MAT 127: Calculus C, Spring 2015
Solutions to Some HW10 Problems

Below you will find detailed solutions to three problems from HW10. Since the first two of them were WebAssign problems, your versions of these problems may have had different numerical coefficients. However, the principles behind the solutions and their structure are as described below. The solution to the last problem relates it to the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Section 8.5, Problems 3

Determine the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

We need to determine for which $x \neq 0$ the series converges. First apply the *Ratio Test* with $a_n = x^n/\sqrt{n} \neq 0$:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{|x|^{n+1}/\sqrt{n+1}}{|x|^n/\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{|x|^{n+1}}{|x|^n} = \sqrt{\frac{n}{n+1}} \cdot \frac{|x|^n \cdot |x|^1}{|x|^n} \\ &= \sqrt{\frac{n/n}{(n+1)/n}} \cdot |x| = \sqrt{\frac{1}{1+1/n}} \cdot |x| \longrightarrow \sqrt{\frac{1}{1+1/\infty}} \cdot |x| = |x|.\end{aligned}$$

Thus, the series converges if $|x| < 1$ and diverges if $|x| > 1$. So, the radius convergence is $R = 1$, and the series converges at least for $x \in (-1, 1)$. It remains to check what happens at the endpoints of this interval, i.e. whether each of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}}$$

converges or diverges. The latter series diverges by the p-Series Test with $p = 1/2 \leq 1$. The former series is alternating, since the odd terms are negative, while the even terms are positive. Furthermore, $1/n \longrightarrow 0$ as $n \longrightarrow \infty$ and $1/(n+1) < 1/n$ for all n. Thus, this series converges by the *Alternating Series Test*. So, $x = -1$ is in the interval of convergence, while $x = 1$ is not. Thus, the radius of convergence is $\boxed{1}$ and the interval of convergence is $\boxed{\text{[-1,1)}}$

Section 8.5, Problem 26

Suppose the series $\sum_{n=0}^{\infty} c_n x^4$ *converges for* $x = -4$ *and diverges for* $x = 6$*. Do the following series converge or diverge?*

$$\text{(a)}\ \sum_{n=0}^{\infty} c_n\,, \qquad \text{(b)}\ \sum_{n=0}^{\infty} c_n 8^n\,, \qquad \text{(c)}\ \sum_{n=0}^{\infty} c_n(-3)^n\,, \qquad \text{(a)}\ \sum_{n=0}^{\infty} (-1)^n c_n 9^n\,.$$

By the assumptions, the radius of convergence R is at least 4 and at most 6. So the series converges if $|x| < 4$ and diverges if $|x| > 6$. Thus, the series $\boxed{\text{(a),(c) converge}}$ while $\boxed{\text{(b),(d) diverge}}$
Note: the series (a) and (d) correspond to $x = 1$ and $x = -9$, respectively.

Problem G

(a) *Show that the series*

$$g(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z - n\pi} + \frac{1}{z + n\pi} \right)$$

converges for every $z \neq m\pi$ *for any nonzero integer* m *and that* $g(0) = 0$*.*

If $z \neq m\pi$ for any nonzero integer m, the series

$$g(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z - n\pi} + \frac{1}{z + n\pi} \right) = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2} = 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2} \tag{1}$$

converges because it looks like $\sum_{n=1}^{\infty} \frac{1}{n^2}$. By the Absolute Convergence Test, it is sufficient to check that $\sum_{n=1}^{\infty} \frac{1}{|z^2 - n^2\pi^2|}$ converges. Apply the Limit Comparison Test with $b_n = 1/n^2 > 0$:

$$\frac{a_n}{b_n} = \frac{1/|z^2 - n^2\pi^2|}{1/n^2} = \frac{n^2}{|z^2 - n^2\pi^2|} = \frac{n^2/n^2}{|z^2 - n^2\pi^2|/n^2} = \frac{1}{|z^2/n^2 - \pi^2|} \longrightarrow \frac{1}{|z^2/\infty - \pi^2|} = \frac{1}{\pi^2}\,;$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does the series (1). By definition,

$$g(0) = \sum_{n=1}^{\infty} \left(\frac{1}{0 - n\pi} + \frac{1}{0 + n\pi} \right) = \sum_{n=1}^{\infty} 0 = 0.$$

Note: It is **wrong** to split the sum given in the statement of the question into two because neither of the two resulting sums converges.

(b) *The function*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z - n\pi} + \frac{1}{z + n\pi} \right)$$

is thus well-defined for every $z \neq m\pi$ for any integer m. Show that

$$\lim_{z \longrightarrow 0} z f(z) = 1, \qquad f(-z) = -f(z), \qquad f(z+\pi) = f(z), \qquad f(\pi/2) = 0, \tag{2}$$

with the middle identities holding whenever either side is defined ($z \neq m\pi$ for any integer m).

