Math 203 - Fall 2018 —— Practice for Final Exam

1. Find the equation for the plane perpendicular to the curve

$$\mathbf{r}(t) = (2t, \cos(\pi t), 4t^2 + 6t)$$

at the point $(1, 0, 4)$.

2. Find the unit tangent vector $\mathbf{T}(\pi)$ and the unit normal vector $\mathbf{N}(\pi)$ to the space curve

$$\mathbf{R}(t) = \left(e^t \cos t, e^t \sin t, \sqrt{2}e^{\pi} t\right).$$

(`Hint:` It is much easier to (i)compute $\mathbf{T}(\pi)$, (ii) compute the acceleration $\mathbf{a}(\pi)$, and then (iii) use the decomposition of acceleration into tangential and normal components to find $\mathbf{N}(\pi)$.)

3. Find the length of the curve

$$\mathbf{r}(t) = (t^2, t^3), \quad 0 \leq t \leq 2.$$

4. Reparametrize the curve

$$\mathbf{r}(t) = (t\cos(\ln t), t\sin(\ln t), 3t)$$

with respect to arc length.

5. Find the directional derivative of the function

$$f(x, y, z) = xye^z$$

at the point $(1, 2, 5)$ and in the direction perpendicular to the plane $5x + 2y + z = 3$.

6. Consider the surface

$$S: \qquad f(x, y, z) := \frac{x^2}{9} + y + \frac{z^3}{8} = 3.$$

   (a) Find a vector that is tangent to S at the point $(3, 1, 2)$.

   (b) Find the vector equation for the line perpendicular to S at $(6, -2, 2)$.

7. Find all of the critical points of the function

$$f(x, y) = (x - 2)^2 + 2(y + 5)^2 + 6xy.$$

Decide which points are local maxima, local minima or neither.

8. Find the absolute maximum and minimum <u>values</u> of the function

$$f(x, y) = x^2 - y^3,$$

over all points (x, y) satisfying the constraint

$$g(x, y) = x^2 - 3 + 2y^2 = 0.$$

9. Calculate the triple integral of the function

$$f(x,y,z) = e^{-(x^2+y^2)(1-x^2-y^2)}.$$

over the region B trapped between the paraboloids

$$z = 1 - x^2 - y^2 \quad \text{and} \quad z = 3x^2 + 3y^2 - 1.$$

10. Find the area of the region D consisting of all points (x,y) such that

$$y \geq 0, \quad y^2 \geq x^2 \quad \text{and} \quad y + x^2 \leq 6.$$

11. Find the mass of the solid

$$\mathbf{E} = \left\{(x,y,z) \ ; \ 0 \leq z \leq 3 - 2x - y^2, \ x^2 + y^2 \leq 1\right\}$$

of mass density $\rho(x,y,z) = \frac{3}{\sqrt{x^2+y^2}}$.

12. Find the volume of the part of the ball of radius 5 that lies outside the cone

$$\mathbf{K} = \left\{(x,y,z) \ ; \ 5z \geq \sqrt{x^2+y^2}\right\}.$$

13. compute the integral

$$\iint_{\mathscr{E}} e^{10(x-y)^2+8xy} dxdy$$

over the ellipse $\mathscr{E} := \{(x,y) \ ; \ 5x^2 + 5y^2 \leq 6xy + 1\}$.
(`Hint:` Use the change of variables

$$x = 2r\cos\theta - r\sin\theta, \quad y = 2r\cos\theta + r\sin\theta$$

to make the integral easier to evaluate.)

14. A thin tube of glass given by the curve

$$\mathbf{r}(t) = (4\cos(\pi t), 4\sin(\pi t), 8t), \quad 0 \leq t \leq 8$$

has density given by the function

$$\rho(x,y,z) = (x^2+y^2)z.$$

Find the mass of the glass tube.

15. Consider the vector field

$$\mathbf{F}(x,y) = \left\langle \frac{y^2}{2}(2x + e^x), e^x y + x^2 y + e^{\cos y}\sin y \right\rangle.$$

Compute the line integral

$$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r}$$

of $\mathbf{F}$ along the curve $\mathscr{C}$ defined by

$$\mathbf{r}(t) = (t^2\sin(\pi t/2), t^3), \qquad 0 \leq t \leq 1.$$

16. Compute the flux

$$\int_{\mathscr{C}} \mathbf{V} \cdot \mathbf{n} ds$$

of the vector field

$$\mathbf{V}(x, y) = \langle x^2 - 2xy, y^2 - 2xy \rangle$$

across the curve

$$\mathscr{C} : \ \mathbf{r}(t) = \left(\ln(1 + t^2), e^t\right), \qquad 0 \le t \le 1$$

in the downward normal direction $\mathbf{n}(\mathbf{t}) = \frac{\sqrt{1+t^2}}{\sqrt{e^t+(4+2e^t)t^2+e^t t^4}} \left\langle e^t, -\frac{2t}{1+t^2} \right\rangle$.

17. Compute the surface area of the domed structure

$$\mathscr{D} := \{(x, y, z) \ ; \ x^2 + y^2 + (z - 1)^2 = 2 \ ; \ z \ge 0\}.$$

18. Compute the flux of the vector field

$$\mathbf{F}(x, y, z) = \frac{1}{x^2 + y^2 + 1}\mathbf{k}$$

out of the conical container

$$\mathbf{K} = \{(x, y, z) \ ; \ 4 + \sqrt{x^2 + y^2} \le z \le 8\}.$$

19. Consider the vector field

$$\mathbf{V}(x, y, z) = \left\langle \frac{1-y}{x^2+(y-1)^2}, \frac{x}{x^2+(y-1)^2} \right\rangle$$

and the curves

$$\mathscr{C}_1 : x^2 + (y - 1)^2 = 1, \quad \mathscr{C}_2 : x^2 + (y - 2)^2 = 4 \quad \text{and} \quad \mathscr{C}_3 : x^2 + (y - 4)^2 = 1.$$

(a) Compute

$$\int_{\mathscr{C}_1} \mathbf{V} \bullet d\mathbf{r}.$$

(b) Compute

$$\int_{\mathscr{C}_2} \mathbf{V} \bullet d\mathbf{r}.$$

(c) Compute

$$\int_{\mathscr{C}_3} \mathbf{V} \bullet d\mathbf{r}.$$

20. Consider the gradient vector field $\mathbf{V} = \nabla g$ of the function

$$g(x, y) = x^2 + y^2 + e^{x^2-y^2} \cos(2xy).$$

Compute the outward flux

$$\int_{\mathscr{E}} \mathbf{V} \cdot \mathbf{n} ds$$

across the ellipse $\mathscr{E} = \{4(x - 3)^2 + 9(y - 1)^2 = 1\}$.