Corrected statement of Bertini from last time. There exists a constructible subset of $\mathbb{P}_S \mathcal{F}^\vee$ that is dense in every S-fiber and such that the incidence scheme is smooth over this constructible subset. If X is proper over S, can choose the constructible subset to be open.

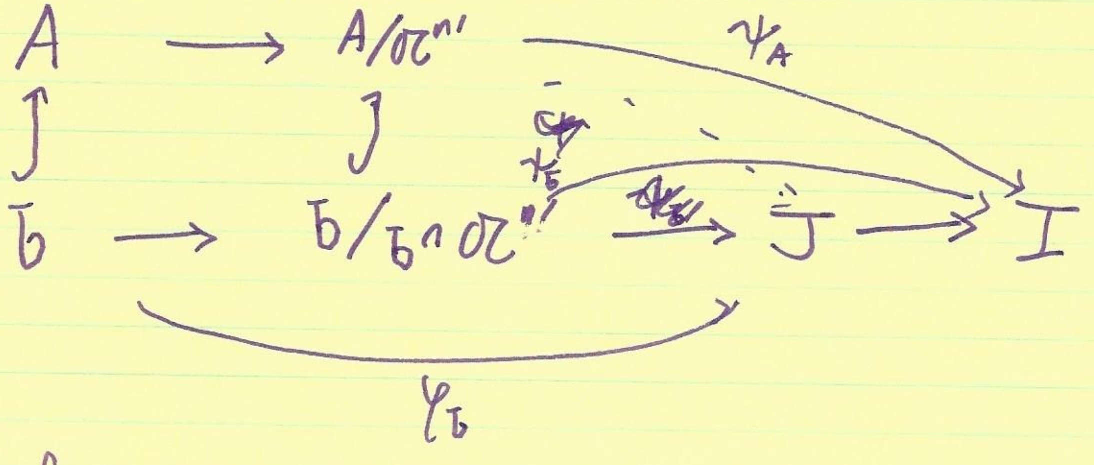
1. By Prop. 5.6, if X is affine & $\mathcal{F}$ is q-coht., $\forall$ s.e.s. of $\mathcal{O}_X$-modules $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, $\mathcal{G}(X) \to \mathcal{H}(X)$ is surj. Now take $\mathcal{G}$ to be inj (or just flasque) to get $H^0(\mathcal{G}) \to H^0(\mathcal{H}) \xrightarrow{\delta} H^1(\mathcal{F}) \to H^1(\mathcal{G}) = 0$. Since $H^0(\mathcal{G}) \to H^0(\mathcal{X})$ is surj, $H^1(\mathcal{F}) = 0$.

If we knew there were a flasque, q-coht. sheaf $\mathcal{G}$ and an injection $\mathcal{F} \to \mathcal{G}$, we could continue this argument to get $H^p(X, \mathcal{F}) = 0$ for every $p > 0$. Unfortunately, if X is a general affine scheme, this does not necessarily hold. But for Noetherian affine schemes, it does hold.

Krull's Intersection Theorem. A Noeth., $M \subset N$ f. gen'd. A-modules, then $\mathfrak{a}$-adic topology on M induced by $\mathfrak{a}$-adic on N, i.e. $\forall\, n>0$, $\exists\, n' \geq n$ s.t. $\mathfrak{a}^n M \supseteq M \cap \mathfrak{a}^{n'} N$.

Lemma 3.2: A a Noeth. ring, $\mathfrak{a}$ an ideal of A, I an inj. A-module. Then $J = \Gamma_{\mathfrak{a}}(I)$ is inj.

Pf: Given $\mathfrak{b} \xrightarrow{\varphi_{\mathfrak{b}}} J$ want to extend to $\varphi_A : A \to J$. By hypoth., $\varphi_{\mathfrak{b}}(\mathfrak{a}^n \mathfrak{b}) = 0$ for some n. So $\exists\, n'$ s.t. $\varphi_{\mathfrak{b}}(\mathfrak{b} \cap \mathfrak{a}^{n'}) = 0$. So have diagram



Define $\psi_{\mathfrak{b}} : \mathfrak{b}/\mathfrak{b}\cap\mathfrak{a}^{n'} \to I$ as above. Because I is inj. & $\mathfrak{b}/\mathfrak{b}\cap\mathfrak{a}^{n'} \hookrightarrow A/\mathfrak{a}^{n'}$ inj, $\exists\, \psi_A : A/\mathfrak{a}^{n'} \to I$. Necessarily this factors (uniquely) through J. This gives φ_A. $\square$

Lem 3.3. Let A be Noeth. & I an inj. A-module. Then $I \to I_f$ is surj.

