MAT 127: Calculus C, Spring 15
Solutions to Midterm I

Problem 1 (15pts)

Solve the following initial-value problem

$$y' = \frac{x}{y^2}\mathrm{e}^x, \qquad y(0) = 0.$$

The above ODE is separable. After writing $y' = \mathrm{d}y/\mathrm{d}x$, we can move everything involving y to the LHS and everything involving x to the RHS and then integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y^2}\mathrm{e}^x \quad\Longleftrightarrow\quad y^2\mathrm{d}y = x\mathrm{e}^x\mathrm{d}x \quad\Longleftrightarrow\quad \int y^2\mathrm{d}y = \int x\mathrm{e}^x\mathrm{d}x\,.$$

In order to integrate the RHS above, we observe that

$$\mathrm{e}^x\mathrm{d}x = \frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x}\mathrm{d}x = \mathrm{d}\mathrm{e}^x\,.$$

Using integration-by-parts, we then find that

$$\int x\mathrm{e}^x\mathrm{d}x \;= \int x\,\mathrm{d}\mathrm{e}^x = x\,\mathrm{e}^x - \int \mathrm{e}^x\mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x + C.$$

Combining with the y-integral above, we obtain

$$\frac{1}{3}y^3 = x\,\mathrm{e}^x - \mathrm{e}^x + C.$$

Plugging in the initial condition $(x, y) = (0, 0)$, we get

$$\frac{1}{3}0^3 = 0\cdot\mathrm{e}^0 - \mathrm{e}^0 + C.$$

Since $\mathrm{e}^0 = 1$, this gives $C=1$ and so

$$\frac{1}{3}y^3 = x\,\mathrm{e}^x - \mathrm{e}^x + C \qquad\Longleftrightarrow\qquad \boxed{y(x) = \big(3(x\,\mathrm{e}^x - \mathrm{e}^x + 1)\big)^{1/3}}$$

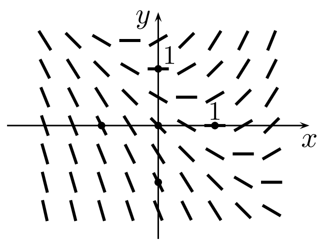
Problem 2 (15pts)

Consider the four differential equations for $y = y(x)$:

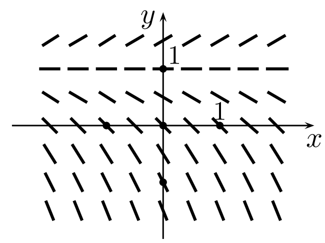
(a) $y' = y - 1$, (b) $y' = y^2 - 1$, (c) $y' = x(y^2 - 1)$, (d) $y' = x + y - 1$.

Each of the two diagrams below shows the direction field for one of these equations:

I

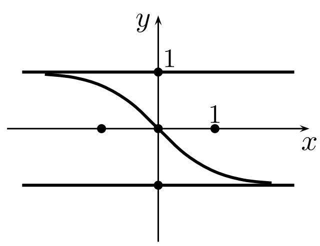


II

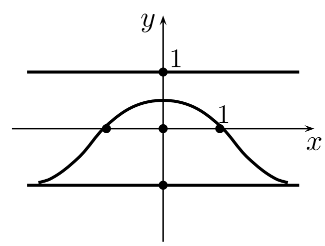


Similarly, each of the two diagrams below shows three solution curves for one of these equations:

III



IV



Match each of the diagrams to the corresponding differential equation (the match is one-to-one):

diagram	*I*	*II*	*III*	*IV*
equation	d	a	b	c

Answer Only: *no explanation is required.*

Note, however, that you will be ***penalized*** *for repeated answers. For example, if you answer (a) for both diagrams I and II, then points will be deducted.*

grading

correct − repeats	0-	1	2	3	4
points	0	3	8	14	15

- III, IV $\neq$ (a),(d) because the constant function $y = -1$ is not a solution of (a) or (d)
- III $\neq$ (c) as a solution curve in III descends when $y^2 < 1$ no matter whether x is positive or negative; IV $\neq$ (b) because the slope of a solution curve in IV (increase/decrease) depends on x, not just y
- II $\neq$ (b), (c), or (d) because the slopes in II are negative for all $y < 1$ no matter what x is
- I $\neq$ (a) or (b) because the slopes depend on x; I $\neq$ (c) as the slope at (0,0) in I is -1, not 0

Problem 3 (15pts)

A radioactive substance, Strontium-90, has a half-life of 30 days. A sample of Strontium-90 initially contains 80 mg. Let $y = y(t)$ denote the amount of this substance remaining after t days.

