<u>Def.</u> For every morphism of locally ringed spaces, $f: X \to S$, the <u>diagonal ideal sheaf</u> $\mathcal{I}_{X/S}$ is the kernel of the surjective morphism of sheaves of commutative unital rings $\Delta_{X/S}^{\#}: \Delta_{X/S}^{-1}(\mathcal{O}_{X\times_S X}) \to \mathcal{O}_X$.
The two splittings, $\Delta^{-1}(pr_i^{\#}): \mathcal{O}_X = \Delta^{-1}pr_i^{-1}\mathcal{O}_X \to \Delta^{-1}(\mathcal{O}_{X\times_S X})$, $i=1,2$, give $\Delta_{X/S}^{-1}(\mathcal{O}_{X\times_S X})$ a structure of $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X$ algebra that is an isomorphism.
For every integer $\ell \geq 0$, the functor $\mathcal{P}^{\ell}_{X/S}: \mathcal{O}_X\text{-mod} \to \mathcal{O}_X\text{-mod}$ associates to every left $\mathcal{O}_X$-module $\mathcal{E}$ the <u>left</u> $\mathcal{O}_X$-module,

$$\mathcal{P}^{\ell}_{X/S}(\mathcal{E}) := \left(\Delta_{X/S}^{-1}(\mathcal{O}_{X\times_S X})/\mathcal{I}_{X/S}^{\ell+1}\right) \otimes_{\mathcal{O}_X} \mathcal{E}.$$

This is ~~func~~ contravariant in the S-scheme X, and compatible with arbitrary base change of S.

In particular, the <u>Atiyah sequence</u> is the short exact sequence (locally split), $0 \to \Omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{P}^1_{X/S}(\mathcal{E}) \to \mathcal{E} \to 0$.
For a quasi-coherent $\mathcal{O}_S$-module $\mathcal{F}$, for $X = \mathbb{P}_S\mathcal{F} = \text{Proj}_S\, \text{Sym}^{\bullet}_{\mathcal{O}_S}\mathcal{F}$ with the universal invertible quotient $f^*\mathcal{F} \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$, the Atiyah sequence of $\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$ is the Euler sequence twisted by $\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$, i.e., $\mathcal{P}^1_{\mathbb{P}\mathcal{F}/S}(\mathcal{O}_{\mathbb{P}\mathcal{F}}(1))$ equals $f^*\mathcal{F}$ and the Atiyah sequence is

$$0 \to \Omega_{\mathbb{P}\mathcal{F}/S} \otimes_{\mathcal{O}_{\mathbb{P}\mathcal{F}}} \mathcal{O}_{\mathbb{P}\mathcal{F}}(1) \xrightarrow{r} f^*\mathcal{F} \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}\mathcal{F}}(1) \to 0$$

<u>For every S-scheme $g: Y \to S$ and S-morphism $h: Y \to \mathbb{P}_S\mathcal{F}$, the Atiyah sequence of $h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$ fits into a push-out diagram,</u>

$$\begin{array}{ccccccccc} 0 & \to & \Omega_{Y/S}\otimes h^*\mathcal{O}(1) & \to & \mathcal{P}^1_{Y/S}(h^*\mathcal{O}(1)) & \to & h^*\mathcal{O}(1) & \to & 0 \\ & & \uparrow \Omega_h & & \uparrow & & \| & & \\ 0 & \to & h^*\Omega_{\mathbb{P}\mathcal{F}/S}\otimes\mathcal{O}(1) & \to & g^*\mathcal{F} & \to & h^*\mathcal{O}(1) & \to & 0 \end{array}$$

