0. Collect PSets.

1. Let $(X, \mathcal{O}_X)$ be a ringed space & let $\mathcal{F}$ be an $\mathcal{O}_X$-module. There are left-exact functors

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -) : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X(X)\text{-mod}$$

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -) : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X\text{-mod}$$

The right derived functors are denoted $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, -)$, resp. $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, -)$.

<u>Observation</u>: (1) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$, thus $\mathcal{E}xt^{i>0}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = 0$.

(2) $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = \Gamma(X, \mathcal{G})$, thus $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = H^i(X, \mathcal{G})$.

(3) For every open im $j : U \to X$, $\mathrm{Ext}^i_{\mathcal{O}_X}(j_! \mathcal{O}_U, \mathcal{G}) = H^i(U, \mathcal{G}|_U)$.

<u>Lemma</u>. $j^{-1}(\text{inj})$ is injective.

<u>Pf</u>:
$$\begin{tikzcd} \mathcal{F}' \arrow[r, hook, "\alpha"] \arrow[dr, "\beta"'] & \mathcal{F} \arrow[d, dotted, "\gamma"] \\ & j^{-1}\mathcal{I} \end{tikzcd} \iff \begin{tikzcd} j_*\tilde{\mathcal{F}}' \arrow[r, "j_*\alpha"] \arrow[dr, "\tilde{\beta}"'] & j_*\mathcal{F} \arrow[d, dotted, "\tilde{\gamma}"] \\ & \tilde{\mathcal{I}} \end{tikzcd}.$$

Since $j_*\alpha$ is inj, $\exists \gamma \ldots$ $\square$

<u>Consequence</u>. $j^{-1} \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathcal{E}xt^i_{\mathcal{O}_U}(j^{-1}\mathcal{F}, j^{-1}\mathcal{G})$

~~Fix an $\mathcal{O}_X$-module $\mathcal{G}$. The collection of functors~~

~~$\mathcal{F} \to \mathrm{Ext}^i$~~

The assignment $(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a bifunctor. <u>Proof:</u> Universality. Same for $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

Let $\mathcal{E}: 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a s.e.s. of $\mathcal{O}_X$-modules. For every i there is a ~~map~~ nat. transf.

$\delta^i_{\mathcal{E}} : \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}', -) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}'', -)$ s.t.

$(\mathrm{Ext}^i_{\mathcal{O}_X}(-, \mathcal{G}), \delta^i)$ form a (contravariant) δ-functor

<u>Construction</u>. $\mathcal{G} \to \mathcal{I}^\bullet$. Because $\mathcal{I}$ is injective,

$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{I}^\bullet) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^\bullet) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{I}^\bullet) \to 0$

is exact ... $\square$.

<u>Prop. 6.5</u>. Compute $\mathcal{E}xt^i_{\mathcal{O}_X}(-, \mathcal{G})$ using locally free sheaves.

<u>Lemma</u>. Injective $\otimes$ locally free of finite rank is injective.

Proof: $\mathcal{E} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\vee, \mathcal{O}_X)$, $\quad \mathcal{E} \otimes \mathcal{I} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\vee, \mathcal{O}_X) \otimes \mathcal{I}$

$$\downarrow \wr$$

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\vee, \mathcal{I})$$

$$\mathrm{Hom}(-, \mathcal{H}om(\mathcal{E}^\vee, \mathcal{I})) = \mathrm{Hom}(- \otimes \mathcal{E}^\vee, \mathcal{I})$$

exact b/c $\mathcal{I}$ inj.

exact b/c $\mathcal{E}$ is flat

(Cons: If $\mathcal{E}$ is a flat $\mathcal{O}_X$-module, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \text{inj})$ is inj.)

Prop. 6.7. $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}))$.

Prop. 6.8. $\mathcal{F}$ locally f. presd $\Rightarrow$ $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x = \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$

Proof: $\mathcal{F}$ locally f. presd $\Rightarrow$ $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ is an isom. $(\cdot)_x$ preserves injectives & is exact $\square$.