(a; 10pts) *Find a differential equation satisfied by $y(t)$.*

Since the decay rate is proportional to y, $y(t)$ satisfies the exponential decay equation:

$$y(t) = y(0)\mathrm{e}^{-rt} = 80 \cdot \mathrm{e}^{-rt},$$

where r is the relative decay rate. Since the half-life is 30 days,

$$y(30) = 80\,\mathrm{e}^{-r\cdot 30} = \frac{1}{2}\cdot 80 \quad\Longleftrightarrow\quad \mathrm{e}^{-30r} = \frac{1}{2} \quad\Longleftrightarrow\quad -30r = \ln\frac{1}{2} = -\ln 2 \quad\Longleftrightarrow\quad r = \frac{\ln 2}{30}.$$

Since $y' = -ry$ is the exponential decay ODE, y satisfies the differential equation

$$\boxed{y' = -\left(\frac{\ln 2}{30}\right)y}$$

Note. There are many other first-order differential equations satisfied by $y = y(t)$.

(b; 5pts) *How much Strontium-90 will remain after 60 days?*

Since every 30 days the sample declines by half, in $60 = 2 \cdot 30$ days this happens 2 times; so the amount drops by the factor of $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. Since the original amount was 80 mg, the amount after 60 days is then

$$80 \cdot \frac{1}{4} = \boxed{20 \text{ mg}}$$

Alternatively, use part (a):

$$y(60) = y(0)\mathrm{e}^{-r\cdot 60} = 80 \cdot \mathrm{e}^{-(\ln 2)60/30} = 80\left(\mathrm{e}^{\ln 2}\right)^{-2} = 80 \cdot 2^{-2} = \boxed{20 \text{ mg}}$$

Problem 4 (15pts)

Consider the autonomous differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (y+2)^2(y-1)(y-2).$$

(a; 5pts) *Find the equilibrium (or constant) solutions.*

If the constant function $y(x) = C$ solves the above ODE, then

$$\begin{aligned} 0 = y'(x) &= \big(y(x)+2\big)^2\big(y(x)-1\big)\big(y(x)-2\big) \\ &= \big(C+2\big)^2\big(C-1\big)\big(C-2\big). \end{aligned}$$

Thus, $C = -2, 1, 2$ and the constant solutions are

$$\boxed{y = -2, \qquad y = 1, \qquad y = 2}$$

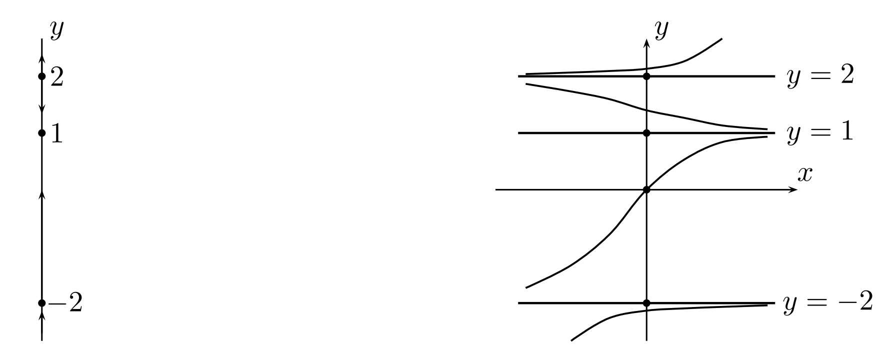
(b; 10pts) *If* $y(0) = 0$*, compute the following limit* $\lim_{x\to\infty} y(x)$.

The graphs of the equilibrium solutions are the horizontal lines $y = -2$, $y = 1$, and $y = 2$, shown in the right diagram below. These lines partition the xy-plane into horizontal bands $y_1^* < y < y_2^*$. Since solution curves of the above ODE do not intersect, no solution curve can cross the graphs of the equilibrium solutions. In particular, if $y = y(x)$ is a solution of this ODE with $y(0) = 0$, then $y(x) \in (-2, 1)$ for all x. In each band, the RHS of the ODE does not change sign. Thus, in each single band, all solution curves of the ODE either descend or ascend. Furthermore, each solution curve approaches either an equilibrium solution curve or $\pm\infty$ as x increases or decreases. In the band $y \in (-2, 1)$,

$$y' = (y+2)^2(y-1)(y-2) > 0$$

and so the solution curves in this band rise toward $y(x) = 1$. In particular, the solution curve with $y(0) = 0$ satisfies

$$\boxed{\lim_{x\to\infty} y(x) = 1}$$



The phase line on the left side of the above diagram shows the equilibrium points for the ODE; these are also the y-intercepts of the graph of the RHS of the ODE as a function of y. The phase line also indicates, using arrows, whether the solution curves in each band cut out by the horizontal equilibrium-solution lines ascend or descend. The arrow corresponding to a segment of the phase line points up (down) if the RHS of the ODE is positive (negative) on this segment.

