MAT 589 Spring 2020
Mid-Semester Summary

Guiding Principles,

I. Use of category theory, particularly adjoint functors, to organize simpler results and combine them into new results.

A. $(f^{-1}: \text{Sheaves}_Y \to \text{Sheaves}_X,\ f_*: \text{Sheaves}_X \to \text{Sheaves}_Y)$ is an adjoint pair for every continuous map $f: X \to Y$

B. For every sheaf of rings $\mathcal{O}_Y$ on Y, there is an induced adjoint pair $(f^{-1}: \mathcal{O}_Y\text{-mod} \to (f^{-1}\mathcal{O}_Y)\text{-mod},\ f_*: (f^{-1}\mathcal{O}_Y)\text{-mod} \to \mathcal{O}_Y\text{-mod})$.

C. For every morphism of ringed spaces, i.e., $f^{\#}: \mathcal{O}_Y \to f_*\mathcal{O}_X$, for $f^*\mathcal{E} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{E}$, $(f^*: \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod},\ f_*: \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod})$

D. Universal property of adjoint pairs and (strict) compatibility of pushforward with composition, imply natural isomorphisms $\tau_{f,g,\mathcal{G}}: f^*g^*\mathcal{G} \xrightarrow{\sim} (g\circ f)^*\mathcal{G}$ for every $(g, g^{\#}): (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ and $\mathcal{O}_Z$-mod $\mathcal{G}$.

E. Example: For Z a singleton set, for every $\mathcal{O}_Z(Z)$-module N, for $\mathcal{G}(Z) = N$, $g^*\mathcal{G} = (\mathcal{O}_Y(Y) \otimes_{\mathcal{O}_Z(Z)} N)^\sim$ and $(g\circ f)^*\mathcal{G} = (\mathcal{O}_X(X) \otimes_{\mathcal{O}_Z(Z)} N)^\sim$. Thus, for every $\mathcal{O}_Y(Y)$-module M, $f^*(\widetilde{M}) \xrightarrow{\sim} (\mathcal{O}_X(X) \otimes_{\mathcal{O}_Y(Y)} M)^\sim$.

F. Adjointness of (g^*, g_*) gives a natural $\mathcal{O}_Y$-mod morphism $\Gamma(Y, \mathcal{E})^\sim \to \mathcal{E}$. Recall that $\mathcal{E}$ is <u>quasi-coherent</u> if Y is covered by the open subsets $\mathcal{U} \subseteq Y$ such that $\Gamma(\mathcal{U}, \mathcal{E}|_\mathcal{U})^\sim \to \mathcal{E}|_\mathcal{U}$ are isomorphisms ("$\mathcal{E}$-quasi-coh. opens"). Then $\Gamma(f^{-1}\mathcal{U}, f^*\mathcal{E}|_{f^{-1}\mathcal{U}})^\sim \to f^*\mathcal{E}|_{f^{-1}\mathcal{U}}$ is also an isomorphism, so $f^*\mathcal{E}$ q-coh.

II. Investigation of notions of covering and basis for topology.

A. For every ringed space $(Y, \mathcal{O}_Y)$, an $\mathcal{O}_Y$-module $\mathcal{L}$ is invertible if locally isomorphic to $\mathcal{O}_Y$. For $\mathcal{O}_Y$-mod morphism $s: \mathcal{O}_Y \to \mathcal{L}$, the open $D_{Y,\mathcal{L}}(s)$ is largest such that s is an isomorphism on $D_{Y,\mathcal{L}}(s)$.

B. Note that $D_{\mathcal{U},\mathcal{L}|_\mathcal{U}}(s|_\mathcal{U})$ equals $\mathcal{U} \cap D_{Y,\mathcal{L}}(s)$ and $D_{Y,\mathcal{L}}(s) \cap D_{Y,\mathcal{L}'}(s')$ equals $D_{Y,\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{L}'}(s \otimes s')$.

C. There is a directed system of $\mathcal{O}_Y$-modules $((\mathcal{L}^{\otimes n})_{n\in\mathbb{Z}_{\geq 0}}, (s^m: \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes(n+m)}))$ giving a functor $\mathcal{O}_Y\text{-mod} \to \mathcal{O}_Y(Y)\text{-mod}$ as $\varinjlim_n \Gamma(Y, \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes n})$.