Prop. 6.9. A Noeth. & X proj. A-scheme. $\mathcal{F}, \mathcal{G}$ coh. $\exists\, n_0 = n_0(\mathcal{F}, \mathcal{G})$ s.t. $\forall n \geq n_0$,

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)) \xrightarrow{\sim} \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))).$$

Pf: $\mathcal{F} = \mathcal{O}_X$ is that $h^{i>0}(X, \mathcal{G}(n)) = 0$ for $n \gg 0$.

So also true for $\mathcal{F}$ locally free.

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

$$0 \to Hom(\mathcal{F}, \mathcal{G}(n)) \to Hom(\mathcal{E}, \mathcal{G}(n)) \to Hom(\mathcal{F}, \mathcal{G}(n)) \to Ext^1(\mathcal{F}, \mathcal{G}(n))$$

$$0 \to \Gamma(\mathcal{H}om(\mathcal{F}, \mathcal{G}(n))) \to \Gamma(\mathcal{H}om(\mathcal{E}, \mathcal{G}(n))) \to \Gamma(\mathcal{H}om(\mathcal{F}, \mathcal{G}(n))) \to \Gamma(\mathcal{E}xt^1) \to 0$$

& $Ext^i(\mathcal{K}, \mathcal{G}(n)) \xrightarrow{\sim} Ext^{i+1}(\mathcal{F}, \mathcal{G}(n))$

(9) Exact sequence $\Rightarrow$ $Ext^1(\mathcal{F}, \mathcal{G}(n)) = \Gamma(X, \mathcal{E}xt^1(\mathcal{F}, \mathcal{G}(n)))$

for $n \gg 0$. Now use $Ext^1(\mathcal{K}, \mathcal{G}(n)) \to Ext^2(\mathcal{F}, \mathcal{G}(n))$

to get 2, etc. $\square$

Recall dualizing pair for degree r

Lemma 7.1 (c)

$Hom_k(Ext^{n-i}(\mathcal{F}, \omega_{\mathbb{P}^n}), k) \cong H^i(X, \mathcal{F})$

Lemma 7.3 $X \overset{\iota}{\subset} \mathbb{P}^n_k$ closed & codim $X \geq c \Rightarrow$

$\mathcal{E}xt^{\ell}_{\mathcal{O}_{\mathbb{P}}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}}) = 0$ for $\ell < c$.

Pf: $\Gamma(\mathbb{P}^n_k, \mathcal{E}xt^{\ell}_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n_k}(d)))$

$$= Ext^{\ell}_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n_k}(d)) = H^{n-\ell}(X, \iota_* \mathcal{O}_X(-d))$$

$$= 0. \quad \square$$

Lemma 7.4. Let ω_X° = $\mathcal{O}_X$-module s.t.

$\iota_* \omega_X^\circ = \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n_k})$ as an $\iota_* \mathcal{O}_X$-module.

Then for every q-coh't $\mathcal{O}_X$-module $\mathcal{F}$,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) \cong \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n_k}).$$

Proof: Let $\omega_{\mathbb{P}^n_k} \to \mathcal{I}^\bullet$ be an inj. resolution.

Then $\mathrm{Ext}^q_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n_k}) = h^q(\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{F}, \mathcal{I}^\bullet))$

$= h^q(\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{F}, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{O}_X, \mathcal{I}^\bullet)))$

$= h^q(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^\bullet))$, $\iota_* \mathcal{J}^\bullet = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{O}_X, \mathcal{I}^\bullet)$

$\mathcal{J}^\bullet$ are injective. By Lemma 7.3, ~~h^q~~

$h^q(\mathcal{J}^\bullet) = 0$ for $q < c$, thus $\mathcal{J} = \mathcal{J}_1^\bullet \oplus \mathcal{J}_2^\bullet$

$\mathcal{J}_1^\bullet$ in $[0, c]$ exact, $\mathcal{J}_2^\bullet$ in $[c, \infty)$.

Thus $\omega_X^\circ = \mathrm{Ker}(d^c : \mathcal{J}_2^c \to \mathcal{J}_2^{c+1})$.

~~So have $\omega_X^\circ[-c] \xrightarrow{\sim}$~~ $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) =$

$h^c(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}_2^\bullet)) = \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n_k})$. $\square$

Prop. 7.5. Every proj. k-scheme has a dualizing pair

Pf: $\iota : X \to \mathbb{P}^n_k$. $c = \text{codim}$.

