Name: SOLUTIONS Id: ______

1. Find the solution $y(x)$ of the initial value problems

(a) (10 points)

$$\frac{dy}{dx} = xy$$
$$y(0) = 1$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

EXPONENTIATE BOTH SIDES,

$$|y| = e^{\frac{1}{2}x^2 + C}$$

$$y = Ae^{x^2/2}$$

SINCE $y(0) = 1$, $A = 1$

$$\boxed{y(x) = e^{x^2/2}}$$

Answer

(b) (10 points)

$$\frac{dy}{dx} = xe^x$$
$$y(0) = 1$$

$$\int dy = \int xe^x\,dx$$

INTEGRATE BY PARTS:

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$y = xe^x - \int e^x dx = xe^x - e^x + C$$

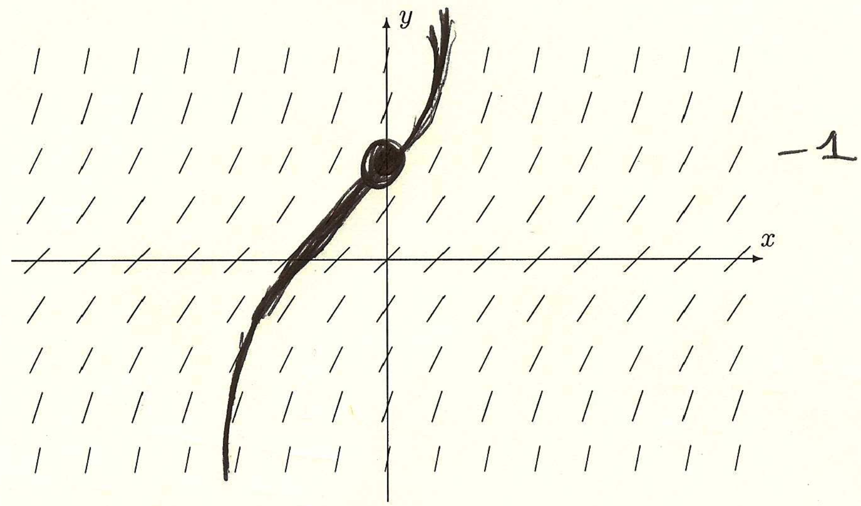
$y(0) = 1 = 0 - 1 + C$, SO $C = 2$.

$$\boxed{y(x) = xe^x - e^x + 2}$$

Answer

2. The diagram below depicts the *slope field* corresponding to a first order differential equation

$$\frac{dy}{dx} = f(x, y).$$




(a) (10 points) Sketch on the diagram above the solution of the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(0) = 1.$$

(b) (10 points) Does the differential equation have any equilibrium (stationary) solutions?

NO, SINCE THERE ARE NO SOLUTIONS OF THE FORM $y(x) = C$.

(c) (10 points) The relevant differential equation is actually one of the equations listed below. Which one? Circle your answer. Give detailed reasons for your answer! (If you need more space use the back of this page.)

(ALL ARE INCR.)

**(i)** $\frac{dy}{dx} = \tan x$

**(ii)** $\frac{dy}{dx} = \arctan y$

**(iii)** $\frac{dy}{dx} = 1 + y^2$

**(iv)** $\frac{dy}{dx} = \frac{\sec^2 x}{1 + y^2}$

**(v)** $\frac{dy}{dx} = 1 - y^2$

(i) & (iv) DONT WORK SINCE SLOPES DON'T VARY BY X, ONLY BY Y.

SINCE ALL SLOPES ARE $\geq 1$, ONLY (iii) IS POSSIBLE.

SOLUTIONS ARE $y = \tan(x + C)$.

3. Bismuth-210 has a half life of 5 days. RADIOACTIVE DECAY, SO $B(t) = Ae^{kt}$

   (a) (10 points) A sample of Bismuth-210 has a mass of 1024 mg. Find a formula for the mass remaining after $t$ days.

$$B(0) = 1024 = A$$

$$512 = B(5) = 1024e^{k5}, \quad \text{so} \quad \frac{1}{2} = e^{5k}$$

$$\text{so} \quad \ln\left(\tfrac{1}{2}\right) = 5k$$

$$\text{so} \quad k = \frac{-\ln 2}{5}$$

$$\boxed{B(t) = 1024\, e^{-\frac{\ln 2}{5} t}}$$

   (b) (5 points) Find the mass remaining after 15 days.

$$B(15) = 1024\, e^{-\frac{\ln 2}{5} \cdot 15} = 1024\, e^{-3\ln 2} = 1024\, e^{\ln\left(\frac{1}{8}\right)}$$

$$\therefore B(15) = \frac{1024}{8} = 128 \text{ mg}$$

   (c) (5 points) When is the mass reduced to 1 mg?