Pf: $(0 :_A f^\infty) = (0 :_A f^r)$ for some r b/c A is Noeth. Let $\frac{m}{f^s}$ be an elt of I_f. Send $\langle f^{r+s} \rangle \to I$ by $f^{r+s} \mapsto f^r m$. This is well-defined because if $a \cdot f^{r+s} = 0$, then $af^r = 0$. Since I is inj, $\exists\, A \xrightarrow{\psi} I$ s.t. $\psi(f^{r+s}) = f^r m$ i.e. $f^{r+s} \cdot \psi(1) = f^r m$. So, in I_f, $\overline{\psi(1)} = \frac{m}{f^s}$. $\square$

Prop 3.4 Let I be an inj. A-module. If A is Noeth., then $\tilde{I}$ is flasque.

Pf. Supp $\mathcal{F}$ = complement of maximal open $\mathcal{U}$ s.t. $\mathcal{F}|_{\mathcal{U}} = (0)$. Induction on Noeth. supp $\tilde{I}$.
If supp $\tilde{I} = \{pt\}$, then $\tilde{I}$ is a skyscraper sheaf, thus flasque.

General case $X_f \subset U \subset X$, supp $\tilde{I}$ intersects U.
$Z = X - X_f$.

$$\begin{array}{ccccc} \Gamma(X, \tilde{I}) & \to & \Gamma(U, \tilde{I}) & \to & \Gamma(X_f, \tilde{I}) \\ \uparrow & & \uparrow & & \\ \Gamma_Z(X, \tilde{I}) & \to & \Gamma_Z(U, \tilde{I}) & & \\ \Gamma(X, \widetilde{\Gamma_{\mathfrak{a}}(I)}) & & \Gamma(U, \widetilde{\Gamma_{\mathfrak{a}}(I)}) & & \end{array}$$

$\Gamma_{\mathfrak{a}}(I)$ inj. & strictly smaller supp.

Thm 3.5. Let X be an ~~Noeth.~~ affine sch. & $\mathcal{F}$ a q-coh. shef. Then $\forall p>0$, $H^p(X,\mathcal{F})=0$.

⌈**Cor. 3.6.** If X is a Noeth. scheme, every q-coh. sheaf admits a monomorphism to a q-coh., flasque shef.⌋

Thm 3.7. Let X be a q-cpct, (q-sepd.[?]) scheme. TFAE

(i) X is affine

(ii) $H^{p>0}(X,\mathcal{F})=0$ for all q-coh $\mathcal{F}$

(iii) $H^{p>0}(X,\mathcal{I})=0$ for all q-coh. idel shea $\mathcal{I}\subset\mathcal{O}_X$.

Pf: (i) $\overset{\text{Thm 3.5}}{\Rightarrow}$ (ii) $\overset{\text{trivial}}{\Rightarrow}$ (iii). Assume (iii).

For every $p\in X$, $\exists\, p\in\mathcal{U}\overset{\text{affine}}{\subset}X$. Let $Y=X-\mathcal{U}$

$0\to\mathcal{I}_{Y\cup\{p\}}\to\mathcal{I}_Y\to k(p)\to 0$

So $\exists\, f\in H^0(X,\mathcal{I}_Y)$ st. $f(p)\neq 0$. Then $p\in D_X(f)\subset\mathcal{U}$, so $D_X(f)=D_{\mathcal{U}}(f)$ is affine.

Q-cptness: need only finitely many.

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus r} \xrightarrow{(f_i)} \mathcal{O}_X \to 0.$$

$$\mathcal{F} = \mathcal{F} \cap \mathcal{O}^r \supset \mathcal{F} \cap \mathcal{O}^{r-1} \supset \cdots \supset \mathcal{F} \cap \mathcal{O}_X \supset (0).$$

Then each assoc. graded is a q-coh. subsheaf of $\mathcal{O}_X^{i-1}/\mathcal{O}_X^i \cong \mathcal{O}_X$.

Čech Cohom.

$\check{C}^p(\mathfrak{U}, \mathcal{F}) = \prod_{\sigma^p} \mathcal{F}(\mathcal{U}_{\sigma^p})$ or

$\check{C}^{p\,\mathrm{red}}(\mathfrak{U}, \mathcal{F}) = \prod_{\sigma^p\,\mathrm{red}} \mathcal{F}(\mathcal{U}_{\sigma^p})$.