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) = \operatorname{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n_k})$$
$$= H^{n-c}(\mathbb{P}^n, \iota_* \mathcal{F})^\vee$$
$$= H^{n-c}(X, \mathcal{F})^\vee \qquad \blacksquare$$

Thm 7.6. (a) There is a map of δ-functors

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X^\circ) \longrightarrow H^{\dim X - i}(X, \mathcal{F})^\vee$$

(b) TFAE

(i) X is equidim'l & CM

(ii) $\forall\ \mathcal{F}$ locally free on X, $H^{i<d}(X, \mathcal{F}(-q)) = 0$ for $q \gg 0$

(iii) θ^i are isos $\forall\ i \geq 0$ & $\mathcal{F}$ coh.

Pf: (a) Universality

(b) (i) $\Rightarrow$ (ii). depth $E = d$. Since $\mathcal{O}_{\mathbb{P}^n, p}$ is reg of dim n,

$\operatorname{pd}(E_p) = n-d$. $\Longleftrightarrow \operatorname{Ext}^q_{\mathcal{O}_{\mathbb{P}^n,p}}(E_p, -) = 0$ for $q > n-d$.

Thus $\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(E, \omega_{\mathbb{P}^n}(\ell)) = 0$ for $q > n-d$

$\Rightarrow \operatorname{Ext}^q_{\mathcal{O}_{\mathbb{P}^n}}(E, \omega_{\mathbb{P}^n}(\ell)) = 0$ for $q > n-d$ & $\ell \gg 0$.

$$H^{n-q}(\mathbb{P}^n, \iota_* E(-\ell)) = H^{n-q}(X, E(-\ell)) = 0$$

for $n-q < d$, $\ell \gg 0$. ✓

(ii) $\Rightarrow$ (i) ~~Let~~ As above, for $\ell \gg 0$,

$\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(E, \omega_{\mathbb{P}^n}(\ell)) = 0 \Rightarrow \operatorname{Ext}^q_{\mathcal{O}_{\mathbb{P}^n,p}}(E_p, \mathcal{O}_{\mathbb{P}^n,p}) = 0$.

$\Rightarrow \operatorname{pd}(E_p) \leq n-d$. Since $\mathcal{O}_{\mathbb{P}^n,p}$ regular,

$\operatorname{depth}(E_p) \geq d$. Since $\dim E_p = d$, E_p is Cohen-Macaulay.

Since $\iota_* \mathcal{O}_{X,p}$ is a factor, it is also Cohen-Ma[caulay]

(ii) $\Rightarrow$ (iii). (ii) $\Rightarrow H^{d-i}(X, \mathcal{F})^\vee$ is coeff.

(iii) $\Rightarrow$ (ii) $H^i(X, \mathcal{F}(-\ell))^\vee = \operatorname{Ext}^{d-i}(\mathcal{F}, \omega_X^\circ(\ell))$

$$= H^{d-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ(\ell)) = 0$$

for $d-i > 0$ & $\ell \gg 0$.

$\square$

0. Recall (again) defn. of dualizing pair: (ω_X°, t) representing the functor $\mathrm{Coh}_X \to k\text{-Vector Spaces}$ by $\mathcal{F} \mapsto H^{\dim(X)}(X, \mathcal{F})^\vee$.
§(*)

(cont'd) ✓

Recall stronger duality thm for $\mathbb{P}^n$: There is an isomorphism of δ-functors $\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_{\mathbb{P}^n}) \to H^{n-i}(\mathbb{P}^n, \mathcal{F})^\vee$.

1. Let $\iota : X \to \mathbb{P}^n_k$ be a closed immersion with $\dim \mathbb{P}^n - \dim X = c$.

<u>Lemma 7.3</u>. For every $i < c$, $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n}) = 0$.

<u>Proof</u>. For all $d \gg 0$, $\Gamma(\mathbb{P}^n_k, \mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n})(d))$

$$= \Gamma(\mathbb{P}^n_k, \mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X(-d), \omega_{\mathbb{P}^n})) = \mathrm{Ext}^q_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X(-d), \omega_{\mathbb{P}^n})$$

$$= H^{n-q}(\mathbb{P}^n, \iota_* \mathcal{O}_X(-d))^\vee = H^{n-q}(X, \iota^* \mathcal{O}_{\mathbb{P}^n}(-d))^\vee = 0$$

if $n - q < \dim X$, i.e., if $q < c$. $\square$.