Since s is invertible on $D_{Y,\mathcal{L}}(s)$, there is a compatible family of $\mathcal{O}_Y(Y)$-mod morphisms, $\Gamma(Y, \mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{L}^{\otimes n}) \xrightarrow{s^{-n}} \Gamma(D_{Y,\mathcal{L}}(s), \mathcal{E}|_{D_{Y,\mathcal{L}}(s)})$, giving a natural transformation of functors, $\rho_{Y,\mathcal{L},s,\mathcal{E}} : \varinjlim_n \Gamma(Y, \mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{L}^{\otimes n}) \to \Gamma(D_{Y,\mathcal{L}}(s), \mathcal{E}|_{D})$.

D. Another notion of "covering" is a torsor over Y for the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 \setminus \mathbb{V}(0)$. For every torsor, $(\pi : E \to Y, \mu)$, consisting of an action $\mu : \mathbb{G}_m \times E \to E$ and a $\mathbb{G}_m$-invariant morphism π that is Zariski locally on Y equal to $pr_Y : \mathbb{G}_m \times Y \to Y$, there is a "universal property": a morphism from X to Y is equivalent to a $\mathbb{G}_m$-torsor over X and a $\mathbb{G}_m$-equivariant morphism from that torsor to E. This is a "glueing" result. Alternatively, for the degree-1 summand $\mathcal{L}$ of the $\mathbb{Z}$-graded $\mathcal{O}_Y$-module $\pi_*\mathcal{O}_E$, then $\mathcal{L}$ is an invertible sheaf and adjointness of π_* and π^* defines an $\mathcal{O}_E$-module morphism $\pi^*\mathcal{L} \xrightarrow{\varphi} \mathcal{O}_E$. This is an isomorphism. For every X, a morphism from X to E is equivalent to a morphism from X to Y and an isomorphism from the pullback of $\mathcal{L}$ to $\mathcal{O}_X$.

E. For an indexed open covering of Y, $\mathfrak{U} = (\mathcal{U}_i)_{i\in I}$, there is an equivalence of the category of sheaves on Y and the category of $\mathfrak{U}$-descent data $((\mathcal{E}_i)_{i\in I}, (\varphi_{ij} : \mathcal{E}_i|_{\mathcal{U}_{ij}} \xrightarrow{\sim} \mathcal{E}_j|_{\mathcal{U}_{ij}})_{(i,j)\in I\times I})$, where each $\mathcal{E}_i$ is a sheaf on $\mathcal{U}_i$, each φ_{ij} is an isomorphism of sheaves on $\mathcal{U}_{ij}$, and for every $(i,j,k) \in I\times I\times I$, the composite $\varphi_{jk}|_{\mathcal{U}_{ijk}} \circ \varphi_{ij}|_{\mathcal{U}_{ijk}}$ equals $\varphi_{ik}|_{\mathcal{U}_{ijk}}$. This "glueing for sheaves" is a special case of inverse limits of sheaves.

F. For the "covering" $(\pi : E \to Y, \mu)$, a descent datum is a sheaf on E with a lifting of the action μ to a $\mathbb{G}_m$-linearization. In particular this is equivalent to a $\mathbb{Z}$-grading of sections on $\mathcal{U}$, compatible with restriction, for every Abelian sheaf and every $\mathbb{G}_m$-invariant open $\mathcal{U}$.

III. The subcategory of locally ringed spaces is the "geometric" subcategory of ringed sp

A. A ringed space is a locally ringed space if & only if each topological point has precisely one "residue field", i.e, surjection from the stalk to a field.

B. A morphism of locally ringed spaces is a morphism of the underlying ringed spaces that is compatible with residue fields.

C. For such a morphism $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, the inverse image $f^{-1} D_{Y,\mathcal{L}}(s)$ equals $D_{X, f^*\mathcal{L}}(f^* s)$.

D. In the category $LRS_{\mathbb{R}}$, resp. $LRS_{\mathbb{C}}$, the category of $C^{\infty}_{\mathbb{R}}$-manifolds, resp. complex analytic spaces, is a full subcategory.