Since the series (1) converges for all z close to 0 and depends continuously on z,

$$\lim_{z \longrightarrow 0} z f(z) = \lim_{z \longrightarrow 0} \big(1 + z g(z)\big) = 1 + 0 \cdot g(0) = 1.$$

By (1),

$$f(-z) = \frac{1}{-z} + 2(-z) \sum_{n=1}^{\infty} \frac{1}{(-z)^2 - n^2\pi^2} = -\left(\frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2} \right) = -f(z).$$

For the third identity, look at partial sums:

$$\begin{aligned}
s_n(z+\pi) &= \frac{1}{z+\pi} + \sum_{k=1}^{k=n} \left(\frac{1}{z+\pi - k\pi} + \frac{1}{z+\pi + k\pi} \right) \\
&= \frac{1}{z+\pi} + \sum_{k=1}^{k=n} \frac{1}{z - (k-1)\pi} + \sum_{k=1}^{k=n} \frac{1}{z + (k+1)\pi} \\
&= \frac{1}{z+\pi} + \sum_{k=0}^{k=n-1} \frac{1}{z - k\pi} + \sum_{k=2}^{k=n+1} \frac{1}{z + k\pi} = \sum_{k=-n+1}^{k=n+1} \frac{1}{z + k\pi} \\
&= \sum_{k=-n}^{k=n} \frac{1}{z + k\pi} + \frac{1}{z + (n+1)\pi} - \frac{1}{z - n\pi} = s_n(z) + \frac{1}{z + (n+1)\pi} - \frac{1}{z - n\pi}\,.
\end{aligned} \tag{3}$$

Thus, taking the limit as $n \longrightarrow 0$,

$$f(z+\pi) = \lim_{n \longrightarrow \infty} s_n(z+\pi) = \lim_{n \longrightarrow \infty} s_n(z) = f(z).$$

The fourth identity follows from the second and third:

$$f(\pi/2) = f(\pi/2 - \pi) = f(-\pi/2) = -f(\pi/2) \qquad \Longrightarrow \qquad f(\pi/2) = 0.$$

Note: there is no issue with splitting the sum in (3) because it is a *finite* sum; no matter in what order you add up *finitely many* terms, their sum will be the same (it *may* change in an infinite sum).

(c) *What is the "simplest" function that satisfies all identities in (2)? (answer only)*

$$\boxed{\cot z = \frac{1}{\tan z} = \frac{\cos z}{\sin z}}$$

In fact,

$$\frac{\cos z}{\sin z} = f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right). \tag{4}$$

Differentiating both sides of this identity with respect to z gives

$$\frac{(-\sin z)(\sin z) - (\cos z)(\cos z)}{\sin^2 z} = -\frac{1}{z^2} + \sum_{n=1}^{\infty}\left(-\frac{1}{(z-n\pi)^2} - \frac{1}{(z+n\pi)^2}\right)$$

$$\Longrightarrow \qquad \sum_{n=1}^{\infty}\left(\frac{1}{(z-n\pi)^2} + \frac{1}{(z+n\pi)^2}\right) = \frac{1}{\sin^2 z} - \frac{1}{z^2}$$

Taking the limit as $z \longrightarrow 0$ of both sides gives

$$\sum_{n=1}^{\infty}\left(\frac{1}{(n\pi)^2} + \frac{1}{(n\pi)^2}\right) = \lim_{z\longrightarrow 0}\left(\frac{1}{(z - \frac{z^3}{6} + \ldots)^2} - \frac{1}{z^2}\right) = \lim_{z\longrightarrow 0}\left(\frac{1}{z^2(1 - \frac{z^2}{6} + \ldots)^2} - \frac{1}{z^2}\right)$$

$$= \lim_{z\longrightarrow 0}\frac{1}{z^2}\left(\left(\sum_{n=0}^{\infty}\left(\frac{z^2}{6} + \ldots\right)^n\right)^2 - 1\right) = \lim_{z\longrightarrow 0}\frac{1}{z^2}\left(\left(1 + \frac{z^2}{3} + \ldots\right) - 1\right) = \frac{1}{3}$$

on the last line $\ldots$ denotes terms involving z^4 and higher power of z. From this, we obtain

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}}$$

From here, we can find the sum of the inverse squares of even integers,

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{24},$$

as well as the sum of the inverse squares of odd integers:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

Taking further derivatives of (4) and then their limits at 0, you can compute $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ for any integer p; this will be a rational multiple of π^{2p}.

The identity (4) is equivalent to

$$h(z) \equiv \frac{f(z)}{\cot z} \;=\; 1.$$

The reason variable z was used in this problem, instead of x, is that everything above makes sense for a *complex* variable $z = x + \mathrm{i}y$. The function $h(z)$ is holomorphic wherever it is defined (depends only on z, not $\bar{z}$, or x and y separately), and by the last three identities in (2) it is defined everywhere and periodic. One of the most important statements you'd learn in MAT 342 is that any such function *must be constant*. In light of the first identity in (2), this constant is 1. This is the reason for (4).