Let ω_X° denote the $\mathcal{O}_X$-module (unique up to isom.) s.t. $\iota_* \omega_X^\circ = \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n_k})$.

~~Lemma 7.4.~~ $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) \to \cancel{\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \iota_* \omega_X^\circ)}$

$\cancel{= \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n_k}))}$
(Relative duality).

<u>Lemma 7.4</u>. There is an element $t_\iota \in \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \omega_X^\circ, \omega_{\mathbb{P}^n})$ such that the induced natural transformation

$$\theta_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) \to \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n})$$

is an isomorphism.

__Proof:__ Let $\omega_{\mathbb{P}^n_k} \to \mathcal{I}^\bullet$ be an inj. resolution. Define $\mathcal{J}^\bullet$ on X by $\iota_* \mathcal{J}^\bullet = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X, \mathcal{I}^\bullet)$. Because $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n}) = 0$ for $i < c$, $\mathcal{J}^\bullet$ is exact for $i < c$. Thus $\mathcal{J}^\bullet = \mathcal{J}_1^\bullet \oplus \mathcal{J}_2^\bullet$ where $\mathcal{J}_1^\bullet \in [0, c]$ is exact, $\mathcal{J}_2^\bullet \in [c, \infty]$. And $\omega_X^\circ \cong \mathrm{Ker}(d^c : \mathcal{J}_2^c \to \mathcal{J}_2^{c+1})$. This isomorphism __is__ ~~an~~ element of

$$h^c(\mathrm{Hom}_{\mathcal{O}_X}(\omega_X^\circ, \mathcal{J}^\circ)) = h^c(\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \omega_X^\circ, \mathcal{I}^\bullet))$$
$$= \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \omega_X^\circ, \omega_{\mathbb{P}^n}).$$

Call it t_ι. Then for every ~~coherent sheaf~~ $\mathcal{O}_X$-module $\mathcal{F}$ ~~on~~

$$\mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n}) = h^c(\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \mathcal{I}^\bullet))$$

$$= h^c(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^\bullet)) = h^c(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}_2^\bullet))$$

$$= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathrm{Ker}(d^c)) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ). \qquad \square$$

~~Prop. 7.4~~ $t_\iota \in \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \omega_X^\circ, \omega_{\mathbb{P}^n}) \overset{t_{\mathbb{P}^n}}{=} H^{n-c}(\mathbb{P}^n, \iota_* \omega_X^\circ)^\vee$
$= H^{\dim(X)}(X, \omega_X^\circ)^\vee$. Call this element t_X.

__Prop. 7.5__. The pair (ω_X°, t_X) is a dualizing pair for X.

__Proof__. $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ) \xrightarrow{t_\iota} \mathrm{Ext}^c_{\mathcal{O}_{\mathbb{P}^n}}(\iota_* \mathcal{F}, \omega_{\mathbb{P}^n}) \xrightarrow{t_{\mathbb{P}^n}} H^{n-c}(\mathbb{P}^n, \iota_* \mathcal{F})^\vee$

$\cong H^{n-c}(X, \mathcal{F})^\vee$. $\square$

(ii) Recall a dualizing pair gives a ~~[illegible]~~ natural transf. of δ-functors $\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{\dim(X)-i}(X, \mathcal{F})^\vee$. The point is that, by Prop III.6.9, $\mathrm{Ext}^i(-, \omega_X^\circ)$ is coeffaceable for $i > 0$. Thus $\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ)$ is a universal δ-functor.

<u>Theorem 7.6</u> [Serre duality for proj. schemes]

Let X be a proj. scheme of dim d & let (ω_X°, t) be a dualizing pair. TFAE

(i) X is equidim. & CM

(ii) $\forall$ locally free $\mathcal{F}$, $\forall i < d$ & $l \gg 0$, $H^i(X, \mathcal{F}(-l)) = 0$

(iii) θ is an isom. of δ-functors.