IV. Emphasis on relative properties of morphisms instead of absolute properties of objects.

A. For a relative property, does it hold after base change? Is it local on the target, resp. on the domain? Is it compatible with composition?

B. For a morphism $X \to Y$, particular emphasis on properties of the associated diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$.

V. Inside LRS, the full subcategory of schemes is "natural".

A. There is an adjoint pair of functors,
$(\Gamma(-, \mathcal{O}) : LRS \to \mathrm{CommUnitalRings}^{opp},\ \mathrm{Spec} : \mathrm{CommUnitalRings}^{opp} \to LRS)$.

B. For affine schemes, the collection of opens $D_{X,\mathcal{L}}(s)$, and even $D_{X,\mathcal{O}_X}(s)$, forms a basis. Schemes are objects of LRS locally isomorphic to Spec of a ring.

C. This adjoint pair extends to a "relative Spec"; for every scheme $(Y, \mathcal{O}_Y)$,
(pushforward to Y : $\mathrm{Schemes}_Y^{qcqs} \to \mathrm{QCoh}_Y^{Alg}$, $\underline{\mathrm{Spec}}_Y : \mathrm{QCoh}_Y^{Alg} \to \mathrm{Schemes}_Y^{qcqs}$).

D. For each quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{A}$, there is an equivalence of the category of quasi-coherent $\mathcal{A}$-modules on Y and quasi-coherent sheaves on $\underline{\mathrm{Spec}}_Y \mathcal{A}$.

E. For every $\mathbb{Z}_{\geq 0}$-graded quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ that is generated in degree 1, the $\mathbb{G}_m$-action on $\underline{\mathrm{Spec}}_Y \mathcal{A} \setminus \mathbb{V}(\mathcal{A}_{>0})$ is a $\mathbb{G}_m$-torsor over $\underline{\mathrm{Proj}}_Y A$. For $\pi: \underline{\mathrm{Spec}}_Y A \setminus \mathbb{V}(A_{>0}) \to \underline{\mathrm{Proj}}_Y A$, the pushforward $\pi_* \mathcal{O}_{\mathrm{Sp}A}$ equals $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}(n)$, where $\mathcal{O}(1)$ is the Y-very ample Serre twisting sheaf.

$\underline{E}$. There is an equivalence of categories of $\mathbb{G}_m$-linearized quasi-coh. sheaves on $\underline{\mathrm{Spec}}_Y A$ and quasi-coherent A-modules with a compatible $\mathbb{Z}$-grading. Localizing this category at the "multiplicatively closed" system of morphisms that are isomorphisms in sufficiently high degree gives an equivalence between the localized category and the category of quasi-coh. sheaves on Proj,
$\Gamma_* : \mathrm{QCh}_{\underline{\mathrm{Proj}}_Y A} \to (\mathbb{Z}\text{-graded } \mathrm{Qcoh}_Y^A)_{\text{high degree}}$.

VI. Čech complexes

$\underline{A}$. The Čech cochain complex is functorial in the Abelian sheaf, and it is also functorial for refinements of indexed open coverings. It depends on the choice of refinement only up to cochain homotopy. Thus Čech cohomology is independent of the choice of refinement.

$\underline{B}$. The Čech complex is naturally homotopy equivalent to the subcomplex of alternating Čech cochains.

$\underline{C}$. For an affine scheme Spec A, for every quasi-coherent sheaf $\widetilde{M}$, for every finite open covering by distinguished basis sets $(D(a_1), \dots, D(a_n))$, the alternating Čech complex, augmented by $M \to \check{C}^\bullet(\{D(a_i)\}, \widetilde{M})$, is the direct limit over integers $e>0$ of the Koszul cochain complex $K^\bullet((a_1^e, \dots, a_n^e), M)$. In particular, higher Čech cohomology vanishes.

$\underline{D}$. By the same method, for every quasi-compact, quasi-separated (qcqs) scheme, for every quasi-coherent sheaf, $\varinjlim_n \Gamma(Y, \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes n}) \to \Gamma(D_{Y,\mathcal{L}}(s), \mathcal{E})$ is an isomorphism.

$\underline{E}$. Finally, for every qcqs morphism $f: X \to Y$, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the pushforward $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. If f is separated, the "Čech higher direct image sheaves" are also quasi-coherent.