If g is smooth, then $\Omega_{Y/S}$ is locally free of finite rank, and the transpose is, $\mathcal{F}$ locally free

$$\begin{array}{ccccccccc} 0 & \to & h^*\mathcal{O}(-1) & \to & \mathcal{P}^1_{Y/S}(h^*\mathcal{O}(1))^{\vee} & \to & \Omega_{Y/S}^{\vee}\otimes_{\mathcal{O}_Y} h^*\mathcal{O}(-1) & \to & 0 \\ & & \downarrow\| & & \downarrow & & \downarrow & & \\ 0 & \to & h^*\mathcal{O}(-1) & \to & g^*(\mathcal{F}^{\vee}) & \xrightarrow{h^*r^t} & h^*(\Omega_{\mathbb{P}\mathcal{F}/S}^{\vee}\otimes_{\mathcal{O}_{\mathbb{P}\mathcal{F}}}\mathcal{O}(-1)) & \to & 0 \end{array}$$

$$\begin{array}{ccccccccc}
0 \to & h^*(\Omega_{\mathbb{P}\mathcal{F}/S} \otimes_{\mathcal{O}_{\mathbb{P}\mathcal{F}}} \mathcal{O}_{\mathbb{P}\mathcal{F}}(1)) & \xrightarrow{h^*r} & g^*\mathcal{F} & \xrightarrow{h^*q} & h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1) & \to 0 \\
& \downarrow \Omega_{h/S}\otimes \mathrm{Id}_{\mathcal{O}(1)} & & \downarrow \mathcal{P}^1_h\mathcal{O}(1) & & \downarrow \| & \\
0 \to & \Omega_{Y/S} \otimes_{\mathcal{O}_Y} h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1) & \xrightarrow{r_h} & \mathcal{P}^1_{Y/S}(h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)) & \xrightarrow{q_h} & h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1) & \to 0
\end{array}$$

If $\mathcal{F}$ is locally free of finite rank and g is smooth, these are all locally free $\mathcal{O}_Y$-modules of finite rank. The transpose is

$$\begin{array}{ccccccccc}
0 \to & h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(-1) & \xrightarrow{q_h^t} & \mathcal{P}^1_{Y/S}(h^*\mathcal{O}(1))^\vee & \to & \Omega^\vee_{Y/S} \otimes_{\mathcal{O}_Y} h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(-1) & \to 0 \\
& \downarrow \| & & \downarrow \mathcal{P}^1_h\mathcal{O}(1)^t & & \downarrow \Omega^t_{h/S}\otimes \mathrm{Id}_{h^*\mathcal{O}(-1)} & \\
0 \to & h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(-1) & \xrightarrow{h^*q^t} & g^*(\mathcal{F}^\vee) & \to & h^*(\Omega^\vee_{\mathbb{P}\mathcal{F}/S} \otimes_{\mathcal{O}_{\mathbb{P}\mathcal{F}}} \mathcal{O}_{\mathbb{P}\mathcal{F}}(-1)) & \to 0
\end{array}$$

The morphism of projective space bundles over $\mathbb{P}_S\mathcal{F}$,

$$\mathbb{P}_{\mathbb{P}\mathcal{F}}(\Omega^\vee_{\mathbb{P}\mathcal{F}/S} \otimes \mathcal{O}_{\mathbb{P}\mathcal{F}}(-1)) \longrightarrow \mathbb{P}_{\mathbb{P}\mathcal{F}}(f^*\mathcal{F}^\vee) = \mathbb{P}_S\mathcal{F} \times_S \mathbb{P}_S\mathcal{F}^\vee$$

is the universal hyperplane, i.e., the partial flag bundle $\mathrm{Flag}_S(1, n-1, \mathcal{F})$. The pullback of this bundle over Y is the "incidence scheme" of hyperplanes intersecting Y. The zero scheme of the composite,

$$\pi^*(\Omega^\vee_{Y/S} \otimes_{\mathcal{O}_Y} h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(-1)) \longrightarrow (h\circ\pi)^*(\Omega^\vee_{\mathbb{P}\mathcal{F}/S} \otimes_{\mathcal{O}_{\mathbb{P}\mathcal{F}}} \mathcal{O}_{\mathbb{P}\mathcal{F}}(-1)) \longrightarrow \pi^*\mathcal{O}_{\mathbb{P}\mathcal{F}^\vee}(+1)$$

is the scheme of "tangent hyperplanes" to Y.