<u>Proof</u>: (i) $\Leftrightarrow$ (ii) and then (ii) $\Leftrightarrow$ (iii).

depth $\mathcal{F}_p = d$. Since $\mathcal{O}_{\mathbb{P}^n, p}$ is reg. of dim n,

proj. dim $(\mathcal{F}_p) = n-d \iff \mathrm{Ext}^q_{\mathcal{O}_{\mathbb{P}, p}}(\mathcal{F}_p, -) = 0$ for $q > n-d$.

Thus $\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(E, \omega_{\mathbb{P}^n}(l)) = 0$ for $q > n-d$

II.6.7.

$\Rightarrow \operatorname{Ext}^q_{\mathcal{O}_{\mathbb{P}^n}}(E, \omega_{\mathbb{P}^n}(l)) = 0$ for $q > n-d$ & $l \gg 0$.

|| Duality for $\mathbb{P}^n$

$H^{n-q}(\mathbb{P}^n, \iota_* E(-l)) = H^{n-q}(X, E \otimes \iota^* \mathcal{O}(-l))$ ✓

(ii) $\Rightarrow$ (i) $\Gamma(\mathbb{P}^n, \mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(\iota_*\mathcal{O}_X, \omega_{\mathbb{P}^n})(l)) = \cdots = H^{n-q}(X, \iota^*\mathcal{O}(-l))^\vee = 0$

for $l \gg 0 \Rightarrow \mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(\iota_*\mathcal{O}_X, \omega_{\mathbb{P}^n}) = 0$ for $q > n-d$

II.6.8

$\Rightarrow \operatorname{Ext}^q_{\mathcal{O}_{\mathbb{P}^n,p}}(\mathcal{O}_{X,p}, \mathcal{O}_{\mathbb{P}^n,p}) = 0$ for $q > n-d$

$\Leftrightarrow$ proj-dim $(\mathcal{O}_{X,p}) \leq n-d$. $\Rightarrow$ depth $\mathcal{O}_{X,p} \geq d$.

Since depth $\leq$ dim $\leq d$, depth $\mathcal{O}_{X,p} = \dim \mathcal{O}_{X,p}$, i.e., X is Cohen-Macaulay of dim d at every $p \in X$. ✓

(ii) $\Rightarrow$ (iii). Suffices to prove $H^{d-i}(X, \mathcal{F})^\vee$ is coeff. for $i > 0$. $\exists$ surj. $E \twoheadrightarrow \mathcal{F}$ given $E = \iota^*\mathcal{O}(-l)^{\oplus m}$. By (ii), $H^{d-i}(X, \mathcal{O}(-l)) = 0$ for $l \gg 0$, $i > 0$. So $H^{d-i}(X, \mathcal{F})^\vee$ is coeff for $i > 0$.

(iii) $\Rightarrow$ (ii). $H^{d-i}(X, E(l))^\vee \cong \mathrm{Ext}^i(E(-l), \omega_X^\circ)$

$\cong$ ~~H^{d-i}~~ $H^i(X, E^\vee \otimes \omega_X^\circ(l))$. B/c $\mathcal{O}(l)$ is ample & $\mathcal{F} = E^\vee \otimes \omega_X^\circ$ is coh't, $H^i(X, E^\vee \otimes \omega_X^\circ(l)) = 0$ for $i > 0$ & $l >> 0$. $\square$.

<u>Rmk.</u> X <u>Gorenstein</u> $\Rightarrow \mathrm{Ext}^r_{\mathcal{O}_{\mathbb{P},p}}(\mathcal{O}_{X,p}, \mathcal{O}_{\mathbb{P},p})$ is a free $\mathcal{O}_{X,p}$-module of rank 1. $\Rightarrow \omega_X^\circ$ is an <u>invertible sheaf</u>. ~~Holds for~~

Regular $\Rightarrow$ Local complete int. $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Ma

<u>Cor. 7.9</u>. X integral, normal proj. variety of dimn ≥ 2 / $k = \bar{k}$. $Y \subset X$ the support of an ample divisor. Then Y is <u>connected</u>.

<u>Proof</u>. Y_q = closed subsch. corresp. to qD.

$0 \to \mathcal{O}_X(-q) \to \mathcal{O}_X \to \mathcal{O}_{Y_q} \to 0$.