$(d\alpha)_{\underline{i}} = \sum_{t=0}^{p+1} (-1)^t \, \alpha_{\hat{\underline{i}}_t} \big|_{\mathcal{U}_{\underline{i}}}.$

$\check{H}^p(\mathfrak{U}, \mathcal{F}) = h^p(\check{C}(\mathfrak{U}, \mathcal{F}))$.

$\check{H}^p(X, \mathcal{F}) = \lim$

Sheafified version.

Prop. 3.4. If A is Noeth. & I is inj then $\tilde{I}$ is flasque on Spec A.

Pf. Noeth. induction on $(\mathrm{Supp}\,\tilde{I})^-$.

Base case. ~~Spec A = pt, all sheaves~~ $(\mathrm{Supp}\,\tilde{I})^- = \{pt\}$, then $\tilde{I}$ is a skyscraper sheaf, thus flasque.

Induction hypothesis. Result is known for all J s.t. $(\mathrm{Supp}\,\tilde{J})^- \subsetneq (\mathrm{Supp}\,\tilde{I})^-$.

Let $U \subset X$ be open. If $U \cap (\mathrm{Supp}\,\tilde{I})^- = \emptyset$, then $\tilde{I}(U) = 0$. ✓

Thus assume $U \cap (\mathrm{Supp}\,\tilde{I})^-$ is not $\emptyset$.

Then $\exists\, a \in A$ s.t. $D(a) \subset U \subset \mathrm{Spec}\,A$.

$$\tilde{I}(X) \to \tilde{I}(U) \to \tilde{I}(D(a)). \quad (\tilde{I}(X) \to \tilde{I}(D(a)) \text{ surj.})$$

So consider $\mathrm{Ker}(\tilde{I}(U) \to \tilde{I}(D(a))) = \mathcal{K}(U)$ where $\mathcal{K} = \mathrm{Ker}(\tilde{I} \to \tilde{I}[\tfrac{1}{a}])$, i.e. $\mathcal{K} = \tilde{J}$ for $J = (0 :_I \mathfrak{a}^{\infty})$. By above, J is inj. & $\mathrm{Supp}\,\tilde{J} \subset (X - D(a)) \cap \mathrm{Supp}\,\tilde{I}$. So $(\mathrm{Supp}\,\tilde{J})^- \subsetneq (\mathrm{Supp}\,\tilde{I})^-$. Thus, by hyp. $\tilde{J}(X) \to \tilde{J}(U)$ is surj. $\square$

This finishes the proof that for A Noeth. & $\tilde{\mathcal{F}}$ q-coht. on Spec A, $H^{p>0}(\mathrm{Spec}\,A, \tilde{\mathcal{F}}) = 0$.

Cor 3.6. Let X be any Noetherian scheme. Every injective object in the category of q-coht. sheaves is flasque.

Pf. Let I be an injective object. If there is a monomorphism $I \hookrightarrow J$, it is split. Thus, if J is flasque, also I is flasque. So it suffices to prove every q-coht. sheaf has a monomorphism to an inj., flasque sheaf. Let $\{U_i\}_{i=1,\dots,n}$ be a finite open aff. covering. There is an injection

$$\tilde{\mathcal{F}} \hookrightarrow \prod_{i=1}^{n} e_{i*} e_i^* \mathcal{F} \quad (e_i : U_i \to X).$$

For each i, there is a monomorphism $e_i^* \mathcal{F} \to \tilde{I}_i$. Since e_i is q-cpct & q-sepd, $e_{i*} \tilde{I}_i$ is q-coht. So

$$\mathcal{F} \to \prod_{i=1}^{n} e_{i*} e_i^* \mathcal{F} \to \prod_{i=1}^{n} e_{i*} \tilde{I}_i$$

is an injection in a flasque q-coh. sheaf. ∎

2. Čech cohom.

There is a category whose objects are open coverings of X, $\mathcal{U} = \{q_a : U_a \to X\}_{a \in A}$ and whose morphisms $\mathrm{Hom}(\mathcal{V}, \mathcal{U}) = \{f \in \mathrm{Hom}_{sets}(A_{\mathcal{V}}, A_{\mathcal{U}}) \mid \forall a \in A_{\mathcal{V}},\ V_a \subset U_{f(a)}\}$, i.e. refinements of coverings.

Also functorial in X: $f : Y \to X$ induces $f^{-1} : \mathrm{Cov}_X \to \mathrm{Cov}_Y$ ...