If g is unramified, the composite is surjective, and the zero scheme ~~Bad~~ is a projective space subbundle of ~~rank~~ codimension equal to the rank of $\Omega^\vee_{Y/S} \otimes_{\mathcal{O}_Y} h^*\mathcal{O}_{\mathbb{P}\mathcal{F}}(-1)$, i.e., the relative dimension of Y/S. Thus, the codimension of the zero scheme in $\mathbb{P}_Y(g^*\mathcal{F}^\vee)$ equals $1 + \dim(Y/S)$. Assuming that h is quasi-compact and quasi-separated, the image in ~~$\mathbb{P}_S\mathcal{F} \times_S \mathbb{P}_S\mathcal{F}^\vee$ of~~ $\mathbb{P}_S\mathcal{F}^\vee$ of this zero scheme is constructible and every irreducible, locally closed subset has fiber dimension over S strictly less than the dimension of $\mathbb{P}_S\mathcal{F}^\vee/S$. Thus the open complement of the closure of the image is dense in every fiber of $\mathbb{P}_S\mathcal{F}^\vee/S$. This is the maximal open subscheme over which $Y \times_{\mathbb{P}\mathcal{F}} \mathrm{Flag}(1, n-1, \mathcal{F})$ is smooth.

1. Criterion for a cohomological / homological δ-functor to be universal: $((F^\bullet), (\delta^\bullet))$ is universal if $\forall i \geq 1$, F^i is <u>effaceable / eraseable</u>, i.e., $\forall$ object M, $\exists$ injection $M \xrightarrow{u} N$ s.t. $F^i(u) = 0$. An object M is <u>F-acyclic</u> if $\forall i > 0$, $F^i(M) = 0$. If every object M has an injection into an F-acyclic object, then F is effaceable, thus universal.

<u>Example</u>. Let B be a flat A-algebra. Then every projective B-module is A-flat. Thus, for every A-module M & B-module N, ${}_A\left(\mathrm{Tor}_p^B(B \otimes_A M, N)\right) \cong \mathrm{Tor}_p^A(M, {}_A N)$.

2. Left adjoint functors (e.g. f^*) preserve right exactness, colimits & projective objects. Right adjoints (e.g. f_*) preserve left exactness, limits (= inverse limits) and injective objects.

<u>Applications</u>. (1) R-mod has enough injective objects. ($\mathrm{Hom}_{Ab}(R, I)$) (2) $\mathcal{O}_X$-mod has enough injectives.

3. Injective objects of $\mathcal{O}_X$-mod are flasque.

Given $V \subset U$ with $i_V : V \to X$, $i_U : U \to X$, since $i_{V!}\mathcal{O}_V \to i_{U!}\mathcal{O}_U$ is injective, $\mathrm{Hom}(i_{U!}\mathcal{O}_U, \mathcal{F}) \longrightarrow \mathrm{Hom}(i_{V!}\mathcal{O}_V, \mathcal{F})$ is surjective.

$\qquad\qquad \mathcal{F}(U) \qquad\qquad\qquad\qquad \mathcal{F}(V)$

Prop. 2.5. Flasque sheaves are $H^p(X,-)$-acyclic.

Proof: Let $\mathcal{F}$ be flasque. Let $\mathcal{F} \to \mathcal{I}$ be a monomorphism to an injective $\mathcal{O}_X$-module.

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

By earlier argument, $\mathcal{G}$ is flasque & $\mathcal{I}(X) \to \mathcal{G}(X)$ is surjective. Since $H^1(X, \mathcal{I}) = 0$, then $H^1(X, \mathcal{F}) = 0$. But then, since $\mathcal{G}$ is flasque, $H^1(X, \mathcal{G}) = (0)$ ... $\square$

Consequence = Prop. 2.6. $H^p(X,-) : \mathcal{O}_X\text{-mod} \to \Gamma(X, \mathcal{O}_X)\text{-mod} \to \mathbb{Z}\text{-mod}$ agrees with $\mathcal{O}_X\text{-mod} \longrightarrow \underline{Ab}_X \xrightarrow{H^p(X,-)} \mathbb{Z}\text{-mod}$.

Vanishing thm. of Groth. X Noeth of dim $\leq n$ $\Rightarrow X$ has cd $\leq n$.

Lemma 2.8 X Noeth $\Rightarrow$ colimit of flasques is flasque

Prop. 2.9. X Noeth $\Rightarrow H^p(X, \varinjlim \mathcal{F}_\alpha) = \varinjlim H^p(X, \mathcal{F}_\alpha)$.