Reduce to proving $h^1(X, \mathcal{O}_X(-q)) = 0$. $= \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(-q))$

$H^1(X, \mathcal{O}_X(-q)) = H^1(\mathbb{P}^n, i_*\mathcal{O}_X(q)) =$

$\mathrm{Ext}^{n-1}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}(q)) = \Gamma(\mathbb{P}, \mathcal{E}xt^{n-1}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})(q))$

for $q >> 0$. Since depth $i_*\mathcal{O}_X \geq 1$,

$\mathcal{E}xt^{n-1}_{\mathcal{O}_{\mathbb{P}^n}}(\iota_*\mathcal{O}_X, -) = 0$. ∎

<u>Thm 7.11</u>. If $X \xhookrightarrow{\iota} \mathbb{P}^n$ is a local complete $\cap$ of codim c, then $\omega_X^\circ \cong \iota^*\omega_{\mathbb{P}^n} \otimes \Lambda^c(\iota^*\mathcal{I}_{X/\mathbb{P}})^\vee$.

[Adjunction formula: $\omega_X^\circ \cong \iota^*\omega_{\mathbb{P}^n} \otimes \det(N_{X/\mathbb{P}^n})$.]

<u>Pf</u>. Use Koszul complex. ∎

Kähler diff. $\Omega_{X/Y} := \Delta^*\mathcal{I}_{\Delta(X)/X\times X}$.

Univ. property: $d: \mathcal{O}_X \to \Omega_{X/Y}$, $s \mapsto s\otimes 1 - 1\otimes s$ is a universal $f^{-1}\mathcal{O}_Y$-d

Fundamental exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ (1): $f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$

(2) $\iota: X \hookrightarrow Y$, $\underbrace{\iota^*\mathcal{I}_{X/Y} \to \iota^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0}$
$\searrow Z \swarrow$

$Z = \mathrm{Spec}\, k$, $Y = \mathbb{P}^n \rightsquigarrow$ If X is lci & reduced, $\boxed{\omega_X = \det(\Omega_X)}$.

Lightning RR for curves:

X a proper, reduced curve

$\chi(X,\mathcal{F}) := h^0(X,\mathcal{F}) - h^1(X,\mathcal{F})$, $p_a(X) := h^1(X,\mathcal{O}_X)$

(a) $\chi(X,\mathcal{F}) = \deg(\mathcal{F}) + \mathrm{rk}(\mathcal{F})(1-p_a(X))$.

(b) If X is Cohen-Macaulay,

$H^1(X,\mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\omega_X^\circ)$. In part., if $\mathcal{F}$ is locally free, $= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X^\circ)$.

$h^0(X,E) - h^0(X,E^\vee \otimes \omega_X^\circ) = \deg(E) + \mathrm{rk}(E)(1-p_a(X))$.

<u>Pf</u>: (a) Choose an isom. $\mathcal{F}_{\eta_X} \xrightarrow{\sim} \mathcal{O}_{X,\eta_X}^{\oplus r}$.

Form $\delta : \mathcal{F}_{\eta_X} \oplus \mathcal{O}_{X,\eta_X}^{\oplus r} \longrightarrow \mathcal{O}_{X,\eta_X}^{\oplus r}$ = diff.

$\Longleftrightarrow \tilde{\delta} : \mathcal{F} \oplus \mathcal{O}_X^{\oplus r} \longrightarrow \iota_* \mathcal{O}_{X,\eta_X}^{\oplus r}$.

Define $\mathcal{K} = \mathrm{Ker}(\tilde{\delta})$.

$0 \to \mathcal{K} \to \mathcal{O}_X^{\oplus r} \to \mathcal{I}_1 \to 0$

$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{I}_2 \to 0$.

Then $\chi(\mathcal{O}_X^{\oplus r}) = \chi(\mathcal{K}) + \chi(\mathcal{J}_1)$

$$= \chi(\mathcal{K}) + \deg(\mathcal{J}_1) = \chi(\mathcal{F}) - \deg(\mathcal{J}_2) + \deg(\mathcal{J})$$

$$\chi(\mathcal{F}) = \chi(\mathcal{O}_X^{\oplus r}) + \deg(\mathcal{J}_2) - \deg(\mathcal{J}_1)$$

$$= \chi(\mathcal{O}_X^{\oplus r}) + \deg(\mathcal{F})$$

$$= r(1 - p_a(X)) + \deg(\mathcal{F}). \qquad \square$$