Problem 5 (20pts)

Alice and Bob each have an investment of $1000 initially in accounts that compound interest continuously. Alice's investment doubles after $20\ln 2$ *days, and Bob's investment triples after* $10\ln 3$ *days. Which account has more money after one year?*

Denote by $A(t)$ the amount of money in Alice's account after t days and by $B(t)$ the amount of money in Bob's account after t days. Let a be the interest rate paid on Alice's account and b be the interest rate paid on Bob's account. Both satisfy the exponential growth equation and so

$$A(t) = A(0)\mathrm{e}^{at} = 1000\mathrm{e}^{at} \qquad \text{and} \qquad B(t) = B(0)\mathrm{e}^{bt} = 1000\mathrm{e}^{bt}\,.$$

By assumption, $A(20\ln 2)=2000$ and $B(10\ln 3)=3000$. Thus,

$$\begin{aligned}
2000 = 1000\mathrm{e}^{(20\ln 2)a} = 1000\big(\mathrm{e}^{\ln 2}\big)^{20a} = 1000\cdot 2^{20a} \quad &\Longrightarrow \quad 2^{20a} = 2 \quad \Longrightarrow \quad \ln(2^{20a}) = \ln 2\\
&\Longrightarrow \quad 20a\,\ln 2 = \ln 2 \quad \Longrightarrow \quad a = \frac{1}{20}\\
3000 = 1000\mathrm{e}^{(10\ln 3)b} = 1000\big(\mathrm{e}^{\ln 3}\big)^{10b} = 1000\cdot 3^{10b} \quad &\Longrightarrow \quad 3^{10b} = 3 \quad \Longrightarrow \quad \ln(3^{10b}) = \ln 3\\
&\Longrightarrow \quad 30b\,\ln 3 = \ln 3 \quad \Longrightarrow \quad b = \frac{1}{10}\,.
\end{aligned}$$

From this, we conclude that the daily interest rate $b=10\%$ paid on Bob's account is higher than the daily interest rate $a=5\%$ paid on Alice's account and so

$$\boxed{\text{Bob's account}}$$

will have more money after 1 year (as well as after any other amount of time).

Alternative solution. We first note that $20\ln 2 > 10\ln 3$. This inequality is equivalent to

$$2\ln 2 > \ln 3 \qquad \Longleftrightarrow \qquad \ln(2^2) > \ln 3 \qquad \Longleftrightarrow \qquad 4 > 3.$$

Thus, Alice's account takes longer to double ($20\ln 2$ days) than Bob's to triple ($10\ln 3$ days). From this, we conclude that the interest rate paid on Bob's account is higher than the interest rate paid on Alice's account and so

$$\boxed{\text{Bob's account}}$$

will have more money after 1 year (as well as after any other amount of time).

Problem 6 (20pts)

A tank contains 10 kg of salt dissolved in 2000 L of water. Brine that contains 0.04 kg of salt per liter of water enters that tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

(a; 10pts) *If $y(t)$ denotes the amount of salt in the tank at time t (measured in minutes), find a differential equation satisfied by y.*

Hint: The total rate of change of the amount of salt in the tank is equal to the rate of salt going in minus the rate of salt going out.

By the hint, $y'(t) = y'_{\text{in}}(t) - y'_{\text{out}}(t)$, where

$$y'_{\text{in}}(t) = (\text{flow rate of salt})_{\text{in}} = (\text{flow rate of solution})_{\text{in}} \cdot (\text{salt concentration})_{\text{in}} = 25 \cdot .04 = 1;$$
$$y'_{\text{out}}(t) = (\text{flow rate of salt})_{\text{out}} = (\text{flow rate of solution})_{\text{out}} \cdot (\text{salt concentration})_{\text{out}}\,.$$

Since the salt in the tank is thoroughly mixed, the outgoing salt concentration is the same as the salt concentration in the tank:

$$(\text{salt concentration})_{\text{out}} = \frac{\text{amount salt in tank}}{\text{volume in tank}} = \frac{y(t)}{2000}\,,$$

since the volume of solution in the tank is kept constant at 2000 gallons. So,

$$y'_{\text{out}}(t) = 25 \cdot \frac{y(t)}{2000} = \frac{y(t)}{80}\,.$$

It follows that $y(t)$ is a solution to the differential equation

$$\boxed{y' = 1 - \frac{y}{80}}$$

(b; 10pts) *How much salt remains in the tank after one hour?*

The function $y = y(t)$ solves the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{80 - y}{80}, \qquad y(0) = 10.$$

First find the general solution to the ODE. Since it is separable, we move everything involving y to the LHS and everything involving t to the RHS and then integrate,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{80-y}{80} \iff \frac{\mathrm{d}y}{80-y} = \frac{\mathrm{d}t}{80} \iff \int \frac{\mathrm{d}y}{80-y} = \int \frac{\mathrm{d}t}{80} \iff -\ln|80-y| = \frac{t}{80} + C,$$
$$\iff \ln|80-y| = -\frac{t}{80} - C \iff \mathrm{e}^{\ln|80-y|} = \mathrm{e}^{-t/80-C} = \mathrm{e}^{-C}\mathrm{e}^{-t/80}$$
$$\iff |80-y| = A\mathrm{e}^{-t/80} \iff 80 - y = \pm A\mathrm{e}^{-t/80} \iff y(t) = 80 - C\mathrm{e}^{-t/80}\,.$$

Plugging in the initial condition $y(0) = (0, 10)$, we obtain $10 = 80 - C\mathrm{e}^{-0/80} = 80 - C$ or $C = 70$. So

$$y(60) = 80 - 70\mathrm{e}^{-60/80} = \boxed{80 - 70\mathrm{e}^{-3/4}\ \text{ kg}}$$