Lemma 2.10. In general, for a closed subset inclusion $Y \xrightarrow{i} X$, $H^p(X, i_* \mathcal{F}) = H^p(Y, \mathcal{F})$.

Reason. i_* is exact & sends flasques to flasques.

Pf of thm 1. Reduction 1. Suffices to prove the result when X is irreducible.

Base case. $\dim X = 0$, known.

Reduction 2. Every sheaf is a filtering colimit of images of $\bigoplus_{U_\alpha} i_! \mathbb{Z}_{U_\alpha} \to \mathcal{F}$. So suffices to prove result for images.

Reduction 3. Using l.e.s., reduce to $\mathcal{F} =$ q.t. of $i_{u!} \mathbb{Z}_u$. $0 \to \mathcal{R} \to i_! \mathbb{Z}_u \to \mathcal{F} \to 0$.

Reduction 4. There exists $\begin{cases} i_! \mathbb{Z}_V \\ 0 \end{cases} \to \mathcal{R}$ inj. whose cokernel has support strictly $\subsetneq X$. Induction reduces to the case $\mathcal{F} = i_{u!} \mathbb{Z}_u$.

Final step. $0 \to i_{u!} \mathbb{Z}_u \to \mathbb{Z}_X \to j_{Y*} \mathbb{Z}_Y \to 0$

Since $\dim Y < \dim X$, $H^p(X, j_{Y*} \mathbb{Z}_Y) = 0$ for $p \geq n \Rightarrow H^{p+1}(X, i_{u!} \mathbb{Z}_u) = H^{p+1}(X, \mathbb{Z}_X)$. Since X is irred., $\mathbb{Z}_X$ is flasque. So $H^q(X, i_{u!} \mathbb{Z}_u) = 0$ for $q > n$. $\square$.

Cohom on an affine scheme. By Prop. 5.6, if X is affine & $\mathcal{F}$ is q-coh, then $H^1(X, \mathcal{F}) = (0)$. What about higher cohomology? There is a boot-strap method/induction argument using Čech cohomology. But it uses spectral sequences, which we will try to avoid. So we will stick to the Noetherian case and give a more direct argument.

Goal. Prove injective objects in the category of q-coh. sheaves are flasque, if X is Noetherian.

Reason. Then the same sort of argument as before proves all $H^i(X, \mathcal{F}) = 0$.

Before proving this:

Thm 3.7. (Serre's criterion for affineness) Let X be a q-cpct scheme. TFAE

(i) X is affine

(ii) $H^{i>0}(X, \mathcal{F}) = 0$ for all q-coh $\mathcal{F}$

(iii) $H^{i>0}(X, \mathcal{I}) = 0$ for all q-coh $\mathcal{I} \subset \mathcal{O}_X$.

Pf: (i) ⇒ (ii) ⇒ (iii).

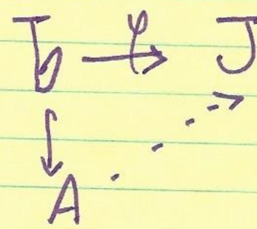
(iii) ⇒ (i), $0 \to \mathcal{I}_{Y \cup \{p\}} \to \mathcal{I}_Y \to k(p) \to 0$

⇝ $\exists f \in \mathcal{I}_Y$ not in $\mathcal{I}_{Y \cup \{p\}}$. So $p \in D(f) \subset X - Y = U$.

So $X_f = U_f$ is affine. q-cpctness. filter

Rough idea: I inj. $\Rightarrow$ I is divisible. $\Rightarrow$
$I \to I_f$ surj.

<u>Lem. 3.2</u>. I inj. $\Rightarrow$ $\Gamma_{\mathfrak{a}}(I)$ inj.



B/c Noeth, $\exists n$ $\mathfrak{a}^n \cdot \varphi(\mathfrak{b}) = 0$.

$\varphi(\mathfrak{a}^n \mathfrak{b}) = 0$.

$\mathfrak{a}^n \mathfrak{b} \subseteq \mathfrak{b} \cap \mathfrak{a}^N$ for $N \gg 0$.

$\mathfrak{b}/\mathfrak{b} \cap \mathfrak{a}^N \longrightarrow J$.