt{a^2 - r^2}} r\,dr\,d\theta = 2\pi a^2$$

20. The surface is the portion of the cone $z = 2\sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 4$

$$S = 4\sqrt{5}\pi$$

See Q38 of this section.

30. $f(x,y) = \cos(x^2 + y^2)$, $R = \{(x,y) : x^2 + y^2 \leq \pi/2\}$ The first partials are

$$f_x = -2\sin(x^2 + y^2)x, \quad f_y = -2\sin(x^2 + y^2)y$$

To set up the double integral for the surface area, we switch to polar coordinates.

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\,dA = \iint_R \sqrt{1 + 4(x^2 + y^2)\sin(x^2 + y^2)}\,dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{\pi/2}} \sqrt{1 + 4r^2 \sin^2 r^2}\, r\,dr\,d\theta$$

38. This is a general case for Q20. We need to show that the surface area of the cone $z = k\sqrt{x^2 + y^2}$ over the region $R = \{(x,y) : x^2 + y^2 \leq r^2\}$ is $\pi r^2 \sqrt{k^2 + 1}$.

$$f_x = \frac{kx}{\sqrt{x^2 + y^2}}, \quad f_y = \frac{ky}{\sqrt{x^2 + y^2}}.$$

The surface area is hence

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\,dA = \iint_R \sqrt{1 + k^2}\,dA = \text{Area(R)}\sqrt{1 + k^2}$$
$$= \pi r^2 \sqrt{k^2 + 1}$$