FIND $t$ WHEN $B(t) = 1$.

$$1 = 1024\, e^{-\frac{\ln 2}{5} t}$$

$$-\ln(1024) = \ln\left(\frac{1}{1024}\right) = \frac{-\ln 2}{5} t$$

$$1024 = 2^{10}, \text{ so } \ln\left(\frac{1}{1024}\right) = -10\ln 2.$$

$$\frac{+10\ln 2}{+\ln 2} \cdot 5 = t, \quad \text{so} \quad \boxed{t = 50 \text{ DAYS}}$$

**Name:** ____________________ **Id:** ____________________

4. (15 points) A tank initially contains 1000 L of brine with 15 kg of dissolved salt. Water is drained from the tank at a rate of 10 liters per minute. Simultaneously, pure water (containing no salt) is added to the tank at a rate of 10 liters per minute. The water in the tank is kept thoroughly mixed, so the salt present is evenly distributed throughout the tank.

   How much salt is left in the tank 100 minutes later?

   **Hint**: The amount of salt in the tank is a function $y(t)$, where $t$ denotes time (measured in minutes after the start of the experiment). Estimate concentration in terms of $y(t)$ and use this to write down a differential equation that describes the amount $y(t)$ of salt in the tank.

$y(t)$ = AMT OF SALT (kg) AT TIME $t$.

SINCE $\frac{1}{100}$ OF THE MIX GOES OUT EACH MIN,

$$y'(t) = \frac{-1}{100} y(t) \underbrace{+0}_{\text{SINCE NO SALT IS ADDED.}}$$

$$\therefore \frac{dy}{y} = \frac{-1}{100} dt$$

$$\ln|y| = \frac{-t}{100} + c \Rightarrow y(t) = Ae^{-t/100}$$

SINCE $y(0) = 15$, $\boxed{y(t) = 15e^{-t/100}}$

$$y(100) = 15e^{\frac{-100}{100}} = \frac{15}{e}$$

A LITTLE MORE THAN 5 kg

$15/e$ kilograms.

*Answer*

5. The position $y(t)$ of a weight hanging on a spring is described by the second-order differential equation

$$y'' + 8y' + 15y = 0$$

(a) (8 points) What is the general form of the solution $y(t)$ to this differential equation?

IF WE PUT $y = e^{kt}$ IN THE D.E., WE GET

$$k^2 e^{kt} + 8ke^{kt} + 15e^{kt} = 0,$$

SO THE CHAR. POLY IS $k^2 + 8k + 15 = 0$

$$(k+5)(k+3) = 0$$

$$k = -5 \text{ OR } k = -3.$$

THUS, THE GENERAL SOLUTION IS

$$\boxed{y(t) = Ae^{-5t} + Be^{-3t}}$$

(b) (7 points) ~~If the initial position of the weight~~ is at $y = 0$ and its initial velocity is given by $y'(0) = -4$, what is the position when $t = 1$? You should not approximate $e$, $\pi$, square-roots, or the like.

SINCE THE INITIAL POSITION IS $y = 0$,

WE HAVE $y(0) = 0$.

SO $0 = A + B$, HENCE $B = -A$.

THEREFORE $y(t) = Ae^{-5t} - Ae^{-3t}$

TAKING THE DERIVATIVE GIVES

$$y'(t) = -5Ae^{-5t} + 3Ae^{-3t}$$

SINCE $y'(0) = -4$, WE HAVE

$$-4 = -5A + 3A$$

SO $-4 = -2A$, SO $A = 2$.

THUS $y(t) = 2e^{-5t} - 2e^{-3t}$

AND SO $\boxed{y(1) = 2e^{-5} - 2e^{-3}}$